The problem of selecting database tuples is central to the task of resolving many forms of queries. The report defines a class of selection queries and presents a set of algorithms for their efficient resolution. The class of queries investigated is those queries which impose a quasi order on the tuples in a database.
Efficient Queries For
Quasi-Ordered Databases

Victor Jon Griswold

WUCS-91-06

May 1, 1991

Department of Computer Science
Washington University
Campus Box 1045
One Brookings Drive
St. Louis, Missouri  63130-4899

Abstract

The problem of selecting database tuples is central to the task of resolving many forms of queries. This report defines a class of selection queries and presents a set of algorithms for their efficient resolution. The class of queries investigated is those queries which impose a quasi order on the tuples in a database.

This work has been supported by NSF grant DCI-8600947, Bellcore, BNR, Italtel, NEC, DEC, SynOptics, and NTT. The author may be reached via email address vjg@wucs1.wustl.edu.
Copy to:

Andreas Bovopoulos
Neil Haller - Bellcore
Alan Kirby - DEC
Ken-ichi Yukimatsu - NTT
Guru Parulkar
Ron Schmidt - SynOptics
Jonathan Turner
Akira Arutaki - NEC America

Abstract to:

Milind Buddhikot
Jerome R. Cox, Jr.
Chuck Cranor
Zubin Dittia
Andy Fingerhut
Larry Gong
Rex Hill
Diamantis Kotoulas
Nader Mirfakhraei
James Sterbenz
Einir Valdimarsson
Ellen Witte
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>No.</th>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Background</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>General Problem</td>
<td>2</td>
</tr>
<tr>
<td>1.3</td>
<td>Definitions</td>
<td>3</td>
</tr>
<tr>
<td>1.4</td>
<td>Primary Issues</td>
<td>4</td>
</tr>
<tr>
<td>1.5</td>
<td>Orthogonal Range Queries</td>
<td>5</td>
</tr>
<tr>
<td>2.</td>
<td>Equivalence Factors</td>
<td>7</td>
</tr>
<tr>
<td>2.1</td>
<td>Requirements</td>
<td>7</td>
</tr>
<tr>
<td>2.2</td>
<td>Approach</td>
<td>8</td>
</tr>
<tr>
<td>3.</td>
<td>Total-Order Factors</td>
<td>11</td>
</tr>
<tr>
<td>3.1</td>
<td>Requirements</td>
<td>11</td>
</tr>
<tr>
<td>3.2</td>
<td>Non-Incremental Algorithm</td>
<td>13</td>
</tr>
<tr>
<td>3.3</td>
<td>Proof and Analysis</td>
<td>16</td>
</tr>
<tr>
<td>3.4</td>
<td>An Incremental Algorithm</td>
<td>18</td>
</tr>
<tr>
<td>3.5</td>
<td>An Algorithm Using Range Query Techniques</td>
<td>21</td>
</tr>
<tr>
<td>3.6</td>
<td>Queries With Both Equivalence and Total-Order Factors</td>
<td>23</td>
</tr>
<tr>
<td>4.</td>
<td>Equivalence-Factor Query Optimization</td>
<td>25</td>
</tr>
<tr>
<td>4.1</td>
<td>Requirements</td>
<td>25</td>
</tr>
<tr>
<td>4.2</td>
<td>NP-Completeness Analysis</td>
<td>27</td>
</tr>
<tr>
<td>4.3</td>
<td>General Heuristics</td>
<td>29</td>
</tr>
<tr>
<td>4.4</td>
<td>Token-Frequency Algorithm</td>
<td>30</td>
</tr>
<tr>
<td>4.5</td>
<td>Set-Overlap Algorithm</td>
<td>33</td>
</tr>
<tr>
<td>4.5.1</td>
<td>Non-Optimal Cases for QOPT_SO</td>
<td>37</td>
</tr>
<tr>
<td>5.</td>
<td>Appendices</td>
<td>41</td>
</tr>
<tr>
<td>5.1</td>
<td>Pseudocode Representation</td>
<td>43</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Control Structures</td>
<td>43</td>
</tr>
<tr>
<td>5.1.2</td>
<td>Operators</td>
<td>44</td>
</tr>
<tr>
<td>5.1.3</td>
<td>Simple and Structured Types</td>
<td>44</td>
</tr>
<tr>
<td>5.1.4</td>
<td>High-level Structured Types</td>
<td>45</td>
</tr>
<tr>
<td>5.2</td>
<td>Orthogonal Range Query Search</td>
<td>47</td>
</tr>
<tr>
<td>6.</td>
<td>Bibliography</td>
<td>49</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>No.</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equivalence-Factor Search Data Structures</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>Precedence Resulting From Three Components Each Ordered By '&lt;'</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>Query by Total-Order Factors (Non-Incremental)</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>Example of $Q_{t_NI}$ Data Structures</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>Example for Incremental Total-Order Query Algorithm</td>
<td>19</td>
</tr>
<tr>
<td>6</td>
<td>Query by Total-Order Factors (Using Range Queries)</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>Combined Structure for Two Queries</td>
<td>26</td>
</tr>
<tr>
<td>8</td>
<td>Optimization of Equivalence-Factor Search Structures</td>
<td>27</td>
</tr>
<tr>
<td>9</td>
<td>Optimized Query With Duplicate Equivalence Factors</td>
<td>27</td>
</tr>
<tr>
<td>10</td>
<td>Optimization Differences Between QOPT and Recursive HITTING SET</td>
<td>29</td>
</tr>
<tr>
<td>11</td>
<td>Examples of Basic Equivalence-Factor Query Set Reductions</td>
<td>30</td>
</tr>
<tr>
<td>12</td>
<td>Token-Frequency Query Optimization Algorithm</td>
<td>31</td>
</tr>
<tr>
<td>13</td>
<td>Example of Non-Optimal Result From Token-Frequency Algorithm</td>
<td>32</td>
</tr>
<tr>
<td>14</td>
<td>Data Structures For Set-Overlap Query Optimization Algorithm</td>
<td>34</td>
</tr>
<tr>
<td>15</td>
<td>Set-Overlap Query Optimization Algorithm</td>
<td>35</td>
</tr>
<tr>
<td>16</td>
<td>QOPT_{SO} Structure $5/4$ Optimal Size</td>
<td>38</td>
</tr>
<tr>
<td>17</td>
<td>QOPT_{SO} Structure $4/3$ Optimal Size</td>
<td>39</td>
</tr>
<tr>
<td>18</td>
<td>Orthogonal Range Query Data Structures</td>
<td>48</td>
</tr>
</tbody>
</table>
Efficient Queries For
Quasi-Ordered Databases

Victor Jon Griswold

1. Introduction

1.1 Background

The project leading to the work presented in this report involves the monitoring of
distributed systems by means of observing "events" generated by the systems being monitored.
By comparing functions of their attributes, these events are ordered in logical time and validated
against user-specified behavior constraints.[6, 7] Within the monitor, incoming events are most
naturally treated as database tuples.* The event attributes correspond directly to tuple
components.

The validation and time-ordering subsystems of the monitor must frequently query the
event database. By far the most common form of these queries picks a single event \( V \) (generally
the event just received by the monitor), finds all the previously-recorded events related to \( V \) by
some criterion (such as "find the transmit corresponding to \( V \), a receive"), and performs some
action on each of those events related to \( V \).

The goal of this report is to find algorithms which respond efficiently to this form of
query on the event database. Towards this goal, we will make use of a novel property of our

* To be precise, the term database refers not just to a set of tuples, but rather to the higher-level structure which
includes the tuple set, indices used to access those tuples, and other auxiliary information. In this report, though,
we will abstract somewhat and use database synonymously with "set of tuples" except where ambiguity may
result.
application. With most database systems, the database is relatively stable and arbitrary queries may be posed against it. The above monitor, however, knows the exact form of each query before any events are received. Tuples, the received events, are subsequently entered into the database incrementally (starting with an empty database). This prior knowledge of the queries to be posed allows significant optimization of the structures used to store the database. Given the same events, two monitor executions with different inference rules (which define which queries will be posed) will generally use different structures to store those events.

1.2 General Problem

The monitor's inference rules compare one event \( V \) at a time with those events in the database "related to" \( V \). The question now becomes "what relationships can be specified which are both useful and can be resolved efficiently?" In general, these relationships are comparisons between event attributes. Consider, for example, events associated with a communications channel. The packet identifiers for a "transmit" and a "receive" event may be compared to determine if they match. Similarly, the "sequence number" attributes of two events may be compared to decide which should have been processed first.

The first example above checks for equality between two event attributes; the second checks for a "less than" or "greater than" ordering between attributes. The relationships supported for the monitor's database queries are conjunctions (combinations via 'and') of such checks. If disjunctions ('or' combinations) are needed, one may simply pose more than one query and combine the results.**

Many, if not most, of the queries posed by the monitor are done in order to determine the temporal (time) orderings between a newly-recorded event and those events already known to the monitor. Time imposes a transitive ordering between events; if \( A \) occurs before \( B \) and \( B \) occurs before \( C \), \( A \) occurs before \( C \). Suppose the monitor has just been notified of \( C \) and is attempting to order its already-known events with \( C \). A database query could be posed which resolves to both \( A \) and \( B \). The inclusion of \( A \) in this result, however, is redundant; the fact that \( A \) occurs before \( C \) may be inferred through other information and does not need to be known explicitly. Though the inefficiency for this example is small, consider what might happen when monitoring

---

* Note that queries are interleaved with these additions.

** This involves reducing the original query \( Q \) to disjunctive normal form and posing separate queries for each of \( Q \)'s conjunctive terms.
a system which generates thousands of events. Given the format of the relationships in the
monitor's database queries, any relationship involving one or more "less than" or "greater than"
checks will be transitive in this manner and thus might produce query results with redundant
information.

For this reason, it is deemed a monitor efficiency requirement that a query \( Q(V) \) (i.e. "find
those events related to \( V \) according to relationship \( Q \)") not return events the relationship of which
may be inferred through an additional posing of \( Q \) with some other event in the result of \( Q(V) \).
Stated another way, assume that \( U \) and \( W \) are in the result of \( Q(V) \). This implies that \( U \) can not
also be in the result of \( Q(W) \), since, if it were, the inclusion of \( U \) in \( Q(V) \)'s result would be
redundant through transitivity.

1.3 Definitions

As stated above, the monitor treats events as database tuples and event attributes as tuple
components. This allows us to formally define the query problem using more traditional database
terminology. We also generalize the query format to allow comparisons of functions of tuple
components (such as "time + 5 milliseconds") in addition to just the component values themselves.
Of course, the monitor can (almost always does) pose many different queries on the database, each
complying with the format below.

\[
Q(V) = \left\{ t \in D \mid \exists u \in D \ [ t < u \land u < V ] \right\}
\]

where: \( D \) is a set of tuples, a database; \( d = |D| \)

< is an irreflexive partial order relation between tuples

Precedes is defined in terms of comparisons between functions of tuple components.
These comparisons may involve the equality or the total (linear) ordering of function values. The
complete definition of precedes is:

* An irreflexive partial order is a quasi order, meaning that \( A \prec A \) is false. Though quasi order is the proper
description of the ordering we require, few people regularly use this term. Throughout the remainder of this
paper, partial order will be used for quasi order except when ambiguity may otherwise result.
\[ t < v \equiv F_1(t) \ R_1 \ G_1(v) \land F_2(t) \ R_2 \ G_2(v) \land \ldots \land F_k(t) \ R_k \ G_k(v) \]

where: \( t, v \) are tuples; \( t \in D \)

there are a finite number \( k \) comparisons making up the relation

for each \( i, 1 \leq i \leq k: \)

Let \( C_i \) be a set whose elements can be partitioned into equivalence classes by a relation \( '=' \) and might additionally be totally (linearly) ordered by a relation \( '<' \)

\( F_i: \) tuple \( \rightarrow \) \( C_i \); \( G_i: \) tuple \( \rightarrow \) \( C_i \)

\( R_i \) is one of \( = \) or \( \neq \) over \( C_i \), or \( <, >, \leq, \geq \) if applicable**

Each 'FRG' group above is called a factor in the definition of precedes. Each \( F_i \) and \( G_i \) can be any function of the designated type. Factors which involve '\=' or '\\neq\' comparisons are called equivalence factors; those which involve '\<', '\>', '\\leq\', or '\\geq\' are called total-order factors.

For convenience, we define the relation follows, denoted by '\>', in terms of precedes. The statement \( "t < u" \) is equivalent to \( "u > t." \) Tuple components are designated by dotted notation on a tuple; for instance, \( t.c_1 \) is the first component of \( t \).

To illustrate such a precedes relation, consider a database of 2-tuples over the integers. We wish to define \( t < v \equiv (t.c_1 = v.c_1) \land (t.c_2 < v.c_2) \). There are two factors \( (k = 2) \), the first of which is an equivalence factor and the second a total-order factor. \( C_1, F_1, R_1, G_1 \) are II, \( c_1(u) \), '\='; and \( c_1(u) \), respectively, \( u \) the tuple parameter of the functions. Likewise, \( C_2, F_2, R_2, G_2 \) are II, \( c_2(u) \), '\<'; and \( c_2(u) \), respectively.

1.4 Primary Issues

Three main issues are distinguished in the resolution of this class of queries. This report devotes a section towards addressing each of these.

The first problem is the efficient evaluation of all equivalence factors of a precedes relation. Though the solution presented is relatively straight-forward, it organizes a framework for the evaluation of total-order factors and presents an enticing optimization problem.

---

* Allowing partial orders was investigated, but difficulties were found with the convergence of several search strategies when applied to "realistic" partial orders.

** We define \( \neq, >, \leq, \) and \( \geq \) according to their classical meanings in terms of \( < \) and \( = \).
The second issue is, naturally, the efficient evaluation of all total-order factors of a *precedes* relation. Each relation can include the conjunction of several total-order factors. Though a single total order is trivial to search for maximal elements (remember the query's non-redundancy requirement), the conjunction of multiple total orders results in a partial order which can be difficult to search for maximal elements.

The third issue concerns equivalence factors and multiple non-identical queries over a database. When several queries are posed, it is desirable to find information common between those queries so that some evaluation need only be performed a single time. This report addresses the use of equivalence factors common between queries to form the basis of such optimizations.

1.5 Orthogonal Range Queries

An orthogonal range query\cite{16\cite{3\cite{11}} over \(D\), a set of \(k\)-tuples, is a query defined as:

\[
\text{RQ}(D, a_1..b_1, a_2..b_2, \ldots, a_k..b_k) = \left\{ t \in D \left| a_i \leq t_i, c_i \leq b_i \right. \right\}
\]

More simply put, an orthogonal range query is a multi-dimensional (or multi-keyed) search across a database in which a range of values for each of the dimensions (components, keys) may be specified.

The class of queries \(Q\) is similar to a class of orthogonal range queries, with the exception of the "maximal elements" requirement. Thus, range query techniques frequently are either incorporated directly into the algorithms presented in this report or are used as a standard for comparison. Since they will soon be important, we take this opportunity to examine the time and space requirements of orthogonal range queries.

Let \(m = \text{the maximum number of distinct values in any tuple component}\). The search time for the best orthogonal range query algorithms is \(O((\log d)^{k-1})\), \(k \geq 2\). Insert/delete time is similar, \(O((\log d)^{k})\). Space requirements are \(O(d(\log m)^{k-1})\). As can readily be seen, both time and space considerations make these algorithms useful only for small dimensionalities (realistically, dimension 2, maybe 3, due to space considerations).

Appendix 5.2 describes how a search is performed using common range query data structures.
2. Equivalence Factors

2.1 Requirements

For a single query $Q$, equivalence factors might be handled by concatenating the factors’ $C_i$ values into a single key and using an ordinary search tree, resulting in $O(\log d)$ resolution time*. The goals of this report, however, include the efficient resolution of a set $Q$ of queries, $q = |Q|$, not all of which employ the same equivalence factors. Queries $Q_1$ and $Q_2$ in $Q$, for example, might have three equivalence factors in common, while $Q_2$ and $Q_3$ might have none in common. Furthermore, when selecting a resolution algorithm, one must keep in perspective that the algorithm must facilitate efficient maintenance of the data structures necessary for query resolution, given an incrementally constructed database $D$.

It is the multi-dimensional aspect of this problem which made algorithms for orthogonal range queries[16][3][11] appear promising. Each equivalence factor in the elements of $Q$ could be represented as a separate dimension to the range query. To resolve a particular query $Q \in Q$, each factor $i$ present in $Q$ would be searched for the single $C_i$ value significant to $Q$, while each factor in some element of $Q$ but not in $Q$ would be searched along its entire range of values (effectively "wildcarding" that factor). Though range queries are capable of far more elaborate searches, the query problem $Q$ has no obvious use for them.

Let the number of equivalence and total-order factors in a query be distinguished by defining $k_e = \text{the number of equivalence factors}$ and $k_t = \text{the number of total-order factors}$ ($k = k_e + k_t$). When discussing sets of queries, it is useful to define $K_e = \text{the number of distinct equivalence factors across all } Q \in Q$. Finally, considering only the equivalence factors of the queries in $Q$, let $m_e = \max_{1 \leq i \leq K_e} |C_i/D|$, where $C_i/D$ is that subset of $C_i$ induced by application of $F_i$ over the tuples in $D$ (i.e. the number of unique $C_i$ values actually used when evaluating queries over $D$). Given the above "brute force" range query approach, the search time for the best orthogonal range query algorithms is $(\log d)^{K_e-1}$. Insert/delete time is similar, $(\log d)^{K_e}$, and space requirements are $d(\log m_e)^{K_e-1}$.

If $Q$ needed the full power of range queries, this would be considered good. The fact, however, is that $Q$ does not. The data structures and algorithms chosen for use in query

* For brevity in the remainder of this report, all time and space complexity measurements shall be assumed to be asymptotic complexities ("$O$") unless otherwise stated.
evaluation are derived from those used for orthogonal range queries via the two-stage reduction presented below. This reduction yields search and update times which remove the exponential dependency on \( K_c \) at the cost of additional multiplicative factors. Space cost is similarly improved.

2.2 Approach

The first optimization to range queries derives from the "one value or all" nature of the searches across each factor. Range queries are designed to allow any range of values to be selected: that is the purpose of the \( \log m_c \) auxiliary tree levels for each dimension, as shown in Appendix 5.2. If one only requires single-value or all-value searches, however, just the top and bottom auxiliary tree levels are required. This corresponds to the auxiliary trees for the entire current database subset and those for the leaves of each subset's dimension's search tree.

The time and space savings for this first reduction are already appreciable. Search time and insert/delete time decrease to \( O(K_c \log d) \) and \( O(2^{K_c-1} \log d) \), respectively. Space requirements diminish to \( O(2^{K_c-1} d) \). There are still, however, factors in these measures which are exponential in \( K_c \). The second reduction removes these complexity factors at the cost of requiring multiple query search structures, perhaps one per query, instead of one structure sufficient for all queries.

The \( K_c \)-exponential complexity factors occur because the range query data structure must take into account two possibilities for each dimension. The first is that a given query will indeed search on that dimension; the second is that a given query will "ignore" that dimension. Considering the application domain of a query set \( Q \) (monitoring a distributed system), it appears quite likely that there will be a large number of equivalence factors, but that only a few factors will be present in each query. For this case, a substantial majority of dimensions will be ignored when resolving each query.

The second optimization, therefore, constructs a separate search data structure for each \( Q_j \in Q \). Each of these structures is of dimension \( k_{c,j} \), not \( K_c \), and is concerned only with searching across the factors actually present in \( Q_j \). The "all values" auxiliary trees are no longer needed, so each dimension has only a single level of auxiliary trees — the level for each distinct value of the dimension.

Figure 1 shows an example of the structure used for resolution of a query with three equivalence factors (three dimensions), the domain of each factor being values in the sets \( A, B, \)
and $J$. Tree roots are shown in boxes. At the root of the base tree, a tuple may have any value for any of the three factors. The base tree sorts by the first dimension, so it has one leaf for each value in set $A$. For each tree in the first level of auxiliary trees, the tuples in that tree share a property similar to those in standard range query data structures. In a standard structure, the first-dimension values of those tuples would all lie in a certain range. With this modified structure, that range is a single value. The first level auxiliary trees all sort by the second dimension and thus have one leaf for each value in $B$ (so long as there is a tuple with that value, of course). The third dimension is correspondingly handled by the second level of auxiliary trees.

Searching this structure is straightforward. For each query, search the base tree by the first factor. Next, search the first-level auxiliary tree just found (leaf of the base tree) by the second factor. To finish, search the appropriate second-level auxiliary tree by the third factor. The end of this search is a leaf of a second-level auxiliary tree, the tuples at which comprise the result of the query.

Since the data structure used by the first range query optimization is used for all queries, care must be taken when comparing it to the second optimization, which requires a separate structure for each query. Search time is still in terms of a single query, but insert/delete time and space requirements must be stated in a manner applying to all queries. Since not every query can be applied to each tuple $t$ added to $D$, $q(t)$ is defined as the number of queries $Q$ in $Q$ for which $t$ is in the domain of every factor of $Q$. Let us call this set of queries $Q(t)$. Generally, $q(t)$ will be much less than $q$. We also define $K_{e}(t)$ in terms of $Q(t)$. Slightly different from $K_{e}$, $K_{e}(t)$ is the maximum $k_{e_{j}}$ for any $Q_{j}$ in $Q(t)$, not the total of all $k_{e_{j}}$. 
The second optimized algorithm has a search time of $O(k_{e_j} \log(d/k_{e_j}))$ for query $Q_j$ (time just slightly lower than the obvious). Time to insert or delete a tuple $t$ is $O(q(t)K_e(t) \log(d/K_e(t)))$. Both of these occur when $|C_i|$ is the same for each dimension (i.e. all dimensions have substantial search time) and an $O(\log m_e)$ search through each dimension is forced by having single tuples rooted in all the leaves of each level's tree structure except for the leaf corresponding to the that level's search key, where all the remaining tuples are rooted for the next dimension. Space cost for the algorithm is at absolute worst $O(q d \max_{t \in D} K_e(t))$, which occurs when $|C_i| = d$ for the first factor of every query (i.e. when tuples have unique first-factor values; all auxiliary trees have only a single node, but there must still be $k_e$ auxiliary tree levels). This measure is terribly pessimistic since it assumes that $q(t) = q$ for all $t \in D$ and uses the largest $K_e(t)$, as well.

Comparing the first and second optimized query algorithms, search time for the second, $O(k_{e_j} \log(d/k_{e_j}))$, is distinctly better than the $O(K_e \log d)$ time required for the first. The $O(q(t)K_e(t) \log(d/K_e(t)))$ insert/delete time for the second algorithm becomes better than that of our first algorithm, $2^{K_e-1} \log d$, as soon as $K_e$ becomes moderately large. Storage cost for the second algorithm, $O(q d \max_{t \in D} K_e(t))$, also becomes better than the $O(2^{K_e-1} d)$ cost of the first algorithm as $K_e$ becomes moderately large.

The only problem with the second algorithm is the waste of time and space forced by entering each tuple $t$ into $q(t)$ data structures. This problem, however, can be substantially avoided by combining the search data structures for queries with equivalence factors in common into a single, though slightly more complex, data structure. Determining the optimal combination of such query structures is the topic of Section 4 in this report.
3. Total-Order Factors

3.1 Requirements

In this section, the problem of evaluating the total-order factors of a precedes relation is addressed, ignoring any equivalence factors. To simplify the discussion without loss of generality, we restrict factor definitions to functions directly on tuple components (this is how precedes relations were first introduced in the early part of Section 1). The specific restrictions necessary for this are listed below; a query meeting these restrictions is referred to as a Q_t(t) query, instead of as the general Q(t) query.

Restrictions on this section’s factor definitions:

1) all (total-order) factors use the '<' relation
2) for each k-tuple t = ⟨c₁, c₂, ..., cₖ⟩, all components cᵢ ∈ I
3) for each i, 1 ≤ i ≤ k, Cᵢ = I, Fᵢ(t) = t.cᵢ, and Gᵢ(t) = v.cᵢ
4) for all components cᵢ and all distinct t,u ∈ D, t.cᵢ ≠ u.cᵢ

With respect to restriction 1 above, the algorithms presented here can be readily modified to handle the relations '>', '<=', and '>=' as well. Explicit handling of these other relations will needlessly complicate discussion. Restrictions 2 and 3 simply make each factor an integer comparison between the values of the same component of two tuples. Restriction 4 is useful both for stating algorithms and for stating LEMMA 1 below. With algorithms, the use of distinct component values obviates the need for tedious list operations for cases of multiple tuples with the same component value. With LEMMA 1, the use of distinct component values allows a much simplified problem definition which can ignore the lengthy but non-substantive case analysis involved when components of two tuples have the same value.

Given these restrictions, an example precedes relation can now be defined as:

\[ t, v \text{ are 2-tuples } (x, y) \]
\[ t < v \equiv t.x < v.x \land t.y < v.y \]

instead of

\[ t, v \in \{ (x, y) \mid x \in I \land y \in I \} \]
\[ t < v \equiv \exists i [1 \leq i \leq 2 \land F_i(t) < G_i(v)] \land \exists j [1 \leq j \leq 2 \land F_j(t) > G_j(v)] \]
\[ F_1(t) = t.x ; G_1(v) = v.x \]
\[ F_2(t) = t.y ; G_2(v) = v.y \]
without misunderstanding. This relation is the partial order which results when \( t \prec v \) means that both components of \( t \) are less than the corresponding components of \( v \) (or, more verbosely, that no \( t.c_i \) equals \( v.c_i \), at least one \( t.c_i \) is less than \( v.c_i \), and no \( t.c_i \) is greater than \( v.c_i \)).

In fact, the result of imposing any ordering relation of this class across all the \( k \)-tuples in a database \( D \), \( k \geq 2 \), is a partial order of those tuples (\( k=1 \) results in a trivial total order). For example, suppose 3-tuples of type \( \langle x, y, z \rangle \) with ordering relation \( \prec \) defined as:

\[
t \prec v \equiv t.x < v.x \land t.y < v.y \land t.z < v.z.
\]

And suppose the database is:

\[
D = \{ \langle 2,1,5 \rangle, \langle 3,3,3 \rangle, \langle 5,5,4 \rangle, \langle 4,6,8 \rangle, \langle 6,2,2 \rangle, \langle 8,7,6 \rangle \}.
\]

Figure 2 shows a representation of \( D \) in the right-hand diagram. A data structure analogous to this is used by all the \( Q_t \) algorithms below and would exist for each query in \( Q \) (thus a given tuple \( t \) is referenced in \( q(i) \) structures). The dotted vertical lines represent the set of values for components \( x, y, \) and \( z \). Each tuple is shown as two solid (non-vertical) lines connecting its three specific component values. With this type of diagram, it is easy to visually determine which tuples precede each other, which follow, and which are incomparable or "cross" each other. The graph \( G \) in Figure 2 shows the transitive reduction of the partial order imposed on \( D \) by \( \prec \). As defined in Section 1, the resolution of the query \( Q(v) \) must include all tuples \( t \in D \) such that \( t \prec v \) and for which there does not exist a tuple \( u \in D \) such that \( t \prec u \prec v \). Simply put, \( Q(v) \) is the tails of all edges with head \( v \) in the graph \( G \). This means:

\[
\begin{align*}
Q(t_1) &= \{ \} \\
Q(t_2) &= \{ t_2 \} \\
Q(t_3) &= \{ t_2 \} \\
Q(t_4) &= \{ t_1, t_2 \} \\
Q(t_5) &= \{ \} \\
Q(t_6) &= \{ t_1, t_3, t_5 \}
\end{align*}
\]

Note that \( Q(t_6) \) does not include \( t_2 \). That ordering is implied through transitivity.

We now investigate three approaches to resolving \( Q_t \) queries. The first algorithm employs classical search techniques. The second is primarily an investigation into the possibility of resolving a query on a tuple \( v \) by using data generated during the resolution of that query on previous tuples (i.e. an incremental approach). The last algorithm resolves a \( Q_t \) query through use of a series of orthogonal range queries. Each of these three algorithms makes use of the observation that all tuples \( t \) in the resolution of \( Q_t(v) \) "cross" each other when viewed as in the right-hand diagram in Figure 2 (more formally, the tuples are incomparable).

**Lemma 1.** Given tuples \( t_r, t_s \in Q_t(v) \), there exists a component \( c_i \) such that \( t_r.c_i < t_s.c_i \) and there exists a component \( c_j \) such that \( t_r.c_j > t_s.c_j \).
\[ D = \{ t_1 = (2, 1.5), t_2 = (3, 3.3), t_3 = (5, 5.4), t_4 = (4, 6.8), t_5 = (6, 2.2), t_6 = (8, 7.6) \} \]

\[ G = \]

![Graph](image)

Figure 2. Precedence Resulting From Three Components Each Ordered By '≤'

**Proof.** This is direct from the definitions of \( Q \) and \( Q_t \). If there is no component \( c_i \ni t_r.c_i < t_s.c_j \), it means that \( t_r > t_s \) (remember: component values are unique in \( Q_t \)) and thus \( t_s \) could not be an element of \( Q_t(v) \) due to the "\( \forall u \in D [u < v] \)" clause of \( Q \)'s definition. Similarly, if there is no component \( c_j \ni t_r.c_j > t_s.c_j \), \( t_r < t_s \) so \( t_r \) could not be an element of \( Q_t(v) \).

Q.E.D.

### 3.2 Non-Incremental Algorithm

Figure 3 shows the first \( Q_t \) algorithm, \( Q_t NI \), using the Pascal-like notation described in Appendix 5.1 and introduced in [6, 7]. As can be seen in the declarations, tuples are treated as simple arrays of integers and the columns in the right-hand diagram of Figure 2 are recorded.
constants
  $k_t$ : natural := the number of (total-order) components in a tuple;

types
  tuple = array [1..$k_t$] of integer;
  C_node = record
    val : integer;
    t : ^tuple;
  end C_node;

globals
  C : array [1..$k_t$] of srt_set of C_node key val;

function next_candidate(i : natural; bound : tuple) : ^tuple;
  candidate : ^tuple := C[i].prev(); // the tuple below the latest one found on C[i]
begin
  while candidate ≠ null and candidate ≠ bound do // make sure candidate is admissible
    candidate := C[i].prev();
  endwhile;
  return candidate;
end next_candidate;

function Q_t_NI(v : tuple) : list of ^tuple;
  Qlist := list of ^tuple := [];
  candidate := ^tuple;
  $t_X$ : ^tuple;
  $i$ : natural;

begin
  $t_X$ := C[1][v[1]]→;$i$; // find the first tuple < v whose $c_i$ value is less than that of v. All other tuples put into Qlist will "cross" $t_X$.
  $t_X$ := next_candidate(1, v); // make sure there is at least one tuple < v
  if $t_X$ ≠ null then
    Qlist += $t_X$;
    for i in [2..$k_t$] do
      candidate := C[i][v[i]]→;$i$; // find the first tuple < v with $c_i$ between $v.c_i$ and $t_X.c_i$
      candidate := next_candidate(i, v);
      while candidate ≠ null and candidate[i] > $t_X[i]$ do // examine all such admissible tuples
        if $\exists u \in Qlist$ [candidate^ ≤ u^] then // make sure candidate is maximal
          Qlist += candidate;
        endif;
        candidate := next_candidate(i, v); // next admissible tuple
      endwhile;
    endfor;
  endif;
  return Qlist;
end Q_t_NI;

Figure 3. Query by Total-Order Factors (Non-Incremental)
as sorted sets* of (integer, tuple) pairs in the array C. Intuitively, each C[i] corresponds to the "used" component values of set C_\text{i}. Since values of a given component are unique across tuples, the nodes in each C[i] point to every tuple in D exactly once. Because the example from Figure 2 will be referenced frequently, specifically Q(t_6), Figure 4 shows the contents of C for this D.

The auxiliary routine next_candidate should be explained before proceeding with the discussion of Q_t_NI. Next_candidate "filters out" those tuples which can not, by definition, be in Q_t(v) because they do not precede v. Specifically, next_candidate returns the next tuple which both precedes v and has c_i less than that of the "current" tuple. This is done by searching down C[i] from the current value until some value is found whose associated tuple precedes v. For instance, if one were evaluating Q(t_6) and had just inspected t_6 itself along component 2, next_candidate(2, t_6) would skip over t_4 because t_4 \neq t_6 and would go on to return t_3.

The approach taken by Q_t_NI is to find a single tuple which is in Q_t(v), then to make use of the observation in LEMMA 1 that all other tuples in Q_t(v) must cross that "seed" tuple. Step by step, the algorithm proceeds as follows to resolve Q_t(v):

- Find a seed tuple which is known to be in Q_t(v). This tuple is called t_X, and is the tuple preceding v which has the highest c_1 value. It is found by scanning down C[1] for decreasing values of C_1, starting at v.c_1 (i.e. t_X < v and no tuple can be between t_X and v).
- Find all tuples which both precede v (are admissible) and "cross" t_X as described in LEMMA 1. These tuples are then added to the query result list, QList. They are found by scanning, down each component i, for tuples whose c_i value is between

\[
D = \{ t_1 = (2, 1, 5), t_2 = (3, 3, 3), t_3 = (5, 5, 4), \\
t_4 = (4, 6, 8), t_5 = (6, 2, 2), t_6 = (8, 7, 6) \}
\]

\[
C = \\
\{ (8, t_6), (6, t_5), (5, t_3), (4, t_4), (3, t_2), (2, t_1) \}, \\
\{ (7, t_6), (6, t_5), (5, t_3), (3, t_2), (2, t_1) \}, \\
\{ (8, t_4), (6, t_5), (5, t_1), (4, t_3), (3, t_2), (2, t_2) \}
\}
\]

Figure 4. Example of Q_t_NI Data Structures

* Specifically, threaded AVL trees[8][15] are used.
those of $v.c_i$ and $t_X.c_i$ (note: this decreasing-value scan is important for proof of the algorithm). Any such tuple, let us say candidate, which precedes $v$ must cross $t_X$ for two reasons. First, candidate does have a $c_i$ greater than $t_X.c_i$. Second, candidate can not have every $c_i$ greater than $t_X.c_i$ because $t_X$ has the highest $c_1$ value of any tuple preceding $v$. Thus, candidate crosses $t_X$.

- Filter out those tuples which violate the $\forall u \in D \ [t \prec u \prec v]$ clause of $Q$’s definition. This is done on a per-tuple basis, before the tuple is added to $Qlist$.  

We now show how $Q_{-t NI}$ operates on the example from Figure 2, $Q(t_6)$. The first action taken by $Q_{-t NI}$ is to find $t_X$. $Q_{-t NI}$ finds $t_6.c_1$, 8, then scans down the values of $c_1$ until it finds the first tuple $t_X < t_6$. This tuple is $t_5$. $Q_{-t NI}$ next goes into a component-scanning loop. It inspects the tuples associated with values between 7 and 2 on $c_2$ (that is, between $t_6.c_2$ and $t_5.c_2$), and then the tuples associated with values between 6 and 2 on $c_3$ ($t_6.c_3$ and $t_5.c_3$).

When scanning $c_2$, $Q_{-t NI}$ first encounters $t_4$ but skips over that tuple (in next_cadidate) because it is inadmissible — it does not precede $t_6$. $Q_{-t NI}$ then encounters $t_3$, finds it admissible, and adds it to $Qlist$. The last tuple encountered in the $c_2$ scan is $t_2$. This is not added to $Qlist$, even though it precedes $t_6$, because it also precedes $t_3$ which is already in $Qlist$. The first tuple encountered in the scan of $c_3$ is $t_1$, which is added to $Qlist$. After that, $t_3$ is encountered a second time and not added to $Qlist$ since it is already there. Finally, $t_2$ is encountered again and not added to $Qlist$ for the same reason as before.

Update of the $Q_{-t NI}$ data structures when adding a new tuple $t$ to $D$ (which $Q_{-t NI}$ only accesses indirectly through tuple pointers) requires the insertion of the values of each of $t$’s components into the corresponding $C[i]$ sorted sets (search trees). It is for this update operation that sorted sets are used in $C$ instead of lists. While list traversal would be slightly faster than thread traversal during the down-scans of the values in each $C[i]$, sorted sets are distinctly more efficient than lists when updating each $C[i]$ with the value from a new tuple.

3.3 Proof and Analysis

The proof of $Q_{-t NI}$, as for other algorithms returning sets, has two obligations. First, it must be shown that $Q_{-t NI}$ never returns tuples not permitted by the definition of $Q_{-t}$. Second, it must be shown that $Q_{-t NI}$ returns all tuples required by the definition of $Q_{-t}$.
Proof of the first obligation is itself in two parts. Part 1 shows that no tuple \( t \) is added to \( Qlist \) such that \( \exists u \in Qlist \ [t < u] \), and Part 2 shows that no tuple \( t \) is added to \( Qlist \) such that \( \exists u \in Qlist \ [u < t] \). Part 1 derives from inspection of \( Q_t_{-NI} \) which reveals that all tuple additions to \( Qlist \) are guarded with this particular invariant. We demonstrate Part 2 by contradiction. If \( u < t \), the value of every component in \( t \) must be greater than that of the corresponding component in \( u \). Candidate tuples to be added to \( Qlist \) are chosen during a descending scan of component values. This contradicts the possibility that \( u < t \); since \( t \) is a candidate for addition to \( Qlist \) after \( u \) was added, at least one of \( t \)'s components must have a value less than that of the corresponding component in \( u \) — somehow, \( u \) was encountered first, but it could not have been.

To begin the second proof obligation, it is known that \( t_X \in Q_{-t}(v) \) because both \( t_X < v \) and there is no tuple \( u \in D \) between \( t_X \) and \( v \). By LEMMA 1, therefore, every other tuple \( t \in Q_{-t}(v) \) must have the value of at least one component greater than that of the corresponding component in \( t_X \), and also must have the value of at least one component less than that of the corresponding component in \( t_X \). Additionally, the value of no component of a tuple \( t \) may be greater than that of the corresponding component in \( v \), since it is required that \( t < v \).

The conjunction of these requirements is that every \( t \in Q_{-t}(v) \) must have at least one component \( c_i \) such that \( t_X.c_i < t.c_i < v.c_i \) and another component \( c_j \) such that \( t.c_j < t_X.c_j \). The \( c_j \) clause is guaranteed on \( c_1 \) by the way in which \( t_X \) is determined and the fact that no tuples are added to \( Qlist \) which do not precede \( v \). The range of component values in the \( c_i \) clause is exactly what \( Q_t_{-NI} \) searches after it determines \( t_X \). During this search, the only tuples which are not added to \( Qlist \) are those which are not allowed by the definition of \( Q_{-t} \). Specifically, these are any tuples which do not precede \( v \) and any which precede some tuple already in \( Qlist \). Thus, given this selection procedure in \( Q_t_{-NI} \), all tuples which should be in \( Q_{-t}(v) \) are added to \( Qlist \).

Q.E.D.

For analysis, it is useful to define \( r = |Q_{-t}(v)| \) and \( b = |\{t \in D \land \exists i[t_X.c_i < t.c_i < v.c_i]\}| \) (that is, the number of tuples with the value of at least one component between the values of the corresponding components of \( t_X \) and \( v \)). \( Q_t_{-NI} \) visits each of the \( k_i \) components exactly once by performing a scan down the appropriate \( C[i] \). Each of these scans can examine the values of up to \( b \) tuples' components. Now, for every value which is associated with an admissible tuple, that tuple will potentially need to be compared with \( r \) tuples in \( Qlist \). Considering that a tuple compare takes \( O(k_i) \) time and remembering that the sorted set operations used by \( Q_t_{-NI} \) to scan each component are initiated with a single \( O(\log d) \) search
followed by a sequence of $O(1)$ thread traversals, we arrive at an $O(k_i^2br + k_i \log d)$ overall time complexity for $Q_{t\_NI}$.

A simple optimization, however, can substantially reduce this time. Much of the above effort is spent performing comparisons between tuples which have already been examined with respect to either $Q\_list$ or $v$. By marking each tuple as it is examined, those repeated comparisons can be eliminated (though $Q_{t\_NI}$ must still scan over the marked tuples). This results in an $O(k_i(br + r + \log d))$ time for $Q_{t\_NI}$, of which $k_i br$ is the remaining comparisons, $k_i r$ is the admissibility checks, and another $k_i b$ is scanning past already-marked tuples.

Update of the $Q_{t\_NI}$ data structures requires simple $O(\log d)$ component value insertions into each element of $C$. There are $k_i$ elements in $C$, so the total update time is $O(k_i \log d)$.

3.4 An Incremental Algorithm

One might notice that three categories of tuples are encountered during the component-value scans made by $Q_{t\_NI}$: tuples which cross $t_X$, tuples which precede some tuple already in $Q\_list$, and tuples which cross $v$. Tuples in all but the first category are ignored. Can this discarded information be used for subsequent queries?

The answer is a qualified "yes." Intuition suggests that the tuples which are found to cross $v$ would be prime candidates for inclusion in $Q\_list$ during subsequent queries in which $v$ takes the role of $t_X$. This might alleviate the problem of encountering a tuple multiple times during the component scans, since the results of a previous series of scans are already known.

Specifically, the available information is a list of all tuples which cross $v$ and have the value of at least one component between those of the corresponding components in $v$ and $t_X$. What is needed to resolve a subsequent query $Q_{t}(v_2)$, for which $v$ takes the role of $t_X$, is all tuples which cross $v$ and for which the value of at least one component is between those of the corresponding components in $v$ and $v_2$. The cross-$v$ tuples for a range of component values below $v$ is known, but what is needed is the cross-$v$ tuples for a range above $v$.

If one were asking queries only of the form of $Q_{t}(v)$, this situation would present a considerable problem. For the monitor application, however, not only is the query $Q_{t}(v)$ made, but also the query $Q_{t\_x}(v)$ is made. $Q_{t\_x}(v)$ finds the minimal elements of the set of tuples in $D$ which follow $v$, instead of the maximal elements of the set of tuples in $D$ which precede $v$.
Efficient Queries For Quasi-Ordered Databases

(Q_\_t(v) = Q_\_t_\_c(v)). The evaluation of Q_\_t_\_c(v) can provide the component value range above v which is needed for the evaluation of subsequent Q_\_t(v) queries.

Figure 5 shows how the cross-v tuple information might be used. In Figure 5a, we see that the tuple t_X_\_c delimits the range of component values scanned during a Q_\_t_\_c(v) query. During this scan, t_c was found to cross v. Later, as shown in Figure 5b, v_2 is added to the database and we wish to resolve Q_\_t(v_2). The t_X_\_c for the new query is v. Because of the results from Q_\_t_\_c(v), it is known that the only tuple which crosses v in the highlighted component range between v and t_X_\_c (and thus might be in Q_\_t(v_2)) is t_c. The evaluation of Q_\_t(v_2) does not need to scan component values in that range.

There are several problems with this approach even before the difficulties of updating v's crossover list are considered. Observe in Figure 5b that t_X_\_c crosses v_2. This, or that t_X_\_c \succ v_2, is guaranteed to occur for every query. Otherwise, t_X_\_c, not v, would be t_X_\_c. For components such as x in Figure 5b, this results in a wider range of some component values being scanned than is necessary. While such a t_X_\_c/v_2 crossing can lead to the wasteful comparison of v_2 with cross-v tuples which do not precede v_2 (whenever t_X_\_c_x_\_c_1 > v_2_c_1), this only increases the time complexity

---

![Diagram](image)

Figure 5. Example for Incremental Total-Order Query Algorithm
of the algorithm by effectively making $b$ larger. What does significantly increase the algorithm's time complexity is that, whenever $t_{X_i} c_i < v_2 c_i$ (as with components $y$ and $z$ in Figure 5b), too narrow a range of component values is scanned than necessary. When this occurs, the algorithm must find some means to finish scanning the "gaps" in the component range between $v_2$ and $v$.

There appear to be two approaches to remedy this problem. The first is to perform a search for some alternative to $v$, let us say $v'$, down each component for which the range of values covered by $t_{X_i}$ is insufficient. Cross-$v'$ tuples would then be checked for addition to $Q_{\bot}(v_2)$. A second approach is to revert to the non-incremental algorithm $Q_{\bot NI}$ for the troublesome components. Both of these approaches raise the possibility of double-checking cross-$v$ tuples, especially if the $t_{X_i}/v'$ and $t_{X_i}/v$ ranges overlap. This means that the overall time complexity of the proposed incremental algorithm can be no better than that of $Q_{\bot NI}$, though the incremental algorithm will have increased storage requirements and a substantially larger constant multiplier factor in the time complexity.

Update of the incremental algorithm's data structures when a new tuple $v$ is added to $D$ appears best performed by a 2-dimensional range query on each component. This query would find those existing tuples whose "scanned" range includes $v$. A total of $k_i$ component-range searches is needed, each requiring $O(\log d)$ time. Following each of these is a series of $O(k_i)$ tuple comparisons to determine if $v$ crosses any of the tuples found in the component-range search (along with marking tuples to avoid double-checking, etc.). It is expected that the number of tuples found in each component-range search will be on the order of $b$. So, the new complexity terms required in order to update the "tuple crossing" lists do not appear to add to the time complexity of the incremental algorithm since $Q_{\bot NI}$ already requires comparable terms. However, the time required to update the component-range structure with each of $v$'s components has a worst-case time of $O((\log d)^2)$ and expected time of $O(\log d)$, putting the incremental algorithm's total worst-case update time at $O(k_i(\log d)^2)$.

Such an update time can easily become appreciable, adding to the algorithm's already considerable constant-multiplier time penalty and increased storage space. For these reasons, further investigation of the incremental algorithm described above has been suspended in favor of the following alternative non-incremental algorithm.
3.5 An Algorithm Using Range Query Techniques

`Q_t ORQ`, like `Q_t NI`, builds `Qlist` by examining all tuples which have the value of at least one component between the values of the corresponding components of `v` and `t_X`. Unlike `Q_t NI`, `Q_t ORQ` finds that set of tuples through use of a sequence of orthogonal range queries instead of through direct traversal of search tree threads. It is known that general-purpose orthogonal range queries\* are quite costly for any application requiring more than a small number of search dimensions. Pleasantly, for the monitor application, any `Q(v)` usually contains no more than two or three total-order factors. Given this, a range query approach to `Q_t` is feasible.

Figure 6 shows how `Q_t NI` is modified to use range queries. `Q_t ORQ` begins by finding `t_X` in the same manner as `Q_t NI`. Note that the C search tree is needed only for `c_1`; `D`'s range query structure performs the role of `C[2]` through `C[k_I]`. For each iteration of `i` from 2 to `k_q`, a range query (whose format is specified below) is performed instead of a search through `C[i]`. The tuples returned from those queries are checked for admissibility\,** and, as in `Q_t NI`, are checked to make sure that they do not precede any tuple already in `Qlist`. If both conditions hold, the new candidate tuple is added to `Qlist`.

For each component `c_i` other than `c_1`, a range query is issued by `Q_t ORQ`. As would be expected, the range specified for `c_i` is those values between `t_X, c_i` and `v, c_i`. For components `c_j` "before" `c_i` (where `j < i`), `Q_t ORQ` needs to exclude tuples it has already examined and so specifies those component values less than `t_X, c_j`. The only reason to specify values for components "after" `c_i` would be to ensure that each tuple returned by the query precedes `v`. Since it is faster to just explicitly check each tuple against `v` than to add more specifications to the query, the trailing components allow the full range of component values.

The leading components of `c_i`'s range query are, except for the change in the range specification of `c_p` the same as those of `c_{i+1}`. Because the queries have these range specifications in common, it is possible to retain the intermediate evaluation data for `c_i`'s query and use it to

---

\* Unlike the optimized equivalence-factor algorithm in Chapter 2, `Q_t ORQ` must use the full power of a general-purpose orthogonal range query.

\** It is possible to avoid the comparison against `v` by explicitly specifying the range of the trailing components in the range query (expressly, use `..v[j+1]`, ..., `..v[k_I]`). Putting this check in `RQ`, however, takes more time than the comparison against `v`. 
constants
\( k_i : \text{natural} \) := the number of (total-order) components in a tuple;

types
tuple = array [1..k_i] of integer;
C_node = record
val : integer;
t : ^tuple;
end C_node;

globals
\( C_1 : \text{srt_set of C_node key val}; \) // unlike with Q.t.NI, D handles the other components
D : range_query_structure of tuple;

// Similar to that of Q.t.NI before, but only applies to C_1
//
function next_candidate(bound : tuple) : ^tuple;

function Q.t.ORQ(v : tuple) : list of ^tuple;
Qlist : list of ^tuple := {};
candidate : ^tuple;
tX : tuple; // inspect all tuples with component values between v and tX
i : natural; // the tuple component on Q.t.ORQ is working

begin
\( t_X := C_1[v[1]] \rightarrow t; \) // Find the first tuple whose \( c_i \) value is less than
\( t_X := \text{next\_candidate}(v); \) // that of \( v \). All other tuples put into \( Qlist \) will "cross" \( t_X \).
if \( t_X \neq \text{null} \) then
Qlist += tX;
for \( i \) in [2..k_i] do
// scanning down values of \( c_i \)
//
// for candidate \( \in \text{RQ}(D, t_X[1], ..., t_X[i-1], t_X[i]..v[i]) \) do
if candidate^ < v andif \( \exists \text{Qlist (candidate^} \leq u^\alpha \) then
Qlist += candidate;
endif;
endfor;
endfor;
return Qlist;
end Q.t.ORQ;

Figure 6. Query by Total-Order Factors (Using Range Queries)
"pre-load" $c_{i+1}$'s query. Given this optimization, the time to evaluate the $k_i$-1 range queries in
$Q_{\text{t}_\text{ORQ}}$ is comparable to the time to evaluate only two queries, not $k_i$-1 queries.

All time and space costs for $Q_{\text{t}_\text{NI}}$ could be expressed in terms of a single $Q_j \in Q$. Unlike
$Q_{\text{t}_\text{NI}}, Q_{\text{t}_\text{ORQ}}$ directly references $D$. Therefore, though some of the $Q_{\text{t}_\text{ORQ}}$ costs
are in terms of a single query, some must be in terms of $Q$. For this purpose, as with Section 2's
equivalence factor algorithm, it is useful to define $K_i = \text{the number of distinct total-order factors}
$ across all $Q \in Q$. As done with $m_e$, consider only the total-order factors of the queries in $Q$ and
let $m_t = \max_{1 \leq i \leq K_i} |C_i/D|$, where $C_i/D$ is that subset of $C_i$ induced by application of $F_i$ over the
tuples in $D$ (i.e. the number of unique $C_i$ values actually used when evaluating queries over $D$).

Analysis yields a time complexity for $Q_{\text{t}_\text{ORQ}}(v)$ of $O(k_{i}(b+br)+(\log d)^{K_{i}-1})$, derived
as follows. The time for checking whether or not a scanned tuple should be put into $Q_{\text{List}}$ is
$k_i br$, the same time as for $Q_{\text{t}_\text{NI}}$. The $k_i br$ admissibility check is also the same as for
$Q_{\text{t}_\text{NI}}$. Range query time is $b + (\log d)^{K_{i}-1}$, which compares to the search time of
$k_i(b + \log d)$ for
$Q_{\text{t}_\text{NI}}$. This time appears favorable for small $k_i$, but must be weighed against the very
high memory cost of the range query data structures and the fact that, for small $k_i$, the $k_i br$ term
common to both algorithms might tend to dominate.

Update of the $Q_{\text{t}_\text{ORQ}}$ data structures requires insertion of each new $v$ into $C_i$ and into
$D$'s range query data structures, an operation requiring $O((\log d)^{K_{i}})$ worst-case time and
$O((\log d)^{K_{i}-1})$ expected time. Since this update applies to all $Q \in Q$, the time compares to
$O(qk_{i}(\log d))$ worst- and expected-case time for $Q_{\text{t}_\text{NI}}$. Space cost for $Q_{\text{t}_\text{ORQ}}$ is a substantial
$O(d(\log m_t)^{K_{i}-1})$.

$Q_{\text{t}_\text{ORQ}}$ would become much more economical if a restricted-domain $D_j$ were searched
for each $Q_j$, thus making $K_e = k_{e_j}$. This is exactly what happens when the equivalence and total-
order factors of a query are re-combined as described in the next subsection.

3.6 Queries With Both Equivalence and Total-Order Factors

As described in Section 2, the equivalence factors of a query $Q_j$ are evaluated through use
of a highly-optimized range query structure which maintains auxiliary trees only for the leaf nodes
of each search dimension. The set of tuples at the leaves of the last set of auxiliary trees have
the same values for all equivalence factors. Quite literally, $D$ has been partitioned not only by
each query (as desired in the previous subsection) but also by the values of that query's
equivalence factors.
Evaluation of total-order factors fully exploits this fine-grained database partitioning. For each query in \( Q \), the "database" to which either \( Q_{t\_NI} \) or \( Q_{t\_ORQ} \) is applied is not \( D \), but is rather one of these auxiliary tree leaves’ tuple sets. There is one \( Q_{t\_NI} \) or \( Q_{t\_ORQ} \) structure per set of equivalence factor values per query. Time and space costs for total-order factor evaluation are significantly reduced. Intuitively, one can view the total-order factor data structures as one additional level of auxiliary trees in the equivalence factor structures.
4. Equivalence-Factor Query Optimization

4.1 Requirements

Generally, a query $Q(x)$ has several equivalence factors in addition to a small number of total-order factors. Other queries, let us say $Q_4(x)$, might have one or more equivalence factors in common with $Q_4(x)$. As mentioned in Section 2, sharing intermediate search results between queries with common equivalence factors would save both time and space when resolving a set $Q$ of queries. The topic of this section is to find how best to optimize the equivalence factor search structures for a set of queries so that intermediate results can be shared between queries.

Since total-order factors are not involved in these optimizations, they are ignored for the rest of this section. Furthermore, it is found useful to abstract the conjunction of equivalence factors in a query into a sequence of tokens (letting adjacency indicate conjunction). For example, suppose that $Q_4(x)$ has three equivalence factors. These can be abstracted as $abd$ (using letters from the beginning of the alphabet for the tokens). If $Q_5(x)$'s equivalence factors were abstracted as $cab$, we would know that $Q_4(x)$ and $Q_5(x)$ have two factors in common, $a$ and $b$.

Continuing this example, how can intermediate results be shared between the resolutions of $Q_4(x)$ and $Q_5(x)$? First, it must be realized that since the factors in a query are part of a conjunction, their order commutes. $Q_4(x)$ and $Q_5(x)$ can thus also be represented as $bad$ and $bac$, respectively. The $ba$ "prefix" is common between the queries and is the basis for the shared information. Now, the query tree structure for $Q_4(x)$ contains a base tree for $b$ and auxiliary tree levels for $a$ and $d$; the structure for $Q_5(x)$ contains a base tree for $b$ and auxiliary tree levels for $a$ and $c$. To optimize the resolution of the pair of queries, their base trees and first level of auxiliary trees are combined. Only the second level of auxiliary trees remains distinct between the queries. During resolution, the search for those tuples for which both factor $b$ and factor $a$ are true is only performed once. The results of this search are then used separately to resolve $Q_4(x)$ and $Q_5(x)$ according to factors $d$ and $c$, respectively. Figure 7 illustrates such a combined query structure. This practice saves space whenever a tuple is in the domain of both queries, and always reduces combined update and query time.

In order to represent query structures more succinctly for optimization purposes, we introduce one more slight abstraction of the problem. Let the $k_{e_j}$ tree levels in a query $Q_j$'s data structure be represented as an undirected path of length $k_{e_j}$ with vertices labeled by the equivalence factors in $Q_j$. The unoptimized query structure for a query set $Q$, therefore, is a forest.
of \( q \) disjoint paths. Optimization of this query structure (if optimization is productive, of course) turns the forest of paths into a forest of trees, where the root vertex of each subtree represents an equivalence factor common between two or more queries. Each query in \( Q \) is still represented by a path originating* at the root of some tree in this forest. Figure 8 uses this notation to show the optimization steps for the \( Q_f(v)/Q_s(v) \) example.

It is not always possible for each equivalence factor to be represented exactly once in an optimized forest. Consider three queries with the factors \( ab, bc, \) and \( ca \). Optimization results in some forest isomorphic (discounting vertex labels) with that illustrated in Figure 9. No matter how the forest is constructed, at least two of \( a, b, \) or \( c \) will be duplicated. To evaluate all the queries in \( Q \), those factors will be encountered as a search dimension more than once.

The query structure optimization problem is formally defined in terms of sets of tokens and a forest of trees. It is stated as a decision problem to aid in a proof in the next subsection.

**QOPT.** Given an alphabet \( E \), a set \( Q \) of sets \( Q_j \) of tokens over \( E \), and a positive integer \( f \). Is there a forest \( F \) with the following properties such that the number of vertices in \( F \) is no greater than \( f \)?

1) Each vertex in the forest is labeled (not necessarily uniquely) with one token \( e \in E \).
2) For each \( Q_j \in Q \), there exists a path \( p_j \) with these two characteristics. First, the origin of \( p_j \) is the root of some tree \( T \) in \( F \).

---

* Since the paths are undirected, origin and terminus are notational conveniences in this discussion. For a query's data structure itself, however, the "origin" of a path has meaning since that identifies the structure's base tree.
Second, the labels of the vertices on \( p_j \) enumerate the tokens in set \( Q_j \) and no tokens not in \( Q_j \).

3) Given the roots of two trees in \( F \), those vertices are labeled with distinct tokens. Similarly, given two subtrees in \( F \) which share the same parent vertex, the roots of those subtrees are labeled with distinct tokens.

### 4.2 NP-Completeness Analysis

Unfortunately, it proves to be the case that the above optimization problem is NP-complete. QOPT turns out to be a multi-level version of the Hitting Set problem as defined in [9] and [5]:

**Hitting Set.** Given a set \( U \) of subsets of a finite set \( S \) and a positive integer \( h \leq |S| \). Is there a subset \( S' \subseteq S \) with \( |S'| \leq h \) such that \( S' \) contains at least one element from each subset in \( U \)?
It will be shown that the HITTING SET problem, subject to the restrictions that \( \forall U_j \in U \ [\|U_j\| = 2] \) and that no two \( U_j \) and \( U_j' \) are identical, can be solved in polynomial time by any algorithm which solves the QOPT problem subject to the restrictions that \( \forall Q_j \in Q \ [\|Q_j\| = 2] \) and that no two \( Q_j \) and \( Q_j' \) are identical. HITTING SET remains NP-complete with these restrictions[5]. To show that QOPT is NP-complete, we must demonstrate that a) it is in NP, and that b) an algorithm which solves QOPT can solve a known NP-complete problem in polynomial time. Let us define \( q = |Q|, K_e = |E|, s = |S|, u = |U|, \) and \( |F| = \) the number of vertices in \( F \).

For part a) of this obligation, consider a nondeterministic Turing machine with two functional blocks: a forest generator, and a recognizer. The forest generator accepts \( E \) and \( f \), then nondeterministically generates a forest with \( f \) vertices, each labeled with some token from \( E \). The NTM can do this in polynomial time.* \( F \), the output of the forest generator, is passed to the recognizer, which also accepts \( Q \). By traversing all paths in \( F \) which originate at the root of some tree in \( F \), the recognizer generates a set \( Q' \) of token sets with which \( F \) is related according to Property 2 in the definition of QOPT. This can be done in deterministic \( O(f^2) \) time. Finally, the recognizer compares \( Q' \) to \( Q \) (deterministic \( O(q^2K_e) \) time) to find if \( Q \subseteq Q' \). If so, \( F \) is a tree of size \( f \) which is related to \( Q \) (not only to \( Q' \)) by Property 2 of QOPT. The NTM has decided QOPT in polynomial time, so QOPT is in NP.

Part b) of the proof obligation is shown by proving that there exists a one-to-many mapping to \( S' \) from the tokens of the roots of the trees in \( F \). Let QOPT's \( Q \) and \( E \) correspond to HITTING SET's \( U \) and \( S \), respectively. Define \( E_R \) as the set of tokens associated with the roots of the trees in \( F \). In QOPT, the enumeration paths \( p_j \) are defined to originate at the root of one of the trees in \( F \). Therefore, \( E_R \) contains at least one token from each enumeration path and thus one token from each \( Q \in Q \): \( E_R \) corresponds to \( S' \). The question now becomes, "how are \( f \) and \( h \) related?"

One of the restrictions above guarantees that no two \( Q_j \) and \( Q_j' \) are identical. This implies that no two \( Q_j \) and \( Q_j' \) have coincident enumeration paths; the paths can share the same origin, but they can not share the same terminus. The size of every \( Q \in Q \) is 2; \( E_R \) contains the distinct label of the origin of every path — the only other vertices in \( F \) are the termini, and they are all

---

* A variety of forest-generation algorithms are available. One such algorithm is used in the test-case generators which provide sample \( Q \) sets to implementations of QOPT_TF and QOPT_SO.
unique. \(|F| = |\text{the path origins}| + |\text{the path termini}| = |E_R| + q.\) In other words, when QOPT specifies a value for \(f\), it is specifying a value of \(f - q\) for \(|E_R|\).

It is now trivial to solve a HITTING SET problem by solving a QOPT problem. Let \(Q = U, S = E,\) and \(f = h + q.\) Use \(E_R\) as a solution to \(S';\) \(|E_R| \leq h.\)

Q.E.D.

Let QOPT and HITTING SET now be treated as optimization, not decision, problems. A major difference between the definitions of QOPT and HITTING SET is that HITTING SET is concerned only with minimizing a set which intersects (at least) a single element in each \(U \in U.\) QOPT is concerned with minimizing, in a way, a group of sets (each "level" of each tree) which must cover every element in each \(Q \in Q.\) Consider attempting to solve a QOPT problem by recursively applying a series of HITTING SET problems to each level of \(F.\) When the token sets in \(Q\) have many tokens in common but have a small number of differentiating tokens, the recursive HITTING SET will fail. Such a situation is illustrated in Figure 10, where \(t\) and \(f\) differentiate several "pairs" of sets which have other tokens in common. Recursive HITTING SET would choose to use \(t\) and \(f\) for the tree roots. QOPT, concerned with total tree size, would recognize the other common tokens, root the trees with those, and put \(t\) and \(f\) at the trees' leaves. It is for this reason that the size of the sets in \(U\) and \(Q\) were restricted to 2 in the above NP-completeness proof; this discrepancy does not arise with such small sets.

4.3 General Heuristics

There are some basic reductions can be made in order to narrow the scope of the QOPT problem. These involve collapsing queries with duplicate equivalence factors, partitioning the

---

**Figure 10. Optimization Differences Between QOPT and Recursive HITTING SET**
problem so that the queries have some equivalence factors in common, collapsing "mated" equivalence factors, and processing any common equivalence factor as a special case. Stated in terms of the QOPT definition above, the reductions are, respectively:

1)  Collapse equivalent sets $Q_j$ and $Q_j'$ into a single set.
2)  Partition the set $Q$ into subsets $Q_p$ so that each $Q_r \in Q_p$ has at least one token in common with some $Q_s \in Q_p$, $Q_r \neq Q_s$. Apply the optimization algorithm separately to each $Q_p$.
3)  Collapse any tokens $e_x$ and $e_x'$ such that $\forall Q_j [e_x \in Q_j \equiv e_x' \in Q_j]$ into a single token.
4)  For any token $e_c$ such that $\forall Q_j \in Q [e_c \in Q_j]$, use $e_c$ as the root of a tree common to all paths $p_p$, construct $Q' = \{Q_j \cdot e_c \mid Q_j \in Q\}$, and apply the optimization algorithm recursively to $Q'$.

Examples of each of these reductions is shown in Figure 11. Their use should enhance the performance of the following two approximate-optimization algorithms.

4.4 Token-Frequency Algorithm

Our first approximate-optimization algorithm, QOPT\_TF, follows the line of intuition which suggests that the token occurring most frequently in $Q$ should be the root of a tree in $F$. Though this intuition is often correct, the previous example in Figure 10 illustrates one scenario
types

token = an identifier (in our case, for an equivalence factor);

token_set = set of token; // here, implement as a list of values

tree_node = record
    e : token;
    children : list of ^tree_node;
end tree_node;

globals

F : tree_node; // root of the query forest QOPT_TF constructs
Q : set of token_set; // the original query set

// Add a new child to node, labeling this child with token.
//
function add_child(node : ^tree_node, child : token) : ^tree_node;
    child_node : ^tree_node;
begin
    child_node := new tree_node;
    child_node->e := child;
    child_node->children := [];
    node->children &:= child_node;
    return child_node;
end add_child;

procedure QOPT_TF(F' : ^tree_node, Q' : set of token_set);
    e_high : token;
    Q_high : set of token_set;
    new_child : ^tree_node;
begin
    e_high := the token occurring most frequently as an element of each Q \in Q';
    Q_high := { Q \mid Q \in Q' \land e_{high} \in Q };
    new_child := add_child(F', e_{high});
    QOPT_TF(new_child, { Q - e_{high} \mid Q \in Q_{high} });
    QOPT_TF(F', Q' - Q_{high});
end QOPT_TF;

Figure 12. Token-Frequency Query Optimization Algorithm
\[ Q = \{ \{a, m, x\}, \{a, n, x\}, \{a, o, y\}, \{a, p, y\}, \{b, q, x\}, \{c, r, y\} \} \]

Token-frequency selects \( a \) and one each from \( \{b, q, x\} \) and \( \{c, r, y\} \) to be roots, leading to 13 tree nodes.

Optimal selects \( x \) and \( y \) as roots, resulting in 12 tree nodes.

Figure 13. Example of Non-Optimal Result From Token-Frequency Algorithm

in which \( \text{QOPT\_TF} \) generates a less-than-optimal query tree. Another example in which \( \text{QOPT\_TF} \) leads to inefficiency is shown in Figure 13.

Nonetheless, \( \text{QOPT\_TF} \) executes quickly and is readily understood. A simple tree node data structure is defined in Figure 12 and a recursive implementation of \( \text{QOPT\_TF} \) is shown which operates on that structure. Execution begins with a call of "\( \text{QOPT\_TF}(\&F', Q) \)" referencing the top-level forest and the original query set. \( \text{QOPT\_TF} \) has four basic steps:

- Find the token, \( e_{\text{high}} \), which occurs most frequently in the sets making up \( Q' \).
- Determine every set in \( Q' \) which contains \( e_{\text{high}} \). This is \( Q_{\text{high}} \).
- Add a new child to \( F' \). Label this child with \( e_{\text{high}} \).
- Apply \( \text{QOPT\_TF} \) to the new child, optimizing the queries which contained \( e_{\text{high}} \) (less \( e_{\text{high}} \), of course, so that there is no endless loop).
- Apply \( \text{QOPT\_TF} \) again to \( F' \), this time for all the sets in \( Q' \) which did not contain \( e_{\text{high}} \).*

\( \text{QOPT\_TF} \) has a relatively low time complexity. Finding \( e_{\text{high}} \) dominates the time in each invocation. Assuming that token sets are represented as lists instead of as bit vectors, this scan takes \( O(qk_{e_{\text{max}}}^2 + K_e) \) time, letting \( k_{e_{\text{max}}} = \max_{j \in Q} |Q_j| \), or just \( O(qk_{e_{\text{max}}}^2) \) since \( K_e \) is no larger than required for the case of unique tokens in each \( Q_j \in Q \).** After determining \( e_{\text{high}} \), an \( O(1) \) add_child takes place and then two recursive calls are made. It is critical for the analysis to observe that, for this next level of recursion, time for the two immediate \( e_{\text{high}} \) scans will be

* Alternatively, this could be implemented as a loop instead of as a second recursive call.

** Even for deeper levels of calls to \( \text{QOPT\_TF} \) where \( q \) and \( k_{e_{\text{max}}} \) change, the effective \( K_e \) will still be no greater than \( qk_{e_{\text{max}}} \) because a linked list of occurring tokens can be made during the frequency scan, eliminating the need to maximize token frequency across each value of the original \( E \).
$O(|Q_{\text{high}}| k_{e_{\text{max}}})$ and $O((q-|Q_{\text{high}}|) k_{e_{\text{max}}})$, for a total of still just $O(q k_{e_{\text{max}}})$. The question now becomes "how many levels deep can QOPT_TF recurse?"

The worst case occurs when all tokens are unique in each $Q_j \in Q$. In that case, only one $Q_j$ will be placed in $Q_{\text{high}}$ for each level of recursion. So, after $q$ levels of recursion, there will be at most one element in $Q'$, allowing QOPT_TF to trivially finish that branch of the forest in $O(k_{e_{\text{max}}})$ time. In total, there are $O(q)$ levels, each at $O(q k_{e_{\text{max}}})$ time, plus at worst $q O(k_{e_{\text{max}}})$ "leaf" branches, for an aggregate $O(q^2 k_{e_{\text{max}}})$ time for QOPT_TF. This is a tight bound, since the unique-tokens case does reach it.

4.5 Set-Overlap Algorithm

Another line of intuition about how to approach QOPT suggests that those sets which have the most elements in common should be combined into one tree in $F$. While this algorithm, QOPT_SO, does take somewhat more time than QOPT_TF, it also produces significantly better results in most test cases.

The revised tree_node data structure is shown in Figure 14. Whereas QOPT_TF combines several sets by a single token they have in common, QOPT_SO combines just two sets at a time by all the tokens that pair has in common. These tokens in common between the sets which were merged to create the node are the node's leading tokens: the tokens which are not common between the node's constituent sets make up the node's children. In QOPT_TF, the basic operation on a tree_node is adding a single token as a child; in QOPT_SO, the basic tree_node operation is combining the leading tokens and children of two nodes.

QOPT_SO, shown in Figure 15, begins with a call of "QOPT_SO($F, Q$)". Initialization involves building separate, uncombined trees for each $Q_j \in Q$ and building a table (common) whose values contain the number of tokens in common between each of the uncombined trees. For each tree[j], QOPT_SO keeps track of the other tree with which tree[j] has the most tokens in common. This is done by means of the ordered_common[j] heap, keyed by the number of common tokens. We keep track of the tree with the most tokens in common with any other tree by means of another heap, max_common.

The main loop of QOPT_SO combines two trees with each pass. Its operation proceeds in three primary phases:
types
  token = an identifier (in our case, for an equivalence factor);
  token_set = set of token; // here, implement as bit vectors
  tree_node = record
    leading : token_set; // tokens in common between all children
    children : list of ^tree_node;
  end tree_node;

  common_node = record
    in_common : natural; // # of leading tokens in common between this tree and tree[idx]
    idx : range [1..q] of natural;
  end common_node;

globals
  F : tree_node; // root of the query forest QOPT_SO constructs
  Q : set of token_set; // the original query set

function combine(node1, node2 : ^tree_node) : ^tree_node;
  new_node : ^tree_node;
begin
  new_node := new tree_node;
  new_node->leading := node1->leading \ node2->leading;
  node1->leading := new_node->leading;
  node2->leading := new_node->leading;
  new_node->children &^ node1;
  new_node->children &^ node2;
  return new_node;
end combine;

---

Figure 14. Data Structures For Set-Overlap Query Optimization Algorithm

- A tree, high_common, is found which has the most tokens in common with some other tree, other_common.*
- These two trees are combined into one and placed in the position formerly occupied by high_common.
- The common data structures are updated to reflect the combined trees. Step one of this cleanup involves removing the other_common tree from reference

* As written in Figure 15, an arbitrary resolution is made between ties. A better heuristic, though, counts the number of other trees with which each tree has one or more tokens in common. Ties are broken in favor of the tree which has some token in common with the fewest number of other trees. This practice combines those trees with the fewest combination options first; early combination of trees with many options might otherwise interfere with some tree’s only option.
procedure QOPT_SO(F : ^tree_node, Q : set of token_set);
  tree : array [1..q] of ^tree_node
   // Yes, this really should include a heap index for ordered_common to assist deletions.
   //
  common : array [1..q, 1..q] of natural := 0;
  ordered_common : array [1..q] of d-heap [q] of common_node ascending key in_common;
  max_common : d-heap [q] of common_node ascending key in_common;
  high_common, other_common, i, j : range [1..q] of natural;

begin
  // Initially, all query sets are uncombined.
  //
  for Q_j in Q do       // j will range in 1..q
    tree[j] := new tree_node;
    tree[j]−>leading = Q_j;
  endfor;

  // Generate initial information about number of common tokens.
  //
  for j in [1..q] do
    for i in [1..q] do
      if j ≠ i then       // don’t combine with self!
        common[j, i] := |Q_j ∩ Q_i|;
      else
        common[j, i] := 0;
      endif;
    endfor;
    ordered_common[j] := makeheap((common[j, i], i));  // implied i in [1..q]
  endfor;

  max_common := makeheap((findmax(ordered_common[j]), j));  // implied j in [1..q]

  while findmax(max_common),in_common > 0 do
    high_common := deletemax(max_common).idx;
    other_common := deletemax(ordered_common[high_common]).idx;
    tree[high_common] := combine(tree[high_common], tree[other_common]);
    tree[other_common] := null;
    remove other_common from max_common and all ordered_common;
    recalculate common[high_common, ..] and common[., high_common];
  endwhile;

  F—children & all non-null elements of tree;
end QOPT_SO;

Figure 15. Set-Overlap Query Optimization Algorithm
throughout the data structures. This requires zeroing-out that tree’s common row and every other tree’s other_common column. These trees’ ordered_common heaps will have to be updated, as well, to delete the heap node for other_common. Step two of the cleanup involves rebuilding the common information for the new tree (including re-inserting a node for it into max_common) and propagating these leading-token intersection counts to all the remaining trees’ rows in common. Again, this might involve changing the other trees’ ordered_common heaps and, perhaps, their nodes in max_common as well.

QOPT_SO takes a bit longer than QOPT_TF, but not terribly so. Note that, unlike for QOPT_TF, it is suggested that token_sets are best implemented as bit vectors for this algorithm. The first phase of initialization requires \( O(qK_q) \) time to build the original uncombined trees. The second initialization phase uses \( O(q^2K_q) \) time to build the \( q \times q \) common structure. Remember that makeheap takes only \( O(q) \) time per tree to build ordered_common, not \( O(q \log q) \), since all key values are known at build time. Similarly, the makeheap for max_common takes only \( O(q) \) time.

Since one tree is combined into another for each pass of QOPT_SO’s main loop, there can be at most \( q - 1 \) passes. Within each pass, the two deletemax operations take \( O(\log q) \) time and the combine just \( O(K_q) \). The significant time is spent in updating common. The first stage of this cleanup requires \( O(\log q) \) time to remove the reference to other_common from max_common, and then \( q \) \( O(\log q) \) deletemax operations to remove other_common from every tree’s ordered_common. The second stage of the cleanup requires \( O(qK_q) \) time to recalculate common[high_common], an \( O(\log q) \) insert into max_common, and up to \( q \) \( O(\log q) \) operations to update other tree’s ordered_common heaps if their common[... high_common] values were changed by this recalculation. Similarly, if these changes altered the maximum value of an ordered_common heap, another \( O(\log q) \) update to max_common is required for each such modification, though it is believed that this frequency is small.

Adding things up, it is found that each of the \( O(q) \) passes through QOPT_SO’s main loop takes \( O(qK_q + \log q) \) time. Initialization takes \( O(q^2K_q) \) time, so total time for QOPT_SO is \( O(q^2(K_q + \log q)) \).
4.5.1 Non-Optimal Cases for QOPT_SO

As does QOPT_TF, QOPT_SO fails to yield optimal query structures in some cases. Since QOPT_SO, in general, performs significantly better than QOPT_TF, QOPT_SO's non-optimal cases have generated sufficient interest to warrant a more detailed investigation. Two such cases are presented here, the first showing how QOPT_SO can produce a query structure of size approaching $\frac{5}{4}$ optimal size, and the second showing a QOPT_SO structure which is $\frac{4}{3}$ optimal size.

In the following discussion, let $X$, $Y$, and $Z$ indicate sets of tokens. Appropriately, let $x = |X|$, $y = |Y|$, and $z = |Z|$. It is given that $X \cap Y = \emptyset$, $Y \cap Z = \emptyset$, and $Z \cap X = \emptyset$.

Figure 16a presents a set of four token sets for which QOPT_SO yields a non-optimal structure if the conditions $x + z > y$, $x \leq y$, and $z \leq y$ hold.* QOPT_SO's initial common (overlap) array is displayed in Figure 16b. This shows that, given the above conditions, it is always possible for QOPT_SO to select sets 2) and 3) for the first set combination. This choice results in the query structure illustrated in Figure 16c, which has size $2x + y + 2z + 4$. The optimal structure, size $x + 2y + z + 4$, would be as shown in Figure 16d.

The maximum size differential between Figure 16c and Figure 16d occurs when $x = y = z$. In this case, QOPT_SO's structure has size $5y + 4$, while the optimal structure has size $4y + 4$. As $y$ increases in size, the ratio of the sizes approaches $\frac{5}{4}$.

One might observe that, at first inspection, the heuristic described on page ? could avoid this non-optimal solution. Though this is sometimes true, test cases have been created which nonetheless yield non-optimal results via this scenario. An even better heuristic than that on page ? is to choose the set combination which keeps the most potential choices available for the next set combination. This is a form of one-move look-ahead, the logical extension of which is to look ahead two or more combinations in advance. Such a practice, however, quickly leads to the combinatorial explosion of decisions which QOPT_SO is designed to avoid.

A variation on Figure 16a can produce a QOPT_SO/optimal size ratio of $\frac{4}{3}$. This is shown in Figure 17a. Two more token sets are required, both of which are supersets of the token set $Y'$. $Y' \subset Y$, and $y' = |Y'|$. For this case, QOPT_SO yields a non-optimal structure when

* The condition $x + z > y$ reflects the difference in size between the optimal structure and that generated by QOPT_SO. The conditions $x \leq y$ and $z \leq y$ guarantee that QOPT_SO can choose to first combine sets 2) and 3), which overlap by $y$ tokens (instead of, for instance, sets 1) and 2) which overlap by $x$ tokens).
the conditions $x + z + 2y' > y$, $x + y' \leq y$, $z + y' \leq y$, $x \geq y'$, and $z \geq y'$ hold. Once more, Figure 17b displays QOPT_SO's initial common (overlap) array.

As with the first example, QOPT_SO can always select sets 2) and 3) for the first set combination (given the above conditions). That combination, followed by two more combinations involving sets 5) and 6), can result in the structure shown in Figure 17c. This is of size $2x+y+2z+2y'+6$. The optimal structure is shown in Figure 17d. It is of size $x+2y+z+6$. 
Figure 17. QOPT_SO Structure 4/3 Optimal Size
For this example, the maximum size differential occurs when $x = z = y'$ and $2x = y$. QOPT_SO's structure size is thus $4y + 6$, while the optimal size is $3y + 6$. A $4/3$ ratio is therefore achieved for large values of $y$. 
5. Appendices
Appendix 5.1 Pseudocode Representation

The representation of algorithms in this report is done using pseudocode which resembles a mixture of Pascal, Ada, and C++. All the standard control structures are available, defined types may be expressed, and a variety of operators may be used.

Below are listed the details of this representation. In pseudocode tradition, however, the more obvious operations in our algorithms are generally expressed with a certain amount of English instead of detailed statements (such as "for every child of..." instead of "child:= foo→child; while child ≠ null do..."). When such use of English is made instead of formal code, this will be clarified by italicizing any English in our algorithms (e.g. "for every child of..." in the above example).

In the following discussion, bold brackets ([ ]) indicate 0 or 1 occurrence of the enclosed item, and bold braces ({ }) indicate 0 or more occurrences. Comments in this pseudocode are as in C++: '//' indicates that the rest of the line is a comment.

5.1.1 Control Structures

How of control is Ada-like. Semicolons are statement terminators, not separators, and loop entry statements are paired with matching loop exit statements. Procedures and functions may be defined and nested, following the usual scope rules. Syntax is:

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Conditional</th>
<th>Alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>statement;</td>
<td>if condition then</td>
<td>case expression of</td>
</tr>
<tr>
<td></td>
<td>sequence;</td>
<td>value_list:</td>
</tr>
<tr>
<td>{statement;}</td>
<td>else</td>
<td>(sequence;);</td>
</tr>
<tr>
<td></td>
<td>sequence;</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>endif;</td>
<td>others:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(sequence;);</td>
</tr>
<tr>
<td></td>
<td></td>
<td>endcase;</td>
</tr>
<tr>
<td>Iteration</td>
<td>Repetition, Test At Entry</td>
<td>Repetition, Test At Exit</td>
</tr>
<tr>
<td>for variable in range do</td>
<td>while condition do</td>
<td>repeat</td>
</tr>
<tr>
<td>sequence;</td>
<td>sequence;</td>
<td>sequence;</td>
</tr>
<tr>
<td>endfor;</td>
<td>endwhile;</td>
<td>until condition;</td>
</tr>
</tbody>
</table>
Procedure
procedure proc_name(formal_parameters);
  declarations;
begin
  sequence;
  return;
end proc_name;

Function
function func_name(formal_parameters) : result_type;
  declarations;
begin
  sequence;
  return value;
end func_name;

where formal_parameters is a list, the elements of which are separated by semicolons and have the form variable_name{, variable_name} : type

5.1.2 Operators

assignment: := \quad // var := value
arithmetic: +, -, *, /, \% \quad // add, subtract, multiply, divide, modulus
arithmetic assign: +=, -=, *=, /=, %= \quad // var op= value = var := var op value
comparison: =, \neq, <, \leq, >, \geq
logical: and, or, xor, not, andif, orelse \quad // two "short circuit" operators

5.1.3 Simple and Structured Types

Basic types include the standard integer, real, Boolean, and character. Derived types include enumerations and subranges of any ordinal type. Structure is expressed by use of array, record, and pointer types which may be arbitrarily nested. As with C++, indexing of an array and of a dereferenced pointer to an array is not distinguished; if a_p is a pointer to an array, a_p^[i] and a_p[i] are equivalent. Records can have Pascal-like variant fields. Syntax is:

Subrange
subrange_type =
  range [first..last]
  of base_type;

Enumeration
enumeration_type =
  (value{, value});

Array
array_type =
  array {range{, range}}
  of base_type;
Efficient Queries For Quasi-Ordered Databases

Record

\[
\text{record} \_\text{type} = \text{record} \\
\text{field} \_\text{name} : \text{type}; \\
\ldots \\
\text{end record}_\text{type};
\]

Variant Record

\[
\text{record} \_\text{type} = \text{record} \\
\{}\{\text{field} \_\text{name} : \text{type};\} \\
\text{case [tag]} : \text{type of} \\
\text{value} \_\text{list} : \\
\quad \text{field} \_\text{name} : \text{type}; \\
\quad \ldots ; \\
\text{others} : \\
\quad \text{field} \_\text{name} : \text{type}; \\
\quad \ldots ; \\
\text{endcase};\} \\
\text{end record}_\text{type};
\]

Pointer

\[
\text{pointer} \_\text{type} = \text{^base}_\text{type};
\]

Pointer Dereference

\[
\text{pointer}_\text{variable}\_\text{^} \\
\text{Also,} \\
\text{pointer}_\text{variable}\_\rightarrow \text{is equivalent to} \\
\text{pointer}_\text{variable}\_\text{^}.
\]

5.1.4 High-level Structured Types

Collections of elements of any other type may be built as sets, lists, and sorted sets (search trees). The syntax for declaring such collections and the operations allowed with them are as follows:

Sets

Sets are defined as unordered collections of objects with no duplicates. Basic set operations of union, intersection, symmetric difference, proper subset and superset, construction, and element containment may be expressed \(\cup\), \(\cap\), \(\subseteq\), \(\supseteq\), \{element, element\} and \(\in\), respectively.

Declaration: \(\text{type}_\text{name} = \text{set of base}_\text{type}\);

Operators: \(\cup, \cap, \setminus, \subseteq, \supseteq, \in\), and the assignment operators \(\cup=\), \(\cap=\), and \(-=\).

Constants: \(\emptyset\) — the empty set

Lists

Lists are defined as collections of objects ordered by their sequence of appearance within the list; duplicates are allowed. Operations include concatenation, construction, element reference, and sublist reference expressed by \&, \{element, element\}, \text{list(element}_\text{number}), and \text{list[element}_\text{range}], respectively.

Declaration: \(\text{type}_\text{name} = \text{list of base}_\text{type}\);

Operators: \&, \(\text{element}_\text{number}\), \{element, range\}, and the assignment operator \&=.

Constants: \([\ ]\) — the empty list
Sorted Sets

Sorted sets are defined as collections of objects ordered by means of a "key" value, with no duplicate key values allowed between two elements. This key may either be the element itself, if the sorted set is of a simple type, or is the value of one field of an element, if the sorted set is of a record type. Operations include insertion and removal of elements and search according to a key.

Insertion of an element into a sorted set either adds an entirely new element or replaces an existing element of the same key. This operation is expressed as set + element. Removal of an element from a sorted set, expressed as set - element, fails if the element is not part of the sorted set. Reference to an element by key has many search criteria and returns a pointer to that element (or null if no such element is found). The search may be for the element with key equal to the search key ('=' search); for the element with the greatest key less than the search key ('<' search); for the element either with the search key or, if not found, with the greatest key less than the search key ('\leq' search); and so on for '>' and '>=' search. Equal-to search is common enough to be expressed as sorted_set[key]; searches with other criteria are expressed as sorted_set(criterion, key).

Algorithms which perform a search for a particular element in a sorted set and then scan successive elements of that set starting at that search point are quite common. To this end, operations next and prev are provided to scan in increasing and decreasing order, respectively. If no further elements exist in that "direction" in the set, these operations return null. So that a scan may begin at either the start or end of a sorted set, the operations first and last are provided. These operations return the appropriate element, or null if the set is empty.

declaration: type_name = srt_set of base_type [ key field_name ];

operators: +, -, e, [key] — equivalent to '=' criterion below,
(criterion, key), where criterion is one of =, <, >, \leq, or \geq,
next(), prev(), first(), last(), and the assignment operators += and -=

costants: \emptyset — the empty sorted set
Appendix 5.2 Orthogonal Range Query Search

An orthogonal range query \([16][3][11]\) \(RQ\) over \(D\), a set of \(k\)-tuples, is defined as:

\[
RQ(D, a_1..b_1, a_2..b_2, \ldots, a_k..b_k) = \{ t \mid \forall i \ [1 \leq i \leq k \Rightarrow a_i \leq t_i \leq b_i] \}
\]

\(RQ\) is a multi-dimensional (or multi-keyed) search across \(D\) in which a range of values for each of the dimensions (keys) may be specified. Let \(d = |D|\) and \(m = \text{the maximum number of distinct values for any tuple component}\). The search time for the best orthogonal range query algorithms is \((\log d)^{k-1}\), insert/delete time is \((\log d)^k\), and space requirements are \(d(\log m)^{k-1}\). Because of both the time and the space requirements of these algorithms, they are useful only for small dimensionalities (realistically, dimension 2, maybe 3, due to space considerations).

Though variations exist, the basic strategy employed for building range query data structures is to have a separate set of component search trees for each of the \(\log m\) levels of the search tree for the previous component. Intuition shows that the space analysis is correct; this uses a tremendous amount of space. Regardless of this, for small dimensions the space penalty might be worth while if speed is essential for the search of a large database.

Figure 18 shows a diagram of a 2-dimensional \((k = 2)\) range query data structure similar to that shown in \([16]\). A search tree for the first component, \(c_1\), makes up the base tree \(B\). For our example, it is easiest to assume that the tuples are stored at the leaves of \(B\). Given a range of values \(a_1..b_1\) for \(c_1\), there will exist certain interior nodes \(x\) of \(B\) for which the \(c_1\) values for all \(x\)'s descendant leaves fall in the range, but not all of \(x\)'s parent's descendant leaves' \(c_1\) values fall in the range. These nodes are called \text{critical} for the range \(a_1..b_1\). In our example, nodes \(b\) and \(f\) are critical for the range \(1..3\). There are at most \(\log d\) of these critical nodes for any range, and they may all be found in \(\log d\) time.

When we have the critical nodes for a given \(c_1\) range, we know the subtrees containing all the tuples which have \(c_1\) values in that range. We now need to know which of the leaves of those subtrees also have a \(c_2\) value in the range \(a_2..b_2\). Each critical node is the root of a subtree of \(B\); the algorithm makes use of this by maintaining an auxiliary search tree (sorted by \(c_2\)) for each possible subtree of \(B\). For the root of \(B\), this is simply the entire database sorted on \(c_2\). For interior nodes \(b\) and \(c\), the auxiliary trees each contain half the tuples in the database, again sorted
on $c_2$. This continues for each level in $B$: for each level in $B$, there is a corresponding level of auxiliary trees which sorts the entire database by $c_2$.

Given the critical nodes for the $c_1$ range, we find those nodes' corresponding auxiliary trees and search each of them for those children with $c_2$ values in the desired range. Since each auxiliary tree contains only those nodes with $c_1 \in [a_1..b_1]$, the result of the $c_2$ search is exactly what we desire. For example, let us continue our example and search for those tuples with $c_1 \in [1..3]$ and $c_2 \in [4..6]$. We know that the critical nodes for the $c_1$ range are $b$ and $f$. Therefore, we search their auxiliary trees for all nodes with $c_2$ values in the range 4..6. We find one such tuple, $(1,6)$, in $b$'s auxiliary tree, and no matching tuples in $f$'s auxiliary tree. This single tuple is thus the answer to our query. Higher dimension queries apply this procedure recursively, treating $c_1$'s auxiliary nodes as new base trees with their own auxiliary nodes.
6. Bibliography


