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Andreas D. Bovopoulos and Saied Hosseini-Khayat

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Department of Computer Science
Washington University
Campus Box 1045
One Brookings Drive
Saint Louis, MO 63130-4899

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Andreas D. Bovopoulos and Saied Hosseini-Khayat
Computer and Communications Research Center
Washington University, St. Louis, MO 63130-4899

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ABSTRACT

It is shown that when a continuous buffer is driven by a semi-Markov modulated fluid flow source(s), the stationary distribution of the buffer content is governed by the same differential equation describing the distribution for continuous time Markov modulated fluid source(s) [1]. It is also shown that the same techniques can be utilized to decompose and solve the eigenvalue problem associated with the differential equation [6]. Finally it is shown that the stationary distribution of buffer content depends only on the mean time spent by each multiplexed semi-Markov source in each state.

Key Words: FLUID SOURCE MODEL; CONTINUOUS BUFFER; SEMI-MARKOV MODULATED FLOW

1 Introduction

One of the important problems in the implementation of broadband telecommunication networks is the selection of appropriate traffic models which capture the important features of complex sources and at the same time allow managable analysis. There has been much effort to analyze the behavior of multiplexers (buffers) under different traffic sources. However, in most cases the exponential distribution with its nice properties have played a key role.

One model that has been studied is the fluid flow model, in which sources generate a continuous flow of traffic in time and buffer content changes by continuous amounts. An advantage of this approach is that the computational complexity of the problem is independent of the buffer size. In [1] the multiplexing of a finite number of independent and identically distributed (i.i.d.) two-state Markov modulated sources is analysed, and it is shown that there is a closed form solution for the stationary distribution of the buffer content. In [3] a problem with m i.i.d. sources and n channels alternating between idle and active periods according to exponential time distributions is analysed. Using Kronecker decomposition it is shown that the computational complexity of the problem is $O(n^3m^3)$. In [6] there are K distinct and independent general multi-state Markov modulated fluid sources multiplexed, and it shown that Kronecker decomposition can vastly reduce the complexity of the problem.

Actual ATM (Asynchronous Transfer Mode) traffic does not always fit the two-state continuous time Markov process model however. Even the assumption that source activity periods are exponentially distributed is not appropriate in ATM networks. Therefore there is a need for models and analyses that go beyond the exponential assupmtion and are still simple. A multiple state semi-Markov

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Address: P.O. Box 1045, Washington University, St. Louis, MO 63130-4899, USA. e-mail: andreas@patti.wustl.edu and saied@wuccrc.wustl.edu, FAX: (314) 935-7302

source model has more flexibility to approximate the actual traffic. In this paper it is demonstrated that the techniques used in the above-mentioned works are still useful when the exponential assumption is relaxed.

The paper is organized as follows. In Section 2, a key property of semi-Markov processes at equilibrium is discussed. In Section 3, the model of a continuous buffer is outlined. In Section 4, the differential equation which yields the distribution of buffer content at steady state is derived. In Section 5 the superposition of a number of independent and identically distributed semi-Markov traffic sources is discussed.

2 Key Idea

Let a traffic source be modelled as a fluid source that is modulated by a finite state irreducible semi-Markov process $S(t)$. The *stationary rate* at which an ergodic semi-Markov process makes a transition from state i to state j, $(i \neq j)$, is defined to be

$$q_{ij} \stackrel{\text{def}}{=} \lim_{h \to 0} \lim_{t \to \infty} \frac{P\{S(t+h) = j | S(t) = i\}}{h} \quad ,$$

It can be easily shown that the above limit exists and is well defined.

Proposition : *If the semi-Markov process is irreducible and nonlattice, then* $q_{ij} = \frac{P_{ij}}{\mu_i}$ *for all distinct* i *and* j.

Proof:

Let $Y(t)$ be the time from t until the next transition, and $Z(t)$ be the state entered at the first transition after t.

$$P\{S(t+h) = j | S(t) = i\} =$$
$$P\{Z(t) = j, Y(t) \leq h | S(t) = i\} + O(h) =$$
$$\frac{P\{Z(t) = j, Y(t) \leq h, S(t) = i\}}{Prob\{S(t) = i\}} + O(h) \quad .$$

It can be easily verified using alternating renewal theorem [5] that as t tends to ∞ the above probability converges to

$$\frac{P_{ij} \int_0^h \bar{F}_{ij}(y) dy}{\mu_i} + O(h) \quad .$$

Dividing the limit by h, letting h go to zero, and noting that $\bar{F}_{ij}(0) = 1$ one finds $q_{ij} = P_{ij}/\mu_i$. •

Now define

$$q_{ii} \stackrel{\text{def}}{=} \frac{P_{ii} - 1}{\mu_i} \quad .$$

Note that, $\sum_j q_{ij} = 0$, for all i. Given the above concept of *stationary rate*, the equilibrium probability of being in state j at time $t + \Delta t$, given that the state is i at time t, equals $q_{ij}\Delta t + O(\Delta t)$. Thus the properties of a semi-Markov process at equilibrium resemble those of Markov process. For example the forward Kolmogorov equation [5] at equilibrium yields

$$\mathbf{PQ} = \mathbf{0} \quad ,$$

where $\mathbf{Q} \stackrel{\text{def}}{=} [q_{ij}]$ and $\mathbf{P} \stackrel{\text{def}}{=} [P_0, P_1, \ldots, P_N]$ are respectively the transition rate matrix and the vector of equilibrium probabilities for the semi-Markov process.

In a later section it is shown that utilizing the above-mentioned concept, one can derive the differential equation of a continuous buffer.

3 Mathematical Model of a Buffer

Consider a buffer whose content can be represented by a real number x such that $0 \leq x \leq X$, where X is the capacity of the buffer.

Assume that there is a fluid source input to the buffer, with flow rate $\alpha(s(t))$, where $s(t)$ is the state of a finite irreducible semi-Markov process at time t. Let $S = \{1, 2, \ldots, N\}$ represent the state space of the process, and let F_{ij} be the distribution of sojourn time of the process in state i before making a transition to state j. Further, let p_{ij} be the transition probabilities of the imbedded Markov chain. The buffer is drained by a channel at rate

$$\gamma(t) \stackrel{\text{def}}{=} \begin{cases} \beta & \text{if } x > 0 \\ \alpha(s(t)) & \text{if } x = 0. \end{cases} \quad .$$

To avoid uninteresting cases, assume that there are intervals of time during which the source flow rate exceeds the channel capacity, *i.e.* $\alpha_i > \beta$ for some i, $1 \leq i \leq N$.

The rate of change of the buffer content is as follows:

$$\frac{dx}{dt} = \begin{cases} \alpha_i - \beta & \text{if } x > 0 \\ (\alpha_i - \beta)^+ & \text{if } x = 0 \end{cases} \quad .$$

Let $p_i(t,x) \stackrel{\text{def}}{=} Prob[x(t) \leq x, s(t) = i]$ and

$$\pi_i(x) \stackrel{\text{def}}{=} \lim_{t \to \infty} p_i(t,x) \quad .$$

Further, let

$$\pi(x) \stackrel{\text{def}}{=} [\pi_1(x), \pi_2(x), \ldots, \pi_N(x)] \quad .$$

The distribution of buffer content can be found as follows:

$$Prob[buffer\ content \leq x] = \sum_{i=1}^{N} \pi_i(x) \quad .$$

4 Differential Equation for a Buffer

In this section it will be shown that using the concept of stationary rates of a semi-Markov process, one can derive a differential equation similar to the one introduced in [1].

$$p_i(t + \Delta t, x) = p_i(t, x - (\alpha_i - \beta)\Delta t)(1 - \sum_{k \neq i} q_{ik}\Delta t)$$

$$+ \sum_{j \neq i} p_j(t, x + o(\Delta t)) q_{ij} \Delta t + o(\Delta t) \quad .$$

Using the previous equation, one can derive the differential equation describing the dynamic behavior of the multiplexer

$$\frac{\partial p_i(t,x)}{\partial t} = (\beta - \alpha_i)\frac{\partial p_i(t,x)}{\partial x} + \sum_j p_j(t,x) q_{ji} \quad .$$

Since the stationary behavior of the system is of interest, $\frac{\partial p_i(t,x)}{\partial t} = 0$, and $\pi_i(x) = \lim_{t\to\infty} p_i(t,x)$. Then, the last equation becomes

$$(\alpha_i - \beta)\frac{d\pi_i(x)}{dx} = \sum_j \pi_j(x) q_{ji} \quad ,$$

for all i. The last set of equations can be cast in a matrix form as follows:

$$\frac{d\pi(x)}{dx}\mathbf{D} = \pi(x)\mathbf{Q} \quad ,$$

where $\mathbf{Q} = [q_{ij}]$ and $\mathbf{D} \stackrel{\text{def}}{=} diag(\alpha_i - \beta)$. If $\mathbf{R} \stackrel{\text{def}}{=} diag(\alpha_i)$ is defined to be the flow rate matrix, $\mathbf{D} = \mathbf{R} - \beta\mathbf{I}$.

The previous differential equation gives the stationary distribution of buffer content x and is formally the same equation as those found in the literature on fluid flow model such as [1], [3] and [6]. In these references the equation is derived under the condition that the sources is described by a N-state continuous time Markov process. Here it is proven that the same equation holds if the source is N-state semi-Markov process.

The method for solving the above differential equation has been extensively discussed in the above mentioned references. If $\{z_i\}$ and $\{\phi_i\}$ is the set of eigenvalues and corresponding eigenvectors of the matrix $\mathbf{QD}^{-1}$, then the solution of the above differential equation is of the form

$$\pi(x) = \sum_i a_i \phi_i e^{z_i x} \quad .$$

When the dimension of the matrix $\mathbf{QD}^{-1}$ is small, there are standard methods for the computation of the eigenvalues and the eigenvectors. When the dimention of the matrix is large, numerical stability problems might arise. As will be demonstrated in the next section, in the important special case in which the source is the superposition of K *independent* semi-Markov processes, computational problems can be avoided because the above equation can be decomposed into K equations of smaller dimension. This property has been known and used in the context of the superposition of continuous time Markov chains [6]. The results will be reported here for completeness.

5 Superposition of Sources

Suppose that there are K semi-Markov modulated fluid flow sources input to the buffer.
Let $S_i = \{1, 2, \cdots, N^{(i)}\}$ represent the state space of the i^{th} source, and let $s_i(t)$ be the state of the i^{th} source at time t. At time t the flow rate of the i^{th} source is $\alpha_i(s_i(t))$.

The flow rate of the superposition of the sources at time t is

$$\alpha(s(t)) = \sum_i \alpha_i(s_i(t)) \quad .$$

The only assumption used in the derivation of the differential equation for a single source

$$\frac{d\pi(x)}{dx}\mathbf{D} = \pi(x)\mathbf{Q}$$

was that the transitions rates between states were well-defined and that the probability that more than one transitions happened in an interval of length h is an $o(h)$ event. When the source is the superposition of K *independent* semi-Markov fluid sources, the probability that more than one transitions take place in an interval of length h is once again an $o(h)$ event.

Let $s(t) = (s_1(t), \cdots, s_K(t))$ be the state of the superposition of the K sources, with values in the state space $S = S_1 \times S_2 \times \cdots \times S_K$. If the vectors describing the superposition of the K sources are arranged in a lexicographical order and then labelled with integers $1, 2, \cdots, N$, where $N = \prod_{i=1}^{K} N^{(i)}$, one can see that the following relations among generators and flow rate matrices of individual sources and of the superposition source exist:

$$\mathbf{Q} = \mathbf{Q}_1 \oplus \mathbf{Q}_2 \oplus \cdots \oplus \mathbf{Q}_K$$

and

$$\mathbf{R} = \mathbf{R}_1 \oplus \mathbf{R}_2 \oplus \cdots \oplus \mathbf{R}_K \quad ,$$

where $\oplus$ is the Kronecker sum [2], and $\mathbf{R}_i$ and $\mathbf{Q}_i$ are the corresponding flow rate and generator matrices of the i^{th} source.

For completeness the definitions of Konecker sum and product are given as follows [4],[2]. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be square matrices of dimension N and M respectively. The Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ is defined to be

$$(\mathbf{A} \otimes \mathbf{B})_{ij,kh} = a_{ik} b_{jh}$$

The Kronecker sum of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$(\mathbf{A} \oplus \mathbf{B})_{ij,kh} = a_{ik}\delta_{jh} + \delta_{ik} b_{jh}$$

or alternatively,

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_M + \mathbf{I}_N \otimes \mathbf{B} \quad ,$$

where δ_{ij} is Kronecker delta and $\mathbf{I}_N$ is an N-dimensional identity matrix. Both Kronecker product and sum are of dimension $NM \times NM$.

The differential equation describing the behavior of the superposed sources is of the form

$$\frac{d\pi(x)}{dx}\mathbf{D} = \pi(x)\mathbf{Q}$$

where $\mathbf{D} = \mathbf{R} - \beta\mathbf{I}$. The solution of the above differential equation is of the form

$$\pi(x) = \sum_{i=1}^{N} a_i \Phi_i e^{z_i x} \quad ,$$

where $\{\Phi_j\}$ and $\{z_j\}$ represent the set of all the eigenvectors and the set of the corrsponding eigenvalues of the matrix $\mathbf{Q}\mathbf{D}^{-1}$.

Assume that $\Phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_K$, where $\otimes$ is the Kronecker product. Then,

$$z\Phi\mathbf{D} = \Phi\mathbf{Q} \quad .$$

Using the matrix identity,

$$(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \cdots \otimes \mathbf{A}_K)(\mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \cdots \oplus \mathbf{B}_K) =$$
$$\sum_{i=1}^{K} \mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \cdots \otimes (\mathbf{A}_i\mathbf{B}_i) \otimes \cdots \otimes \mathbf{A}_K \ ,$$

and assuming that $\beta = \sum_{i=1}^{K} \beta_i$, the previous equation becomes

$$\begin{aligned}(z\phi_1\mathbf{R}_1 - z\beta_1\phi_1 - \phi_1\mathbf{Q}_1) \otimes \phi_2 \cdots \otimes \phi_K + \\ \phi_1 \otimes (z\phi_2\mathbf{R}_2 - z\beta_2\phi_2 - \phi_2\mathbf{Q}_2) \otimes \phi_3 \cdots \otimes \phi_K + \cdots \\ + \phi_1 \otimes \phi_2 \otimes \cdots \otimes (z\phi_K\mathbf{R}_K - z\beta_K\phi_K - \phi_K\mathbf{Q}_K) = 0 \quad .\end{aligned}$$

The above equality holds if

$$z\phi_i(\mathbf{R}_i - \beta_i\mathbf{I}) = \phi_i\mathbf{Q}_i \quad ,$$

or equivalently,

$$z\phi_i\mathbf{D}_i = \phi_i\mathbf{Q}_i$$

for all i, $1 \le i \le K$. Thus it is shown that the eigenvalue problem corresponding to the superposition of K independent sources can be equivalent to K smaller scale eigenvalue problems.

6 Conclusion

In this paper the stationary transition rate matrix **Q** and the flow rate matrix **R** of a semi-Markov process were derived. Given a semi-Markov modulated fluid source, the differential equation describing the distribution of the buffer content in steady state was then derived. The application of Kronecker decomposition to the reduction of the computatinal complexity of the problem was mentioned. Finally it was observed that the stationary distribution of buffer content depends only on the mean time spent by each of the multiplexed semi-Markov sources in each state.

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