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Systemic Risk in Financial Networks

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Dedicated to my parents.

ABSTRACT OF THE DISSERTATION

Systemic Risk in Financial Networks

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In this dissertation, I have used the network model based approach to study systemic risk in financial networks. In particular, I have worked on generalized extensions of the Eisenberg--Noe [2001] framework to account for realistic financial situations viz. pricing of corporate debt while accounting for network effects, asset liquidation mechanisms during fire sale, dynamic clearing, and impact of contingent payments such as insurance and credit default swaps.

First, I present formulas for the valuation of debt and equity of firms in a financial network under comonotonic endowments. I demonstrate that the comonotonic setting provides a lower bound to the price of debt under Eisenberg--Noe financial networks with consistent marginal endowments. Special consideration is given to the setting in which firms only invest in a risk-free bond and a common risky asset following a geometric Brownian motion.

Next, I develop a framework for price-mediated contagion in financial systems wherein banks are forced to liquidate assets to satisfy a risk-weight based capital ratio requirement. I

consider the case of multiple illiquid assets and develop conditions for the existence and uniqueness of equilibrium prices. I show that the sensitivity analysis of these prices with respect to the system parameters can be written as a fixed point problem and prove the existence and uniqueness of a solution to this problem. I also develop a methodology to quantify the cost of regulation faced by different banks in this setting. Numerical case studies are provided to study the application of this model to data.

Furthermore, I extend the network model of financial contagion to allow for time dynamics in both discrete and continuous time. Emphasis is placed on the continuous-time framework and its formulation as a differential equation driven by the operating cash flows. I provide results on existence and uniqueness of firm wealths under the discrete and continuous-time models and discuss the financial implications of time dynamics. In particular, I focus on how the dynamic clearing solutions differ from those of the static Eisenberg--Noe model.

Finally, I study the implications of contingent payments on the clearing wealth in a network model of financial contagion. I first consider the problem in a static framework and develop conditions for existence and uniqueness of solutions as long as no firm is speculating on the failure of other firms. In order to achieve existence and uniqueness under more general conditions, I introduce a dynamic framework and demonstrate how this setting can be applied to problems that are ill-defined in the static framework.

Chapter 1

Introduction

1.1 Background

1.1.1 The interconnected financial system and the 2008 financial crisis

The modern day financial system is a highly interconnected network. The connections that exist within this network might be through direct channels such as interbank debt linkages or through indirect channels such as overlapping portfolios. These shared connections open up avenues for shared prosperity but also introduce potential channels for contagion. These avenues of contagion become particularly significant when initial losses for one bank or a particular asset class propagate through these linkages and affect other organizations, assets, economic sectors and countries and result in a financial crisis, such as the 2008 financial crisis.

The 2008 financial crisis began with losses in the housing market in the United States. These losses led to a rapid devaluation of financial instruments such as mortgage-backed securities and precipitated a crisis in the subprime mortgage market. Banks with a large exposure to such assets faced a liquidity crunch and the contagion spread to the financial sector with the collapse of the investment bank Lehman Brothers on September 15, 2008. As a consequence, AIG faced bankruptcy due to the large payouts it was required to make on its Credit Default Swap(CDS) contracts referencing Lehman and mortgage backed securities. The sudden call to pay out the CDS contracts put great pressure on AIG, which traditionally had a thin capital base. The panic that ensued caused a severe liquidity crisis and a run on the money market funds. The U.S. Treasury was eventually required to provide a 700 billion dollar bail-out to prevent a possible collapse of the world financial system. The 2008 financial crisis was followed by the The Great Recession, a global economic downturn. This resulted in tremendous losses across all sectors of the economy such as income, stock values, home values and employment and caused massive hardships for people.

The 2008 financial crisis prompted action from the policymakers. The U.S. Congress passed the Dodd-Frank Wall Street Reform and Consumer Protection Act to reorganize the financial regulatory system. Some of the provisions in the act included creating agencies such as the Financial Stability Oversight Council (to identify threats to the financial stability of the United States) and the Office of Financial Research (to conduct research and standardize data collection), providing the Federal Reserve with additional powers to regulate systemically important institutions and restricting banks from certain kinds of speculative investments. On a global scale, the Bank for International Settlements developed the Basel III Accords to promote stability in the international financial system.

From an academic standpoint, the 2008 financial crisis has prompted the need to understand the mechanisms and channels which propagate financial contagion, to construct indicators which identify such contagion beforehand and to develop measures for the effective prevention and mitigation of such contagion. It is hoped that such research will aid in the development of prudent policy and regulations that will prevent financial crises from happening in the future.

1.1.2 Systemic risk and channels of contagion

A major characteristic of the 2008 financial crisis was the amplification of losses through interconnections within the financial system. The field of systemic risk studies these channels and mechanisms within the financial system through which an initial shock to the system gets greatly exacerbated and can ultimately threaten the stability of the entire system. Thus *systemic risk* refers to the "risk or probability of breakdowns (losses) in an entire system as opposed to breakdowns in individual parts or components and is evidenced by comovements (correlation) among most or all the parts." ([76]). The researchers in this field try to understand and model contagion channels and loss amplification mechanisms as have been observed in past crises. The goal of this is to develop policy and regulatory measures to prevent such crises in the future.

In the systemic risk literature, researchers have identified different channels through which financial contagion spreads across the network. In the remainder of this subsection, I describe some of these interconnections and mechanisms which might act as loss-amplifying channels in financial systems. This will provide the necessary context for the mathematical modeling undertaken in this dissertation. I want to stress that this is by no means an exhaustive list; rather, it is intended to provide a flavor of the myriad interconnections in a financial system.

The first of these channels is the direct linkages or local interactions among financial organizations. For example, if we consider a network with bidirectional *debt linkages*, the failure of a bank will result in losses for its counterparties which may now default on their obligations. Thus the initial shock may propagate across the entire network through these direct linkages and result in cascading failures.

A different form of linkages exist between banks and insurance companies and are formed and resolved in a way that is different from normal debt contracts. A typical example of such a linkage is a credit default swap [CDS]. A credit default swap is a contract in which a buyer pays a premium to a seller in order to protect itself against a potential loss due to the occurrence of a credit event that affects the value of the contract's underlying reference obligation, e.g., a corporate or sovereign bond. The contract specifies the credit events that will trigger payment from the seller to the buyer. Whereas such instruments can be used to hedge risks, they may also be used for speculative purposes to put a short position on the credit markets. The important role that such *contingent linkages* plays is demonstrated by the financial crisis of 2008. As that crisis unfolded, AIG faced bankruptcy after the failure of Lehman Brothers due to the large payouts it was required to make on its CDS contracts referencing Lehman and mortgage-backed securities. Eventually AIG had to be rescued by the U.S. Department of Treasury so as to avoid jeopardizing the financial health of firms that had bought CDSs from AIG.

The banks in a financial system are also connected through *cross-holdings*. Banks often own the equity of other banks and these linkages can potentially trigger a financial vicious cycle. For example if Bank A owns a part of Bank B, the deterioration of the financial health of Bank B due to a default by Bank A will result in the damage of the financial health of

Bank A itself due to its ownership of Bank B. Such cyclical dependencies within the financial network will result in amplification of losses due to feedback effects.

A different channel of contagion comes through *fire sale* spillovers. Those originate when a firm is forced to liquidate its assets to meet some obligation or regulation. As firms hold overlapping portfolios, this causes impacts globally to all other firms due to marked-to-market accounting. These firms are now forced to liquidate their assets, exacerbating the crisis. An important factor in the origin of fire sale is the unintended consequence of capital regulations in the form of capital ratio or leverage ratio. Due to these regulatory constraints, banks might be forced to deleverage, setting off a vicious cycle of contagion: the develeraging results in the depreciation of prices, causing marked-to-market losses and further weakening the position of banks, which are forced to sell more. Such deleveraging occured in a large scale in the 2008 financial crisis, resulting in amplification of losses.

The loss amplification due to the cascading defaults during a financial crisis can be further exacerbated through default mechanisms such as *bankruptcy costs*. When a bank defaults, there are legal and administrative costs associated with such an event. Additionally, the bankruptcy proceedings will result in delays in payment to the creditors. Thus the combination of these costs and delays will increase the likelihood of defaults in the system and magnify the overall losses.

In a financial network, all of these channels may become active simultaneously, and hence a minor initial shock may be severely amplified and potentially bring down the entire financial system, thereby causing significant damage to the economy.

1.1.3 Network Models: Approach and Challenges

The 2008 financial crisis showed the severe impacts that systemic crises can have on the financial sector and the economy as a whole. As the costs of such events is tremendous, the modeling of such events is imperative. Post 2008, there has been a surge in the research in the domain of systemic risk. A number of modeling frameworks and interdisciplinary approaches have been proposed. One of the most significant modeling approaches in systemic risk is the network model based framework. In the remainder of this subsection, I discuss this particular approach. A more detailed discussion on individual papers is provided in Section 1.2.

The network model of systemic risk has been pioneered by the seminal paper of [45]. [45] proposes a weighted graph framework to model the spread of defaults in the financial system. In this approach, banks' liabilities are modeled through the edges. The banks use their liquid assets to pay off these liabilities; unpaid liabilities may cause other banks to default as well. Under simple conditions, [45] proves the existence and uniqueness of the clearing payments and develops an algorithm for computing the same. [45] provides a simple foundational model and boosts of attractive features from mathematical and financial standpoints. From a mathematical standpoint, it provides results on existence and uniqueness of solutions under a general network setting. It further provides an elegant algorithm for the computation of the solution. From a financial viewpoint, the Eisenberg--Noe framework is able to capture the heterogeneity of real world networks and encode an intuitive notion of the cascading default process, which makes it suitable for the design of stress testing algorithms. Thus the Eisenberg--Noe model provides financial intuition and is suitable for use in an operational framework, without compromising on mathematical rigor.

Despite being elegant, the framework of [45] is undoubtedly simple and it only considers local interactions among banks in the form of debt contracts. Hence there have been efforts to use this baseline model and adopt it for the study of other channels of contagion and more realistic financial mechanisms such as bankruptcy costs, cross-holdings, fire sale etc. These extensions form the crux of the network model of systemic risk. In these extensions, the focus is on deriving the clearing solution (payment/price) and to study which market participants default during the clearing process. Thus, much emphasis is placed on developing conditions for the existence and uniqueness of the clearing solution. The stress tests, designed using this approach, typically consist of applying shocks to different parts of the financial network and identifying contagion through a pre-defined measure such as the number of defaults or the resultant wealth. These models typically inherit the benefits of [45]: they are able to capture properties of real world financial networks, they are comparatively easier to operationalize, and they can be used to develop stress tests based on rigorous mathematical results instead of relying on heuristics. In fact, many central banks and regulatory bodies have incorporated these network models into their stress tests of the financial system (see, e.g., [9, 73, 25, 50, 101, 61]).

However, these models are not free from challenges. One such challenge is that in models with more realistic financial mechanisms a consolidation of the desirable mathematical and financial features is often difficult to achieve. In fact, in many of these works incorporating practical financial systems, results on uniqueness and, sometimes, existence can not be obtained. Despite being challenging from a mathematical standpoint, this has provided us additional insights into financial systems. First, it might be perfectly natural to expect multiple equilibria in several situations owing to the non-linearity of the financial system and often, even though there is not a unique solution, a worst-case analysis can be performed to bound the system behaviour. Secondly, it is often observed that restricting the system parameters to certain domains can result in uniqueness of solutions. Hence, this can be seen as a method to calibrate system parameters instead of using heuristics.

A second challenge stems from the fact that the results of the network models are very much dependent on the system parameters, which are often not exactly known. Despite the progress that has been made in recent years, the regulators still face significant legal and logistical hurdles in the data collection process. A second factor which hinders this process is the fact that most of the systemic risk models are very nascent and the discovery of which data is actually required in this analysis is very much an ongoing process. In that context, the stress testing results using network models are very much dependent upon how these parameters are calibrated. Owing to this uncertainty, it is imperative that we have an understanding of how a variation in these parameters might affect the results. Thus it is very important to perform *sensitivity analysis* with respect to the system parameters. A different way to look at this problem is that the models will inform the data collection process. Thus regulators can look at these models and identify which data they are required to collect for efficient oversight.

In this dissertation, I have employed the network model based approach for the study of systemic risk in financial networks. In particular, I have extended the baseline model of [45] to study corporate debt pricing and dynamic clearing as well as model contagion channels such as contingent payments and fire sale. The main contributions of this dissertation are discussed in Section 1.4.

1.2 Literature Review

In this section, I review the literature on systemic risk. In particular, my focus will be on the network model of systemic risk that has been pioneered by the seminal paper of Eisenbeg--Noe ([45]). The majority of this literature review will revolve around [45] and its extensions. Other approaches such as mean-field models are not discussed separately but highlighted as a comparison to the network models in particular situations.

1.2.1 Eisenberg--Noe Model(2001)

Interbank networks were studied first in [45] to model the spread of defaults in the financial system. This work considers debt linkages between different banks in a financial system and models these linkages using a bidirectional graph. The failure of a bank will result in losses for its counterparties, which may now default on their obligations. Thus the initial shock may propagate across the entire network through these debt linkages and result in cascading failures. This interdependency of realized (clearing) payments is modeled as a fixed point problem. [45] proves the existence and uniqueness of the clearing payments and provides an elegant algorithm for the computation of the same. The mathematical framework used in this model is discussed in Section 1.3. The Eisenberg--Noe model provides a foundational framework in the systemic risk literature. The simple but elegant baseline model has been extended in many directions to capture complexities in the financial system.

1.2.2 Network Valuation Models

An important question in financial mathematics is the valuation of corporate debt and determination of the market capitalization. Traditionally, two alternative approaches have been followed. The first of these approaches is the *structural approach*, which was introduced in the seminal paper of [86]. The second approach is the *reduced form* approach where the modeling is done from a more statistical perspective, taking into account the historical default intensities and the underlying stochastic factors ([44]).

The valuation in structural models is generally done from the perspective of an individual bank. However, this does not take into account the myriad interconnections that exist within a financial system and might result in gross misspecification in firm health. The Eisenberg--Noe framework provides an elegant framework to incorporate these interconnections and has been considered in multiple works. [99] considers the problem in a setting with cross-holdings and [58] generalizes that setting to include multiple seniority classes and derivatives; these works rely solely on Monte Carlo simulation for numerical computations. [69] considers a model with random exogenous shocks on the assets of each bank. [70] gives an interpretation of this problem as one with a "hidden" Collaterized Debt Obligations (CDO). [16] considers the problem of network valuation and the effect of a random exogenous shock in the external assets. A different approach is considered in [75] where a PDE method is used for the case where the banks' assets are driven by correlated multidimensional Brownian motions with drift. A reduced form model for studying distress contagion and marked-to-market write-downs of debt contagion has been studied by [103].

Network valuation adjustment mechanisms, using a SDE approach, have been studied in [21, 22].

1.2.3 Dynamic Contagion Models

[45] considers a static framework. Several works have considered the extension of this framework to include multiple clearing dates. This has been studied directly in [27, 57]. Additionally, [80] considers a similar approach to model financial networks with multiple maturities. [52] provides another approach to financial networks with multiple maturities by considering each clearing date as a different asset. All of these works, however, only consider clearing at discrete times. [98] presents a continuous-time clearing model that exactly replicates the static Eisenberg--Noe framework.

Dynamic contagion has been studied using mean field models, which represent an alternative approach to the network models in the systemic risk literature. [60] provides a model of agents who revert to the ensemble mean to provide understanding of "systemic risk events" in which many firms fail. Similar mean field diffusion models without controls were studied in, e.g., [59, 65, 66]. In contrast, mean field and stochastic games have been proposed for the study of systemic risk in, e.g., [30, 29]. In such models the firms are allowed to borrow from (or lend to) a central bank, the amount of which is optimized to minimize a quadratic cost function. Thus the choice of borrowing and lending provides an optimal control problem beyond the simpler mean field model of [60]. [90] proposes a separate particle system model with mean field interactions.

1.2.4 Contingent Payment Models

[45] considers interconnections through bilateral debt contracts. However, linkages formed between banks and insurance companies can also act as potential channels of financial contagion. These linkages are formed and resolved in a way that is different from normal bank loans, e.g. Credit Default Swaps. The framework in [45] is not suitable to deal with such contingent payments and several extensions have been proposed to incorporate such interconnections in the Eisenberg--Noe framework. [19, 18] show that the clearing vector in the presence of generalized CDS contracts is not well-defined and need not exist. They further propose a static setting to model CDS payments and give sufficient conditions on the network topology for existence of a clearing solution. [81] considers such a model in a static framework and proposes a method to rewrite some classes of network topologies as an Eisenberg--Noe system. [17, 36] model CDS payments, but most of the reference entities are required to be external to the financial system. [78] models reinsurance networks and studies the implications of network topologies on existence and uniqueness of the liabilities and clearing payments. A different approach has been taken in [74] in which a stochastic setting is used to analyze contagion caused by credit default swaps. The role of credit default swaps in causing financial contagion has been captured in several empirical studies, see e.g. [92, 84].

1.2.5 Models with bankruptcy cost

In [45], the defaulting banks can use the entire proceeds from other banks to pay off their obligations. However, in reality this is not the case. The defaulting banks suffer additional losses due to legal and administrative fees and there are often delays in payments to creditors. This additional loss is termed as bankruptcy cost. [96] extends the framework of [45] by incorporating bankruptcy costs in the default mechanism. It shows the existence of a greatest and least fixed point and presents an algorithm for the efficient computation of the same. [104] considers a joint framework to study the impact of cross-holdings, bankruptcy cost and fire sale and concludes that bankruptcy costs are a main driver of systemic risk. [67] finds that "bankruptcy costs must be quite large in order to have an appreciable impact on expected losses as they propagate through the network". Other notable works which consider bankruptcy costs in their models include [46, 47].

1.2.6 Models with cross-holding

Beyond debt linkages, banks are connected with each other via equity claims. [47] considers an extension of [45] to accommodate cross-holdings and develops an algorithm for the determination of clearing solutions. Furthermore, it discusses the impact of seniority structure of debt on the clearing solutions. [46] considers a model with cross-holdings and studies the impact of diversification and integration on the clearing solutions. [104] finds that cross-holdings can stabilize the system against default contagion.

1.2.7 Fire sale models

An important channel of contagion in financial networks comes through indirect connections or global interactions among organizations, e.g., fire sale spillovers. [33] considers the liquidation problem in the context of a capital adequacy ratio. [6] studies the fire sale problem in a single asset setting when banks are forced to liquidate assets to meet debt obligations and shows the existence and uniqueness of a solution to the resultant fixed point problem. [51] considers a multi-asset extension to [6] and shows the existence of a Nash equilibrium to the the joint payment-pricing-liquidation problem. [54] develops an extension to [51] where banks, in addition to meeting their debt obligations, must satisfy a leverage ratio. [26] considers the fire sale problem in a single asset setting where banks are required to satisfy a risk-weighted capital ratio. [53] considers the price-mediated contagion problem in a continuous time setting and provides results on existence and uniqueness as well as analytical bounds under a random setting. [71] develops empirical measures of the vulnerability and connectivity of banks and discusses intervention measures to reduce the vulnerability to fire sale contagion. [38] develops an operational framework for quantifying the effects of deleveraging and "shows that such indirect contagion effects may modify the outcome of bank stress tests and lead to heterogeneous bank-level losses which cannot be replicated in a stress test without deleveraging effects". [43] constructs an index of aggregate vulnerability and discusses the connection of this index to SRISK, one of the most prominent systemic risk measures.

1.2.8 Implications of network topology

The importance of the network topology in the Eisenberg--Noe framework has been explored to identify structures that tend to propagate default or alternatively dampen it. [1] shows the 'robust-yet-fragile' tendency of financial networks. [28] considers the problem of network topology using majorization-based tools. [46] studies the effect of diversification and integration on financial contagion.

An alternative approach to study the effect of network topology is the random graph model. [5] derives rigorous asymptotic results for the magnitude of contagion and gives an analytical expression for the asymptotic fraction of defaults, in terms of network characteristics. [41] derives conditions under which local shocks can propagate through the network. [8] gives bounds on the size of the cascade and derives testable conditions for this cascade to be small.

1.2.9 Calibration of network models

The results on financial contagion in the network models is dependent upon the system parameters. Unfortunately, these parameters might not be exactly known due to logistical and legal issues. For example, in the Eisenberg--Noe framework ([45]), the nominal liabilities matrix L is not often exactly known and thus needs to be estimated ([72, 50, 10]). An entropy maximizing approach has been proposed in [102] and has been used to estimate the liabilities matrix from balance sheet data in [102, 49, 40]. However, empirical literature has shown that the real world liabilities matrix L might be quite different in contrast to the homogeneous network that is produced by that approach ([88, 37]). A minimum density method, based on minimizing the total number of edges consistent with the aggregated assets and liabilities, has been proposed in [10]. [63] proposes a Bayesian method to estimate the liabilities matrix, given the total liabilities. This method has been applied to reconstruct credit default swap markets in [64].

1.2.10 Sensitivity analysis

As discussed in the previous subsection, the results on identifying contagion and designing stress tests using network models is dependent upon system parameters that are often not exactly known and need to be calibrated. Hence, sensitivity analysis with respect to the system parameters becomes an important exercise. [82] performs sensitivity analysis of the clearing vector with respect to the initial net worth of each bank in the framework of [45]. [55] considers the sensitivity problem with respect to estimation errors in the relative liabilities matrix in the Eisenberg--Noe framework and poses it as a fixed point problem. It further studies worst case and probabilistic interpretations of the perturbation analysis.

1.2.11 Adoption of network models in stress tests

Central banks and regulatory bodies have incorporated these network models into their stress tests of the financial system. [9] discusses the results of using MacroFinancial Risk Assessment Framework (MFRAF), a stress testing framework, to the stress testing scenario used in the 2013 Canada Financial Sector Assessment Program led by the International Monetary Fund. [61] uses the network model for a study of the UK banks. [73] studies the emergence of interbank networks using data of the European banks. Other notable works in this domain include [25, 50, 101]).

1.3 A Brief Review of the Eisenberg--Noe Model

In this section, I review the mathematical framework used in [45]. This will provide insights into the type of mathematical modeling and arguments that are typically employed in the network model based approach. Additionally, I show that the Eisenberg--Noe problem can be reformulated as a fixed point in the clearing wealth, rather than the clearing payment.

1.3.1 Notation

I begin with some simple notation that will be consistent for the entirety of this dissertation. Let $x, y \in \mathbb{R}^n$ for some positive integer n, then

$$x \wedge y = \left(\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n)\right)^\top,$$

$x^- = -(x \wedge 0)$, and $x^+ = (-x)^-$. Further, to ease notation, I will denote $[x, y] := [x_1, y_1] \times [x_2, y_2] \times \ldots \times [x_n, y_n] \subseteq \mathbb{R}^n$ to be the n-dimensional compact interval for $y - x \in \mathbb{R}^n_+$. Similarly, I will consider $x \leq y$ if and only if $y - x \in \mathbb{R}^n_+$.

Throughout this dissertation I will consider a network of n financial institutions. I will denote the set of all banks in the network by $\mathcal{N} := \{1, 2, \ldots, n\}$.

1.3.2 Mathematical framework of Eisenberg--Noe Model (2001)

[45] considers linkages in the form of bilateral debt contracts. These are represented mathematically by the *nominal liability matrix* $L \in \mathbb{R}^{n \times n}_+$. Any bank $i \in \mathcal{N}$ may have obligations $L_{ij} \geq 0$ to any other firm $j \in \mathcal{N}$. It is assumed that no firm has any obligations to itself, i.e., $L_{ii} = 0$ for all firms $i \in \mathcal{N}$. Thus the *total liabilities* for bank $i \in \mathcal{N}$ is given by $\bar{p}_i := \sum_{j \in \mathcal{N}} L_{ij}$ and relative liabilities $\pi_{ij} := \frac{L_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and arbitrary otherwise; for simplicity, in the case that $\bar{p}_i = 0$, we will let $\pi_{ij} = \frac{1}{n-1}$ for all $j \in \mathcal{N} \backslash \{i\}$ and $\pi_{ii} = 0$ to retain the property that $\sum_{j \in \mathcal{N}} \pi_{ij} = 1$. On the other side of the balance sheet, all firms are assumed to begin with some amount of external assets $x_i \geq 0$ for all firms $i \in \mathcal{N}$. Thus the complete financial system can be characterized by the pair (L, x) or equivalently by the triple $(\Pi, \bar{p}, x)$.

The central question explored in this work is the determination of a clearing payment p. To accomplish this, [45] considers the following stylized rules of bankruptcy:

(i) *Limited liabilities*: the total payment made by any firm will never exceed the total assets available to the bank.

(ii) *Priority of debt claims*: the shareholders of a firm receive no value unless all its debts are paid in full.

(iii) *All debts are of the same seniority*: in case a bank defaults, debts are paid out in proportion to the size of the nominal claims.

I will assume that these rules hold for the remainder of this dissertation. The resultant *clearing payments*, is represented by the following mapping $\Phi : [0, \bar{p}] \mapsto [0, \bar{p}]$, which is given by

$$\Phi(p) = \bar{p} \wedge \left(x + \Pi^\top p\right). \tag{1.1}$$

That is, each bank pays the minimum of what it owes ($\bar{p}_i$) and what it has ($x_i + \sum_{j\in\mathcal{N}} \pi_{ji} p_j$). A clearing payment is defined by a fixed point p^* of Equation (1.1) satisfying

$$p^* = \Phi(p^*). \tag{1.2}$$

The existence of a clearing payment is proved in the following proposition which is taken from Theorem 1 of [45].

Proposition 1.3.1. *Consider a financial system given by $(\Pi, \bar{p}, x)$. There exists a greatest and least clearing payment $p^\uparrow \geq p^\downarrow$.*

Proof. $\Phi(p)$ is monotonically non-decreasing in p and its domain is compact. So the Tarski fixed point theorem can be used to get a greatest and least clearing payment $p^\uparrow \geq p^\downarrow$. □

The clearing payment, given by (1.2) is not generally unique. However, [45] provides very simple conditions under which uniqueness is guaranteed. For this, the following definitions need to be considered which is given in [45].

Definition 1.3.2. *A set $S \subset \mathcal{N}$ is a* surplus set *if no node in the set has any obligations to any node outside the set and the set has positive operating cash flows, that is, for all $(i, j) \in S \times S^c$, $\Pi_{ij} = 0$ and $\sum_{i\in S} x_i > 0$.*

Definition 1.3.3. *For each node $i \in \mathcal{N}$, define the* risk orbit *of node i, given by $o(i)$, as follows: $o(i) = \{j \in \mathcal{N} | \text{there exists a directed path from } i \text{ to } j\}$.*

The preceding definitions can now be used to define a *regular* financial system.

Definition 1.3.4. *A financial system is* regular *if every risk orbit, $o(i)$, is a surplus set.*

An economic interpretation of *regularity* is the existence of some value somewhere in the financial system that can reach all nodes in the system. A sufficient condition to ensure regularity is to set $x > 0$. The uniqueness condition is now stated in the following proposition which is replicated from Theorem 2 of [45].

Proposition 1.3.5. *Consider a regular financial system. Then the clearing payment p is unique.*

Proof. The proof is given in Appendix 1 of [45]. □

[45] gives an elegant algorithm, termed as the *Fictitous Default Algorithm*, for the determination of the clearing payment. This algorithm converges to the clearing solution in, at most, n steps. It is replicated as follows:

Algorithm 1.3.6. Initialize $k = 0$, $D^0 = \emptyset$, and $p^0 = \bar{p}$.

(i) Increment $k = k + 1$;

(ii) Denote the set of insolvent banks by $D^k = \{i \in \mathcal{N} \mid x_i + \sum_{j \in \mathcal{N}} \pi_{ji} p_j^k < \bar{p}\}$;

(iii) If $D^k = D^{k-1}$ then terminate;

(iv) Define the matrix $\Lambda \in \{0, 1\}^{n \times n}$ so that

$$\Lambda_{ij} = \begin{cases} 1 & \text{if } i = j \in D^k \\ 0 & \text{else} \end{cases};$$

(v) $p^k = \hat{p}$ is the solution of the following fixed point problem:

$$\hat{p} = (I - \Lambda)\bar{p} + \Lambda(x + \Pi^\top \hat{p})$$

(vi) Go back to step (i).

[45] gives results on the comparative statics of the clearing vector in Lemma 5 which is restated in the following proposition.

Proposition 1.3.7. *The clearing payment vector p^* is a concave, increasing function of the operating cash flow vector x and the level of nominal liabilities L.*

Proof. The proof is given in Lemma 5 of [45]. □

1.3.3 Reformulation as a fixed point in clearing wealth

The resultant vector of *wealths* for all firms, in the Eisenberg--Noe setting, is given by

$$V = x + \Pi^\top p - \bar{p}. \tag{1.3}$$

Noting that payments can be written as a simple function of the wealths ($p = \bar{p} - V^-$), I provide the following proposition. I refer also to [103, 16] for similar notions of utilizing clearing wealth instead of clearing payments.

Proposition 1.3.8. *A vector $p \in [0, \bar{p}]$ is a clearing payment in the Eisenberg--Noe setting* (1.1) *if and only if $p = [\bar{p} - V^-]^+$ for some $V \in \mathbb{R}^{n+1}$ satisfying the following fixed point problem*

$$V = x + \Pi^\top [\bar{p} - V^-]^+ - \bar{p}. \tag{1.4}$$

Vice versa, a vector $V \in \mathbb{R}^{n+1}$ is a clearing wealth (i.e., satisfying (1.4)*) if and only if V is defined as in* (1.3) *for some clearing payment $p \in [0, \bar{p}]$ as defined in the fixed point problem* (1.1).

Proof. I will prove the first equivalence only, the second follows similarly.

Let $p \in [0, \bar{p}]$ be a clearing payment vector. Define the wealth vector V by (1.3), then it is clear that $V^- = \bar{p} - p$ by definition as well, i.e., $p = \bar{p} - V^- \geq 0$. Thus from (1.3) I immediately recover that the wealth vector V must satisfy (1.4).

Let $p = [\bar{p} - V^-]^+$ for some wealth vector $V \in \mathbb{R}^{n+1}$ satisfying (1.4). By construction I find

$$p = [\bar{p} - V^-]^+ = \bar{p} - \left(x + \Pi^\top [\bar{p} - V^-]^+ - \bar{p}\right)^- = \bar{p} - \left(x + \Pi^\top p - \bar{p}\right)^- = \bar{p} \wedge \left(x + \Pi^\top p\right).$$

Note that $\bar{p} \geq \left(x + \Pi^\top [\bar{p} - V^-]^+ - \bar{p}\right)^-$ can be shown trivially. □

Due to the equivalence of the clearing payments and clearing wealths provided in Proposition 1.3.8, I am able to consider the Eisenberg--Noe system as a fixed point of equity and losses rather than payments.

1.4 Main contributions

In this dissertation, I have used the network model based approach to study systemic risk in financial networks. In particular, I have proposed extensions to the base model of [45] to study channels of contagion such as fire sale and contingent payments and account for realistic financial situations such as corporate debt pricing and dynamic clearing. These models are presented under general settings without presupposing the nature of the network or system parameters. Hence, these results can be used to augment stress tests, calibrate system parameters and improve the data collection process. The main contributions of this dissertation are summarized as follows:

(i) *Valuation of debt and equity in a financial network with comonotonic endowments:* I study the problem of valuation of corporate debt and the determination of market capitalization. I follow the structural approach, introduced in the seminal paper of Merton [86]. The valuation in structural models is generally done from the perspective of an individual bank. However, this does not take into account the interconnections that exist within a financial system. As evidenced by the 2008 financial crisis, considering the risk of a single firm alone can cause gross misspecification in firm health. As such, valuing claims that take the full network effects into account is imperative in ensuring that the true risk of the claims are taken into account. To account for the network effects, I consider this valuation problem in the Eisenberg--Noe framework [45]. A common thread of the existing literature in this domain ([99, 58, 69, 70, 16]) is that explicit, analytical solutions are considered only for cases with either no direct interconnection between firms or where the number of firms in the system is very low. This is because the valuation problem, under a general endowment vector, will require partitioning the endowment space into 2^n possible default scenarios in a system with n banks. Therefore, computing the expectation suffers from the curse of dimensionality and is typically computationally intractable for realistic systems. Further complications arise if bankruptcy costs are considered along the lines of [96]. Without bankruptcy costs, the 2^n default scenarios result in the partition of the bank endowment space into 2^n mutually exclusive and convex regions. However, in the presence of bankruptcy costs, these partitioned regions are not, in general, convex. Providing any analytical solution in this case becomes very challenging even in small systems. In contrast, I focus on the setting wherein the banks have comonotonic endowments. This setting follows from a portfolio optimization perspective. Under the assumption of comonotonic endowments, the default regions can be characterized by at most n intervals on $\mathbb{R}_+$ and become tractable analytically. This tractability extends to the case where bankruptcy costs are considered. Thus the comonotonic setting allows me to explore the network effects from an analytical perspective. Furthermore, I show that the comonotonic setting provides a lower bound for the price of debt under the framework of [45]. This is particularly valuable in the context of systemic risk. I derive the closed-form expressions for the price of debt, market capitalization and and the survival probability in the case where the firms invest in a risk-free bond and a risky asset following a geometric Brownian motion. This allows me to compare this setting to the the one taken by Merton [86]. Using the 2011 European Banking Authority data, I show that the price of debt should be significantly higher under the full network effects compared to the scenario in which it is assumed that the banks will be able to pay in full. Incorporating this risk properly in the valuation of assets will ensure that there are no pricing shocks to the system at a future date.

(ii) *Price-mediated contagion through capital ratio requirements:* Fire sale is a critical source of contagion in financial systems. An important factor in the origin of fire sale is the unintended consequence of capital regulations in the form of capital ratio or leverage ratio. Due to these regulatory constraints, banks might be forced to deleverage, setting off a vicious cycle of contagion: the develeraging results in the depreciation of prices, causing marked-to-market losses and further weakening the position of banks, which are forced to sell more. Such deleveraging occured in a large scale in the 2008 financial crisis, resulting in amplification of losses ([39, 77, 83]). I develop a general mathematical and economic framework to study price-mediated contagion in a multi-asset setting where the firms liquidate assets during a crisis due to risk-weighted capital requirements. This multi-asset setting provides a suitable framework to model cross-asset contagion, which was widely observed in the 2008 financial crisis. One of the major challenges of the multi-asset setting is the effective modeling of the strategic aspect, that presents itself while performing liquidation to satisfy the capital requirements. Existing literature (e.g. [71, 43, 38]) considers proportional liquidation for this analysis. In contrast, I consider a general liquidation function with mild continuity and monotonicity requirements and am thus able to study liquidation schemes beyond proportional liquidation (e.g. utility maximizing liquidation) and encode the strategic aspect in my framework. Hence this model can be used to design stress tests beyond proportional liquidation and linear price impact. This is particularly important as the modeling of the contagion is dependent upon the choice of the liquidation function (as shown in this work). In contrast to the other static models of fire sale ([26, 54, 51, 7, 33, 38, 43, 71]), I consider two notions of pricing: the current liquidation price and the volume weighted average liquidation price. The adoption of these two separate notions of pricing, besides being realistic, significantly aides the mathematical analysis, particularly in the development of conditions for uniqueness. In fact, in contrast to existing literature on price-mediated contagion due to leverage/capital requirements ([26, 54]), I am able to prove the uniqueness of the pricing equilibrium, conditional on certain properties of the inverse demand function. I also provide an alternative characterization of the uniqueness condition in terms of a monotonicity condition on the inverse demand function and a lower bound on the risk-weights, which can be used in the calibration of the risk-weights. In contrast to other works on fire sale, I develop a mathematical framework to perform sensitivity analysis of the equilibrium prices with respect to the system parameters. This is particularly important as the system parameters are often not exactly known and hence the results are very much dependent on how the system parameters are calibrated. Using these results on sensitivity analysis, I develop a method to study the cost of regulation incurred by each bank and highlight this through an empirical study with the six banks participating in the 2015 Federal Reserve CCAR with the largest trading operations. Through a numerical simulation with a two bank two asset system, I study the effect of diversification of bank portfolio in our framework and find that diversification does not uniformly lead to a more stable system. In fact, under certain liquidation regimes, the cross-asset contagion might outweigh the benefits of diversification.

(iii) *Dynamic clearing and contagion in financial networks:* The Eisenberg--Noe model and the majority of the literature in the network model based approach consider a static framework. Hence, they fail to account for dynamic clearing in financial networks which can give an incorrect assessment of the health of the financial system. I consider a generalized extension of the Eisenberg--Noe model to allow for cash flows and obligations to be dynamic in time. I present this model in both discrete and continuous time, thus extending the frameworks of [80, 27, 57], which consider only discrete time clearing. However, my emphasis is on the derivation and the characterization of the continuous-time model as a differential equation driven by stochastic cash flows. In particular, I consider existence and uniqueness of the clearing solutions, and a numerical algorithm for finding sample paths of this clearing solution, under cash flows modeled by Itô processes. The proof of existence and uniqueness in the continuous time framework is approached in an entirely different manner than the traditional fixed point approach used in the network models. This is in contrast to the other works on multiple maturity models such as [80, 27, 57]. The main benefit of my approach is that it no longer requires strong monotonicity assumptions for existence and uniqueness which are generally required for static and discrete-time systems (that typically employ the Tarski fixed point theorem). This is also valuable for future works that may model network formation and payments as a non-cooperative game; such games may not satisfy the strong monotonicity assumptions usually considered in static and discrete-time systems, but would likely satisfy the sufficient conditions for the continuous-time framework. I provide a discussion of the financial implications of time dynamics in interbank networks and show that the dynamical system for the Eisenberg--Noe contagion model may include an inherent prioritization scheme. In particular, I find that the static Eisenberg--Noe clearing solution can be recovered in the continuous-time setting by choosing the network parameters precisely. This allows for a notion of determining the true order of defaults as opposed to the fictitious default order discussed in the static literature based on [45]. However, if the continuous-time network parameters are determined to not follow the rules for recreating the static Eisenberg--Noe setting, then the dynamic and static clearing solutions will generally not coincide. In fact, the set of defaulting and solvent institutions can be altered by rearranging the timing of obligations. As such, using the static Eisenberg--Noe framework for stress testing may result in an incorrect assessment of the health of the financial system.

(iv) *Impact of contingent payments on systemic risk in financial networks:* Contingent payments such as credit default swaps and insurance are an important part of the financial system. These instruments introduce additional linkages in financial networks, which played a vital role in the 2008 financial crisis. As that crisis unfolded, AIG faced bankruptcy after the failure of Lehman Brothers due to the large payouts it was required to make on its CDS contracts referencing Lehman and mortgage-backed securities and eventually had to be rescued by the U.S. Department of Treasury. However, despite the importance of these linkages, current models are unable to account for the conditional payment that an insurance or credit default swap contract would require. I provide a generalized theoretical framework in which to study credit default swaps and other contingent payments in the Eisenberg--Noe setting. I focus on existence and uniqueness of the clearing vectors under contingent payments without presupposing the nature of those payments or making strong assumptions on the network topology. This is in contrast to the existing literature on CDS network models ([19, 18, 81]) in which there is no guarantee that the realized networks would obey the required conditions. Hence it is paramount to develop results for a general network, irrespective of the topology. I do this by first considering the problem in a static framework where all claims are settled simultaneously and develop conditions to provide existence and uniqueness of the clearing wealth. Further, sensitivity analysis and financial implications are considered in this setting. I find that the static framework is suitable only for a certain class of networks and I cannot guarantee the existence of a clearing solution beyond these systems. Indeed, the problem often becomes ill-defined from a financial standpoint. Hence I introduce a dynamic framework. This setting ensures both existence and uniqueness of a clearing solution under the usual conditions from [45] and makes the problem well-defined from a financial viewpoint. Further, I show that the problems which could not be solved in the static framework can be studied with this dynamic approach. This approach can be used to design stress tests that take into account the conditional nature of the payments under contingent claims.

1.5 Organization

The remainder of this dissertation is organized as follows:

- **Chapter 2** presents the problem of valuation of debt and equity in a financial network with comonotonic endowments. I provide the expectation of the equilibrium payments, equity, and wealth and prove that these expected values can provide lower bounds for the general random endowment setting. I consider the special case where the firms invest in a risk-free bond and a common risky asset following a geometric Brownian motion and present the explicit solutions for the risk neutral price of debt, the market capitalization, and the survival probability. Furthermore, I consider simple comparative statics of the provided valuations with respect to the different system parameters.

- **Chapter 3** discusses price-mediated contagion through capital ratio requirements. I develop a framework for price-mediated contagion in financial systems where banks are forced to liquidate assets to satisfy a risk-weight based capital ratio requirement and derive conditions for the existence and uniqueness of equilibrium prices. I show that the sensitivity analysis of these prices with respect to the system parameters can be written as a fixed point problem and prove the existence and uniqueness of a solution to this fixed point problem. Furthermore, I develop a methodology to quantify the cost of regulation faced by different banks in this setting. Numerical case studies are provided to study the application of this model to data.

- **Chapter 4** considers dynamic clearing and contagion in financial networks. I propose a discrete-time formulation for the Eisenberg--Noe model and provide results on existence and uniqueness, as well as a numerical algorithm based on the fictitious default algorithm of [45]. I then extend our model to a continuous-time setting and consider existence and uniqueness of the clearing solutions, and a numerical algorithm for finding sample paths of this clearing solution, under cash flows modeled by Itô processes. Finally, I discuss the financial implications of time dynamics in interbank networks.

- **Chapter 5** presents the impact of contingent payments on systemic risk in financial networks. I develop the static framework for incorporating contingent payments such as insurance and CDS and provide results on existence and develop conditions for uniqueness that are intimately related to considerations of insurance versus speculation. Further I demonstrate some shortcomings inherent to the static framework with contingent payments. Then I introduce a discrete time dynamic framework and discuss existence and uniqueness results. Additionally I demonstrate how this framework can be applied to problems that are ill-defined in the static framework through numerical examples.

- **Chapter 6** concludes and offers directions for future research.

Chapter 2

Pricing of debt and equity in a financial network with comonotonic endowments

This chapter is based on [13] which is joint work with Zachary Feinstein.

2.1 Introduction

In this chapter, I consider the problem of valuation of corporate debt and determination of market capitalization. Traditionally, two alternative approaches have been followed. The first of these approaches is the *structural approach*, which was introduced in the seminal paper of Merton [86]. Merton considered the debt and equity of a firm to be derivatives on the value of the assets of the firm where the strike price is given by the firm's liabilities. Merton's model has been extended in many directions and forms the basis of modern structural credit risk models, used in the Basel 2 framework and KMV. In these models, the value of the firm's assets is given by an underlying stochastic process and a credit event occurs when the value of the assets fall below the liabilities of the firm. The second approach is the *reduced form* approach where the modeling is done from a more statistical perspective, taking into account the historical default intensities and the underlying stochastic factors ([44]). In this chapter, I will focus exclusively on the structural approach.

The valuation in structural models is generally done from the perspective of an individual bank. However, this does not take into account the myriad interconnections that exist within a financial system. As evidenced by the 2008 financial crisis, considering the risk of a single firm alone can cause gross misspecification in firm health. In this work, I will focus on interconnections through correlated assets as well as interbank debt claims. These interconnections effectively link the balance sheets of different banks and make the value of a firm dependent on the performance of other firms. These shared connections might open up avenues for shared prosperity but also introduce potential channels for contagion. These avenues of contagion become particularly significant during a financial crisis where the default of one firm might cause the failure in other firms. This effect is also referred to as cascading defaults. As such, valuing claims that take the full network effects into account is imperative so that the true risk of the claims are also taken into account. In the Eisenberg--Noe framework for interbank payments, this valuation problem has been considered in multiple works by means of a structural approach. [99] considers the problem in a setting with cross-holdings and [58] generalizes that setting to include multiple seniority classes and derivatives; these works rely solely on Monte Carlo simulation for numerical computations. [69] considers a model with random exogenous shocks on the assets of each bank. [70] gives an interpretation of this problem as one with a "hidden" CDO. [16] considers the problem of network valuation and the effect of a random exogenous shock in the external assets. A different approach is considered in [75] where a PDE method is used for the case where the banks' assets are driven by correlated multidimensional Brownian motions with drift. A reduced form model for studying distress contagion and mark-to-market write-downs of debt contagion has been studied by [103]. A common thread of the existing literature is that explicit, analytical solutions are considered only for cases with either no direct interconnection between firms or where the number of firms in the system is very low. I wish to note that the general methodology of [69] requires partitioning the endowment space into 2^n possible default scenarios in a system with n banks. Therefore computing the expectation suffers from the curse of dimensionality and is typically computationally intractable for realistic systems. Further complications arise if I consider bankruptcy costs along the lines of [96]. Without bankruptcy costs, the 2^n default scenarios result in the partition of the bank endowment space into 2^n mutually exclusive and convex regions. However, in the presence of bankruptcy costs, these partitioned regions are not, in general, convex. I refer the reader to Appendix A for elucidation of this point. Providing any analytical solution in this case becomes very challenging even in small systems. Hence a numerical approach has been generally followed, i.e. via Monte Carlo simulations.

In light of the above, I focus on the setting where the banks have comonotonic endowments. This setting follows from a portfolio optimization perspective. If all firms are portfolio optimizers, then each will position their endowments to be countermonotonic to the pricing kernel ([93]). Hence the endowments of the banks will be comonotonic to each other. Under the assumption of comonotonic endowments, the default regions can be characterized by at most n intervals on $\mathbb{R}_+$ and the problem becomes tractable analytically. This tractability extends to the case where bankruptcy costs are considered. Thus the comonotonic setting allows me to explore the network effects from an analytical perspective. Furthermore, I am able to provide a lower bound on the expectation of the system behavior under any general random endowment with the comonotonic framework. This is particularly valuable from a stress testing perspective. I highlight the closed-form expressions for the price of debt and market capitalization in the case where the firms invest in a risk-free bond and a risky asset following a geometric Brownian motion. I compare this setting to that taken by [86] using numerical case studies.

The rest of the chapter is organized as follows. In Section 2.2, I provide a description of my mathematical setting. Section 2.3.1 considers network clearing when firms have comonotonic endowments. I provide the expectation of the equilibrium payments, equity, and wealth. Further, I prove that these expected values can provide lower bounds for the general random endowment setting of, e.g., [69]. Section 2.3.2 considers the special case where the firms invest in a risk-free bond and a common risky asset following a geometric Brownian motion as in [86]. I provide the explicit solutions for the risk neutral price of debt, the market capitalization, and the survival probability. Section 2.4 considers simple comparative statics of the provided valuations with respect to the different system parameters.

2.2 Setting

Throughout this chapter, I will consider a network of n financial institutions. Often I will consider an additional node $n+1$, which encompasses the entirety of the financial system outside of the n banks; this node $n+1$ will also be referred to as society or the societal node.

I refer to [56, 67] for further discussion of the meaning and concepts behind the societal node.

In this chapter, I consider obligations with a single maturity date, as considered in [45]. Any bank $i \in \{1, 2, ..., n\}$ may have obligations $L_{ij} \geq 0$ to any other firm or society $j \in \{1, 2, ..., n+1\}$. I will assume that no firm has any obligations to itself, i.e., $L_{ii} = 0$ for all firms $i \in \{1, 2, ..., n\}$, and the society node has no liabilities at all, i.e., $L_{n+1,j} = 0$ for all firms $j \in \{1, 2, ..., n+1\}$. Thus the *total liabilities* for bank $i \in \{1, 2, ..., n\}$ is given by $\bar{p}_i := \sum_{j=1}^{n+1} L_{ij} \geq 0$ and *relative liabilities* from bank $i \in \{1, 2, ..., n\}$ to bank $j \in \{1, 2, ..., j\}$ is given by $\pi_{ij} := \frac{L_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and arbitrary otherwise; for simplicity, in the case that $\bar{p}_i = 0$, I will let $\pi_{ij} = 0$ for all $j \in \{1, 2, ..., n\}$. Note that, for any firm i, I recover the property that $\sum_{j=1}^{n} \pi_{ij} \leq 1$. Throughout this work I will consider the square matrix $\Pi \in [0, 1]^{n \times n}$; the relative liabilities from firm i to the societal node $n+1$ can be defined as being $1 - \sum_{j=1}^{n} \pi_{ij} \geq 0$. On the other side of the balance sheet, all firms are assumed to begin with some endowments $x_i \geq 0$ for all $i \in \{1, 2, ..., n\}$, with $x_0 > 0$.

The central question explored in the network models is the determination of the firm wealths after network clearing. Let the clearing wealths be given by $V \in \mathbb{R}^n$. To determine the clearing wealths, I assume the stylized rules of bankruptcy viz., limited liabilities, priority of debt claims and same seniority as discussed in Chapter 1.3. Throughout this work, I consider a system with some exogenous recovery rate in case of default, i.e. a special case of [96]. This means if bank i has negative wealth $V_i < 0$ then it is defaulting and its assets are reduced with recovery rate $\beta \in [0, 1]$.

With the rules set, I formalize the clearing process $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ in wealths to describe this system. I can define the payments and equity from the wealths V as $p = (\bar{p} - V^-)^+$ and $E = V^+$ respectively. The clearing process is defined for all firms i as

$$\begin{aligned} \Psi_i(V) := \mathbb{I}_{\{V_i \geq 0\}} & \left[x_i + \sum_{j=1}^{n} \pi_{ji} (\bar{p}_j - V_j^-)^+ - \bar{p}_i \right] \\ & + \mathbb{I}_{\{V_i < 0\}} \left[\beta \left(x_i + \sum_{j=1}^{n} \pi_{ji} (\bar{p}_j - V_j^-)^+ \right) - \bar{p}_i \right]. \end{aligned} \tag{2.1}$$

As such, the clearing procedure Ψ implies: if bank i has nonnegative wealth $V_i \geq 0$ then it is solvent and its wealth is equal to its total assets minus its total liabilities; if bank i has negative wealth $V_i < 0$ then it is defaulting and its assets are reduced by the recovery rate β. I note that with $\beta = 1$ (i.e. under no bankruptcy costs) I recover the model of [45].

I will now consider existence and uniqueness results on the clearing wealth V. In general, I can get existence by applying Tarski's fixed point theorem.

Proposition 2.2.1. *There exists a greatest and least clearing solution to $V = \Psi(V)$ for $V \in \mathbb{R}^n$ and any finite clearing solution falls within the lattice $[-\bar{p}, x + \Pi^\top \bar{p} - \bar{p}]$.*

Proof. First note that Ψ is nondecreasing in wealths V. Now I will prove that $-\bar{p} \leq V \leq x + \Pi^\top \bar{p} - \bar{p}$ for any $V = \Psi(V) \in \mathbb{R}^n$.

- For any bank i: $V_i \geq \mathbb{I}_{\{V_i \geq 0\}}[-\bar{p}_i] + \mathbb{I}_{\{V_i < 0\}}[-\bar{p}_i] = -\bar{p}_i$ by construction.
- By monotonicity of the clearing procedure I recover $V \leq \Psi(V^+) = x + \Pi^\top \bar{p} - \bar{p}$.

The proof is completed by an application of Tarski's fixed point theorem on the lattice $[-\bar{p}, x + \Pi^\top \bar{p} - \bar{p}]$. □

In general, however, the clearing wealth V is not unique. In the special case without bankruptcy costs ($\beta = 1$), this reduces to the network described in [45]. In that setting one can get uniqueness under very mild assumptions.

Corollary 2.2.2. *Consider a setting with no bankruptcy costs ($\beta = 1$) and all firms have obligations to the societal node $n + 1$ (i.e. $\sum_{j=1}^n \pi_{ij} < 1$ for all firms i), then there exists a unique clearing solution $V = \Psi(V)$.*

Proof. The external node $n + 1$ implies the system is a regular network [45, Definition 5]. Thus by Theorem 2 of [45] I recover the uniqueness of the clearing solution. □

Assumption 2.2.3. *For the remainder of this chapter, I will assume that all firms have obligations to the societal node $n + 1$ (i.e. $\sum_{j=1}^n \pi_{ij} < 1$ for all firms i).*

Proposition 2.2.4. *Let $V^{\uparrow} = \Psi(V^{\uparrow})$ denote the greatest clearing solution from Proposition 2.2.1. Then $V^{\uparrow} = \Psi^*(V^{\uparrow})$ and is the greatest real-valued fixed point of Ψ^* where:*

$$\begin{aligned}\Psi_i^*(V) := \mathbb{I}_{\{V_i \geq 0\}} &\left[x_i + \sum_{j=1}^n \pi_{ji}(\bar{p}_j - V_j^-) - \bar{p}_i\right] \\ &+ \mathbb{I}_{\{V_i < 0\}} \left[\beta\left(x_i + \sum_{j=1}^n \pi_{ji}(\bar{p}_j - V_j^-)\right) - \bar{p}_i\right].\end{aligned} \tag{2.2}$$

Proof. By Proposition 2.2.1, $V^{\uparrow} \geq -\bar{p}$ and thus $V^{\uparrow} = \Psi^*(V^{\uparrow})$ as well. Similarly to the proof of Proposition 2.2.1, I can apply Tarski's fixed point theorem to (2.2) on the lattice $[-\infty, x + \Pi^{\top}\bar{p} - \bar{p}]$. Let $V^* = \Psi^*(V^*)$ be the greatest real-valued fixed point of Ψ^* and assume $V^* \geq V^{\uparrow}$ with $V_i^* > V_i^{\uparrow}$ for some bank i. Then it must follow that $V^* \geq V^{\uparrow} \geq -\bar{p}$, which implies $V^* = \Psi(V^*)$. However this is a contradiction to $V^{\uparrow}$ being the greatest clearing solution to Ψ. $\square$

I can compute the maximal clearing solution, as discussed in the previous proposition, through an application of the fictitious default algorithm as described in [96].

Corollary 2.2.5. *The following algorithm converges to the maximal clearing solution $V^{\uparrow} = \Psi(V^{\uparrow})$:*

(i) Initialize $V^{(0)} = x + \Pi^{\top}\bar{p} - \bar{p}$, $z^{(0)} = 0 \in \mathbb{R}^n$, and $k = 0$.

(ii) Iterate $k = k + 1$ and define $z^{(k)} = \mathbb{I}_{\{V^{(k-1)} < 0\}} \in \{0, 1\}^n$.

(iii) If $z^{(k)} = z^{(k-1)}$ then $V^{\uparrow} = V^{(k-1)}$ and terminate.

(iv) Define $\Lambda = \operatorname{diag}(z^{(k)})$ to be the diagonal matrix with main diagonal defined by $z^{(k)}$ and

$$\begin{aligned}V^{(k)} &= (I - \Lambda)\left[x + \Pi^{\top}\bar{p} + \Pi^{\top}\Lambda V^{(k)} - \bar{p}\right] \\ &\quad + \Lambda\left[\beta\left(x + \Pi^{\top}\bar{p} + \Pi^{\top}\Lambda V^{(k)}\right) - \bar{p}\right] \\ &= \left(I - (I - (1 - \beta)\Lambda)\,\Pi^{\top}\Lambda\right)^{-1}\left[(I - (1 - \beta)\Lambda)\left(x + \Pi^{\top}\bar{p}\right) - \bar{p}\right].\end{aligned}$$

(v) Go to step (ii).

Proof. The convergence of this algorithm to the greatest clearing wealth solution follows from the logic of the fictitious default algorithm in [45]. The nonsingularity of the matrix $I-(I-(1-\beta)\Lambda)\Pi^\top\Lambda$ follows from input-output results as detailed in [55, Theorem 2.6]. □

Throughout this work I will focus on the greatest clearing wealths solution $V^\uparrow$. I choose this equilibrium as all firms and regulators, if given the choice, would prefer these clearing wealths to all others as no firm can improve on their performance beyond that given by $V^\uparrow$. I wish to note that if a different clearing solution were chosen, and in particular the least clearing wealths, all subsequent results of this chapter would follow comparably.

Definition 2.2.6. *Define the mapping $V:\mathbb{R}^n_+\to\mathbb{R}^n$ so that $V(x)$ is the maximal clearing wealth solution under endowments $x\in\mathbb{R}^n_+$. Further, define $x\mapsto p(x):=\bar{p}-V(x)^-$ and $x\mapsto E(x):=V(x)^+$ to be the associated payments and equity.*

Proposition 2.2.7. *The greatest clearing wealth mapping V, and thus also the payment and equity mappings p and E, is nondecreasing in the endowments x.*

Proof. The monotonicity of the clearing wealths in the endowments follow from Theorem 3 of [87]. The results for the payments and equity follow directly from the definition of those mappings from the clearing wealths. □

From Corollary 2.2.5 I am able to give a linear construction for the clearing vector provided the defaulting set is known. This is given by the following construction. I compare this linear structure to the directional derivative proposed in [82] and the "network multipliers" from [32] when considering only the model of [45], i.e. with full recovery ($\beta=1$). For the remainder of this chapter I will use the following definitions:

$$\Delta(z):=\left(I-(I-(1-\beta)\operatorname{diag}(z))\Pi^\top\operatorname{diag}(z)\right)^{-1}(I-(1-\beta)\operatorname{diag}(z)) \tag{2.3}$$
$$\bar{\delta}(z):=\left(I-(I-(1-\beta)\operatorname{diag}(z))\Pi^\top\operatorname{diag}(z)\right)^{-1}\left[I-(I-(1-\beta)\operatorname{diag}(z))\Pi^\top\right]\bar{p} \tag{2.4}$$

for $z\in\{0,1\}^n$ denoting the set of defaulting institutions as in the fictitious default algorithm above. Thus

$$V(x):=\Delta(\mathbb{I}_{\{V(x)<0\}})x-\bar{\delta}(\mathbb{I}_{\{V(x)<0\}})$$

for any endowment $x \in \mathbb{R}^n_+$ by construction.

2.3 Expectations and pricing of debt and equity

Consider now a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ all measurable random variables. Further, denote by $L^1 \subseteq L^0$ those random variables that have finite absolute expectation, i.e. $X \in L^1$ if X is $(\Omega, \mathcal{F}, \mathbb{P})$ measurable and $\mathbb{E}[|X|] < \infty$. Further I will denote by L^1_+ those random variables that are almost surely nonnegative.

I note, first, that the problem of finding the expectation of the clearing wealths and payments under random endowments $X \in (L^1_+)^n$ was considered in [69] in the case of no bankruptcy costs ($\beta = 1$) but with cross-ownership. I replicate those results and extend them to consider the case with bankruptcy costs in the Appendix A. As finding the expectations in such a general setting requires computing the measure of 2^n regions in $\mathbb{R}^n$, this would typically require Monte Carlo simulation and suffer greatly from the curse of dimensionality. Furthermore, if all firms are portfolio optimizers and do not take any other firm's investments into account, the chosen endowments will all be countermonotonic to the pricing kernel (see, e.g., [93]). As such, a general endowment space is not necessary for understanding systemic risk and financial contagion. This motivates me to consider comonotonic endowments as these are often tractable analytically and require the consideration of at most n intervals. In fact, I will demonstrate that, in the setting of [45], I am able to provide a bound on the expectation of the system behavior under any random endowment using the comonotonic setting. I will present below, first, the expectations and probability distributions under general comonotonic endowments and, second, a special case related to Merton's model for credit pricing ([86]). I wish to emphasize that in this chapter, I consider a single maturity model along the lines of [45, 96], i.e. the network is formed and fixed at time 0 and all claims mature (and solvency is determined) at time $T > 0$.

2.3.1 Expectations under comonotonic endowments

In this section I will consider the endowments to be random and comonotonic. As such, I wish to recall a definition of comonotonicity of random variables which comes from Proposition 7.18 of [85].

Definition 2.3.1. *$X \in (L^1)^n$ is comonotonic if $X \overset{(d)}{=} f(q)$ (i.e. equal in distribution) for some random variable $q \in L^0$ and $f : \mathbb{R} \to \mathbb{R}^n$ nondecreasing.*

Assumption 2.3.2. *Throughout this chapter, I will restrict my consideration to comonotonic nonnegative random vectors of endowments $X \in (L^1_+)^n$ that are equal in distribution to $f(q)$ for some nonnegative random variable $q \in L^0_+$ and nondecreasing map $f : \mathbb{R}_+ \to \mathbb{R}^n_+$.*

For any random endowment satisfying Assumption 2.3.2, I can now consider the defaulting regions, i.e. the regions in which different combinations of banks are deemed to be defaulting on a portion of their liabilities. In fact, under the comonotonic setup considered herein, all such regions in the q-space must be convex intervals in $\mathbb{R}_+$. This is in contrast to a general endowment space in which the regions need not be convex if the recovery rate is strictly less than 1 ($\beta < 1$); I refer to [69] and Appendix A for more on this analysis. Thus, with the comonotonicity assumption I can uniquely define the regions of q under which different firms default, which I will do so with the vector $q^* \in \mathbb{R}^n_+$.

Definition 2.3.3. *Fix some random endowments satisfying Assumption 2.3.2. Define $q^* \in \mathbb{R}^n_+$ so that q^*_i is the minimal value such that firm i is solvent, i.e.*

$$q^*_i = \inf \{q \geq 0 \mid V_i(f(q)) \geq 0\}.$$

With this comonotonic setting described in Assumption 2.3.2 I can provide an iterative representation for the lowest prices $q^* \in \mathbb{R}^n_+$ such that each firm is solvent.

Proposition 2.3.4. *The lowest prices such that the various firms are solvent, defined by q^* in Definition 2.3.3, can be defined explicitly by the following iterative relation of decreasing values. Initialize $q^*_{[0]} = \infty$ and $z^{(0)} = 0 \in \mathbb{R}^n$. Then for any $k = 1, 2, ..., n$:*

$$[k] \in \operatorname*{arg\,max}_{i:\, z_i^{(k-1)}=0} \sup \left\{q \geq 0 \mid e_i^\top \Delta(z^{(k-1)}) f(q) < \delta_i(z^{(k-1)})\right\}^+$$

$$q^*_{[k]} := \min\left\{q^*_{[k-1]}\ ,\ \sup\left\{q \geq 0 \mid e_{[k]}^\top \Delta(z^{(k-1)}) f(q) < \delta_{[k]}(z^{(k-1)})\right\}^+\right\}$$
$$z^{(k)} := z^{(k-1)} + e_{[k]}.$$

As a convention, the supremum of the empty set is assumed to be $-\infty$. If the set definition of $[k]$ has cardinality greater than one, then only a single argument is chosen arbitrarily. In particular, if f is continuous and strictly increasing in the single factor then $q^*_{[1]} = \left[\max_i f_i^{-1}(\bar{p}_i - \sum_{j=1}^n L_{ji})\right]^+$ and $q^*_{[k]}$ can be found via bisection search between 0 and $q^*_{[k-1]}$.

Proof. This follows directly from the monotonicity of the wealths as given in Proposition 2.2.7 and the construction of Δ, δ in (2.3) and (2.4). The level $q^*_{[k]}$ is chosen exactly to be the largest price q so that the $[k]^{th}$ bank would have 0 wealth (i.e. the lowest price so that it is solvent) given that the prior [1] through $[k-1]$ banks have already been deemed insolvent. The minimum taken with $q^*_{[k-1]}$ is necessary only in the case of contagious defaults, i.e. from bankruptcy costs if the jump in payments from bank $[k-1]$ causes bank $[k]$ to also become insolvent at the same time. □

Assumption 2.3.5. *Without loss of generality I will assume for the remainder of this chapter (except where explicitly mentioned otherwise) that the banks are placed in descending order of q^*, i.e. so that $q^*_1 \geq q^*_2 \geq ... \geq q^*_n$. Additionally, define $q^*_0 = \infty$ and $q^*_{n+1} = 0$.*

Thus the value q^*_i is, in some sense, indicative of the financial stability of bank i. Immediately with this construction of minimal values q^* for which each firm is solvent, I am able to deduce formulations for the defaulting probabilities for each bank as well as the expectations of the wealth, payments, and equity for each firm. In contrast to the general formulation in [69] which requires a partition of the endowment space into 2^n defaulting regions, the formulations given in the below theorem require only a partition into n intervals. Further, as demonstrated in Lemma 2.3.8 below, in the setting of [45], this formulation provides a tractable bound on the general expectations given in, e.g., [69, 16] for large networks.

Theorem 2.3.6. *Let the endowments be defined by $X \in (L^1_+)^n$ satisfying Assumption 2.3.2. The probability of default, the expected wealth, the expected payment, and the expected equity for firm i are given, respectively, by:*

$$\mathbb{P}(V_i(X) < 0) = \mathbb{P}(q < q^*_i)$$

$$\mathbb{E}[V_i(X)] = e_i^\top \sum_{k=0}^{n} \left[\Delta_k \mathbb{E}[f(q)\mathbb{I}_{\{q\in[q^*_{k+1},q^*_k)\}}] - \bar{\delta}_k \mathbb{P}(q \in [q^*_{k+1}, q^*_k))\right]$$

$$\mathbb{E}[p_i(X)] = \bar{p}_i + e_i^\top \sum_{k=i}^{n} \left[\Delta_k \mathbb{E}[f(q)\mathbb{I}_{\{q\in[q^*_{k+1},q^*_k)\}}] - \bar{\delta}_k \mathbb{P}(q \in [q^*_{k+1}, q^*_k))\right]$$

$$\mathbb{E}[E_i(X)] = e_i^\top \sum_{k=0}^{i-1} \left[\Delta_k \mathbb{E}[f(q)\mathbb{I}_{\{q\in[q^*_{k+1},q^*_k)\}}] - \bar{\delta}_k \mathbb{P}(q \in [q^*_{k+1}, q^*_k))\right]$$

where I define Δ_k and $\bar{\delta}_k$ by:

$$\Delta_k := \begin{cases} \Delta\left(\sum_{j=1}^k e_j\right) & \textit{if } k = 1, 2, ..., n \\ I & \textit{if } k = 0 \end{cases} \quad \textit{and } \bar{\delta}_k := \begin{cases} \bar{\delta}\left(\sum_{j=1}^k e_j\right) & \textit{if } k = 1, ..., n \\ (I - \Pi^\top)\bar{p} & \textit{if } k = 0 \end{cases}.$$

Proof. This follows directly from the construction of V in the fictitious default algorithm of Corollary 2.2.5 and the construction of q^*. □

Remark 2.3.7. I will now compare the expectation results above to those from [69] with cross-ownership of equity but no bankruptcy costs. In this setting the ownership of equity is denoted by $\Gamma \in [0,1]^{n\times n}$ where bank j owns γ_{ij} of bank i's equity. Under the assumption that no firm has sold off all of its equity, I consider only the case that $\sum_{j=1}^n \gamma_{ij} < 1$. I can consider this setting in the same way by redefining $\Delta, \bar{\delta}$ as:

$$\begin{aligned} \Delta(z) &:= \left(I - (I - (1-\beta)\operatorname{diag}(z))\left[\Pi^\top \operatorname{diag}(z) + \Gamma^\top(I - \operatorname{diag}(z))\right]\right)^{-1} \\ &\quad \times (I - (1-\beta)\operatorname{diag}(z)) \\ \bar{\delta}(z) &:= \left(I - (I - (1-\beta)\operatorname{diag}(z))\left[\Pi^\top \operatorname{diag}(z) + \Gamma^\top(I - \operatorname{diag}(z))\right]\right)^{-1} \\ &\quad \times \left[I - (I - (1-\beta)\operatorname{diag}(z))\,\Pi^\top\right]\bar{p}. \end{aligned}$$

The comonotonic case that I consider would, as discussed previously, solve the curse of dimensionality issue that exists in the work of [69].

I now wish to consider how the formulas above for the expectations of wealth and debt under comonotonic endowments can provide a bound for the more general random endowments. As previously mentioned, the expectations of debt and equity were studied in [69, 16], but the formulations required suffer from the curse of dimensionality. More generally, if the correlations between firm endowments is unknown, the following lemma is useful from a stress test viewpoint as I find that the comonotonic case is a lower bound on the health of the system.

Lemma 2.3.8. *Consider the setting of [45], i.e.* $\beta = 1$*. Let* $X \in (L^1_+)^n$ *and* $Z = (F^{-1}_{X_1}(U), ..., F^{-1}_{X_n}(U))$ *for uniform random variable* U *on the support* $[0,1]$ *and marginal distributions* $F_{X_1}, ..., F_{X_n}$ *for* $X_1, ..., X_n$ *respectively. Then*

$$\mathbb{E}[V_i(Z)] \leq \mathbb{E}[V_i(X)] \leq V_i(\mathbb{E}[X]) \quad \textit{and}$$
$$\mathbb{E}[p_i(Z)] \leq \mathbb{E}[p_i(X)] \leq p_i(\mathbb{E}[X])$$

for any bank i*.*

Proof. By construction $X \leq_{sm} Z$ with respect to the supermodular order (see Section 9.A.4 and, in particular, Theorem 9.A.21 of [97]). I further note that, under the setting of [45], I can consider this system as a fixed point in the payments $p(x) = \bar{p} \wedge (x + \Pi^\top p(x))$ with $V(x) = x + \Pi^\top p(x) - \bar{p}$. Thus if $p : \mathbb{R}^n_+ \to [0, \bar{p}]$ is submodular then the lower-bounding result is proven. To prove this result I consider that the payment function is the pointwise limit of the mappings $p^k : \mathbb{R}^n_+ \to [0, \bar{p}]$ defined iteratively as:

$$p^0(x) := \bar{p} \quad \text{and} \quad p^{k+1}(x) := \bar{p} \wedge (x + \Pi^\top p^k(x)) \; \forall k \in \mathbb{N}, \; \forall x \in \mathbb{R}^n_+.$$

As $p(x) = \lim_{k \to \infty} p^k(x)$ by construction (where convergence follows from the monotonicity and boundedness of the arguments $0 \leq p^{k+1}(x) \leq p^k(x)$), if p^k is submodular for all k then the same must be true for the clearing payments p. Trivially p^0 is submodular. Now by induction assume that p^{k-1} is submodular. Take $x, y \in \mathbb{R}^n_+$ and $i \in \{1, 2, ..., n\}$; there are three cases that must be considered:

(i) If $p^k_i(x) = p^k_i(y) = \bar{p}_i$ then $p^k_i(x) + p^k_i(y) \geq p^k_i(x \wedge y) + p^k_i(x \vee y)$ by construction.

(ii) If $p^k_i(x) < p^k_i(y) = \bar{p}_i$ then $p^k_i(x) \geq p^k_i(x \wedge y)$ and $p^k_i(y) = p^k_i(x \vee y)$ by monotonicity (Proposition 2.2.7); thus $p^k_i(x) + p^k_i(y) \geq p^k_i(x \wedge y) + p^k_i(x \vee y)$.

(iii) If $p_i^k(x) < \bar{p}_i$ and $p_i^k(y) < \bar{p}_i$ then $p_i^k(x) = x_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(x)$ and $p_i^k(y) = y_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(y)$. Therefore I find

$$\begin{aligned}
p_i^k(x) + p_i^k(y) &= \left[x_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(x)\right] + \left[y_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(y)\right] \\
&= x_i + y_i + \sum_{j=1}^n \pi_{ji} \left[p_j^{k-1}(x) + p_j^{k-1}(y)\right] \\
&\geq x_i \wedge y_i + x_i \vee y_i + \sum_{j=1}^n \pi_{ji} \left[p_j^{k-1}(x \wedge y) + p_j^{k-1}(x \vee y)\right] \\
&= \left[(x \wedge y)_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(x \wedge y)\right] + \left[(x \vee y)_i + \sum_{j=1}^n \pi_{ji} p_j^{k-1}(x \vee y)\right] \\
&\geq p_i^k(x \wedge y) + p_i^k(x \vee y).
\end{aligned}$$

Further, by p concave (see Lemma 5 of [45]), the upper bound follows by Jensen's inequality.

□

In the below simple network, I demonstrate some counterexamples to extending the results of Lemma 2.3.8. For instance, I first consider the setting with bankruptcy costs, i.e. with recovery rate $\beta < 1$. Second I demonstrate that, even in the Eisenberg--Noe setting, no consistent bounds for firm equity exists. Intriguingly, though the payments and wealth attain their worst-case under comonotonic endowments, the equity of the different firms in the financial system may actually be higher under the comonotonic endowments than other correlation structures. However, I wish to note that the societal node would exhibit the same lower bound property as given in Lemma 2.3.8 since its equity is equal to its wealth by construction.

Example 2.3.9. I wish to consider the cases not proven to have bounds in Lemma 2.3.8 above. Specifically, I will demonstrate that the lower bounds do not hold when bankruptcy costs exist ($\beta < 1$) and that the equity does not satisfy the bounding property with comonotonicity. For these counterexamples I will consider the 2 bank and societal node system depicted in Figure 2.1. That is, firm 1 owes 1 to both firm 2 and the societal node and firm 2 owes 1 to firm 1 and 2 to the societal node.

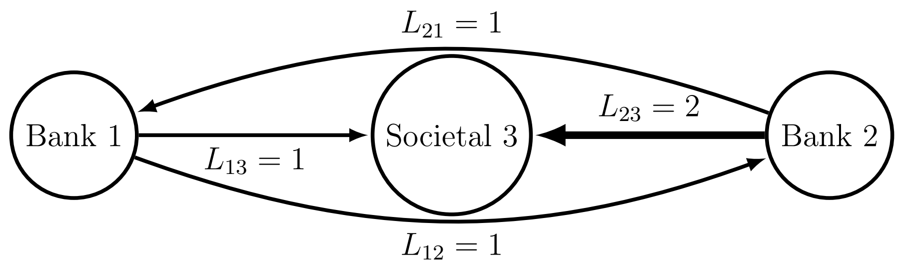


Figure 2.1: Example 2.3.9: A sample network topology used to determine counterexamples to expanding the results of Lemma 2.3.8.

- First, I wish to consider a counterexample to the bounds provided in Lemma 2.3.8 for a network with bankruptcy costs ($\beta < 1$). Consider the countermonotonic and comonotonic random endowments (including for the societal node 3)

$$X = \begin{cases} (0,2,0) & \text{with } \mathbb{P}(X = (0,2,0)) = \frac{1}{2} \\ (1,0,0) & \text{with } \mathbb{P}(X = (1,0,0)) = \frac{1}{2} \end{cases}$$
$$Z = \begin{cases} (0,0,0) & \text{with } \mathbb{P}(Z = (0,0,0)) = \frac{1}{2} \\ (1,2,0) & \text{with } \mathbb{P}(Z = (1,2,0)) = \frac{1}{2} \end{cases}.$$

With recovery rate $\beta \in [0,1]$ I find that the four possible payment vectors are:

$$\begin{aligned} p((0,0,0)) &= (0,0,0) \\ p((0,2,0)) &= \left(\frac{4\beta^2}{6-\beta^2}, \frac{12\beta}{6-\beta^2}, 0\right) \\ p((1,0,0)) &= \left(\frac{6\beta}{6-\beta^2}, \frac{3\beta^2}{6-\beta^2}, 0\right) \\ p((1,2,0)) &= (2,3,0). \end{aligned}$$

Thus I can conclude

$$\mathbb{E}[p(X)] = \left(\frac{\beta(2\beta+3)}{6-\beta^2}, \frac{3\beta(\beta+4)}{2(6-\beta^2)}, 0\right) \leq \left(1, \frac{3}{2}, 0\right) = \mathbb{E}[p(Z)].$$

In fact $\mathbb{E}[p_i(X)] < \mathbb{E}[p_i(Z)]$, and thus also $\mathbb{E}[V_i(X)] < \mathbb{E}[V_i(Z)]$, for both firms i if $\beta \in [0,1)$.

- Second, I wish to consider a counterexample to a bound for the equity. Again I will consider two simple random endowments

$$X = \begin{cases} (1+\epsilon, 2, 0) & \text{with } \mathbb{P}(X = (1+\epsilon, 2, 0)) = \frac{1}{2} \\ (2, 1+\epsilon, 0) & \text{with } \mathbb{P}(X = (2, 1+\epsilon, 0)) = \frac{1}{2} \end{cases}$$
$$Z = \begin{cases} (1+\epsilon, 1+\epsilon, 0) & \text{with } \mathbb{P}(Z = (1+\epsilon, 1+\epsilon, 0)) = \frac{1}{2} \\ (2, 2, 0) & \text{with } \mathbb{P}(Z = (2, 2, 0)) = \frac{1}{2} \end{cases}$$

with $\epsilon \in (0, \frac{1}{4})$. I find that the four possible payment and equity vectors are:

$$\begin{aligned} p((1+\epsilon, 1+\epsilon, 0)) &= \left(\frac{8}{5}(1+\epsilon)\ ,\ \frac{9}{5}(1+\epsilon)\ ,\ 0\right) \\ p((1+\epsilon, 2, 0)) &= (2, 3, 0) \\ p((2, 1+\epsilon, 0)) &= (2, 2+\epsilon, 0) \\ p((2, 2, 0)) &= (2, 3, 0). \end{aligned}$$

Thus I can conclude from $E(x) = x + \Pi^\top p(x) - p(x)$ that

$$\begin{aligned} \mathbb{E}[E(X)] &= \left(\frac{1+2\epsilon}{3}\ ,\ 0\ ,\ \frac{8+\epsilon}{3}\right) \\ \mathbb{E}[E(Z)] &= \left(\frac{1}{2}\ ,\ 0\ ,\ \frac{5}{2}+\epsilon\right). \end{aligned}$$

Notably I find that I cannot compare $\mathbb{E}[E(X)]$ and $\mathbb{E}[E(Z)]$ since $\mathbb{E}[E_1(X)] < \mathbb{E}[E_1(Z)]$ but $\mathbb{E}[E_3(X)] > \mathbb{E}[E_3(Z)]$.

2.3.2 Pricing under geometric Brownian motion with common noise

In this section, I will consider a specific case of the comonotonic structure. Here I consider a market with a risk-free bond and a single risky asset in which the different firms have heterogeneous investments. The risky asset, which can be viewed as an investment in the market portfolio, will be assumed to follow a geometric Brownian motion. This setting is chosen for the purposes of comparison to [86]. Due to the heterogeneity of portfolios, but with a common risky asset, this system can be considered equivalent to a vector of geometric Brownian motions with differing volatilities but a single common noise term.

Assumption 2.3.10. *Let $r \geq 0$ be the risk-free rate. Let $b, s \in \mathbb{R}^n_+$ denote the investments of the firms in the risk-free bond and in a risky asset. Consider a terminal time $T > 0$. Define $X = f(q_T) := be^{rT} + sq_T$ almost surely where*

$$\frac{dq_t}{q_t} = \mu dt + \sigma dW_t$$

for Brownian motion W and with initial price of the risky asset $q_0 > 0$.

I wish to compare this setting to the single firm setting proposed in [86] in which the assets of a firm follow a geometric Brownian motion and the pricing of debt and equity is considered. I note that the comonotonic construction from the prior section would allow me to consider more complicated underlying market models, e.g. a jump diffusion model. But for simplicity and due to its use in the seminal work by Merton, I will restrict myself to the geometric Brownian motion setting herein.

With the setting proposed in Assumption 2.3.10 I can explicitly give an iterative representation for the lowest prices $q^* \in \mathbb{R}^n_+$ such that each firm is solvent. I note that this construction does not rely on Assumption 2.3.5.

Proposition 2.3.11. *The lowest prices such that the various firms are solvent, defined by q^* in Definition 2.3.3, can be defined explicitly by the following iterative relation of decreasing values. Initialize $q^*_{[0]} = \infty$ and $z^{(0)} = 0 \in \mathbb{R}^n$. Then for any $k = 1, 2, ..., n$:*

$$\begin{aligned}
[k] &\in \operatorname*{arg\,max}_{i:\, z_i^{(k-1)}=0} \left[e_i^\top \operatorname{diag}(\Delta(z^{(k-1)})s)^{-1} \left(\bar{\delta}(z^{(k-1)}) - \Delta(z^{(k-1)})^{-1} be^{rT}\right)\right]^+ \\
q^*_{[k]} &:= \min\left\{q^*_{[k-1]} \,,\, \max_{i:\, z_i^{(k-1)}=0} \left[e_i^\top \operatorname{diag}(\Delta(z^{(k-1)})s)^{-1} \left(\bar{\delta}(z^{(k-1)}) - \Delta(z^{(k-1)})^{-1} be^{rT}\right)\right]^+\right\} \\
z^{(k)} &:= z^{(k-1)} + e_{[k]}.
\end{aligned}$$

If the maximizing argument in the definition of $[k]$ is non-unique, then only a single argument is chosen arbitrarily.

Proof. This follows directly from the monotonicity of the wealths as given in Proposition 2.2.7 and the construction of $\Delta, \bar{\delta}$ in (2.3) and (2.4). The level $q^*_{[k]}$ is chosen exactly to be the largest price q so that the $[k]^{th}$ bank would have 0 wealth (i.e. the lowest price so that it is solvent) given that the prior [1] through $[k-1]$ banks have already been deemed insolvent. The minimum taken with $q^*_{[k-1]}$ is necessary only in the case of contagious defaults, i.e. from bankruptcy costs if the jump in payments from bank $[k-1]$ causes bank $[k]$ to also become insolvent at the same time. I wish to note that, if $e_i^\top \Delta(z)s = 0$ for some $z \in \{0,1\}^n$, I take $1/(e_i^\top \Delta(z)s) = \infty$. This can only occur if firm i holds no risky assets ($s_i = 0$) and all firms j with obligations to firm i ($L_{ji} > 0$) either are solvent ($z_j = 0$) or hold no risky assets ($s_j = 0$), i.e. if firm i's solvency is independent from the behavior of the risky asset under the default set z. □

For the remainder of this chapter I will consider Assumption 2.3.5 to hold, i.e. $q_1^* \geq ... \geq q_n^*$. This is for simplicity of notation and does not restrict the results of this work. With this construction I now present a corollary to Theorem 2.3.6 to provide the risk-neutral price of debt and value of the market capitalization for the firms in the financial system in this common investment scenario. These formulations are analytical expressions with respect to the vector q^* which was algorithmically provided in Proposition 2.3.11.

Corollary 2.3.12. *Let $\mathbb{Q}$ be the risk-neutral measure in the market, i.e.*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \left(\frac{\mu - r}{\sigma}\right) W_T\right).$$

Under the risk-neutral measure $\mathbb{Q}$, the probability of firm i's default $\mathbb{Q}(V_i(X) < 0)$, the discounted price of firm i's debt $\mathbb{E}^{\mathbb{Q}}[e^{-rT} p_i(X)/\bar{p}_i]$, and firm i's market capitalization $\mathbb{E}^{\mathbb{Q}}[e^{-rT} E_i(X)]$ are given by:

$$\mathbb{Q}(V_i(X) < 0) = \Phi\left(-d_i^2\right) \tag{2.5}$$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\frac{p_i(X)}{\bar{p}_i}\right] = e^{-rT} + \frac{1}{\bar{p}_i} e_i^\top \sum_{k=i}^{n} \Big[& \left(\Delta_k b - e^{-rT}\bar{\delta}_k\right)\left[\Phi\left(-d_k^2\right) - \Phi\left(-d_{k+1}^2\right)\right] \\ & + \Delta_k s q_0 \left[\Phi\left(-d_k^1\right) - \Phi\left(-d_{k+1}^1\right)\right]\Big] \end{aligned} \tag{2.6}$$

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}E_i(X)] = e_i^\top \sum_{k=0}^{i-1} \left[\left(\Delta_k b - e^{-rT}\bar{\delta}_k\right) \left[\Phi\left(-d_k^2\right) - \Phi\left(-d_{k+1}^2\right)\right] \right. \\ \left. + \Delta_k s q_0 \left[\Phi\left(-d_k^1\right) - \Phi\left(-d_{k+1}^1\right)\right]\right] \tag{2.7}$$

$$d_k^1 = \frac{\log\left(\frac{q_0}{q_k^*}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad \forall k = 0, 1, ..., n+1 \tag{2.8}$$

$$d^2 = d^1 - \sigma\sqrt{T} \tag{2.9}$$

where $\Phi : \mathbb{R} \to [0,1]$ *is the standard normal cumulative distribution function.*

Proof. This follows from Theorem 2.3.6. Particularly,

$$\mathbb{E}^{\mathbb{Q}}[e^{-rT}(be^{rT} + sq_T)\mathbb{I}_{\{q_T \in [q_{k+1}^*, q_k^*)\}}] = b\mathbb{Q}(q_T \in [q_{k+1}^*, q_k^*)) + sq_0\widehat{\mathbb{Q}}(q_T \in [q_{k+1}^*, q_k^*))$$

where $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{e^{-rT}q_T}{q_0}$. In fact, I can explicitly provide these probabilities by, first, noting that $W_t^{\widehat{\mathbb{Q}}} = W_t^{\mathbb{Q}} - \sigma t$ is a Brownian motion under $\widehat{\mathbb{Q}}$. Therefore, letting Z and $\hat{Z}$ follow a normal distribution under $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ respectively, for any $\bar{q} \geq 0$:

$$\mathbb{Q}(q_T \leq \bar{q}) = \mathbb{Q}(q_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right) \leq \bar{q}) = \Phi\left(-\frac{\log(q_0/\bar{q}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$
$$\widehat{\mathbb{Q}}(q_T \leq \bar{q}) = \widehat{\mathbb{Q}}(q_0 \exp\left((r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\hat{Z}\right) \leq \bar{q}) = \Phi\left(-\frac{\log(q_0/\bar{q}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

□

Remark 2.3.13. As an external investor may wish to hedge her counterparty risk, I wish to consider the Greeks for debt for firm i. In particular, from a pricing and hedging perspective, I find the delta of the debt in a financial network as the most interesting. By direct computation I find that, for the price process $P_i(t) := \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}p_i(X)/\bar{p}_i \mid \mathcal{F}_t]$, the delta at time t is given by:

$$\frac{\partial P_i(t)}{\partial q} = \frac{1}{\bar{p}_i q_t \sigma\sqrt{T-t}} e_i^\top \sum_{k=i}^{n} \left[\left(e^{-r(T-t)}\bar{\delta} - \Delta_k b e^{rt}\right) \left[\phi(-d_k^1) - \phi(-d_{k+1}^2)\right] \right. \\ \left. + (\Delta_k s)\left[\left(\Phi(-d_k^2) - \Phi(-d_{k+1}^2)\right) - q_t\left(\phi(-d_k^2) - \phi(-d_{k+1}^2)\right)\right]\right]$$

$$d_k^1 = \frac{\log\left(\frac{q_t}{q_k^*}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \forall k = 0, 1, ..., n+1$$
$$d^2 = d^1 - \sigma\sqrt{T-t}.$$

In the above formulation I define $\phi = \Phi'$ as the standard normal probability density function. In a similar way I can compute the other Greeks.

Under this geometric Brownian motion setting I am able to provide the explicit distribution for the wealths (and therefore also payments and equity). These joint distributions would be particularly useful for stress testing a financial system as it would allow the regulatory authority to assign probabilities to the outcomes of the stress tests. This is more explicitly considered in the numerous works on systemic risk measures, see e.g. [31, 79, 56, 20, 11], in which some "aggregate" of the financial system needs to be deemed acceptable. This acceptability criterion would commonly be constructed from a law-invariant framework, e.g. value-at-risk or expected shortfall, which explicitly requires the distribution of the system outcomes.

Lemma 2.3.14. *Take $v^* \in -\bar{p} + \mathbb{R}^n_+$, $p^* \in [0, \bar{p}]$, and $e^* \in \mathbb{R}^n_+$:*

$$\mathbb{Q}(V(X) \leq v^*) = \sum_{k=0}^{n} \left[\min\left\{ \Phi(-d_k^2), \min_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\} - \min\left\{ \Phi(-d_{k+1}^2), \min_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\} \right] \tag{2.10}$$

$$\mathbb{Q}(V(X) \geq v^*) = \sum_{k=0}^{n} \left[\max\left\{ \Phi(-d_k^2), \max_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\} - \max\left\{ \Phi(-d_{k+1}^2), \max_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) \right\} \right] \tag{2.11}$$

$$\mathbb{Q}(p(X) \leq p^*) = \mathbb{Q}(V(X) \leq \operatorname{diag}(\mathbb{I}_{\{p^* < \bar{p}\}})[p^* - \bar{p}] + \mathbb{I}_{\{p^* = \bar{p}\}}\infty) \tag{2.12}$$
$$\mathbb{Q}(p(X) \geq p^*) = \mathbb{Q}(V(X) \geq p^* - \bar{p}) \tag{2.13}$$
$$\mathbb{Q}(E(X) \leq e^*) = \mathbb{Q}(V(X) \leq e^*) \tag{2.14}$$
$$\mathbb{Q}(E(X) \geq e^*) = \mathbb{Q}(V(X) \geq e^*) \tag{2.15}$$

where $\bar{q}_i^k(v^*) := e_i^\top \operatorname{diag}(\Delta_k s)^{-1}\left(v^* + \bar{\delta}_k - \Delta_k b e^{rT}\right)$.

Proof. Consider first the probability distribution for the wealths V given in (2.10). Take $v^* \in -\bar{p} + \mathbb{R}^n_+$ (if v^* is taken otherwise then $\mathbb{Q}(V(X) \leq v^*) = 0$ by the bound given in Proposition 2.2.1).

$$\begin{aligned}
\mathbb{Q}(V(X) \leq v^*) &= \sum_{k=0}^{n} \mathbb{Q}(V(X) \leq v^*,\ q_T \in [q^*_{k+1}, q^*_k)) \\
&= \sum_{k=0}^{n} \mathbb{Q}(\Delta_k[be^{rT} + sq_T] - \bar{\delta}_k \leq v^*,\ q_T \in [q^*_{k+1}, q^*_k)) \\
&= \sum_{k=0}^{n} \mathbb{Q}(q_T \leq e_i^\top \operatorname{diag}(\Delta_k s)^{-1}\left(v^* + \bar{\delta}_k - \Delta_k b e^{rT}\right)\ \forall i,\ q_T \in [q^*_{k+1}, q^*_k)) \\
&= \sum_{k=0}^{n} \mathbb{Q}\left(q_T \in [\min\left\{q^*_{k+1}, \min_{i=1,2,\ldots,n} \bar{q}_i^k(v^*)\right\}\ ,\ \min\left\{q^*_k, \min_{i=1,2,\ldots,n} \bar{q}_i^k(v^*)\right\}]\right) \\
&= \sum_{k=0}^{n} \left[\min\left\{\Phi(-d^2_k), \min_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right\}\right. \\
&\qquad \left. - \min\left\{\Phi(-d^2_{k+1}), \min_{i=1,2,\ldots,n} \Phi\left(\frac{\log\left(\bar{q}_i^k(v^*)/q_0\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right\}\right].
\end{aligned}$$

As before, I take $1/(e_i^\top \Delta_k s) = \infty$ if $e_i^\top \Delta_k s = 0$. This cannot occur if, by construction of Δ_k for any k, all firms are long the risky asset ($s \in \mathbb{R}^n_{++}$). The construction of the explicit form of the probabilities in the final equation come from the same logic as in the proof of Corollary 2.3.12 above.

Now consider the survival distribution for the wealths V given in (2.11). Again take $v^* \in -\bar{p} + \mathbb{R}^n_+$ (if v^* is taken otherwise then $\mathbb{Q}(V(X) \geq v^*) = \mathbb{Q}(V(X) \geq [-\bar{p}] \wedge v^*)$ by the bound given in Proposition 2.2.1). The logic for the representation in (2.11) follows similarly to that for the distribution function for wealths given above.

Finally, consider the distribution for the payments p and equity E given in (2.12)-(2.15). Take $p^* \in [0, \bar{p}]$. I note that $p \leq \bar{p} + V$ with equality for any *defaulting* bank i. Thus I can conclude:

$$\mathbb{Q}(p(X) \leq p^*) = \mathbb{Q}(p_i(X) \leq p_i^*\ \forall i:\ p_i^* < \bar{p}_i)$$

$$= \mathbb{Q}(V_i(X) \leq p_i^* - \bar{p}_i \; \forall i : \; p_i^* < \bar{p}_i)$$
$$= \mathbb{Q}(V(X) \leq \text{diag}(\mathbb{I}_{\{p^* < \bar{p}\}})[p^* - \bar{p}] + \mathbb{I}_{\{p^* = \bar{p}\}} \infty)$$
$$\mathbb{Q}(p(X) \geq p^*) = \mathbb{Q}(V(X) \geq p^* - \bar{p}).$$

Take $e^* \in \mathbb{R}^n_+$. Then $E \leq e^*$ if and only if $V \leq e^*$ and similarly for $E \geq e^*$. Thus the result is proven. □

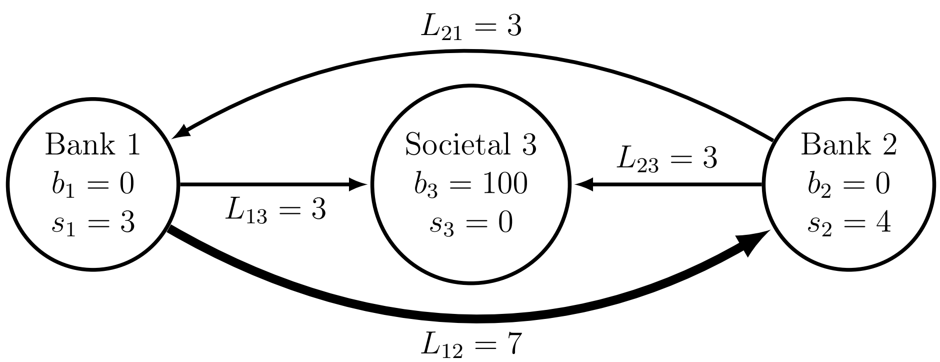
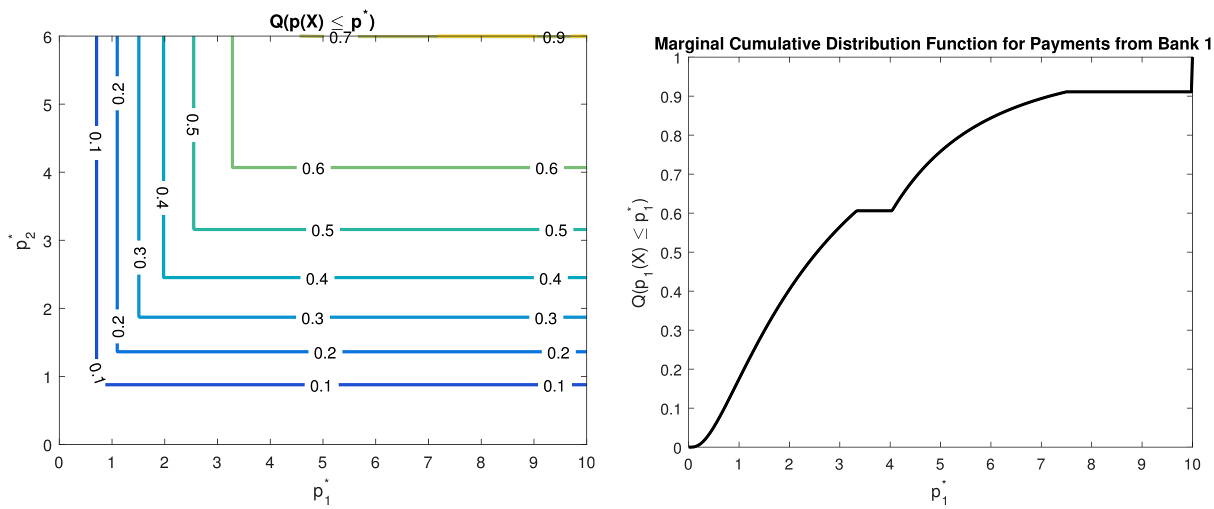


Figure 2.2: A simple network topology for Examples 2.3.15, 2.3.16, and 2.4.2.

I wish to conclude this section by providing simple examples to demonstrate the results considered so far in this chapter. Namely, I wish to show the analytical distribution for the payments from the firms followed by an empirical example showing the bounding properties first proposed in Lemma 2.3.8. In both of these examples I will focus on a simple 2 bank network with societal node as depicted in Figure 2.2.

Example 2.3.15. Consider the financial system depicted in Figure 2.2 under geometric Brownian motion assets with a common noise. This two bank system with an additional societal node will be considered with bankruptcy costs described by the recovery rate $\beta = 0.75$ and is such that bank 1 owes 7 units to bank 2 and 3 to the societal node and bank 2 owes 3 units to both bank 1 and the societal node. Further, for simplicity, the risk-free rate is assumed to be $r = 0$. Additionally, the maturity is set so that $T = 1$ and volatility so that $\sigma = 1$. Finally, the initial price of the risky asset is set to $q_0 = 1$. With this financial system, I will demonstrate the distributions of the clearing payments under the risk-neutral measure. Figure 2.3a displays the cumulative distribution function of the clearing payments as a contour plot. Notably, the joint cumulative distribution function has fairly stark right-angles in the individual contour lines. This structure follows from the use of minimums and maximums in the construction of the joint distribution for the clearing payments. The marginal cumulative distribution function for the payments from bank 1 is displayed in Figure 2.3b. The sections over which no probability is assigned are due to the bankruptcy costs in this setting. Notably, though piecewise similar to the cumulative distribution function of the lognormal, the marginal distribution clearly exhibits behavior that can only be found by accounting for the network effects.



(a) The joint cumulative distribution function of payments.

(b) Marginal cumulative distribution function of payments from bank 1.

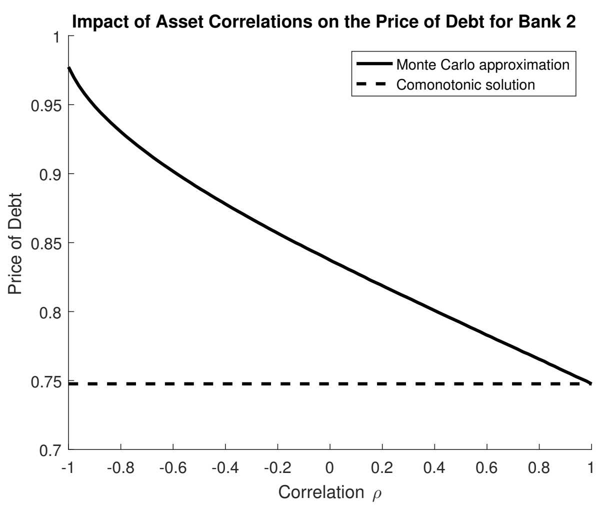
Figure 2.3: Example 2.3.15: The probability distribution of the clearing payments of the financial network provided by Figure 2.2 under bankruptcy costs.

Finally, I wish to numerically consider the case in which firm assets are correlated lognormals, but not comonotonic. I will use this numerical example to illustrate the bounding properties found in Lemma 2.3.8.

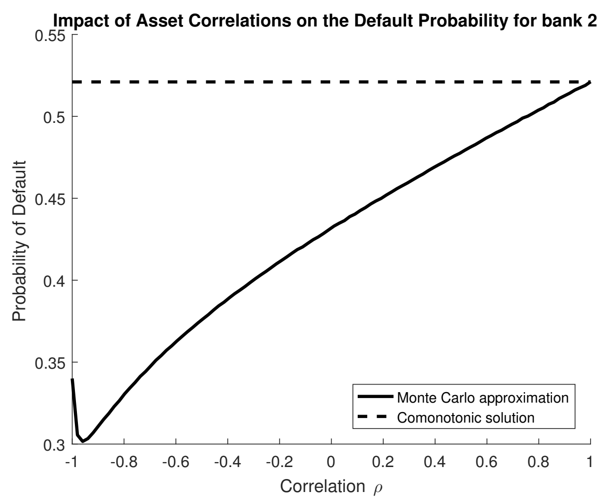
Example 2.3.16. Consider again the financial system with 2 banks and an additional societal node presented in Figure 2.2 and discussed in Example 2.3.15. Herein I wish to demonstrate the accuracy of the bounds found in Lemma 2.3.8 compared with varying correlations between the risky investments of the two firms. As such I will restrict myself to the Eisenberg--Noe setting without bankruptcy costs ($\beta = 1$). To construct the assets of the two banks I consider separate pricing processes for each bank. These pricing processes follow the correlated geometric Brownian motions with correlations $\rho \in [-1, 1]$ under the risk-neutral measure $\mathbb{Q}$:

$$\frac{dq_t^1}{q_t^1} = rdt + \sigma dW_t^1, \quad \frac{dq_t^2}{q_t^2} = rdt + \sigma dW_t^2, \quad \mathbb{E}^{\mathbb{Q}}[dW_t^1 dW_t^2] = \rho dt.$$

I will primarily focus on bank 2 in this example as it has a typical response. I first draw attention to Figure 2.4a. In this figure I see that the price of debt is bounded from below by the comonotonic case as proven in Lemma 2.3.8. However, as previously discussed in Example 2.3.9, the market capitalization does not need to have its worst-case under the comonotonic asset scenario. In fact, in this example, I find that under the comonotonic scenario the price of debt is lowest but the market capitalization is at its maximum for bank 2. In contrast, the societal node finds its equity worth the least when the firms have comonotonic assets. Finally, in Figure 2.4b I plot the (risk-neutral) probability of default of bank 2 as a function of asset correlations. Notably, the comonotonic case provides the greatest probability of default for bank 2, but the countermonotonic setting is *not* the lowest probability of default.



(a) Clearing payments from bank 2 under changes in asset correlations (solid line) and comonotonic assets (dashed line).



(b) Risk-neutral probability of default for bank 2 under changes in asset correlations (solid line) and comonotonic assets (dashed line).

Figure 2.4: Example 2.3.16: Demonstration of that clearing payments under comonotonic assets provide a lower bound, but the probability of default is not monotonically increasing in correlation of assets.

2.4 Comparative statics

In this section I provide the comparative statics for the performance of the system with respect to important system parameters through numerical examples. In these numerical examples I will assume the geometric Brownian motion setting of Section 2.3.2.

Definition 2.4.1. *Define the effective interest rate on firm i's debt by:*

$$R_i = \frac{1}{T}\left[\log(\bar{p}_i) - \log\left(\mathbb{E}^{\mathbb{Q}}[e^{-rT}p_i(X)]\right)\right].$$

For bank i, I can define $R_i - r$ as the risk premium along the lines of [86]. Please note that the risk premium can be taken as an effective measure of the price of debt.

Example 2.4.2. Consider again the financial system with 2 banks and an additional societal node as depicted in Figure 2.2 and used in Examples 2.3.15 and 2.3.16. Additionally, for simplicity and where otherwise I am not varying that parameter, I consider the risky asset to have initial price $q_0 = 1$, volatility $\sigma = 1$, and the claims to have maturity at time $T = 1$. Further, the risk-free rate is assumed to be $r = 0$. I consider this simple, illustrative, example so as to demonstrate the effects of the financial network (in comparison to the same system in two baseline systems without interbank debt as in [86]). For a clear comparison I will take this system without bankruptcy costs ($\beta = 1$) and with a common risky asset following a geometric Brownian motion. Specifically, I will consider the same comparative statics on the risk premium as undertaken by [86], i.e. by varying the debt-firm value ratios, the volatility of the risky asset, and the maturity of the debt claims. Please note that since I have assumed $r = 0$ I will use the terms risk premium and effective interest rate interchangeably.

(i) First, I will consider the impact of the debt-firm value ratios $d = \operatorname{diag}\left(b + sq_0 + \Pi^\top \bar{p}\right)^{-1}\bar{p}$ on the risk premium and thus the price of debt for each firm. In [86] in which no firm holds any risk-free assets ($b_i = 0$) nor any interbank assets ($\sum_{j=1}^n \pi_{ji}\bar{p}_j = 0$), it was shown that an individual firm's debt-firm value ratio can completely determine its own risk premium $R - r$. However, herein I consider explicitly the effects of the interbank assets. In my case, I can vary d by either:

(a) altering the liabilities $\bar{p}$ and keeping endowments b and s constant, i.e.

$$\bar{p} = \begin{bmatrix} 1 & -\pi_{21}d_1 \\ -\pi_{12}d_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} (b_1 + s_1 q_0)d_1 \\ (b_2 + s_2 q_0)d_2 \end{bmatrix}$$

given the desired debt-firm value ratio $d \in \mathbb{R}^2_+$ constrained by $d_1 d_2 \pi_{12}\pi_{21} < 1$; or

(b) altering the assets s and keeping liabilities $\bar{p}$ constant, i.e.

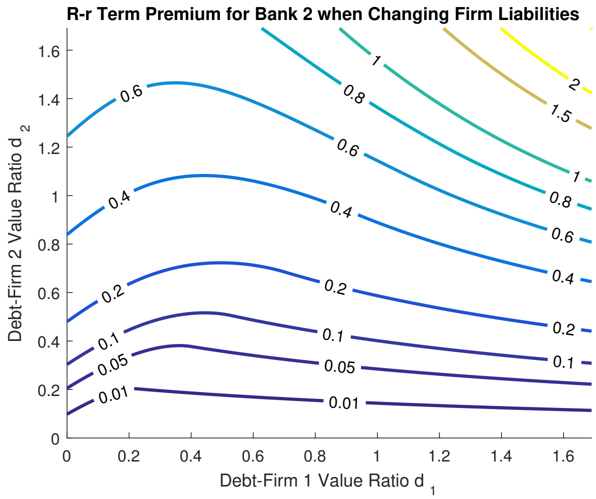
$$s = \text{diag}(d)^{-1} \left[\bar{p} - \text{diag}(d)\Pi^\top \bar{p}\right]$$

given the desired debt-firm value ratio $d \in \mathbb{R}^2_{++}$ so that $d \leq \text{diag}(\Pi^\top \bar{p})^{-1}\bar{p}$.

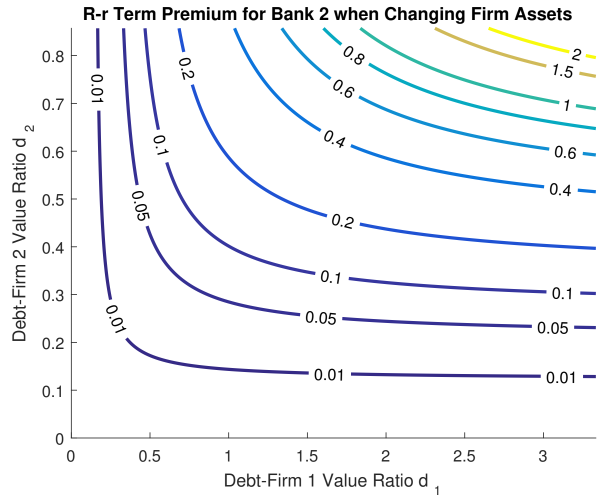
The distinction between the two approaches to varying d is important because I find that the manner in which the debt-firm value ratio is modified can greatly affect the price of debt as measured by the risk premium. The contour plots of the risk premium of bank 2 with respect to debt-firm 1 value ratio d_1 and debt-firm 2 value ratio d_2 are shown in Figure 2.5a and Figure 2.5b for varying the debt-firm values by altering liabilities and altering assets respectively. To provide further clarity on how the individual debt-firm values affect each other, I consider three slices of this data, by fixing the level of d_1 and varying d_2 through either altering the liabilities or the assets in Figure 2.5c and Figure 2.5d respectively. Notably, if firm 1 has a lower debt-firm value ratio d_1 constructed through the change in assets, then firm 2 consistently has a lower effective interest rate for any debt-firm ratio chosen. However, there is no such monotonicity when the debt-firm value ratios are constructed through changes in the liabilities.

(ii) Second I will consider the impact of the volatility σ of the risky asset on the risk premium (and thus the price of debt) and the market capitalization for each firm. In this consideration I wish to compare the network effects with two baseline models without a network. In order to consider the system without network effects I consider two settings:

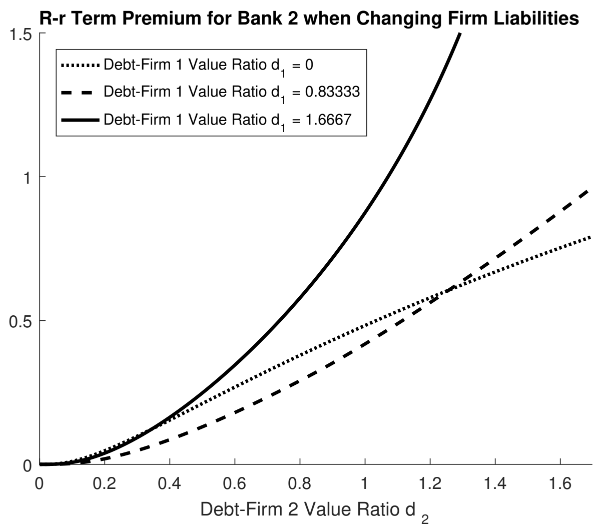
(a) with the assumption that all interbank assets are paid off in full in units of the risk-free asset; and



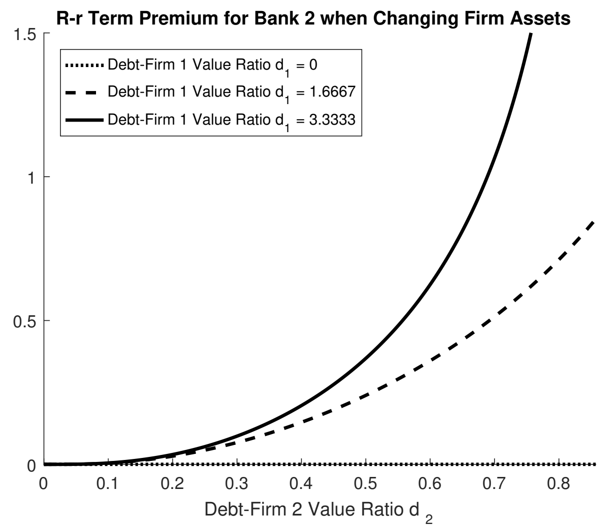
(a) Contour plot of the effective interest rate for firm 2 under different debt-firm value ratios when determined by the total liabilities.



(b) Contour plot of the effective interest rate for firm 2 under different debt-firm value ratios when determined by the total endowments.



(c) Cross-section of the effective interest rate for firm 2 under changes in its own debt-firm value ratio and with three levels of debt-firm 1 value ratio when changes are accomplished through modifications in total obligations.



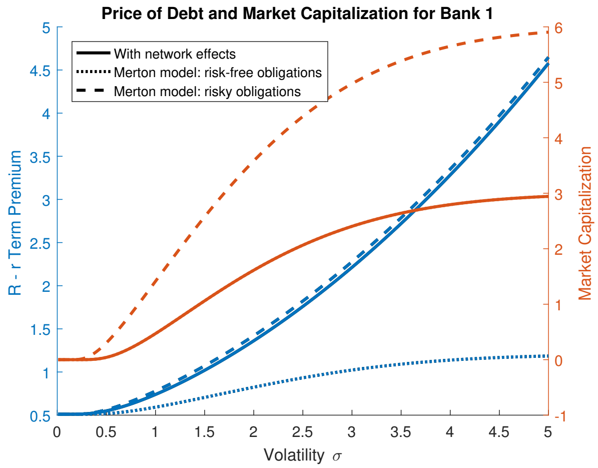
(d) Cross-section of the effective interest rate for firm 2 under changes in its own debt-firm value ratio and with three levels of debt-firm 1 value ratio when changes are accomplished through modifications in total assets.

Figure 2.5: Example 2.4.2(i): The effective interest rate of firm 2 versus changes in the debt-firm value ratios.

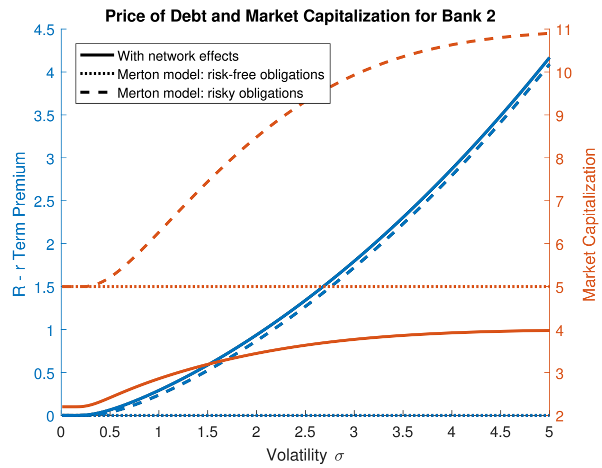
(b) with the assumption that all interbank assets are treated no differently than other risky assets (i.e. following the market geometric Brownian motion and not capped by the total obligations).

Due to the risk of the interbank assets, and as verified numerically, it is clear that the risk premium would be lower when neglecting counterparty risk and all interbank assets are treated as if they are risk-free than in the full networked system. Further, the market capitalization would be higher when neglecting the network effects and debts are treated as being paid in full in the risk-free asset. However, when interbank assets are treated as if they follow the market, the effective interest rates are comparable with the network effects. In contrast, the equity is significantly higher in this setting as the limited value that interbank assets can obtain is removed when treated as being an investment in the market. I display the results for the risk premium and the market capitalization for firm 1 and firm 2 under the networked system and the two baseline models in Figure 2.6a and Figure 2.6b respectively. As depicted in Figure 2.6a, firm 1 has greater effective interest rate when including the network effects or risk; in fact, firm 1 has its highest interest rates (asymptoting to approximately a 7% higher interest rate) when interbank assets are treated as risky assets. However, as shown in Figure 2.6a, firm 1 has equal market capitalization with network effects and when all interbank assets are treated as being paid in full; the market capitalization is significantly higher when interbank assets follow the market geometric Brownian motion. Firm 2, as depicted in Figure 2.6b, has orders of magnitude higher effective interest rates under the network effects than if they had no counterparty risk, and highest effective interest rates (asymptoting to approximately an 8% higher interest rate) when full network effects are taken into account. Here the market capitalization is distinct under all three considerations with the networked effects having the lowest market capitalization. This distinction makes clear that network effects can and should not be neglected when considering the price of debt and market capitalization.

(iii) Finally I will consider the impact of the maturity T for the claims on the risk premium (and thus the price of debt) and the market capitalization for each firm. As with the consideration on volatility above, I wish to compare the network effects with two baseline models without a network. I accomplish this in exactly the same way as considered previously. Again, due to the risk of the interbank assets in a network, it is clear that the effective interest rate would be higher and market capitalization lower when including the network effects than when all interbank assets are treated as in baseline model (iia). As depicted in Figure 2.7a, firm 1 has similar effective interest rate in all three scenarios, though the market capitalization is nearly double when



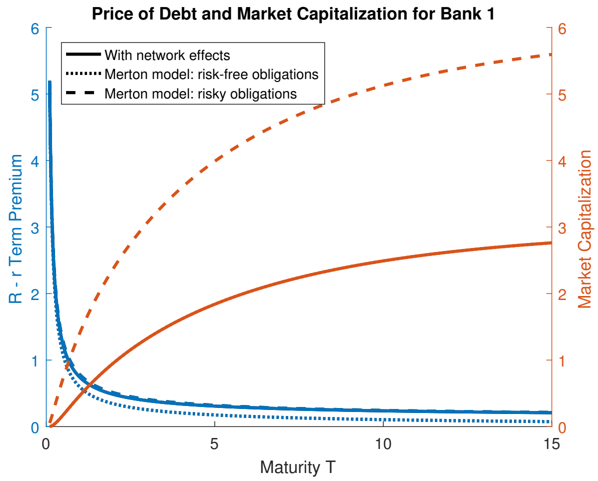
(a) The effective interest rate and market capitalization for firm 1 under changes to the volatility of the risky asset with network effects and under single firm effects only.



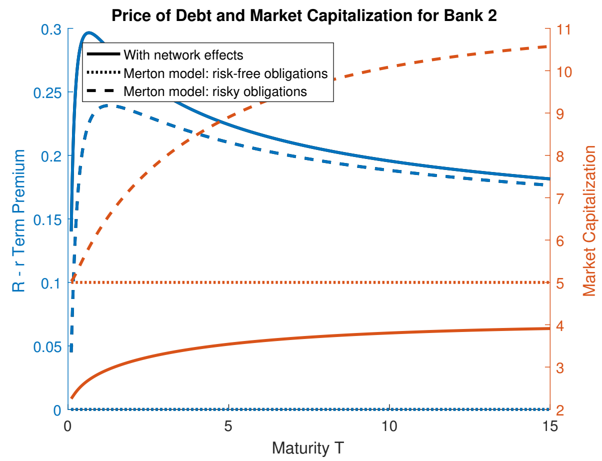
(b) The effective interest rate and market capitalization for firm 2 under changes to the volatility of the risky asset with network effects and under single firm effects only.

Figure 2.6: Example 2.4.2(ii): The effective interest rate and market capitalization versus changes in the volatility of the risky asset.

interbank assets are treated as the market asset (baseline model (iib)). Firm 2, as depicted in Figure 2.7b, has orders of magnitude higher effective interest rates under the network effects than if they had no counterparty risk and noticeably higher effective interest rate when full network effects are taken into account than if interbank assets are treated no differently than other risky assets (i.e. following the market model). As with all prior discussions on the market capitalization, the network effects greatly reduce the market capitalization compared to the two single-firm scenarios considered herein. I wish to conclude by considering the shapes of the interest rates as a function of the maturity of the claims under network effects. In [86] the hyperbolic shape of firm 1's effective interest rate would only occur if its debt-firm value ratio was greater than or equal to 1; similarly the shape exhibited by firm 2's effective interest rate would only occur if its debt-firm value ratio was strictly less than 1. However, as discussed above, the debt-firm value ratio does not have as unique a property under network effects as it did in [86] without counterparty risk. Thus I find that the change in shape need not (and in this numerical example, does not) change shape at the individual debt-firm value ratios of 1.



(a) The effective interest rate and market capitalization for firm 1 under changes to the maturity of the debt claims with network effects and under single firm effects only.



(b) The effective interest rate and market capitalization for firm 2 under changes to the maturity of the debt claims with network effects and under single firm effects only.

Figure 2.7: Example 2.4.2(iii): The effective interest rate and market capitalization versus changes in the maturity of the debt claims.

Example 2.4.3. I will now consider a larger financial network consisting of $n = 87$ banks. This large network provides clear reasoning for considering the comonotonic approach taken within this chapter. As previously discussed, with 87 banks, there are $2^{87} > 10^{26}$ potential combinations of defaulting banks $z \in \{0,1\}^{87}$. As such, the general framework for considering expected payments from [69] would be computationally intractable. However, the comonotonic framework presented herein (and which, under the setting of [45], provides a worst-case for the general setting) is computationally tractable as only 87 defaulting regions need to be considered.

For this example, I will consider these 87 banks to come from the 2011 European Banking Authority EU-wide stress tests.[1] This dataset has been used in multiple prior empirical case studies (e.g. [63, 32]) of financial contagion in interbank networks. To calibrate this system, I will take the same approach from [52]. I note, however, that though I am calibrating the financial network to a real dataset, the marginal distribution for bank endowments are *not* calibrated and as such this example is for *illustrative purposes only*. I believe that there

[1] Due to complications with the calibration methodology, I only consider 87 of the 90 institutions. DE029, LU45, and SI058 were not included in this analysis.

would be significant value in a further, detailed, case study to empirically determine the marginal distributions of the bank endowments and, with that result, consider yield rates and bond prices to compare with the realized prices in the market. This is, however, beyond the scope of the current example. In fact, the primary purpose of using this dataset in this example, as opposed to a large fictional network, is to demonstrate the order of magnitude that the price of debt and effective interest rates can achieve (in comparison to the values presented in the prior case studies on the 2 bank system).

For this network calibration, I consider a stylized balance sheet for each bank. I consider banks with only two types of assets: *interbank assets* $\sum_{j=1}^n L_{ji}$ and *external (risky) assets* s_i. Similarly, I consider three types of liabilities for each bank: *interbank liabilities* $\sum_{j=1}^n L_{ij}$, *external liabilities* $L_{i,n+1}$, and *capital* C_i. In contrast, the EBA dataset provides the total assets A_i, capital C_i, and interbank liabilities $\sum_{j=1}^n L_{ij}$ for each bank i.

Therefore, to calibrate the interbank network, I will need to make a few simplifying assumptions and take advantage of techniques from prior literature. In particular, as in [52, 32, 67], the external (risky) assets are the difference between the total assets and interbank assets, the external obligations (owed to the societal node $L_{i,n+1}$) are equal to the total liabilities less the interbank liabilities and capital, and the interbank assets will be assumed equal to the interbank liabilities, i.e., $\sum_{j=1}^n L_{ij} = \sum_{j=1}^n L_{ji}$ for all banks i. Thus, I can construct the remainder of my stylized balance sheet through the system of equations

$$s_i = A_i - \sum_{j=1}^n L_{ij}, \quad L_{i0} = A_i - \sum_{j=1}^n L_{ij} - C_i, \quad \bar{p}_i = L_{i0} + \sum_{j=1}^n L_{ij}.$$

To verify the consistency of this calibration, I note that firm i's net worth is equal to its capital, i.e., $C_i = A_i - \bar{p}_i$.

Finally, for my calibration, I need to consider the full nominal liabilities matrix $L \in \mathbb{R}_+^{87\times 87}$ and not just the total interbank assets and liabilities. In order to accomplish this task I consider the methodology of [63]. That paper presents an MCMC methodology to construct the nominal liabilities matrix consistent with the total interbank assets and liabilities and which allows for a (randomized) sparsity structure. As noted previously, this example is for illustrative purposes only and thus I will consider only a single calibration of the interbank network.

In order to complete my model, I need to consider the remaining parameters of the system. First, as all economic data pulled from the EBA EU-wide stress test dataset are already in a consistent unit (millions of euros), I will consider $q_0 = 1$ (million). Further, during the period over which this data was collected, central banks were setting a low interest rate environment. Therefore I estimate that the risk-free interest rate is $r = 0$. Additionally, as this is data from a single year's stress test, I will consider maturity on all debt claims to be $T = 1$ (year). Finally, the volatility of the risky asset is estimated to be $\sigma = 20\%$ from comparisons to annualized historical volatility of European markets in 2011.

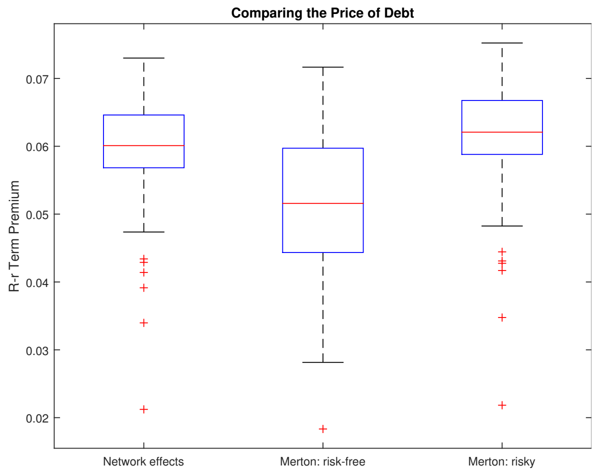
First, I wish to consider the impact of the full network effects on the effective interest rates and market capitalization in the setting without bankruptcy costs ($\beta = 1$). For this analysis I consider the same two baseline models as in Example 2.4.2 above, i.e.

(i) with the assumption that all interbank assets are paid off in full in units of the risk-free asset; and

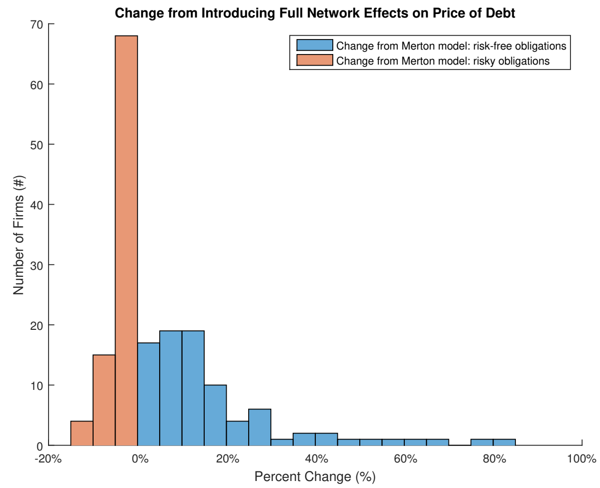
(ii) with the assumption that all interbank assets are treated no differently than other risky assets (i.e. following the market geometric Brownian motion and not capped by the total obligations).

The data for these are provided in Figures 2.8 and 2.9 respectively. I note that, as in Example 2.4.2 above, the price of debt with full network effects is generally comparable to the single firm effect case with all interbank assets treated as the risky asset. In fact, the interest rate of debt with full network effects is lower than if all interbank assets are treated as the risky asset, but significantly higher than when interbank assets are treated as the risk-free asset. In contrast, and again comparable to that in Example 2.4.2 above, the market capitalization for firms is strikingly similar between the full network effects and the single firm effects with interbank assets treated as the risk-free asset. The single firm effects with interbank assets treated as the risky asset can differ by a large degree from the network effects for the market capitalization of the individual firms.

Second, though above I consider the setting without bankruptcy costs, I now wish to consider how the price of debt and equity are affected by the bankruptcy costs. Analytically, I can conclude before any simulations, that the effective interest rates will decrease and the market

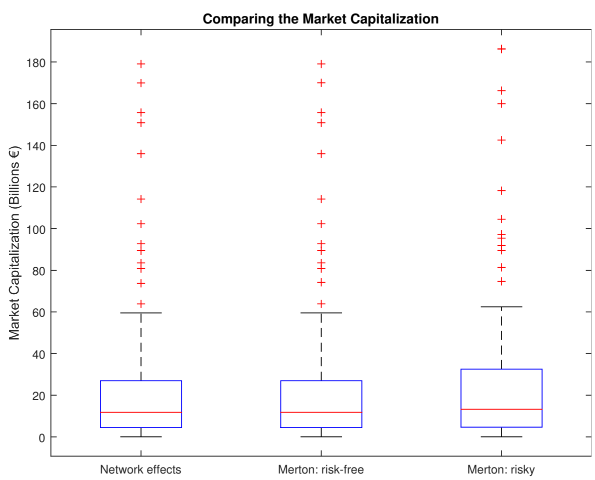


(a) Box plot of effective interest rates under network effects and under single firm effects only.

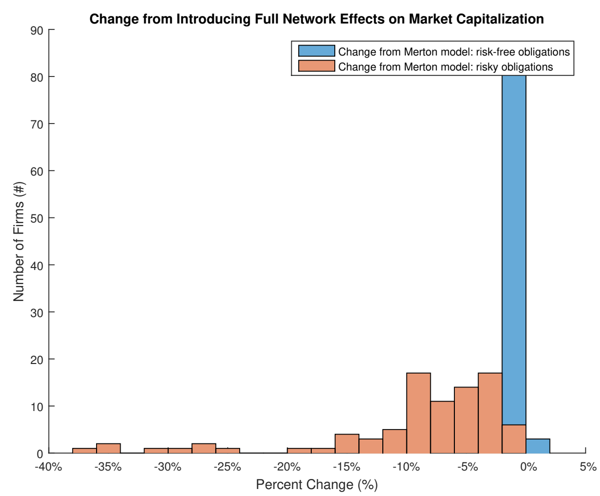


(b) Histogram of the relative change in the effective interest rates by including full network effects.

Figure 2.8: Example 2.4.3: Comparison of the price of debt under network effects and under single firm effects only without bankruptcy costs ($\beta = 1$).



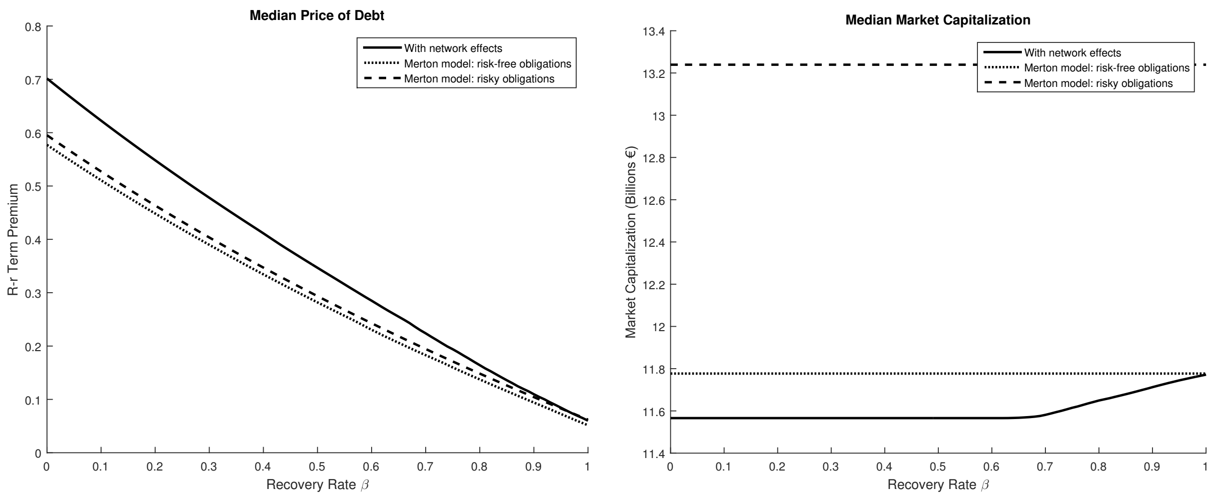
(a) Box plot of the market capitalization under network effects and under single firm effects only.



(b) Histogram of the relative change in the market capitalizations by including full network effects.

Figure 2.9: Example 2.4.3: Comparison of market capitalization under network effects and under single firm effects only without bankruptcy costs ($\beta = 1$).

capitalization will increase with the recovery rate β. Figure 2.10 depicts the median price of debt and market capitalization for the 87 banks under consideration; this demonstrates that the comparisons made at high values of β that appear accurate lose their predictive power if $\beta < 1$. Thus, if bankruptcy costs exist, considering the interbank assets as either the risk-free or risky asset can cause mispricing of risk. Given the simulated network below, the full network effects cause higher risk in both debt and equity than if each firm were treated individually.



(a) Effect of recovery rate on the median effective interest rate.

(b) Effect of recovery rate on the median market capitalization.

Figure 2.10: Example 2.4.3: Comparative statics on the recovery rate (β) to the market prices of debt and equity.

Chapter 3

Price-mediated contagion through capital ratio requirements

This chapter is based on [15] which is joint work with Zachary Feinstein.

3.1 Introduction

An important channel of contagion in financial systems comes through indirect connections or global interactions among organizations, e.g., fire sale spillovers. Those originate when a firm is forced to liquidate its assets to meet some obligation or regulation. As firms hold overlapping portfolios this causes impacts globally to all other firms due to marked-to-market accounting. These firms are now forced to liquidate their assets, thus exacerbating the crisis. An important factor in the origin of fire sale is the unintended consequence of capital regulations in the form of capital ratio or leverage ratio. Due to these regulatory constraints, banks might be forced to deleverage, setting off a vicious cycle of contagion. Such deleveraging occured in a large scale in the 2008 financial crisis, resulting in amplification of losses. For further discussion on such mechanisms see [26, 38].

The literature in the study of fire sale may be broadly divided into two different bodies depending on the focus of the study. The first among these two bodies places more emphasis on the development of a general mathematical framework and exploring questions about existence and uniqueness in this general setting. Such works maybe considered as extensions of [45] in the setting of price-mediated contagion. Among these works, [33] considers the liquidation problem in the context of a capital adequacy ratio. [7, 51] study the fire sale problem when banks are forced to liquidate assets to meet debt obligations. [54] develops an extension to [51] where banks, in addition to meeting their debt obligations, must satisfy a leverage ratio. [26] considers the problem where banks are required to satisfy a risk-weighted capital ratio. [53] considers the price-mediated contagion problem in a continuous time setting and provides results on existence and uniqueness as well as analytical bounds under a random setting. The second body of work in the domain of fire sale focuses more on the development of an operational modeling framework and the design of stress tests using this approach. Typically these results depend on a particular liquidation strategy (e.g. proportional liquidation) and linear price impacts. Some of the notable works in this domain include [71, 43, 38].

The primary goal of this chapter is to develop a general mathematical and economic framework to study price-mediated contagion in the case in which firms liquidate assets during a crisis due to risk-weighted capital requirements. The risk-weighted capital ratio is defined as the ratio of the capital of a bank divided by the risk-weighted assets of that bank. Such type of regulatory requirements have been considered in the Basel regulation to assess the health of financial institutions. In particular, Basel III mandates that this risk- weighted capital ratio be greater than 8% for any bank. In order to adhere to such regulation during a crisis, banks will liquidate their assets thus setting off fire sale and causing widespread losses, as documented in [39, 77, 83]. Hence it is imperative that we accurately model such contagion. The main highlights and contributions of my analysis are summarized as follows:

(i) I consider a multi-asset setting to model the price-mediated contagion due to risk-weighted capital requirement constraints. The multi-asset setting provides a suitable framework to model cross-asset contagion, which was widely observed in the 2008 financial crisis. This consideration of multiple assets, though realistic, offers a much more challenging setting in comparison to a single asset. One of the major challenges is the strategic component that presents itself while performing liquidation to satisfy the capital requirements. Existing literature (e.g. [71, 43, 38]) considers proportional liquidation for this analysis. In contrast, I consider a general liquidation function with mild continuity and monotonicity properties and am thus able to study liquidation schemes beyond proportional liquidation and encode the strategic component in my framework. An interesting example that I consider is utility maximizing liquidation in which banks choose which assets to liquidate in order to maximize their utility. I highlight the importance of the choice of the liquidation function through numerical examples.

(ii) In contrast to other static models of fire sale ([26, 54, 51, 7, 33, 38, 43, 71]), I consider two notions of pricing: the current price, which gives the current valuation for the unliquidated assets, and the volume weighted average liquidation price, which I use to determine the proceeds for liquidation. The adoption of these two separate notions of pricing is important from financial and mathematical standpoints. While liquidating, the prices of the assets which have already been sold will be higher than the current price of the assets. The average liquidation price enables me to encode this dynamic aspect in my modeling. Beyond this financial motivation, the adoption of the average liquidation price significantly aides the mathematical analysis, particularly in the development of conditions for uniqueness.

(iii) In the existing literature on price-mediated contagion due to leverage/capital requirements, results on the existence of the equilibrium prices have been explored in [26, 54]. However, results on uniqueness have not been explored in this setting. In contrast, in this work I develop conditions for the uniqueness of the pricing equilibrium. This condition is characterized by certain properties on the inverse demand function. I also provide an alternative characterization of the uniqueness condition in terms of a monotonicity condition on the inverse demand function and a lower bound on the risk-weights which depends on the liquidity of the assets under consideration. Equivalently, this result provides a way to calibrate the risk-weights properly depending on the liquidity of the assets rather than using heuristics. I compare my result to the analysis performed in [53] for a continuous time framework.

(iv) An important consideration in the study of fire sale is how prices vary as more assets are liquidated. This is often modeled by an inverse demand function. Existing literature on fire sale mostly consider a linear inverse demand function ([71, 26, 38, 43]). Although such an inverse demand function will be easier to calibrate, this is a strong assumption. In contrast, in my analysis I consider a general inverse demand function and develop conditions on the inverse demand function to get uniqueness of solutions.

(v) The equilibrium pricing is dependent upon a number of system parameters. In reality, it is very difficult to have a complete and accurate knowledge of all these system parameters. Despite the progress that has been made in recent years, the regulators face significant legal and logistical hurdles in the data collection process. A second factor which hinders this process is the fact that most of the systemic risk models are very nascent and the discovery of which data is required in this analysis is very much an ongoing process. In that context, the determination of the equilibrium prices and hence the stress testing results is very much dependent upon how these parameters are calibrated. Owing to this uncertainty, it is imperative that we have an understanding of how a variation in these parameters might affect the equilibrium prices. I develop a mathematical framework to perform sensitivity analysis of the equilibrium prices with respect to the system parameters. In the systemic risk literature, sensitivity analysis has been studied in the case where the banks have bilateral debt linkages in [82, 55]. As far as I am aware, this is the first work to perform sensitivity analysis in the fire sale literature.

(vi) I develop a method to study the cost of regulation incurred by each bank. This is based on the fact that tightening the regulatory threshold, that the banks have to satisfy, will result in increased loss for the banks. Thus the marginal rate of change of this loss with respect to the marginal change in the threshold gives a measure of the cost of regulation. Based on my result in the sensitivity analysis, I can compute this loss. In fact, I develop two notions of this cost depending on the type of loss that a bank might suffer. The first one is based on the actual realized loss a bank may suffer and it depends directly on the increased liquidation of that bank as a result of the increased regulation. I develop a second notion of loss based on the marked-to-market impact. This encodes the notion that even if a bank might not need to liquidate anything as a result of the increased regulation, and hence do not have any realized loss, it is still susceptible to contagion through global interactions brought about by price depreciation and overlapping portfolios. I highlight these ideas through an empirical study and focus on the six banks participating in the 2015 Federal Reserve CCAR with the largest trading operations.

(vii) Through a numerical simulation with a two bank two asset system, I study the effect of diversification of bank portfolio in this framework. I vary the portfolio of these institutions as the system moves from fully diverse to fully diversified and study the price-mediated contagion under different liquidation schemes. I find that diversification does not uniformly lead to a more stable system, measured in this case by the total market capitalization. In fact, under certain liquidation regimes, the cross-asset contagion might outweigh the benefits of diversification.

The remainder of this chapter is organized as follows: In Section 3.2, I develop the mathematical framework of my model and characterize the fire sale as a fixed point problem. In Section 3.3, I develop conditions for the existence and uniqueness of the equilibrium prices. In Section 3.4, I formulate the sensitivity analysis of the equilibrium prices with respect to the system parameters as a fixed point problem and prove the existence and uniqueness of a solution to this problem. Further, I develop a methodology to evaluate the cost of regulation. Numerical case studies highlighting the applications of this model are presented in Section 3.5.

3.2 Mathematical framework

3.2.1 Balance sheet and risk-weighted Capital Ratio

Throughout this chapter I will consider a network of n financial institutions. I will denote the set of all banks in the network by $\mathcal{N} := \{1, 2, \ldots, n\}$.

I will consider two time points $t = \{0, 1\}$. At $t = 0$, each firm $i = 1, ..., n$ holds $x_i \geq 0$ in liquid assets (e.g. cash). I will assume without loss of generality that the price of this asset stays constant at 1 at all times. In addition to this liquid asset, the bank portfolio comprises of illquid assets. In line with [26, 53], I consider two classes of illiquid assets: marketable (stocks or bonds issued by a non-financial corporation) or non-marketable (loans). The distinction between these two classes of illiquid assets is that non-marketable assets are difficult to sell in the short-run, and hence those cannot be liquidated. For further discussion on non-marketable assets see [26, 42]. I will consider $m \in \mathbb{Z}^+$ marketable illquid assets. Specifically I allow for the situation where banks hold multiple marketable illiquid assets. This represents a more realistic setting in comparison to single asset models, such as [26]. A particularly significant aspect of the multi-asset setting is that this allows me to model cross-asset contagion. However, the multi-asset case is much more challenging to deal with mathematically compared to the single asset case, as has been demonstrated in [51, 54]. I assume that each bank holds $s_i \in \mathbb{R}^m_+$ shares of marketable iilquid assets and $l_i \geq 0$ of non-marketable assets. Without loss of generality, I assume that the price of all the illiquid assets are 1 at time 0.

On the other side of the balance-sheet each firm i has $\bar{p}_i \geq 0$ in liabilities. I assume that the liabilities are long-term and these are not owed to any other bank within the system. Thus, at time 0, bank i has a capital of $x_i + l_i + 1^\top s_i - \bar{p}_i$. This is depicted in Figure 3.1.

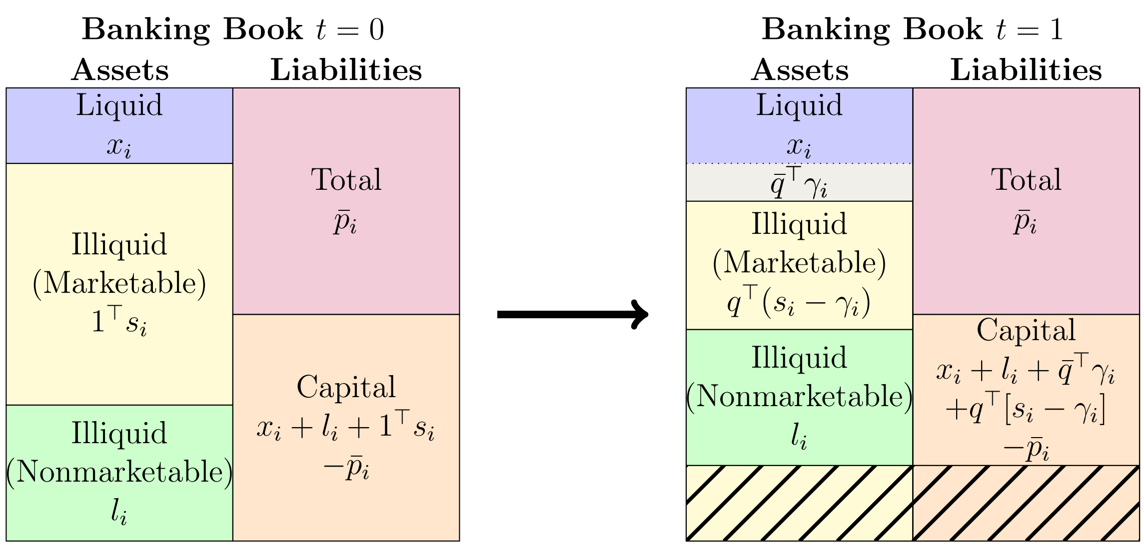


Figure 3.1: Stylized banking book for a firm before and after liquidation updates for bank i

In terms of vector notation, at time $t = 0$, the banks are holding an amount $x \in \mathbb{R}^n_+$ of liquid assets, $l \in \mathbb{R}^n_+$ shares of non-marketable illiquid assets, $S = (s_{ik}) \in \mathbb{R}^{n \times m}_+$ shares of marketable illquid assets and an amount $\bar{p} \in \mathbb{R}^n_+$ in liabilities.

The Basel Regulation mandates the use of a risk-weighted capital ratio to assess the solvency of banks. The risk-weighted capital ratio is defined as

$$\text{Risk-weighted capital ratio} = \frac{\text{Total Capital}}{\text{Risk-Weighted Assets}}$$

The determination of the risk-weights of different assets requires the consideration of a number of complex factors. In this case, I make the assumption that these risk-weights are known to me and given by 0 for the liquid asset and α_k for $k = 1, ..., m$ for the marketable illiquid assets for all the banks. For the non-marketable asset I let the risk-weight to be dependent on each bank and let α_{li} be the risk-weight for bank i. Let me define $A \equiv \text{diag}(\alpha_1, ..., \alpha_m)$.

Thus at time $t = 0$, the risk-weighted capital ratio θ_i of bank i is given by

$$\theta_i = \frac{x_i + l_i + 1^\top s_i - p_i}{1^\top A s_i + \alpha_{li} l_i} \tag{3.1}$$

According to banking regulations, banks are required to maintain a minimum capital ratio θ_{min} e.g. 8% in Basel III regulations. I assume that at $t = 0$, all banks comply with this regulatory constraint i.e. $\theta_i \geq \theta_{min}$ for all $i \in \mathcal{N}$.

3.2.2 Leverage targeting and asset liquidation

At time $t = 0^+$, the system is subject to an exogenous shock. This might be

(i) shock in prices, where the prices of any or all of the assets are hit as assumed in the standard literature viz. [26, 71, 43].

(ii) shock in the risk-weight, where the risk-weight of an asset jumps due to a credit downgrade.

Depending on the size of the shock and the state of the balance-sheet at $t = 0$, the capital ratio of some banks may fall below the regulatory minimum θ_{min}. In this situation, banks typically have two options:

(i) issue new stocks

(ii) sell existing assets

At the time of a crisis, issuing new stocks might not be feasible. In this situation, banks will be forced to liquidate assets to meet the regulatory constraint and might set off fire sale.

Assumption 3.2.1. *Banks will sell assets to restore their risk-weighted capital ratio to* θ_{min}.

This practice, known as leverage-targeting, is in line with the existing literature [26, 71, 43, 53] as well as empirical evidence [3].

To effectively model the fire sale, I need to consider how prices vary with the amount of assets that need to be liquidated. In line with the systemic risk literature I assume that the current price of the marketable illiquid assets is given by the current inverse demand function $F : \mathbb{R}^m_+ \mapsto [0,1]^m \subset \mathbb{R}^m_+$.

In a break with the existing literature on static fire sale models ([26, 54, 51, 7, 33, 38, 43, 71]), I consider a second notion of pricing: the volume weighted average price (VWAP) of liquidation. This is the average over all liquidation prices where the average is taken over the total volume sold. This average liquidation price is given by the volume weighted average inverse demand function $\bar{F} : \mathbb{R}^m_+ \mapsto [0,1]^m \subset \mathbb{R}^m_+$. Intuitively, the average liquidation price encodes the notion that the price of the assets that have already been sold will be higher than the current price of the assets. Thus considering these two separate prices makes this model more realistic and enables me to encode a dynamic notion which is absent in the existing literature. This is further discussed in Remark 3.2.4.

Assumption 3.2.2. *The current inverse demand function* $F : \mathbb{R}^m_+ \mapsto [0,1]^m \subset \mathbb{R}^m_+$ *and the average inverse demand function* $\bar{F} : \mathbb{R}^m_+ \mapsto [0,1]^m \subset \mathbb{R}^m_+$ *are continuous and non-increasing. Further the components of the inverse-demand functions are independent i.e.* $F(s) = [F_1(s_1)\ F_2(s_2)\ \ldots\ F_m(s_m)]^\top$ *and* $\bar{F}(s) = [\bar{F}_1(s_1)\ \bar{F}_2(s_2)\ \ldots\ \bar{F}_m(s_m)]^\top$.

The assumption about continuity and monotonicity is standard and intuitive. The component-wise assumption implies that there are no cross-impacts, i.e., the sale of one asset does not directly affect the prices of other assets. I make this assumption for mathematical and computational simplicity. It should be noted that I do not require an an explicit assumption on the linearity of the inverse demand function, in contrast to existing literature such as [26, 43, 38, 71]. Under this assumption, the current and the average inverse demand functions are related by the following equation:

$$\bar{F}_k(s_k) = \frac{1}{s_k}\int_0^{s_k} F_k(a_k)da_k \;\forall k \tag{3.2}$$

Since I am dealing with multiple assets, the banks will often have a number of ways in which they can liquidate the assets. I encode this strategic component in my model through the liquidation function γ. Let me define $D = \{q \times \bar{q} \in [0,1]^{2m} | \bar{q} \geq q\}$. Then the liquidation function is given by $\gamma_{ik} : D \mapsto \mathbb{R}_+$ and defined as the number of units of asset $k = 1, 2, ..., m$ that firm $i = 1, 2, ..., n$ wishes to sell. For a further discussion on liquidation function see [51]. In the standard literature on fire sale due to regulatory requirements, this strategic component does not present itself in single asset models [26] or the banks are assumed to follow a particular strategy, e.g., proportional liquidation [71, 43, 38].

Remark 3.2.3. It might be entirely possible that even when a bank liquidates all its assets it cannot restore its capital ratio to θ_{min}. In this situation I will assume that such a bank is insolvent and costlessly liquidated at $t = 1$ along the lines of [26].

Given a marked-to-market average liquidation price $\bar{q} \in \mathbb{R}^m_+$ and current liquidation price $q \in \mathbb{R}^m_+$, the liquidation is given by $\gamma(q, \bar{q})$ and the capital of bank i is given by $x_i + l_i + \bar{q}^\top \gamma_i + q^\top [s_i - \gamma_i] - \bar{p}_i$. The risk-weighted capital ratio for bank i is then given by

$$\frac{x_i + l_i + \bar{q}^\top \gamma_i + q^\top [s_i - \gamma_i] - \bar{p}_i}{q^\top A[s_i - \gamma_i] + \alpha_{li} l_i}$$

This situation is depicted in Figure 3.1. If bank i needs to perform liquidation but is solvent, by Assumption(3.2.1) it will perform liquidation γ_i such that

$$\frac{x_i + l_i + \bar{q}^\top \gamma_i + q^\top [s_i - \gamma_i] - \bar{p}_i}{q^\top A[s_i - \gamma_i] + \alpha_{li} l_i} = \theta_{min} \tag{3.3}$$

Let me define shortfall of bank i as $h_i \equiv \bar{p}_i - x_i - (1 - \theta_{min}\alpha_{li}) l_i$ for $i = 1, 2, .., n$. Then (3.3) can be reformulated as,

$$\bar{q}^\top \gamma_i + q^\top [I - \theta_{min} A](s_i - \gamma_i) = h_i^+ \wedge \bar{q}^\top s_i \tag{3.4}$$

I refer to (3.4) as the *liquidation condition.* It encodes the liquidation constraint for bank i when bank i is solvent but needs to perform some liquidation.

More generally, a bank $i \in \mathcal{N}$ can belong to any of the following three mutually exclusive and exhaustive sets:

- Solvent and do not need to liquidate: Let me denote this set by $S(q, \bar{q})$. In this case $\gamma_i = 0$ and $h_i \leq q^\top [I - \theta_{min} A] s_i$.
- Solvent but needs to liquidate: Let me denote this set by $L(q, \bar{q})$. This is characterized by $q^\top [I - \theta_{min} A] s_i < h_i < \bar{q}^\top s_i$.
- Defaults: Let me denote this set by $D(q, \bar{q})$. In this case $\gamma_i = s_i$ and $h_i \geq \bar{q}^\top s_i$.

To reflect these three possible states of a bank, I can rewrite (3.4) as the general *liquidation condition*

$$\bar{q}^\top \gamma_i + q^\top [I - \theta_{min} A](s_i - \gamma_i) = h_i^+ \wedge \bar{q}^\top s_i \tag{3.5}$$

Equation (3.5) states that the number of units liquidated by a bank is either enough to meet the risk-weighted capital requirement or all assets are liquidated. Additionally, it ensures that the banks are adhering to the principle of *leverage targeting* i.e. no bank is selling more than it is necessary to meet the threshold θ_{min}. This is similar to the liquidity constraint used in [51, 54].

Remark 3.2.4. In Equation 3.3, while computing the capital of bank i, the proceeds from liquidation is given by $\bar{q}^\top \gamma_i$. This is in contrast to existing literature in fire sale (e.g [26, 54]) where no distinction is made between liquidation price and current price and thus the proceeds would be computed as $q^\top \gamma_i$. This is a more realistic scenario and it offers more favourable conditions to analyze the uniqueness of solutions (as discussed in the succeeding section).

Assumption 3.2.5. $\alpha_k \theta_{min} < 1$ *for* $k = 1, 2, .., m$.

If $\alpha_k \theta_{min} \geq 1$ for any k, then the setting of this chapter implies that as price drops in that asset, the bank will always satisfy the capital regulation which is opposite to the scenario that I am modeling. For further discussion see Remark 2.2 of [53].

3.2.3 Fixed point formulation

Equation (3.5) gives the liquidation that bank i performs under a given marked-to-market price$(q, \bar{q})$. However this liquidation might result in further price depreciation which necessitates the consideration of a different γ_i. Thus this situation can be accurately modeled using a fixed point equation, as is common in the systemic risk literature.

Let me define $\Gamma(q, \bar{q}) \in \mathbb{R}_+^m$ as the vector of total illiquid assets sold i.e $\Gamma_k(q, \bar{q}) = \sum_{i=1}^n \gamma_{ik}(q, \bar{q})$ for $k = 1, 2, .., m$. Then the equilibrium price is defined by the function $\Phi : [0,1]^m \times [0,1]^m \mapsto [0,1]^m \times [0,1]^m$ where

$$\Phi(q, \bar{q}) = (F^\top(\Gamma(q, \bar{q})),\ \ \bar{F}^\top(\Gamma(q, \bar{q})))^\top \tag{3.6}$$

and $\gamma_i(q, \bar{q})$ must satisfy the liquidation condition (3.5) for $i = 1, 2, .., n$.

The value of the equilibrium liquidation price $\bar{q}^* \in [0, 1]$ and current price $q^* \in [0, 1]$ is given by the fixed point of Φ defined in Equation (3.6), i.e.

$$(q^*, \bar{q}^*) = \Phi(q^*, \bar{q}^*)$$

3.3 Existence and uniqueness

In this section, I develop conditions for existence and uniqueness of the equilibrium prices described by the fixed point equation (3.6). In the existing literature on price-mediated contagion due to regulatory requirements, existence of solutions has been explored for the one asset case in [26] and for the multi-asset case in [54]. However in these works, uniqueness has not been explored.

3.3.1 Existence and uniqueness theorems

Theorem 3.3.1. *Consider the general setting as described above. Let* $M \geq \sum_{i=1}^n s_i$ *:*

(i) If the liquidation function γ is jointly continuous in $(q, \bar{q})$, there exists an equilibrium price $(q^*, \bar{q}^*)$.

(ii) If the liquidation function γ is non-increasing in $(q, \bar{q})$, there exists a greatest and least equilibrium price $(q^\uparrow, \bar{q}^\uparrow) \geq (q^\downarrow, \bar{q}^\downarrow)$.

(iii) If additionally, $\bar{F}(\Gamma)^\top \Gamma + F(\Gamma)^\top [I - \theta_{min} A](M - \Gamma)$ is strictly increasing in Γ, then there exists a unique equilibrium price $(q^*, \bar{q}^*)$.

Proof. (i) This is a straight-forward application of Brouwer fixed point theorem.

(ii) This is a straight-forward application of Tarski fixed point theorem.

(iii) As discussed in the preceding section, a bank i can belong to any of the following three mutually exclusive and exhaustive sets:

- $S(q, \bar{q}) = \{i \in \mathcal{N} | h_i \leq q^\top [I - \theta_{min} A] s_i\}$.
- $L(q, \bar{q}) = \{i \in \mathcal{N} | q^\top [I - \theta_{min} A] s_i < h_i < \bar{q}^\top s_i\}$.
- $D(q, \bar{q}) = \{i \in \mathcal{N} | h_i \geq \bar{q}^\top s_i\}$.

Then using the liquidation condition (3.5) and under $(q^*, \bar{q}^*)$,

$$\bar{q}^{*\top} \gamma_i(q^*, \bar{q}^*) + q^{*\top} [I - \theta_{min} A](s_i - \gamma_i(q^*, \bar{q}^*)) = \begin{cases} q^{*\top} [I - \theta_{min} A] s_i & \text{if } i \in S(q^*, \bar{q}^*) \\ h_i & \text{if } i \in L(q^*, \bar{q}^*) \\ \bar{q}^{*\top} s_i & \text{if } i \in D(q^*, \bar{q}^*) \end{cases}$$

Using (ii), there exists a greatest and least clearing price $(q^\uparrow, \bar{q}^\uparrow) \geq (q^\downarrow, \bar{q}^\downarrow)$. Further from (ii), Γ is non-increasing in $(q, \bar{q})$, hence $\Gamma^\uparrow = \Gamma(q^\uparrow, \bar{q}^\uparrow)) \leq \Gamma(q^\downarrow, \bar{q}^\downarrow)) = \Gamma^\downarrow$.

Since, $(q^\uparrow, \bar{q}^\uparrow) \geq (q^\downarrow, \bar{q}^\downarrow)$, $\exists k \in \{1, 2, .., m\}$ such that $q_k^\uparrow > q_k^\downarrow$ or $\bar{q}_k^\uparrow > \bar{q}_k^\downarrow$. Now $(q^\uparrow, \bar{q}^\uparrow)$ and $(q^\downarrow, \bar{q}^\downarrow)$ are equilibrium solutions, so using (iii),

$$\begin{aligned} 0 &\geq [\bar{q}^{\uparrow\top} \Gamma^\uparrow + {q^\uparrow}^\top [I - \theta_{min} A](M - \Gamma^\uparrow)] - [\bar{q}^{\downarrow\top} \Gamma^\downarrow + {q^\downarrow}^\top [I - \theta_{min} A](M - \Gamma^\downarrow)] \\ &\geq [\bar{q}^{\uparrow\top} \Gamma^\uparrow + {q^\uparrow}^\top [I - \theta_{min} A](\sum_{i=1}^n s_i - \Gamma^\uparrow)] - [\bar{q}^{\downarrow\top} \Gamma^\downarrow + {q^\downarrow}^\top [I - \theta_{min} A](\sum_{i=1}^n s_i - \Gamma^\downarrow)] \end{aligned}$$

$$= \sum_{i \in D^\uparrow \cap D^\downarrow} (\bar{q}^\uparrow - \bar{q}^\downarrow)^\top s_i + \sum_{i \in L^\uparrow \cap D^\downarrow} (h_i - \bar{q}^{\downarrow\top} s_i) + \sum_{i \in S^\uparrow \cap D^\downarrow} (q^\uparrow [I - \theta_{min} A] - \bar{q}^\downarrow)^\top s_i$$
$$+ \sum_{i \in L^\uparrow \cap L^\downarrow} (h_i - h_i) + \sum_{i \in S^\uparrow \cap L^\downarrow} (\bar{q}^{\uparrow\top} [I - \theta_{min} A] s_i - h_i) + \sum_{i \in S^\uparrow \cap S^\downarrow} (q^\uparrow - q^\downarrow)^T [I - \theta_{min} A] s_i$$
$$> 0$$

This is a contradiction.

□

Remark 3.3.2. The adoption of the average liquidation price in this framework, besides providing a more realistic financial framework, offers significant mathematical advantages, particularly in the analysis of uniqueness as is evident from the preceding theorem.

Remark 3.3.3. I want to point out the similarity in the uniqueness condition presented in this work to one of the very few uniqueness result in the fire sale literature as presented in [7]. In that paper, the analysis was restricted to the sale of a single asset to satisfy short term interbank liabilities. Considering the same setting, my uniqueness condition is exactly similar to Assumption (iii) on the inverse demand function in [7].

Theorem 3.3.1 provides a condition for the uniqueness of solution for an equilibrium price $(q^*, \bar{q}^*)$ in terms of the inverse demand function F. However this condition also depends on the risk-weight α. I can make this dependence explicit by stating the uniqueness condition in terms of the inverse demand function F and the risk-weight α along the lines of [53]. This is described in the following corollary.

Corollary 3.3.4. *Let the inverse demand function F be such that $\frac{(M_k - \Gamma_k) F_k'(\Gamma_k)}{F_k(\Gamma_k)}$ be non-decreasing in Γ_k $\forall$ $\Gamma_k \in [0, M_k]$ for any asset $k = 1, 2, .., m$. If $\alpha_k \in (-\frac{1}{\theta_{min}} \frac{M_k F_k'(0)}{1 - M_k F_k'(0)}, \frac{1}{\theta_{min}})$ $\forall$ k then there exists a unique equilibrium price $(q^*, \bar{q}^*)$.*

Proof. The uniqueness condition in Theorem 3.3.1 requires that:

$\sum_{k=1}^{m} \Gamma_k \bar{F}_k(\Gamma_k) + \sum_{k=1}^{m} (M_k - \Gamma_k)(1 - \alpha_k \theta_{min}) F_k(\Gamma_k)$ is increasing in Γ_k for $k = 1, 2, .., m$.

Taking partial derivative of the above expression with respect to Γ_k, I have for $k = 1, 2, .., m$,

$$\alpha_k > -\frac{1}{\theta_{min}} \frac{(M_k - \Gamma_k) F_k'(\Gamma_k)}{F_k(\Gamma_k) - (M_k - \Gamma_k) F_k'(\Gamma_k)} \;\; \forall \; \Gamma_k \in [0, M_k] \tag{3.7}$$

.

Then the condition that $\frac{(M_k-\Gamma_k)F'_k(\Gamma_k)}{F_k(\Gamma_k)}$ is non-decreasing in $\Gamma_k \forall\ \Gamma_k \in [0, M_k]$ is a sufficient condition to ensure that the right-hand side of (3.7) i.e. $-\frac{1}{\theta_{min}}\frac{(M_k-\Gamma_k)F'_k(\Gamma_k)}{F_k(\Gamma_k)-(M_k-\Gamma_k)F'_k(\Gamma_k)}$ is non-increasing in $\Gamma_k\ \forall\ \Gamma_k \in [0, M_k]$ for $k = 1, 2, ..., m$.

Thus to ensure (3.7), I require α_k to satisfy the inequality at $\Gamma_k = 0$. Using this fact and Assumption 3.2.5, uniqueness is ensured if

$$\alpha_k \in \Big(-\frac{1}{\theta_{min}}\frac{M_kF'_k(0)}{1 - M_kF'_k(0)}, \frac{1}{\theta_{min}}\Big)\ \forall\ k \tag{3.8}$$

. □

Remark 3.3.5. Proposition 3.3.4 is a sufficient condition for Theorem 3.3.1, and hence represents a stronger condition than Theorem 3.3.1. However, Proposition 3.3.4 is easier to deal with in terms of the analysis of practical inverse demand functions and provides clearer financial interpretations.

Remark 3.3.6. Proposition 3.3.4 requires the exact same condition on the risk-weight α and the inverse demand function F for uniqueness as Lemma 3.11 of [53] which deals with price-mediated contagion for multiple assets in a *continuous time setting.*

Remark 3.3.7. The assumed monotonicity property in Proposition 3.3.4 implies that for every illiquid asset, the firm need not increase the speed it is selling the asset solely to counteract its own market impacts. For a further discussion on this see Remark 3.5 of [53].

Remark 3.3.8. The bound on α as given by (3.8) can be viewed as a method to calibrate the risk-weight properly in terms of the illiquidity of the asset (as measured by the derivative of the inverse demand function). This is similar to the notion introduced in [53].

Example 3.3.9. The monotonicity condition on F in Proposition 3.10 is readily satisfied by the linear inverse demand function $F(\Gamma) = 1 - \beta\Gamma$ with $\beta \in [0, \frac{1}{M})$ and the exponential inverse demand function $F(\Gamma) = \exp(-\beta\Gamma)$ with $\beta > 0$. The risk-weight α in these cases can be readily calibrated using (3.8).

Remark 3.3.10. In the existing literature on fire sale, the linear price impact has been mostly used ([71, 26, 38, 43]). Although such inverse demand functions are relatively easier to calibrate, those represent very strong assumptions on the system behavior. However in this work I can work with more general inverse demand functions that satisfy the conditions of Theorem 3.3.1 or Corollary 3.3.4.

3.3.2 Examples

In this subsection, I consider several examples of liquidation functions and explore their properties.

One Asset:

My first example is the case with one asset, i.e. $m = 1$. This has been explored in details in [26]. The liquidation function γ, for $m = 1$ is entirely decided by the liquidation condition (3.5). In this case, I have

$$\gamma_i(q, \bar{q}) = \Big(\frac{h_i - q(1 - \alpha\theta_{min})s_i}{\bar{q} - (1 - \alpha\theta_{min})q}\Big)^+ \wedge s_i \tag{3.9}$$

Proposition 3.3.11. *$\gamma_i(q, \bar{q})$ is continuous and non-increasing in $(q, \bar{q})$.*

Proof. The proof of continuity is trivial. $\gamma_i(q, \bar{q})$ is clearly non-increasing in $\bar{q}$.

For the case where $\gamma_i = s_i$ or $\gamma_i = 0$,

$$\frac{\partial \gamma_i(q, \bar{q})}{\partial q} \equiv 0$$

For the case where $0 < \gamma_i < s_i$,

$$\frac{\partial \gamma_i(q, \bar{q})}{\partial q} = \frac{(h_i - \bar{q}s_i)(1 - \alpha\theta_{min})}{(\bar{q} - (1 - \alpha\theta_{min})q)^2}$$

Thus $\frac{\partial \gamma_i(q,\bar{q})}{\partial q} \geq 0$ is possible only if $h_i \geq \bar{q}s_i$. In this case bank i will default i.e. $\gamma_i = s_i$. Hence in this case, $\frac{\partial \gamma_i(q,\bar{q})}{\partial q} = 0$.

Hence, $\gamma_i(q,\bar{q})$ is non-increasing in q.

□

Proportional Liquidation:

My second example is Proportional Liquidation where the banks liquidate the assets in proportion to the initial holding i.e. for each bank i, and $j = 1, 2, ..m$, $k = 1, 2, ..., m$.

$$\frac{\gamma_{ij}(q,\bar{q})}{\gamma_{ik}(q,\bar{q})} = \frac{s_{ij}}{s_{ik}} \tag{3.10}$$

Proportional liquidation has been widely explored in the existing literature (e.g. [43, 71, 38]) for the analysis of fire sale. For each bank, if I consider the first asset to be the numeriare I can express s_i in terms of s_{i1} and the ratios of the amount of shares in the other assets to the first asset. In case some bank is holding none of asset 1, I can use the second asset as my numeriare.

Then for $i = 1, 2, .., n$,

$r_i \equiv [r_{i1} \ r_{i2} \ r_{i3} \ ....]^\top$ where $r_{ik} \equiv \frac{s_{ik}}{s_{i1}}$ for $k = 1, 2, .., m$.

Then using (3.10) and the definition of r_i, for $i = 1, 2, ..n$,

$$\gamma_i \equiv \gamma_{i1}[r_{i1} \ r_{i2} \ r_{i3} \ ....]^\top \tag{3.11}$$

Then the liquidation condition Equation (3.5) reduces to

$$\gamma_{i1} = \left(\frac{h_i - q^\top [I - \theta_{min} A] s_i}{(\bar{q} - [I - \theta_{min} A] q)^\top r_i)} \right)^+ \wedge s_{i1} \tag{3.12}$$

From γ_{i1} I can easily recover γ_i using (3.11).

Proposition 3.3.12. *$\gamma_i(q, \bar{q})$ is continuous and non-increasing in $(q, \bar{q})$.*

Proof. The proof of continuity is trivial. $\gamma_{i1}(q, \bar{q})$ and hence $\gamma_i(q, \bar{q})$ is clearly non-increasing in $\bar{q}$.

For the case where $\gamma_i = s_i$ or $\gamma_i = 0$, for any asset $k = 1, 2, .., m$ I have

$$\frac{\partial \gamma_{i1}(q, \bar{q})}{\partial q_k} \equiv 0$$

For the case where $0 < \gamma_i < s_i$,

$$\frac{\partial \gamma_{i1}(q, \bar{q})}{\partial q_k} = \frac{(r_{ik}h_i - s_{ik}\bar{q}^\top r_i) + (s_{ik}q^\top[I - \theta_{min}A]r_i - r_{ik}q^\top[I - \theta_{min}A]s_i)}{((\bar{q} - [I - \theta_{min}A]q)^\top r_i)^2}$$

Now, $s_{ik}r_i \equiv r_{ik}s_i$. Hence,

$$\frac{\partial \gamma_{i1}(q, \bar{q})}{\partial q_k} = \frac{r_{ik}(h_i - \bar{q}^\top s_i)(1 - \alpha_k \theta_{min})}{((\bar{q} - [I - \theta_{min}A]q)^\top r_i)^2}$$

Thus $\frac{\partial \gamma_{i1}(q, \bar{q})}{\partial q_k} \geq 0$ is possible only if $h_i \geq \bar{q}^\top s_i$. In this case bank i will default i.e. $\gamma_i = s_i$. Hence in this case, $\frac{\partial \gamma_{i1}(q, \bar{q})}{\partial q_k} = 0$. This argument holds for $k = 1, 2, ..., m$.

Hence, $\gamma_i(q, \bar{q})$ is non-increasing in q.

□

Utility Maximizing:

My final example is the Utility Maximizing liquidation function. In this case each bank i decides on its liquidation strategy γ_i to maximize its utility u_i. This represents a realistic scenario as a bank can choose an appropriate utility representing its priorities as opposed to a mechanically imposed condition. Thus γ_i is decided by the following optimization problem:

$$\begin{aligned} \text{argmax}_{\gamma_i} \quad & u_i(\gamma_i) \\ \text{s.t.} \quad & \gamma_i \in [0, s_i] \\ & \bar{q}^\top \gamma_i + q^\top [I - \theta_{min} A](s_i - \gamma_i) \geq h_i \end{aligned} \tag{3.13}$$

Assumption 3.3.13. *$u_i(\gamma_i)$ is strictly concave in γ_i.*

Example 3.3.14. An example of a feasible utility function is the Cobb-Douglas utility. Thus $u_i(\gamma_i) = \prod_{k=1}^{m} (s_{ik} - \gamma_{ik})^{t_k}$ where $0 < t_k < 1$.

Let me denote the constraint set in (3.13) as $S(q, \bar{q})$ and let $\gamma_i^*(q, \bar{q})$ to be the solution to (3.13).

Proposition 3.3.15. *(i) (3.13) admits a unique solution.*

(ii) $\gamma^*(q, \bar{q})$ is continuous in $(q, \bar{q})$.

Proof. (i) $u_i(\gamma_i)$ is strictly concave in γ_i. Further the constraints are linear in γ_i. Hence there is a unique solution to (3.13).

(ii) This follows from Berge Maximum Principle.

□

3.4 Sensitivity analysis

In this section, I perform sensitivity analysis of the equilibrium prices with respect to the system parameters. This is a critical exercise as the exact system parameters are often unknown and the results depend on how these parameters are calibrated. I characterize the sensitivity analysis as a fixed point problem and prove the existence and uniqueness of the solution to this problem. Sensitivity analysis for systems with debt linkages has been studied in [82, 55]. However, as far as I am aware, this work is the the first to attempt such an analysis in the context of fire sale.

3.4.1 Fixed point formulation

I want to start off by considering the sensitivity of the equilibrium prices with respect to the risk-weights α_k. Later, I show that the framework that I develop in this exercise is generally applicable to sensitivity analysis with respect to the other parameters as well.

Considering (3.6) and using the chain rule of differentiation, at equilibrium I have,

$$\begin{aligned}\frac{\partial \bar{q}_j(\alpha)}{\partial \alpha_k} &= \frac{\partial \bar{F}_j(\Gamma_j(\bar{q}(\alpha), q(\alpha), \alpha))}{\partial \alpha_k} \\ &= \bar{F}_j'((\nabla^{\bar{q}}\Gamma_j)^\top \nabla^{\alpha_k}\bar{q}(\alpha) + \bar{F}_j'(\nabla^{q}\Gamma_j)^\top \nabla^{\alpha_k} q(\alpha) + \bar{F}_j'(\nabla^{\alpha}\Gamma_j)^\top (\nabla^{\alpha_k}\alpha)\end{aligned} \tag{3.14}$$

where $\bar{F}_j' = \frac{\partial \bar{F}_j}{\partial \Gamma_j}$, $\nabla^{\bar{q}}\Gamma_j = [\frac{\partial \Gamma_j}{\partial \bar{q}_1} \ \frac{\partial \Gamma_j}{\partial \bar{q}_2} \ ...]^\top$ and $\nabla^{\alpha_k}\bar{q} = [\frac{\partial \bar{q}_1}{\partial \alpha_k} \ \frac{\partial \bar{q}_1}{\partial \alpha_k} \ ...]^\top$. In a similar way, at equilibrium,

$$\begin{aligned}\frac{\partial q_j(\alpha)}{\partial \alpha_k} &= \frac{\partial F_j(\Gamma_j(\bar{q}(\alpha), q(\alpha), \alpha))}{\partial \alpha_k} \\ &= F_j'((\nabla^{\bar{q}}\Gamma_j)^\top \nabla^{\alpha_k}\bar{q}(\alpha) + F_j'(\nabla^{q}\Gamma_j)^\top \nabla^{\alpha_k} q(\alpha) + F_j'(\nabla^{\alpha}\Gamma_j)^\top (\nabla^{\alpha_k}\alpha)\end{aligned} \tag{3.15}$$

From (3.14), I note that $\frac{\partial \bar{q}_j(\alpha)}{\partial \alpha_k}$ is in both sides of the equation. A similar observation can be made about $\frac{\partial q_j(\alpha)}{\partial \alpha_k}$ in (3.15). Hence the sensitivity analysis may be characterized using a fixed point equation in a similar way as [55]. In particular, considering the same argument for $j = 1, 2, ..., m$, the sensitivity of q_j and $\bar{q}_j$ with respect to α_k can be considered as the solution set of a system of joint fixed point equations in $\frac{\partial q_j(\alpha)}{\partial \alpha_k}$ and $\frac{\partial \bar{q}_j(\alpha)}{\partial \alpha_k}$ for $j = 1, 2, .., m$. Note that with some minor readjustment this set of $2m$ joint fixed point equations can be described by the following set of $2m$ linear equations in $\frac{\partial q_j(\alpha)}{\partial \alpha_k}$ and $\frac{\partial \bar{q}_j(\alpha)}{\partial \alpha_k}$ for $j = 1, 2, .., m$.

$$\begin{pmatrix} 1 - \bar{F}_1{}'\frac{\partial \Gamma_1}{\partial \bar{q}_1} & -\bar{F}_1{}'\frac{\partial \Gamma_1}{\partial \bar{q}_2} & ... & -\bar{F}_1{}'\frac{\partial \Gamma_1}{\partial q_1} & ... & -\bar{F}_1{}'\frac{\partial \Gamma_1}{\partial q_m} \\ -\bar{F}_2{}'\frac{\partial \Gamma_2}{\partial \bar{q}_1} & 1 - \bar{F}_2{}'\frac{\partial \Gamma_2}{\partial \bar{q}_2} & ... & -\bar{F}_2{}'\frac{\partial \Gamma_2}{\partial q_2} & ... & -\bar{F}_2{}'\frac{\partial \Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_1{}'\frac{\partial \Gamma_1}{\partial \bar{q}_1} & -F_2{}'\frac{\partial \Gamma_1}{\partial \bar{q}_2} & ... & 1 - F_1{}'\frac{\partial \Gamma_1}{\partial q_1} & ... & -F_1{}'\frac{\partial \Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_m{}'\frac{\partial \Gamma_m}{\partial \bar{q}_1} & -F_m{}'\frac{\partial \Gamma_m}{\partial \bar{q}_2} & ... & -F_m{}'\frac{\partial \Gamma_m}{\partial q_1} & ... & 1 - F_m{}'\frac{\partial \Gamma_m}{\partial q_m} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \bar{q}_1}{\partial \alpha_k} \\ \frac{\partial \bar{q}_2}{\partial \alpha_k} \\ ... \\ \frac{\partial q_1}{\partial \alpha_k} \\ ... \\ \frac{\partial q_m}{\partial \alpha_k} \end{pmatrix} = \begin{pmatrix} \bar{F}_1{}'\frac{\partial \Gamma_1}{\partial \alpha_k} \\ \bar{F}_2{}'\frac{\partial \Gamma_2}{\partial \alpha_k} \\ ... \\ F_1{}'\frac{\partial \Gamma_1}{\partial \alpha_k} \\ ... \\ F_m{}'\frac{\partial \Gamma_m}{\partial \alpha_k} \end{pmatrix} \tag{3.16}$$

In (3.16), I consider the sensitivity of the prices q and $\bar{q}$ with respect to the risk-weight α_k. This can be similarly done for all the risk-weights $k = 1, 2, .., m$.

More generally, I can consider the sensitivity of the prices with respect to other system parameters viz. the shortfall of the banks h_i for $i = 1, 2, .., n$ as well as the regulatory threshold θ_{min}. In fact, the sensitivity analysis with respect to θ_{min} allows me to consider an interesting application in evaluating the cost of regulation which is discussed later. In a similar way as the sensitivity analysis with respect to the risk-weights, the sensitivity with respect to other parameters can be characterized by a system of linear equations. I consider the general case in the following equation.

$$\begin{pmatrix} 1-\bar{F_1}'\frac{\partial\Gamma_1}{\partial\bar{q}_1} & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial\bar{q}_2} & ... & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial q_1} & ... & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial q_m} \\ -\bar{F_2}'\frac{\partial\Gamma_2}{\partial\bar{q}_1} & 1-\bar{F_2}'\frac{\partial\Gamma_2}{\partial\bar{q}_2} & ... & -\bar{F_2}'\frac{\partial\Gamma_2}{\partial q_2} & ... & -\bar{F_2}'\frac{\partial\Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_1'\frac{\partial\Gamma_1}{\partial\bar{q}_1} & -F_2'\frac{\partial\Gamma_1}{\partial\bar{q}_2} & ... & 1-F_1'\frac{\partial\Gamma_1}{\partial q_1} & ... & -F_1'\frac{\partial\Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_m'\frac{\partial\Gamma_m}{\partial\bar{q}_1} & -F_m'\frac{\partial\Gamma_m}{\partial\bar{q}_2} & ... & -F_m'\frac{\partial\Gamma_m}{\partial q_1} & ... & 1-F_m'\frac{\partial\Gamma_m}{\partial q_m} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\bar{q}_1}{\partial\#} \\ \frac{\partial\bar{q}_2}{\partial\#} \\ ... \\ \frac{\partial q_1}{\partial\#} \\ ... \\ \frac{\partial q_m}{\partial\#} \end{pmatrix} = \begin{pmatrix} \bar{F_1}'\frac{\partial\Gamma_1}{\partial\#} \\ \bar{F_2}'\frac{\partial\Gamma_2}{\partial\#} \\ ... \\ F_1'\frac{\partial\Gamma_1}{\partial\#} \\ ... \\ F_m'\frac{\partial\Gamma_m}{\partial\#} \end{pmatrix} \tag{3.17}$$

where $\# \in \{h_1, h_2, ..., h_n, \theta_{min}, \alpha_1, ..., \alpha_m\}$.

Let me define

$$I - W = \begin{pmatrix} 1-\bar{F_1}'\frac{\partial\Gamma_1}{\partial\bar{q}_1} & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial\bar{q}_2} & ... & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial q_1} & ... & -\bar{F_1}'\frac{\partial\Gamma_1}{\partial q_m} \\ -\bar{F_2}'\frac{\partial\Gamma_2}{\partial\bar{q}_1} & 1-\bar{F_2}'\frac{\partial\Gamma_2}{\partial\bar{q}_2} & ... & -\bar{F_2}'\frac{\partial\Gamma_2}{\partial q_2} & ... & -\bar{F_2}'\frac{\partial\Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_1'\frac{\partial\Gamma_1}{\partial\bar{q}_1} & -F_2'\frac{\partial\Gamma_1}{\partial\bar{q}_2} & ... & 1-F_1'\frac{\partial\Gamma_1}{\partial q_1} & ... & -F_1'\frac{\partial\Gamma_1}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_m'\frac{\partial\Gamma_m}{\partial\bar{q}_1} & -F_m'\frac{\partial\Gamma_m}{\partial\bar{q}_2} & ... & -F_m'\frac{\partial\Gamma_m}{\partial q_1} & ... & 1-F_m'\frac{\partial\Gamma_m}{\partial q_m} \end{pmatrix}$$

From (3.17), I note that regardless of the parameter with respect to which I perform the analysis, the matrix $I - W$ will remain same. I have to only vary the right hand side of the equation accordingly. The existence and uniqueness of a solution for the sensitivity of the prices is thus entirely decided by the invertibility of the matrix $I - W$.

Proposition 3.4.1. *Consider the setting of Theorem 3.3.1 (iii). If additionally, γ is strictly decreasing in $(q, \bar{q})$ then $I - W$ is invertible.*

Proof. The proof is given in the Appendix. □

Thus under the condition of the previous proposition, $I - W$ is invertible and (3.17) admits a unique solution given by

$$\begin{pmatrix} \frac{\partial \bar{q}_1}{\partial \#} \\ \frac{\partial \bar{q}_2}{\partial \#} \\ \cdots \\ \frac{\partial q_1}{\partial \#} \\ \cdots \\ \frac{\partial q_m}{\partial \#} \end{pmatrix} = (\text{I-W})^{-1} \cdot \begin{pmatrix} \bar{F}_1{}' \frac{\partial \Gamma_1}{\partial \#} \\ \bar{F}_2{}' \frac{\partial \Gamma_2}{\partial \#} \\ \cdots \\ F_1{}' \frac{\partial \Gamma_1}{\partial \#} \\ \cdots \\ F_m{}' \frac{\partial \Gamma_m}{\partial \#} \end{pmatrix}$$

3.4.2 Cost of regulation

A particularly interesting application of the sensitivity analysis is in the development of a scheme for computing the cost of regulation incurred by each bank. This is based on the idea that a tightened regulatory threshold θ_{min} will result in an increased loss for a bank and hence computing the loss incurred for a marginal increase in the regulatory threshold gives a measure of the regulatory cost for a bank. The loss incurred, and hence the cost of regulation, may be quantified in two different ways:

- *Cost of regulation based on realized loss:* Let me consider the situation where under the current regulatory regime, a bank has to liquidate a part of its assets. Then as the threshold θ_{min} increases, the bank has to liquidate more of its assets. Then I can use the marginal change in liquidation loss for a marginal change in θ_{min} to quantify the cost of regulation. Let $(q^*, \bar{q}^*)$ be the equilibrium pricing vectors and let ${\gamma_i}^*$ be the liquidation strategy of bank i under the current θ_{min}. Mathematically, for bank i, I represent cost of regulation based on realized loss (CRL_i) as

$$CRL_i = \frac{\partial (1 - \bar{q}^*)^\top \gamma_i^*}{\partial \theta_{min}} \tag{3.18}$$

For computation purposes, $\frac{\partial (1-\bar{q}^*)^\top \gamma_i^*}{\partial \theta_{min}} = \sum_{k=1}^{m}((1 - \bar{q}_k^*)\frac{\partial \gamma_{ik}^*}{\partial \theta_{min}} - \gamma_{ik}^* \frac{\partial \bar{q}_k^*}{\partial \theta_{min}})$. $\frac{\partial \bar{q}_k^*}{\partial \theta_{min}}$ for $k = 1, 2, .., m$ can be obtained from (3.17) and $\frac{\partial \gamma_{ik}^*}{\partial \theta_{min}}$ depends on the chosen liquidation function. I note that for the situation where banks are not liquidating any assets, increasing θ_{min} will not result in increased liquidation losses and indeed in such instances CRL_i will be equal to 0.

- *Cost of regulation based on marked to market impact:* As I noted in the earlier paragraph, when banks are not liquidating any asset, increasing θ_{min} will not result in increased liquidation losses. However, increasing θ_{min} might cause some other bank to liquidate more and hence depreciate the price. As banks hold overlapping portfolios this causes impacts globally to all other banks due to marked-to-market accounting, even if they are not performing direct liquidation. Mathematically, for bank i, I represent cost of regulation based on marked-to-market impact (CMI_i) as

$$CMI_i = -\frac{\partial (x_i + l_i + \bar{q^*}^\top \gamma_i^* + {q^*}^\top [s_i - \gamma_i^*] - \bar{p}_i)}{\partial \theta_{min}} \tag{3.19}$$

Thus for bank i, CMI_i is the negative of the partial derivative of the equity of bank i with respect to θ_{min}. Hence this will capture the losses that are not reflected in CRL_i. The computation can be done in a similar fashion as CRL_i. I note that in the situation, where *none* of the banks need to liquidate any of their assets, increasing θ_{min} will not result in further price depreciation and hence marked-to-market losses. Hence in that case for all banks $CMI_i = 0$.

3.5 Case studies

In this section I consider three case studies to discuss the implications of my model. For simplicity, each of the case studies is undertaken with a linear inverse demand function. I restrict the risk-weight α to the bound discussed in (3.8). The three case studies are as follows:

(i) First, I consider the case where all banks are symmetric, i.e., their assets and liabilities are exactly similar. In this case, I am able to provide an analytical solution to the equilibrium price problem. I use this framework to explore the effect of splitting up a bank into smaller, symmetric components.

(ii) I consider a two asset two bank system and explore the implications of diversification under different liquidation functions.

(iii) Finally, I consider a system of six large banks participating in the 2015 CCAR stress test as considered in [26]. I use this data to study the cost of regulation (as discussed in Section 3.4.2).

3.5.1 Symmetrical system

Pricing Equilibrium:

The simplest example to consider is the case for symmetrical banks i.e. where the assets and liabilities of the banks are exactly similar (e.g. [5, 23]). In this case I am able to provide an analytical solution to the pricing problem.

Here I consider a system of n symmetrical banks i.e. $x_i = x$, $\bar{p}_i = \bar{p}$, $l_i = l$ for all $i = 1, 2, ..n$. Then the shortfall $h_i = h = \bar{p} - x - (1 - \theta_{min}\alpha_l)l$ for all i. Also let me consider only one marketable asset and let each bank has s shares of this asset. So $M = ns$.

If any bank has to sell its asset, then all bank will sell the same amount of assets by symmetry. Let this amount be $\gamma \in [0, s]$. Then $\Gamma = n\gamma$.

I shall consider a linear inverse demand function $F(\Gamma) = 1 - 2\beta\Gamma$ for $\Gamma \in [0, M]$. Hence $\bar{F}(\Gamma) = 1 - \beta\Gamma$ for $\Gamma \in [0, M]$. In this example, I let $\beta \in (0, \frac{1}{2M})$. Using (3.8), I restrict the risk-weight to

$$\alpha \in \Big(\frac{1}{\theta_{min}} \frac{2\beta M}{1 + 2\beta M}, \frac{1}{\theta_{min}}\Big). \tag{3.20}$$

.

Then at equilibrium I have,

$$q = 1 - 2\beta n\gamma \tag{3.21}$$

$$\bar{q} = 1 - \beta n\gamma \tag{3.22}$$

$$\gamma = \Big(\frac{h - q(1 - \alpha\theta_{min})s}{\bar{q} - (1 - \alpha\theta_{min})q}\Big)^+ \wedge s \tag{3.23}$$

There are three possible scenarios:

- The banks are solvent and do not need to liquidate. This happens if and only if $h - q(1 - \alpha\theta_{min})s \leq 0$ i.e. $\gamma = 0$ if $h \leq (1 - \alpha\theta_{min})s$.
- The banks are insolvent. This happens if and only if $\frac{h - q(1-\alpha\theta_{min})s}{\bar{q} - (1-\alpha\theta_{min})q} \geq s$ i.e. $\gamma = s$ if $h \geq (1 - \beta M)s$.
- The banks are solvent, but liquidate some of their assets. In this case, I have the following quadratic equation in γ.

$$(1 - 2\alpha\theta_{min})\beta n\gamma^2 + [\alpha\theta_{min} - 2(1 - \alpha\theta_{min})\beta M]\gamma - [h - (1 - \alpha\theta_{min})s] = 0 \tag{3.24}$$

  I make the observation that using (3.20), $[\alpha\theta_{min} - 2(1 - \alpha\theta_{min})\beta M] > 0$. Three cases need to be considered:

  (i) $\alpha\theta_{min} = \frac{1}{2}$.
  In this case $\gamma = \frac{2[h-(1-\alpha\theta_{min})s]}{1-2\beta M}$.

(ii) $\alpha\theta_{min} < \frac{1}{2}$.

Then the only root of the quadratic equation (3.24) that gives a feasible solution is $\frac{[-\alpha\theta_{min}+2(1-\alpha\theta_{min})\beta M]+\sqrt{[-\alpha\theta_{min}+2(1-\alpha\theta_{min})\beta M]^2+4(1-2\alpha\theta_{min})\beta n[h-(1-\alpha\theta_{min})s]}}{2(1-2\alpha\theta_{min})\beta n}$.

The other root of the quadratic equation is less than 0 and hence not feasible.

(iii) $\alpha\theta_{min} > \frac{1}{2}$.

Then the only root of the quadratic equation (3.24) that gives a feasible solution is $\frac{[-\alpha\theta_{min}+2(1-\alpha\theta_{min})\beta M]-\sqrt{[-\alpha\theta_{min}+2(1-\alpha\theta_{min})\beta M]^2+4(1-2\alpha\theta_{min})\beta n[h-(1-\alpha\theta_{min})s]}}{2(1-2\alpha\theta_{min})\beta n}$.

The other root of the quadratic equation is greater than s and hence not feasible.

Let $T \equiv [-\alpha\theta_{min}+2(1-\alpha\theta_{min})\beta M]$. Then, combining all the cases, the solution set is given by:

$$\gamma = \begin{cases} 0 & \text{if } h \leq (1-\alpha\theta_{min})s \\ s & \text{if } h \geq (1-\beta M)s \\ \frac{T-\sqrt{T^2+4(1-2\alpha\theta_{min})\beta n[h-(1-\alpha\theta_{min})s]}}{2(1-2\alpha\theta_{min})\beta n} & \text{if } (1-\beta M)s \geq h \geq (1-\alpha\theta_{min})s,\ \alpha\theta_{min} > \frac{1}{2} \\ \frac{T+\sqrt{T^2+4(1-2\alpha\theta_{min})\beta n[h-(1-\alpha\theta_{min})s]}}{2(1-2\alpha\theta_{min})\beta n} & \text{if } (1-\beta M)s \geq h \geq (1-\alpha\theta_{min})s,\ \alpha\theta_{min} < \frac{1}{2} \\ \frac{2[h-(1-\alpha\theta_{min})s]}{1-2\beta M} & \text{if } (1-\beta M)s \geq h \geq (1-\alpha\theta_{min})s,\ \alpha\theta_{min} = \frac{1}{2} \end{cases} \tag{3.25}$$

Remark 3.5.1. Using (3.20) and the fact that $2\beta M \leq 1$, $1-\alpha\theta_{min} < \frac{1}{1+2\beta M} \leq 1-\beta M$, so the partition of h where $\gamma = 0$ and $\gamma = s$ are clearly disjoint.

Remark 3.5.2. From (3.25), it is evident the strong influence that regulatory requirements have in the liquidation process. First, it influences when the liquidation starts. Secondly in the case where banks are forced to liquidate a part of their assets, even under the same shortfall the precise liquidation is affected by the condition whether $\alpha\theta$ is greater, equal or less than 0.5.

I highlight the results obtained in this setting through a numerical example. I consider a system with 100 banks. I assume that $s = 1$ and the price is given by a linear inverse demand function with $\beta = 0.002$. For simplicity, I assume that $x = 0$ and $l = 0$. Thus the shortfall h is only determined by the liabilities $\bar{p}$. In this example I vary $\bar{p}$, thus varying h. This enables me to study the effect of the shortfall h on the price q. To study the regulatory impact on the

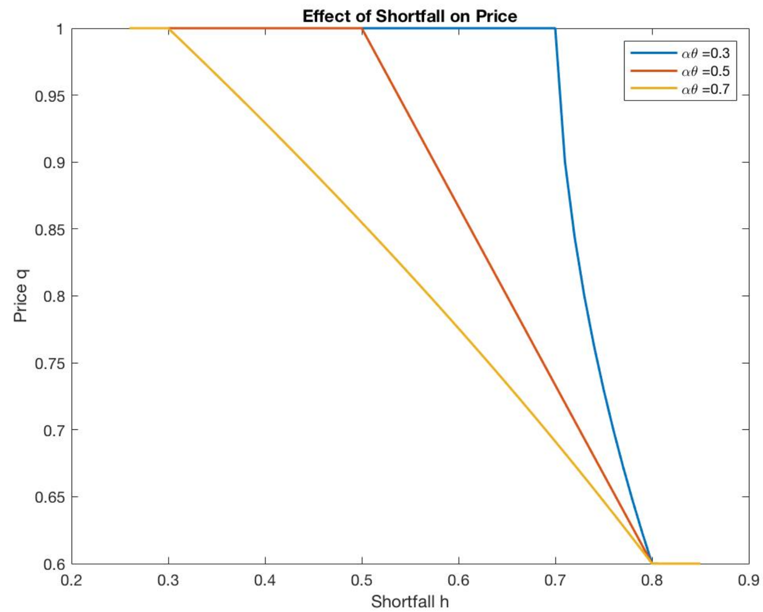


Figure 3.2: Case Study 3.5.1 Effect of shortfall h on the price q in a symmetrical system

liquidation process, I consider three settings: $\alpha\theta = 0.3$, $\alpha\theta = 0.5$ and $\alpha\theta = 0.7$. This enables me to contrast the setting of $\alpha\theta < 0.5$, $\alpha\theta = 0.5$ and $\alpha\theta > 0.5$ which becomes important when the banks are liquidating but are not insolvent as I saw in (3.25).

The results of this exercise is shown in Figure 3.2. I start from a scenario with no fire sale. As h increases banks are forced to liquidate some of their assets until at $h = 0.8$, the banks become insolvent and the price reaches its nadir. The effect of $\alpha\theta$ becomes evident if I compare across the three regulatory settings. First, as discussed in Remark (3.5.2), it influences at which h liquidations start. Secondly, it strongly influences the manner in which the liquidation occurs which is evident from the shape of the curves at different values of $\alpha\theta$. For $\alpha\theta = 0.5$, the curve is linear, for $\alpha\theta < 0.5$ I get a strictly convex curve whereas for $\alpha\theta > 0.5$, I get a strictly concave curve.

Splitting into symmetrical sub components:

An interesting application of the symmetrical system is to study the effect of splitting up a bank into smaller symmetrical components. Initially consider $n = 1$ and this bank has cash

x^0, non-marketable assets l^0 and liabilities $\bar{p}^0$. Let the shortfall be given by h^0. The bank has $s^0 = M$ illiquid assets.

Now let me consider the situation where this large bank has been split into n symmetric banks. I assume that the assets and liabilities have been split equally, i.e., each bank has liquid assets $x^1 = \frac{x^0}{n}$, non-marketable assets $l^1 = \frac{l^0}{n}$ and liabilities $\bar{p}^1 = \frac{\bar{p}^0}{n}$. Thus the shortfall $h^1 = h^0/n$. Each bank has $s^1 = \frac{s^0}{n} = \frac{M}{n}$ in illiquid assets.

I note that $\alpha\theta$ and βM remains constant across both cases so the partitions of h in (3.25) corresponds to exactly same condition in either case. Further examining (3.25), I find that $\gamma^1 = \frac{\gamma^0}{n}$. Then $\Gamma^1 = n\gamma^1 = \gamma^0 = \Gamma^0$. Thus under both conditions, the prices q and $\bar{q}$ are exactly same.

Thus under the regulatory condition described in this chapter and a linear inverse demand function, splitting up a bank into *symmetrical* sub components does not alter the pricing equilibrium.

Sensitivity Analysis:

I conclude this case study with a discussion of sensitivity analysis. Similar to the equilibrium pricing, I am able to provide an analytical solution to the sensitivity problem. In this particular case, I consider the sensitivity analyis of the prices with respect to the risk-weight α. Sensitivity analysis with respect to the other system parameters will follow in the exact same manner.

Let q^*, $\bar{q}^*$ be the solution to the pricing equilibrium. Under the symmetric setting, Equation (3.16) reduces to

$$\begin{pmatrix} 1 - \bar{F}' \frac{\partial \Gamma}{\partial \bar{q}} & -\bar{F}' \frac{\partial \Gamma}{\partial q} \\ -F' \frac{\partial \Gamma}{\partial \bar{q}} & 1 - F' \frac{\partial \Gamma}{\partial q} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \bar{q}}{\partial \alpha} \\ \frac{\partial q}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \bar{F}' \frac{\partial \Gamma}{\partial \alpha} \\ F' \frac{\partial \Gamma}{\partial \alpha} \end{pmatrix} \tag{3.26}$$

In a similar manner to the pricing problem, three cases need to be considered.

(i) $q^* = \bar{q}^* = 1$. This corresponds to the case where $\gamma = 0$. Then from (3.26), $\frac{\partial \bar{q}}{\partial \alpha} = 0$ and $\frac{\partial q}{\partial \alpha} = 0$.

(ii) $q^* = 1 - 2\beta M$, $\bar{q}^* = 1 - \beta M$. This corresponds to the case where $\gamma = s$. Then from (3.26), $\frac{\partial \bar{q}}{\partial \alpha} = 0$ and $\frac{\partial q}{\partial \alpha} = 0$.

(iii) $1 - 2\beta M < q^* < \bar{q}^* < 1$. This corresponds to the case where $0 < \gamma < s$.

$F' = -2\beta$, $\bar{F}' = -\beta$

$\frac{\partial \Gamma}{\partial \alpha} = \frac{n q^* \theta_{min}(s\bar{q}^* - h)}{[\bar{q}^* - (1-\alpha\theta_{min})q^*]^2}$, $\frac{\partial \Gamma}{\partial q} = \frac{n(1-\alpha\theta_{min})(h - s\bar{q}^*)}{[\bar{q}^* - (1-\alpha\theta_{min})q^*]^2}$, $\frac{\partial \Gamma}{\partial \bar{q}} = \frac{-n(h - s(1-\alpha\theta_{min})q^*)}{[\bar{q}^* - (1-\alpha\theta_{min})q^*]^2}$

Let me define

$$D(q, \bar{q}) = \alpha^2 \theta_m^2 q^2 - 2\alpha\theta_{min} q^2 + 2\alpha\theta_{min} q\bar{q} - \beta M \bar{q} \alpha\theta_{min} - 2\beta h n \alpha\theta_{min} + q^2 - 2q\bar{q} + \beta M q + \bar{q}^2 - 2\beta M \bar{q} + \beta h n$$

Solving (3.26), I have

$$\frac{\partial \bar{q}}{\partial \alpha} = \frac{\beta n q^* \theta_{min}(h - \bar{q}^* s)}{D(q^*, \bar{q}^*)} \tag{3.27}$$

$$\frac{\partial q}{\partial \alpha} = \frac{2\beta n q^* \theta_{min}(h - \bar{q}^* s)}{D(q^*, \bar{q}^*)} \tag{3.28}$$

3.5.2 Diversity vs Diversification

In this case study, I consider a two bank ($n = 2$) and two asset ($m = 2$) system. I assume that the banks do not hold any liquid or non-marketable asset i.e., $x_i = l_i = 0$ for $i = 1, 2$. I assume that both banks have liabilities $\bar{p}_i = 1$ and the total market capitalization of each asset is 2, i.e., $M_k = s_{1k} + s_{2k} = 2$ for $k = 1, 2$.

I study the impact of diversity vs diversification by varying the composition of the illquid assets held by each bank. I use a similar setting as [53]. I use a parameter $\lambda \in [0, 2]$ and set $s_{11} = \lambda$, $s_{12} = M_2 - \lambda$, $s_{21} = M_1 - \lambda$ and $s_{22} = \lambda$. When $\lambda = \{0, 2\}$, the banks are holding non-overlapping portfolios and this corresponds to a *fully diverse system.* When $\lambda = 1$, the portfolios of the banks are exactly same and this corresponds to a *fully diversified system.* Due to symmetry, I will only consider $\lambda \in [0, 1]$. Thus as λ increases, the system moves from fully diverse to fully diversified.

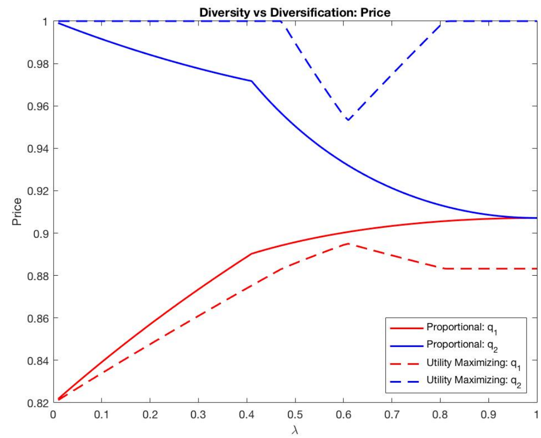
I will use the linear inverse demand function $F_k = 1 - 0.2\Gamma_k$ for $k = 1, 2$. The liquidation will depend on the choice of the liquidation function. In this example, I will explore two such liquidation functions:

- Proportional Liquidation as discussed in Section (3.3.2).
- Utility Maximizing Liquidation as discussed in section (3.3.2). In this example, I use the Cobb-Douglas utility i.e. $u_i(\gamma_i) = (s_{i1} - \gamma_{i1})^{0.5}(s_{i2} - \gamma_{i2})^{0.5}$ for $i = 1, 2$.

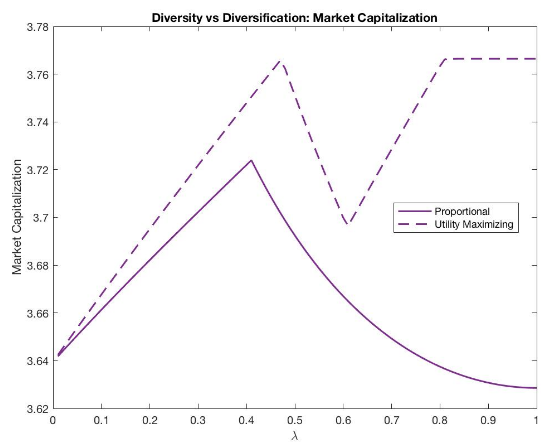
Initially, the regulatory environment is set by $\theta = 0.2$ and $\alpha_1 = \alpha_2 = 2$. Under this setting, there is no liquidation under both Proportional and Utility Maximizing liquidation. At this point, I assume that asset 1 has been downgraded and its risk-weight has been doubled i.e. $\alpha_1 = 4$. I will consider this setting for the remainder of this example. Under this condition, significant liquidation happens and is shown in Figure 3.3. The fact that downgrading one asset sets off significant fire sale reinforces the crucial role played by these system parameters.

Let me first examine the case of Proportional Liquidation. From Figure 3.3, I see that the diversification of assets improves the market capitalization till a particular peak is reached at around $\lambda = 0.4$. Thereafter, the market capitalization decreases till it reaches its lowest point at $\lambda = 1$ i.e. the fully diversified system. This highlights the fact that, under certain liquidation regimes, contagion effects from holding similar portfolios can surpass the benefits of diversification. In fact, even though the risk-weight of asset 2 has not been downgraded, it reaches its lowest point under the fully diversified system thus highlighting *cross-asset contagion.* In contrast, in the fully diverse system, the fire sale is limited to Asset 1, even though it can be extremely significant.

Next I examine the case of Utility Maximizing liquidation function. I see that the diversification of assets does not uniformly improve market capitalization. In fact, the market capitalization increases till $\lambda = 0.47$, starts to decrease till $\lambda = 0.6$ approximately and then again continues to increase till it reaches its peak at $\lambda = 1$ i.e. the fully diversified system. In



(a) Price under Proportional and Utility Maximizing Liquidation with varying diversification



(b) Market Capitalization under Proportional and Utility Maximizing Liquidation with varying diversification

Figure 3.3: Case Study 3.5.2: Diversity vs Diversification under Proportional, Utility Maximizing and Equilibrium Liquidation

this case, the fully diverse system corresponds to the worst case scenario in terms of market capitalization with very significant fire sale in Asset 1. The fully diversified situation i.e. $\lambda = 1$ corresponds to the best case in terms of market capitalization as the fire sale remain confined to Asset 1 but the banks enjoy the stabilizing effect of a more diversified portfolio comprising of the less risky Asset 2 which reduces the detrimental effect of the fire sale in Asset 1.

I conclude this Case Study by comparing the situations under Proportional and Utility Maximizing liquidation functions. The market capitalization, under the Utility Maximizing is uniformly better under the Proportional regime. The fully diverse system, i.e. $\lambda = 0$ fares equally under both liquidation functions in terms of market capitalization with the fire sale remaining confined to the downgraded Asset 1. In contrast, the fully diversified system $\lambda = 1$ corresponds to the best case for the Utility Maximizing liquidation and the worst case for Proportional liquidation in terms of market capitalization. In the Proportional case, the fire sale spreads to Asset 2 and hence outweighs the benefits of diversification whereas in the Utility Maximizing case the fire sale remain confined to Asset 1 thus enabling the banks to reap the dividends of diversification. This shows that price-mediated contagion is very much dependent on the choice of the liquidation function.

3.5.3 Cost of regulation

In this case study, I explore the cost of regulation as developed in Section 3.5.2. For this, I use the Comprehensive Capital Analysis and Review (CCAR) 2015 data. In their website the Board of Governors of the Federal Reserve System reports that "The Federal Reserve conducts the annual Comprehensive Capital Analysis and Review (CCAR) exercise to assess capital positions and planning practices of large firms consistent with Regulation YY (12 CFR part 252) and the capital plan rule (12 CFR 225.8)". For a detailed discussion on CCAR see [26]. For this case study, I consider the six Global Systemically Important Banks (GSIBs) with large trading operations viz. Bank of America, Citigroup, The Goldman Sachs, JP Morgan Chase Co., Morgan Stanley and Wells Fargo Company along the lines of [26]. The data for these organizations is shown in Table 3.1 which has been replicated from Table 7 and an unnumbered table titled "Calibrated quantities (in billion)" from Page 67 of [26].

Banks	Total Capital	Total Assets	V^{Trad}	V^{Bank}	RWA^{Trad}	RWA^{Bank}
Bank of America	161.623	2104.534	565.20	1400.70	279.40	1185.60
Citigroup	165.454	1842.181	596.90	1213.17	203.50	1089.10
The Goldman Sachs	90.978	856.240	473.97	324.69	335.91	234.50
JP Morgan Chase & Co	206.594	2572.274	857.40	1687.90	313.40	1305.60
Morgan Stanley	74.972	801.510	430.72	349.40	204.04	251.98
Wells Fargo & Company	192.900	1687.155	355.95	1311.61	130.24	1115.26

Table 3.1: Assets (in billion of dollars) for the six banks under consideration in Case Study 3.5.3

I calibrate the system parameters using the data in Table 3.1. For each bank i, I set

$$\begin{aligned} \bar{p}_i &= \text{Total Assets} - \text{Total Capital} \\ x_i &= \text{Total Assets} - V^{Trad} - V^{Bank} \\ l_i &= V^{Bank} \\ \alpha_{li} &= \frac{RWA^{Bank}}{V^{Bank}} \end{aligned}$$

For calibrating s, I make use of the risk-weights for commonly traded assets. I assume that there are $m = 16$ illiquid marketable asset and choose α for these 16 assets. This is shown in Table 3.2. I choose assets with a wide array of risk-weights. I note that the purpose of this calibration is to provide a demonstrative data set for the case study. An accurate calibration of the financial system is an interesting problem in itself and beyond the scope of the current work.

Asset	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
α	0.07	0.08	0.1	0.12	0.15	0.18	0.2	0.25	0.35	0.5	0.6	0.75	1	2.5	4.25	6.5

Table 3.2: Risk-weight α for the assets in Case Study 3.5.3

For each bank i, the individual portfolio s_i is chosen by using the following optimization problem

$$
\begin{aligned}
\min \quad & ||s_i||_2 \\
\text{s.t.} \quad & s_i \geq 0 \\
& \sum_{k=1}^{m} s_{ik} = V_i^{Trad} \\
& \sum_{k=1}^{m} \alpha_k s_{ik} = RWA_i^{Trad}
\end{aligned}
$$

I consider $\theta_{min} = 0.08$ in accordance with Basel III norms. Under the setting considered I find that the banks do not need to liquidate and hence no fire sale occurs. For demonstrative purposes, I consider a shock of 8% to the non-marketable illquid asset l_i of each bank. Under this stress regime, I find that four banks: Citigroup, The Goldman Sachs, Morgan Stanley and Wells Fargo Company do not need to perform any liquidation. JP Morgan Chase and Co needs to perform some liquidation and Bank of America is insolvent. Under this condition I consider the cost of regulation. This is plotted in Figure 3.4. I note that while only Bank of America and JP Morgan Chase and Co show a non-zero CRL, all the 6 banks have a non-zero CMI. This highlights that even though the four banks are not liquidating under the current stress regime, they will incur marked-to-market losses if θ_{min} is increased.

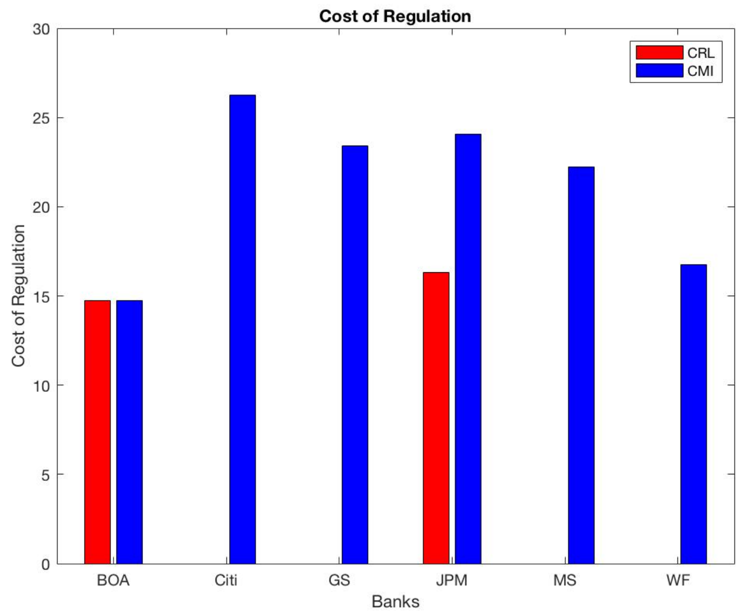


Figure 3.4: Case Study 3.5.3: Cost of Regulation

Chapter 4

Dynamic clearing and contagion in financial networks

This chapter is based on [12] which is joint work with Alex Bernstein and Zachary Feinstein.

4.1 Introduction

The Eisenberg--Noe model [45] and the majority of the literature in the network model based approach consider a static framework. Hence, they fail to account for liabilities with different maturity periods as well as dynamic clearing in financial networks. Consequently, in the stress tests developed using the network models, only the aggregate liabilities are taken into consideration. This can give an incorrect assessment of the health of the financial system.

This chapter will focus on adding the time dynamics to the setting of [45]. In fact, the conclusion of [45] provides a discussion of future extensions, one of which is the inclusion of multiple clearing dates. This has been studied directly in [27, 57]. Additionally, [80] considers a similar approach to model financial networks with multiple maturities. [52] further provides another approach to financial networks with multiple maturities by considering each clearing date as a different asset. All of these works, however, only consider clearing at discrete times. [98] presents a continuous-time clearing model that exactly replicates the static Eisenberg--Noe framework. In this chapter, I will present both discrete and continuous-time clearing models. However, my emphasis is on the derivation and the characterization of the continuous-time model as a differential equation driven by stochastic cash flows. In particular, I consider existence and uniqueness of the clearing solutions, and a numerical algorithm for finding sample paths of this clearing solution, under cash flows modeled by Itô processes. The proof of existence and uniqueness in the continuous time framework is approached in an entirely different manner than the traditional fixed point approach used in the network models. This is in contrast to the other works on multiple maturity models such as [80, 27, 57]. The main benefit of this approach is that it no longer requires strong monotonicity assumptions for existence and uniqueness which are generally required for static and discrete-time systems (that typically employ the Tarski fixed point theorem). This is also valuable for future works that may model network formation and payments as a non-cooperative game; such games may not satisfy the strong monotonicity assumptions usually considered in static and discrete-time systems, but would likely satisfy the sufficient conditions for the continuous-time framework. Further, I discuss the implications of the time dynamics on the clearing process. In particular, I find that the static Eisenberg--Noe clearing solution can be recovered in the continuous-time setting by choosing the network parameters precisely. This allows for a notion of determining the true order of defaults as opposed to the fictitious default order discussed in the static literature based on [45]. However, if the continuous-time network parameters are determined to not follow the rules for recreating the static Eisenberg--Noe setting, then the dynamic and static clearing solutions will generally not coincide. In fact, the set of defaulting and solvent institutions can be altered by rearranging the timing of obligations. As such, using the static Eisenberg--Noe framework for stress testing may result in an incorrect assessment of the health of the financial system.

The organization of this chapter is as follows. In Section 4.2, I will provide a review of the static clearing systems; I consider the clearing to be in terms of the equity and losses of the firms, as considered in, e.g., [103, 16] rather than payments as originally studied in [45]. In Section 4.3, I propose a discrete-time formulation for the Eisenberg--Noe model. In discrete time I provide results on existence and uniqueness, as well as a numerical algorithm based on the fictitious default algorithm of [45]. I then extend my model to a continuous-time setting in Section 4.4. For continuous time I consider existence and uniqueness of the clearing solutions, and a numerical algorithm for finding sample paths of this clearing solution, under cash flows modeled by Itô processes. I additionally provide conditions for the discrete-time setting to converge to the continuous-time solution as the time step limits to 0. Section 4.5

provides discussion on the financial implications of time dynamics in interbank networks. The proofs of the main results are provided in the Appendix.

4.2 Static clearing systems

I begin this chapter by reviewing some notation for static clearing systems. For a detailed discussion of the mathematical framework for static systems, I refer the reader to Chapter 1.3. The goal of this chapter is to extend this framework to include dynamic clearing.

Throughout this chapter, I will consider a network of n financial institutions. I will denote the set of all banks in the network by $\mathcal{N} := \{1, 2, \ldots, n\}$. I will consider an additional node 0, which encompasses the entirety of the financial system outside of the n banks; this node 0 will also be referred to as society or the societal node. The full set of institutions, including the societal node, is denoted by $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$.

I will be extending the model from [45] in this chapter. In that work, any bank $i \in \mathcal{N}$ may have obligations $L_{ij} \geq 0$ to any other firm or society $j \in \mathcal{N}_0$. I will assume that no firm has any obligations to itself, i.e., $L_{ii} = 0$ for all firms $i \in \mathcal{N}$, and the society node has no liabilities at all, i.e., $L_{0j} = 0$ for all firms $j \in \mathcal{N}_0$. Thus the *total liabilities* for bank $i \in \mathcal{N}$ is given by $\bar{p}_i := \sum_{j \in \mathcal{N}_0} L_{ij} \geq 0$ and relative liabilities $\pi_{ij} := \frac{L_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and arbitrary otherwise; for simplicity, in the case that $\bar{p}_i = 0$, I will let $\pi_{ij} = \frac{1}{n}$ for all $j \in \mathcal{N}_0 \backslash \{i\}$ and $\pi_{ii} = 0$ to retain the property that $\sum_{j \in \mathcal{N}_0} \pi_{ij} = 1$. On the other side of the balance sheet, all firms are assumed to begin with some amount of external assets $x_i \geq 0$ for all firms $i \in \mathcal{N}_0$. In particular, the societal node has $x_0 > 0$. The resultant *clearing payments*, under a no priority of payments assumption, satisfy the fixed point problem in payments $p \in [0, \bar{p}]$

$$p = \bar{p} \wedge \left(x + \Pi^\top p\right). \tag{4.1}$$

That is, each bank pays the minimum of what it owes ($\bar{p}_i$) and what it has ($x_i + \sum_{j \in \mathcal{N}} \pi_{ji} p_j$). The resultant vector of *wealths* for all firms is given by

$$V = x + \Pi^\top p - \bar{p}. \tag{4.2}$$

Due to the equivalence of the clearing payments and clearing wealths as discussed in Chapter 1.3.3, I am able to consider the Eisenberg--Noe system as a fixed point of clearing wealth, as given by (4.3), rather than payments. For a detailed discussion on this point, I refer the reader to Chapter 1.3.3.

$$V = x + \Pi^\top[\bar{p} - V^-]^+ - \bar{p}. \tag{4.3}$$

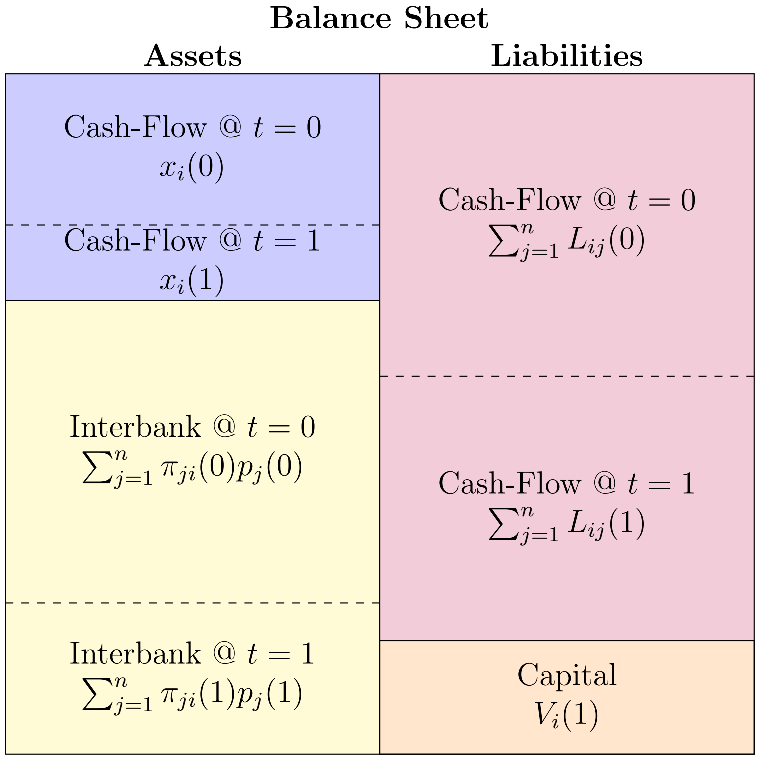
In Chapter 1.3, results for the existence and uniqueness of the clearing payments (and thus for the clearing wealths as well) are provided. In fact, it can be shown that there exists a unique clearing solution in the Eisenberg--Noe framework so long as $L_{i0} > 0$ for all firms $i \in \mathcal{N}$. I will take advantage of this result later in this paper. This is a reasonable assumption (as discussed in, e.g., [67]) as obligations to society include, e.g., deposits to the banks.

4.3 Discrete-time clearing systems

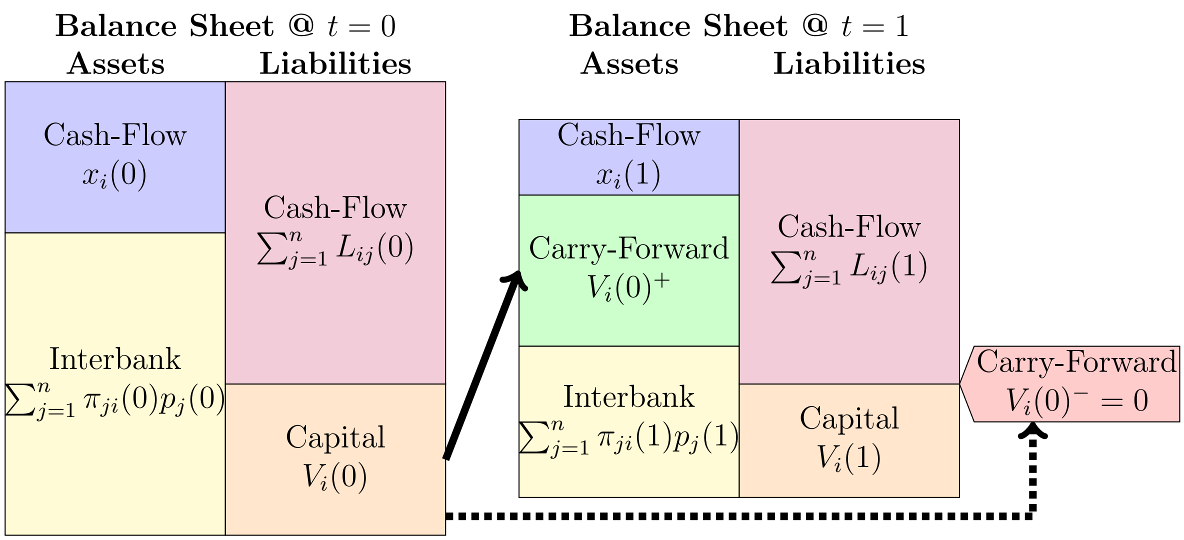
Consider now a discrete set of clearing times $\mathbb{T}$, e.g., $\mathbb{T} = \{0, 1, \ldots, T\}$ for some (finite) terminal time $T < \infty$ or $\mathbb{T} = \mathbb{N}$. Such a setting is presented in [27]. For processes I will use the notation from [35] such that the process $Z : \mathbb{T} \to \mathbb{R}^n$ has value of $Z(t)$ at time $t \in \mathbb{T}$ and history $Z_t := (Z(s))_{s=0}^t$.

In this setting, I will consider the external (incoming) cash flow $x : \mathbb{T} \to \mathbb{R}^{n+1}_+$ and nominal liabilities $L : \mathbb{T} \to \mathbb{R}^{(n+1)\times(n+1)}_+$ to be functions of the clearing time, i.e., as assets and liabilities with different maturities. The external cash in-flows and nominal liabilities can explicitly depend on the clearing results of the prior times (i.e., $x(t, V_{t-1})$ and $L(t, V_{t-1})$) without affecting the existence and uniqueness results I present, but for simplicity of notation I will focus on the case where the external assets and nominal liabilities are independent of the health and wealth of the firms. Throughout I am considering the discounted cash flows and liabilities so as to simplify notation.

In contrast to the static Eisenberg--Noe framework, herein I need to consider the results of the prior times. In particular, if firm i has positive equity at time $t-1$ (i.e., $V_i(t-1) > 0$) then these additional assets are available to firm i at time t in order to satisfy its obligations.



(a) Stylized actualized balance sheet for firm i with two time periods.



(b) Stylized "snapshot" of actualized balance sheet for firm i at times 0 and 1.

Figure 4.1: Comparison of the full balance sheet to the snapshot of maturities utilized for Section 4.3.

Similarly, if firm i has negative wealth at time $t-1$ (i.e., $V_i(t-1) < 0$) then the debts, that the firm has not yet paid, will roll-forward in time and be due at the next period. For example, consider a network in which obligations come due throughout the day at, e.g., opening, mid-day, and closing, but that all debts must be cleared by the end of the day. In such a way, the current unpaid liabilities may be paid at a future time, but before the terminal time. That is, a firm can be considered in *distress* at a time if it is unable to satisfy its obligations at that time, but only *defaults* if it has negative wealth at the terminal time. Thus in this chapter I primarily focus on the intra-day dynamics rather than the inter-day dynamics. See Figure 4.1b for a stylized (snapshot of the) balance sheet example for a firm that has positive wealth at time 0 that rolls forward to time 1. The full (actualized) balance sheet for this example with only those two time periods is displayed in Figure 4.1a; Note that the full balance sheet as depicted considers actualized payments rather than the book value of the obligations.

Remark 4.3.1. To incorporate the inter-day dynamics in this framework I can "zero out" a firm before the terminal date if it is deemed to default in much the same as in [14]. A broader framework for dealing with various default mechanisms is discussed in Remark 4.3.7. I can further consider the Nash game in which firms decide if they will allow debts to be rolled forward in time. In such a setting, if I include a delay for payment due to, e.g., bankruptcy court so that defaulting firms do not pay any obligations until after the terminal time T, then the optimal strategy for all firms (up until the terminal time T) would be to always allow other firms to roll all debts forward so as to maximize payments.

Assumption 4.3.2. *Before the time of interest, all firms are solvent and liquid. That is,* $V_i(-1) \geq 0$ *for all firms* $i \in \mathcal{N}_0$.

I can now construct the total liabilities and relative liabilities at time $t \in \mathbb{T}$ as

$$\bar{p}_i(t, V_{t-1}) := \sum_{j \in \mathcal{N}_0} L_{ij}(t) + V_i(t-1)^-$$

$$\pi_{ij}(t, V_{t-1}) := \begin{cases} \frac{L_{ij}(t) + \pi_{ij}(t-1, V_{t-2}) V_i(t-1)^-}{\bar{p}_i(t, V_{t-1})} & \text{if } \bar{p}_i(t, V_{t-1}) > 0 \\ \frac{1}{n} & \text{if } \bar{p}_i(t, V_{t-1}) = 0, \; j \neq i \\ 0 & \text{if } \bar{p}_i(t, V_{t-1}) = 0, \; j = i \end{cases} \quad \forall i, j \in \mathcal{N}_0.$$

In this way, coupled with the accumulation of positive equity over time, the clearing wealths must satisfy the following fixed point problem in time t wealths:

$$V(t) = V(t-1)^+ + x(t) + \Pi(t, V_{t-1})^\top \left[\bar{p}(t, V_{t-1}) - V(t)^-\right]^+ - \bar{p}(t, V_{t-1}). \tag{4.4}$$

That is, all firms have a clearing wealth that is the summation of their positive equity at the prior time, the new incoming external cash flow, and the payments made by all other firms minus the total obligations of the firm (including the prior unpaid liabilities). In this way I can construct the wealths of firms forward in time. This can be considered a discrete-time extension of Equation 4.2.

I now wish to consider a reformulation of (4.4). To accomplish this, I consider a process of cash flows c and functional relative exposures A. These I define by

$$\begin{aligned} c(t) &:= x(t) + L(t)^\top \vec{1} - L(t)\vec{1} \\ a_{ij}(t, V_t) &:= \begin{cases} \pi_{ij}(t, V_{t-1}) & \text{if } \bar{p}_i(t, V_{t-1}) \geq V_i(t)^- \\ \frac{L_{ij}(t) + a_{ij}(t-1, V_{t-2}) V_i(t-1)^-}{V_i(t)^-} & \text{if } \bar{p}_i(t, V_{t-1}) < V_i(t)^- \end{cases} \quad \forall i, j \in \mathcal{N}_0. \end{aligned} \tag{4.5}$$

In the above, $\vec{1} := (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$ is the vector of ones. Here I consider $c(t) = x(t) + L(t)^\top \vec{1} - L(t)\vec{1} \in \mathbb{R}^{n+1}$ to be the vector of book capital levels at time t, i.e., the new wealth of each firm assuming all other firms pay in full. I wish to note that the new total liabilities are given by $L(t)\vec{1}$ and the new incoming interbank obligations are given by $L(t)^\top \vec{1}$. I can also consider $c_i(t)$ to be the *net cash flow* for firm i at time t. Further, I introduce the functional matrix $A : \mathbb{T} \times \mathbb{R}^{n+1} \to [0,1]^{(n+1)\times(n+1)}$ to be the relative exposure matrix. That is, $a_{ij}(t, V_t) V_i(t)^-$ provides the (negative) impact that firm i's losses have on firm j's wealth at time $t \in \mathbb{T}$. This is in contrast to Π, the relative liabilities, in that it endogenously imposes the limited exposures concept. In this work the two notions will generally coincide, but for mathematical simplicity I introduce this relative exposure matrix. For the equivalence I seek, I define the relative exposures so that

$$L(t)^\top \vec{1} + A(t-1, V_{t-1})^\top V(t-1)^- - A(t, V_t)^\top V(t)^- = \Pi(t, V_{t-1})^\top [\bar{p}(t, V_{t-1}) - V(t)^-]^+$$

for any $V(t) \in \mathbb{R}^{n+1}$. This formulation is such that if the positive part were removed from the right hand side, the relative exposures A would be defined exactly as the relative liabilities Π

by construction. In particular, I will define the relative exposures element-wise and pointwise so as to encompass the limited exposures as in (4.5). If $\bar{p}_i(t, V_{t-1}) > 0$ then I can simplify this further as $a_{ij}(t, V_t) = \frac{L_{ij}(t) + a_{ij}(t-1, V_{t-1}) V_i(t-1)^-}{\max\{\bar{p}_i(t, V_{t-1}), V_i(t)^-\}}$.

Using the notation and terms above I can rewrite (4.4) with respect to the cash flows c and relative exposures A as

$$\begin{aligned}
V(t) &= V(t-1)^+ + x(t) + \Pi(t, V_{t-1})^\top [\bar{p}(t, V_{t-1}) - V(t)^-]^+ - \bar{p}(t, V_{t-1}) \\
&= V(t-1)^+ + x(t) + L(t)^\top \vec{1} + A(t-1, V_{t-1})^\top V(t-1)^- \\
&\quad - A(t, V_t)^\top V(t)^- - L(t)\vec{1} - V(t-1)^- \\
&= V(t-1) + x(t) + L(t)^\top \vec{1} + A(t-1, V_{t-1})^\top V(t-1)^- - A(t, V_t)^\top V(t)^- - L(t)\vec{1} \\
&= V(t-1) + c(t) - A(t, V_t)^\top V(t)^- + A(t-1, V_t)^\top V(t-1)^-. \qquad (4.6)
\end{aligned}$$

For the remainder of this chapter I will utilize the cash flow c rather than the external (incoming) cash flow x. That is, I will consider financial networks defined by the joint parameters (c, L) as given by the state equations (4.6) and (4.5) for wealths and relative exposures.

With this setup I now wish to extend the existence and uniqueness results of [45] to discrete time.

Theorem 4.3.3. *Let $(c, L) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ define a dynamic financial network such that every bank has cash flow at least at the level dictated by nominal interbank liabilities, i.e., $c_i(t) \geq \sum_{j\in\mathcal{N}} L_{ji}(t) - \sum_{j\in\mathcal{N}_0} L_{ij}(t)$, and so that every bank owes to the societal node at all times $t \in \mathbb{T}$, i.e., $L_{i0}(t) > 0$ for all banks $i \in \mathcal{N}$ and times $t \in \mathbb{T}$. Under Assumption 4.3.2, there exists a unique solution of clearing wealths $V : \mathbb{T} \to \mathbb{R}^{n+1}$ to (4.6).*

Remark 4.3.4. The assumption that all firms have obligations to the societal node 0 at all times $t \in \mathbb{T}$ guarantees that the financial system is a "regular network" (see [45, Definition 5]) at all times.

The analysis of the discrete-time framework can be extended to a probabilistic setting over the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in\mathbb{T}}, \mathbb{P})$. That is, I can consider the clearing wealths in the same manner assuming the cash flow $c : \mathbb{T} \times \Omega \to \mathbb{R}^{n+1}$ and nominal liabilities $L : \mathbb{T} \times \Omega \to \mathbb{R}_+^{(n+1)\times(n+1)}$ be adapted processes. Let $\mathcal{L}_t^0(\mathbb{R}^m)$ be the space of $\mathcal{F}_t$-measurable random vectors in $\mathbb{R}^m$. Let $\mathcal{L}_t^p(\mathbb{R}^m) \subseteq \mathcal{L}_t^0(\mathbb{R}^m)$ for $p \in (0,\infty]$ be the space of equivalence classes of $\mathcal{F}_t$-measurable functions $X : \Omega \to \mathbb{R}^m$ such that $\|X\|_p := \left(\int_\Omega |X(\omega)|^p d\mathbb{P}(\omega)\right)^{1/p} < \infty$ for $p < \infty$ and $\|X\|_\infty := \operatorname{ess\,sup}_{\omega \in \Omega} |X(\omega)|$ for $p = \infty$. The following corollary considers the boundedness and measurability properties of the discrete-time clearing wealths. Though I will not utilize this discrete-time result in this chapter, I consider it important to discuss random events to more closely match reality. Further, this result will implicitly appear in the construction and analysis of the continuous-time Eisenberg--Noe formulation of the next section.

Corollary 4.3.5. *Consider the setting of Theorem 4.3.3 where the random network parameters (c, L) adapted to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\in\mathbb{T}}, \mathbb{P})$. If $c(s) \in \mathcal{L}_s^p(\mathbb{R}^{n+1})$ and $L(s) \in \mathcal{L}_s^p(\mathbb{R}_+^{(n+1)\times(n+1)})$ for all times $s \leq t$ for some $p \in [0,\infty]$, then the unique clearing solution at time t has finite p-norm, i.e., $V(t) \in \mathcal{L}_t^p(\mathbb{R}^{n+1})$.*

With the construction of the existence and uniqueness of the solution I now want to emphasize the *fictitious default algorithm* from [45] to construct this clearing wealths vector over time. This algorithm is presented for the deterministic setting; if a stochastic setting is desired then Algorithm 4.3.6 provides a method for computing a single sample path. I note that at each time t this algorithm takes at most n iterations. Thus with a terminal time T, this algorithm will construct the full clearing solution over $\mathbb{T}$ in nT iterations.

Algorithm 4.3.6. Under the assumptions of Theorem 4.3.3 in a deterministic setting the clearing wealths process $V : \mathbb{T} \to \mathbb{R}^{n+1}$ can be found by the following algorithm. Initialize $t = -1$ and $V(-1) \geq 0$ as a given. Repeat until $t = \max \mathbb{T}$:

(i) Increment $t = t + 1$.

(ii) Initialize $k = 0$, $V^0 = V(t-1) + c(t)$, and $D^0 = \emptyset$. Repeat until convergence:

  (a) Increment $k = k + 1$;

  (b) Denote the set of insolvent banks by $D^k := \left\{i \in \{1, 2, ..., n\} \mid V_i^{k-1} < 0\right\}$.

  (c) If $D^k = D^{k-1}$ then terminate and set $V(t) = V^{k-1}$.

  (d) Define the matrix $\Lambda^k \in \{0,1\}^{n\times n}$ so that $\Lambda_{ij}^k = \begin{cases} 1 & \text{if } i = j \in D^k \\ 0 & \text{else} \end{cases}$.

(e) Define $V^k = (I - \Pi(t, V_{t-1})^\top \Lambda^k)^{-1} \left(V(t-1) + c(t) + A(t-1, V_{t-1})^\top V(t-1)^-\right)$.

Remark 4.3.7. Note that in the construction of V^k in step (iie) of the fictitious default algorithm I utilize the relative liabilities $\Pi(t, V_{t-1})$ in the matrix inverse rather than the relative exposures $A(t, (V_{t-1}, V^k))$. This has the added benefit that this definition of V^k is *not* a fixed point problem, which it would be if the relative exposures matrix at time t were considered. This change is possible since, as discussed in the proof of Theorem 4.3.3, any clearing solution must be in the domain so that the relative liabilities and exposures coincide. This additionally provides the invertibility of this matrix using standard input-output results as discussed in [45, 55].

I wish to finish up my discussion of the discrete-time Eisenberg--Noe framework by considering some extensions involving loans.

Remark 4.3.8. The theoretical framework presented in this chapter can be easily extended to incorporate the concepts of loans until some (deterministic) insolvency condition is hit. In particular, I will consider loans made from a central bank or lender of last resort who I will assume are part of the societal node 0. From this perspective I consider three cases that a firm might be in:

- **solvent and liquid** in which case the firm has positive equity and pays off its obligations in full;
- **solvent and distressed** in which case the firm has negative equity, but receives an overnight loan (with interest rate set at the risk-free rate for simplicity) to cover all obligations due on that date; and
- **insolvent** in which the firm will not receive any loans and is sent to a bankruptcy court.

The determination whether a firm is solvent can be done with an appropriate exogenous solvency function. I will assume that once a firm is deemed insolvent it can never recover to solvency again. Two possible systems for considering insolvent firms are:

(i) **Receivership:** In such a system, when a firm is deemed insolvent it is placed in receivership so that obligations are payed out on a first-come first-serve basis.

(ii) **Auctions:** In such a system, when a firm is deemed insolvent its future assets are auctioned off in order to pay the future liabilities (in a proportional scheme) at the next time point. This will then affect the cash flows c and nominal liabilities L, as such I would need to consider $c(t, V_{t-1})$ and $L(t, V_{t-1})$ to truly consider this case. I refer to [27] for a detailed discussion of the auction model for insolvency. The auction system can be interpreted as an internal mechanism for determining bankruptcy costs in contrast to the exogenous parameter in, e.g., [96].

The existence and uniqueness of the clearing solutions in these scenarios require an additional monotonicity property; I can use the notion of a speculative system from Chapter 5 to get the desired results. This condition encodes the notion that a firm does not benefit from any firm's distress.

4.4 Continuous-time clearing systems

Consider now a continuous set of clearing times $\mathbb{T}$, e.g., $\mathbb{T} = [0, T]$ for some (finite) terminal time $T < \infty$ or $\mathbb{T} = \mathbb{R}_+$. As before, for processes I will use the notation from [35] such that the process $Z : \mathbb{T} \to \mathbb{R}^n$ has value of $Z(t)$ at time $t \in \mathbb{T}$ and history $Z_t := (Z(s))_{s \in [0,t]}$. I will now construct an extension of the continuous-time setting of [98] in that I allow for liabilities to change over time and for firms to have stochastic cash flows.

In order to construct a continuous-time model I will begin by considering my network parameters of cash flows and nominal liabilities. Instead of considering $c(t)$ to be the net cash flow at time $t \in \mathbb{T}$, I will consider the term $dc(t)$ of marginal change in cash flow at time t. Similarly I will consider $dL(t)$ to be the marginal change in nominal liabilities matrix at time t; I note that by assumption $dL_{ij}(t) \geq 0$ for all firms $i, j \in \mathcal{N}_0$ as, without any payments made, total liabilities should accumulate over time. My main result in this section (Theorem 4.4.5) provides existence and uniqueness of the clearing wealths driven by (dc, dL) when $c(t) = \int_0^t dc(s)$ is an Itô process and $L(t) = \int_0^t dL(s)$ is deterministic and continuous (e.g., dL does not include any Dirac delta functions). This setting, and the results on the continuous-time Eisenberg--Noe model, can be extended to the case in which the cash flows and liabilities are additionally functions of the wealths V. For simplicity, in this section I will restrict myself so that the parameters are independent of the current wealths. In order to construct a continuous-time differential system, I will consider again the discrete-time setting with explicit time steps Δt.

Assumption 4.4.1. *The cash flows c are defined by the Itô stochastic differential equation $dc(t) = \mu(t, c(t))dt + \sigma(t, c(t))dW(t)$ for $(n+1)$-vector of Brownian motions W over some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$. Additionally, the drift and diffusion functions $\mu : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\sigma : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ are jointly continuous and satisfy the linear growth and Lipschitz continuous conditions, i.e., there exist constants $C, D > 0$ such that for all times $t \in \mathbb{T}$ and cash flows $c, d \in \mathbb{R}^{n+1}$*

$$\|\mu(t,c)\|_1 + \|\sigma(t,c)\|_1^{op} \leq C(1 + \|c\|_1)$$
$$\|\mu(t,c) - \mu(t,d)\|_1 + \|\sigma(t,c) - \sigma(t,d)\|_1^{op} \leq D\|c - d\|_1$$

where $\|\cdot\|_1$ is the 1-norm and $\|\cdot\|_1^{op}$ is the corresponding operator norm. The nominal liabilities $L : \mathbb{T} \to \mathbb{R}_+^{(n+1)\times(n+1)}$ are deterministic and twice differentiable; for notation I will define $dL(t) = \dot{L}(t)dt$ and $d^2L(t) = \ddot{L}(t)dt^2$. Further, the relative liabilities to society is bounded from below by a level $\delta > 0$, i.e., $\inf_{t \in \mathbb{T}} \frac{dL_{i0}(t)}{\sum_{k \in \mathcal{N}_0} dL_{ik}(t)} = \delta > 0$ for all banks $i \in \mathcal{N}$.

The assumption on the cash flows can be relaxed so long as the stochastic differential equation has a unique strong solution on $\mathbb{T}$ and μ, σ satisfy a local linear growth condition and are locally Lipschitz. This relaxation will be applied in Examples 4.5.3 and 4.5.7.

In the prior section on a discrete-time model for clearing wealths, I implicitly assumed a constant time-step between each clearing date of $\Delta t = 1$ throughout. In order to construct a continuous-time clearing model I will begin by making a discrete-time model with an explicit $\Delta t > 0$ term. In fact, this is immediate from the prior construction with a minor alteration to the cash flow term. Herein I construct the net cash flow at time t to be given by $\Delta c(t, \Delta t) := \int_{t-\Delta t}^t dc(s)$ and the nominal liabilities at time t are similarly provided by $\Delta L(t, \Delta t) := \int_{t-\Delta t}^t dL(s)$ where both dc and dL are discussed above (additionally, I set $dc(-t) = 0$ and $dL(-t) = 0$ for any times $t < 0$). The choice of notation for Δc and ΔL are to make explicit the "change" inherent in the construction.

With these parameters I can construct the Δt-discrete-time clearing process $V(t, \Delta t)$ and exposure matrix $A(t, \Delta t, V_t(\Delta t))$ by:

$$\begin{aligned} V(t, \Delta t) &= V(t - \Delta t, \Delta t) + \Delta c(t, \Delta t) - A(t, \Delta t, V_t(\Delta t))^\top V(t, \Delta t)^- \\ &\quad + A(t - \Delta t, \Delta t, V_{t-\Delta t}(\Delta t))^\top V(t - \Delta t, \Delta t)^- \end{aligned} \tag{4.7}$$

$$\begin{aligned} a_{ij}(t, \Delta t, V_t(\Delta t)) &= \frac{\Delta L_{ij}(t, \Delta t) + a_{ij}(t - \Delta t, \Delta t, V_{t-\Delta t}(\Delta t)) V_i(t - \Delta t, \Delta t)^-}{\max\{\sum_{k \in \mathcal{N}_0} \Delta L_{ik}(t, \Delta t) + V_i(t - \Delta t, \Delta t)^-, V_i(t, \Delta t)^-\}} 1_{\{i \neq 0\}} \\ &\quad + \frac{1}{n} 1_{\{i=0,\, j \neq 0\}} \qquad \forall i, j \in \mathcal{N}_0. \end{aligned} \tag{4.8}$$

Here I assume that $V(t) = V(-1) \geq 0$ for every time $t < 0$ as in Assumption 4.3.2. This construction can be computed either in continuous time $t \in \mathbb{T}$ with sliding intervals of size Δt or at the discrete times $t \in \{0, \Delta t, ..., T\}$. The existence and uniqueness of this system follow exactly as in Theorem 4.3.3 under Assumption 4.4.1.

Corollary 4.4.2. *Let $(dc, dL) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ define a dynamic financial network satisfying Assumption 4.4.1 such that every bank has cash flow at least at the level dictated by nominal interbank liabilities, i.e., $\Delta c_i(t, \Delta t) \geq \sum_{j \in \mathcal{N}} \Delta L_{ji}(t, \Delta t) - \sum_{j \in \mathcal{N}_0} \Delta L_{ij}(t, \Delta t)$ for all banks $i \in \mathcal{N}_0$, times $t \in \mathbb{T}$, and step-sizes $\Delta t > 0$. Under Assumption 4.3.2, there exists a unique solution of clearing wealths $V : \mathbb{T} \times \mathbb{R}_{++} \to \mathbb{R}^{n+1}$ to (4.7). Further, the clearing wealths are jointly continuous in time and step-size.*

Now I want to consider the limiting behavior of this discrete-time system as Δt tends to 0. To do so, first, I will consider the formulation of the relative exposures a_{ij} from bank i to j. From Corollary 4.4.2 and Assumption 4.4.1, I know that for any time $t \in \mathbb{T}$ and bank $i \in \mathcal{N}$ it must follow that $\sum_{k \in \mathcal{N}_0} \Delta L_{ik}(t, \Delta t) + V_i(t - \Delta t, \Delta t)^- \geq V_i(t, \Delta t)^-$ for $\Delta t > 0$ small enough due to the joint continuity of the wealths in time and step-size. Thus in the limiting case, as $\Delta t \searrow 0$, I find that I can consider the relative liabilities rather than the relative exposures, i.e., for Δt small enough

$$\begin{aligned} a_{ij}(t, \Delta t, V_t(\Delta t)) &= \frac{\Delta L_{ij}(t, \Delta t) + a_{ij}(t - \Delta t, \Delta t, V_{t-\Delta t}(\Delta t)) V_i(t - \Delta t, \Delta t)^-}{\sum_{k \in \mathcal{N}_0} \Delta L_{ik}(t, \Delta t) + V_i(t - \Delta t, \Delta t)^-} 1_{\{i \neq 0\}} \\ &\quad + \frac{1}{n} 1_{\{i=0,\, j \neq 0\}} \qquad \forall i, j \in \mathcal{N}_0. \end{aligned} \tag{4.9}$$

Rearranging these terms I am able to deduce that, for any firm $i \in \mathcal{N}$,

$$\begin{aligned}&[a_{ij}(t,\Delta t,V_t(\Delta t)) - a_{ij}(t-\Delta t,\Delta t,V_{t-\Delta t}(\Delta t))]V_i(t-\Delta t,\Delta t)^- \\ &\quad = \Delta L_{ij}(t,\Delta t) - a_{ij}(t,\Delta t,V_t(\Delta t)) \sum_{k\in\mathcal{N}_0} \Delta L_{ik}(t,\Delta t).\end{aligned} \tag{4.10}$$

Coupled with the assumption that the societal node always has positive wealth, I am thus able to consider the limiting behavior of (4.7) as the step-size Δt tends to 0. To do so, consider

$$\begin{aligned}V(t,\Delta t) &= V(t-\Delta t,\Delta t) + \Delta c(t,\Delta t) - A(t,\Delta t,V_t(\Delta t))^\top V(t,\Delta t)^- \\ &\qquad + A(t-\Delta t,\Delta t,V_{t-\Delta t})^\top V(t-\Delta t,\Delta t)^- \\ &= V(t-\Delta t,\Delta t) + \Delta c(t,\Delta t) \\ &\qquad - A(t,\Delta t,V_t(\Delta t))^\top V(t,\Delta t)^- + A(t,\Delta t,V_t(\Delta t))^\top V(t-\Delta t,\Delta t)^- \\ &\qquad - A(t,\Delta t,V_t(\Delta t))^\top V(t-\Delta t,\Delta t)^- + A(t-\Delta t,\Delta t,V_{t-\Delta t})^\top V(t-\Delta t,\Delta t)^- \\ &= V(t-\Delta t,\Delta t) + \Delta c(t,\Delta t) - A(t,\Delta t,V_t(\Delta t))^\top [V(t,\Delta t)^- - V(t-\Delta t,\Delta t)^-] \\ &\qquad - \Delta L(t,\Delta t)^\top \vec{1} + A(t,\Delta t,V_t(\Delta t))^\top \Delta L(t,\Delta t)\vec{1}.\end{aligned}$$

Consider the notation for the matrix of distressed firms from the fictitious default algorithm (Algorithm 4.3.6), i.e., $\Lambda(V) \in \{0,1\}^{(n+1)\times(n+1)}$ is the diagonal matrix of banks in distress

$$\Lambda_{ij}(V) = \begin{cases} 1 & \text{if } i = j \neq 0 \text{ and } V_i < 0 \\ 0 & \text{else} \end{cases} \qquad \forall i,j \in \mathcal{N}_0.$$

I am able to set $\Lambda_{00}(V) = 0$ without loss of generality since, by assumption, the outside node 0 has no obligations into the system. Thus, as with (4.9), by continuity of the clearing wealths and Δt small enough, I can conclude that except at specific event times (to be considered later, see Algorithm 4.4.8) it follows that $\Lambda(V(t,\Delta t)) = \Lambda(V(t-\Delta t,\Delta t))$. Thus, with this added notation I can reformulate the clearing wealths equation (4.7) as

$$\begin{aligned}V(t,\Delta t) &= V(t-\Delta t,\Delta t) + A(t,\Delta t,V_t(\Delta t))^\top \Lambda(V(t,\Delta t))[V(t,\Delta t) - V(t-\Delta t,\Delta t)] + \Delta c(t,\Delta t) \\ &\qquad - \Delta L(t,\Delta t)^\top \vec{1} + A(t,\Delta t,V_t(\Delta t))^\top \Delta L(t,\Delta t)\vec{1}.\end{aligned}$$

For the construction of a differential form I can consider the equivalent formulation

$$V(t,\Delta t) - V(t-\Delta t,\Delta t) = \\ [I - A(t,\Delta t, V_t(\Delta t))^\top \Lambda(V(t,\Delta t))]^{-1} \begin{pmatrix} \Delta c(t,\Delta t) - \Delta L(t,\Delta t)^\top \vec{1} \\ + A(t,\Delta t, V_t(\Delta t))^\top \Delta L(t,\Delta t)\vec{1} \end{pmatrix}. \tag{4.11}$$

Note that $I - A(t,\Delta t, V_t(\Delta t))^\top \Lambda(V(t,\Delta t))$ is invertible by standard input-output results and as proven in Proposition C.2.1.

Utilizing (4.11) and (4.9) and taking the limit as $\Delta t \searrow 0$, I am thus able to construct the joint differential system:

$$dV(t) = [I - A(t)^\top \Lambda(V(t))]^{-1} \left(dc(t) - dL(t)^\top \vec{1} + A(t)^\top dL(t)\vec{1} \right) \tag{4.12}$$

$$da_{ij}(t) = \begin{cases} \frac{d^2 L_{ij}(t) - a_{ij}(t)\sum_{k\in\mathcal{N}_0} d^2 L_{ik}(t)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(t)} & \text{if } i \in \mathcal{N},\ V_i(t) \geq 0 \\ \frac{dL_{ij}(t) - a_{ij}(t)\sum_{k\in\mathcal{N}_0} dL_{ik}(t)}{V_i(t)^-} & \text{if } i \in \mathcal{N},\ V_i(t) < 0 \\ 0 & \text{if } i = 0 \end{cases} \quad \forall i,j \in \mathcal{N}_0 \tag{4.13}$$

with initial conditions $V(0) \geq 0$ given and $a_{ij}(0) = \frac{dL_{ij}(0)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(0)} 1_{\{i\neq 0\}} + \frac{1}{n} 1_{\{i=0,\ j\neq 0\}}$ for all firms $i,j \in \mathcal{N}_0$. As in (4.11), $I - A(t)^\top \Lambda(V(t))$ is invertible by standard input-output results and as proven in Proposition C.2.1. The first case in (4.13) is constructed by noting that $a_{ij}(t) = \frac{dL_{ij}(t)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(t)}$ if $V_i(t) \geq 0$ and $i \in \mathcal{N}$ and $da_{0j}(t) = 0$ for any firm $j \in \mathcal{N}_0$ for all times t; the second case in (4.13) follows from (4.10) and taking the limit as $\Delta t \searrow 0$. Note that this differential system is discontinuous, with events at times when firms cross the 0 wealth boundary, i.e., when $\Lambda(V(t)) \neq \Lambda(V(t^-))$. As such, I will consider the differential system on the inter-event intervals, then update the differential system between these intervals. This is made more explicit in the proof of Theorem 4.4.5 and in Algorithm 4.4.8. As with the discrete-time system (4.8), the relative exposures follow the incoming proportional obligations if a firm has a surplus wealth. When a firm is in distress, the relative exposures follow a path that provides the average relative obligations between new liabilities and the prior unpaid liabilities.

Remark 4.4.3. As in the discrete-time section I consider the debt to roll forward in this case. In this way I encode the notion of either intra-day dynamics in this model or when bankruptcy court would not settle debts before the terminal time T for the system. To allow for insolvencies, I can consider some (deterministic) mechanism to determine when a bank becomes insolvent and restart the differential system with updated parameters from that time point, e.g., using an instantaneous auction as in [27]; see also Remarks 4.3.7 and 4.4.9.

I will complete my discussion of the construction of this differential system by providing some properties on the relative liabilities and exposures matrix A. Notably, these properties are those that would be expected from the discrete-time setting for the relative exposures. Namely, as a firm recovers from a distressed state its relative liabilities return to be only the fraction of incoming liabilities, that the relative exposures are bounded from below by 0 (and to society by δ as provided in Assumption 4.4.1), and the relative exposure matrix is row stochastic at all times.

Proposition 4.4.4. *Let* $(dc, dL) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ *define a dynamic financial network satisfying Assumption 4.4.1. Let* $(V, A) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}^{(n+1)\times(n+1)}$ *be any solution of the differential system* (4.12) *and* (4.13) *satisfying Assumption 4.3.2. The relative exposure matrix* $A(t)$ *satisfies the following properties:*

(i) For any bank $i \in \mathcal{N}$, *if* $V_i(t) \nearrow 0$ *as* $t \nearrow \tau$ *then* $\lim_{t \nearrow \tau} a_{ij}(t) = \frac{dL_{ij}(\tau)}{\sum_{k \in \mathcal{N}_0} dL_{ik}(\tau)}$.

(ii) For all times $t \in \mathbb{T}$ *and for any bank* $i \in \mathcal{N}$, *the elements* $a_{ij}(t) \geq 0$ *for all banks* $j \in \mathcal{N}$ *and* $a_{i0}(t) \geq \delta$;

(iii) For all times $t \in \mathbb{T}$ *and for any bank* $i \in \mathcal{N}_0$, *the row sums* $\sum_{k \in \mathcal{N}_0} a_{ik}(t) = 1$;

With this differential construction (4.12) and (4.13), I seek to prove existence and uniqueness of the clearing solutions. For notational simplicity, define the space of relative exposure matrices

$$\mathbb{A} := \left\{ A \in [0,1]^{(n+1)\times(n+1)} \mid A\vec{1} = \vec{1},\ a_{ii} = 0,\ a_{i0} \geq \delta\ \forall i \in \mathcal{N},\ a_{0j} = \frac{1}{n}\ \forall j \in \mathcal{N} \right\}.$$

From Proposition 4.4.4, I have already proven that if $(V, A) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}^{(n+1)\times(n+1)}$ is a solution to the continuous-time Eisenberg--Noe system then $A(t) \in \mathbb{A}$ for all times $t \in \mathbb{T}$.

Theorem 4.4.5. *Let* $\mathbb{T} = [0, T]$ *be a finite time period and let* $(dc, dL) : \mathbb{T} \to \mathbb{R}^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ *define a dynamic financial network satisfying Assumption 4.4.1. There exists a* *unique strong solution to the clearing wealths and relative exposures* (V, A) *satisfying* (4.12) *and* (4.13) *if* $V(0) \in \mathbb{R}^{n+1}_{++}$.

Proof. The proof is presented in the appendix. □

Remark 4.4.6. The proof of existence and uniqueness, presented in the preceding theorem, is approached in an entirely different manner than the traditional approach used in the network models. In particular, I make use of the results on the uniqueness of differential equations. This is in contrast to other works on multiple maturity models [80, 27, 57] which employ the Tarski fixed point theorem. The main benefit of my approach is that it no longer requires the strong monotonicity assumptions for Tarski fixed point theorem to hold.

Remark 4.4.7. The restrictions on the cash flows dc made in Assumption 4.4.1 can be relaxed to depend explicitly on the wealths and relative exposures, i.e.,

$$dc(t) = \mu(t, c(t), V(t), A(t))dt + \sigma(t, c(t), V(t), A(t))dW(t).$$

This would still guarantee a unique strong solution of the clearing wealths and relative exposures as in Theorem 4.4.5 so long as μ, σ satisfy a local linear growth condition, local Lipschitz condition, and $c(t)$ can be bounded above and below by elements of $\mathcal{L}^2_t(\mathbb{R}^{n+1})$ for all time t.

I now present an algorithm for numerically computing an approximation of a single sample path for the continuous-time Eisenberg--Noe clearing system. To do so I consider Euler's method for differential equations with an event finding algorithm.

Algorithm 4.4.8. Under the assumptions of Theorem 4.4.5 for a fixed event $\omega \in \Omega$ the clearing wealths process $V : \mathbb{T} \to \mathbb{R}^{n+1}$ and relative exposures $A : \mathbb{T} \to \mathbb{A}$ can be found by the following algorithm. Fix a step-size $\Delta t_0 > 0$. Initialize $t = 0$, $V(0) \geq 0$ given, $a_{ij}(0) = \frac{dL_{ij}(0)}{\sum_{k \in \mathcal{N}_0} dL_{ik}(0)} 1_{\{i \neq 0\}} + \frac{1}{n} 1_{\{i=0,\, j \neq 0\}}$, and $\Lambda = \{0\}^{(n+1)\times(n+1)}$. Repeat until $t \geq T$:

(i) Initialize $\Lambda^0 \neq \Lambda$ and $\Delta t = \Delta t_0$.

(ii) Sample $Z \sim N(0, I)$.

(iii) Repeat until $\Lambda^0 = \Lambda$:

(a) Set $\Lambda^0 = \Lambda$.

(b) Compute

$$\bar{\mu}(t) = (I - A(t)^\top \Lambda)^{-1} \left(\mu(t, c(t)) - \dot{L}(t)^\top \vec{1} + A(t)^\top \dot{L}(t) \vec{1} \right)$$
$$\bar{\sigma}(t) = (I - A(t)^\top \Lambda)^{-1} \sigma(t, c(t)) Z.$$

(c) Loop through each bank $i \in \mathcal{N}$:

i. If $V_i(t) > 0$, $\bar{\mu}_i(t) < 0$, and $\bar{\sigma}_i(t)^2 - 4\bar{\mu}_i(t)V_i(t) \geq 0$ then

$$\Delta t = \min \left\{ \Delta t \ , \ \left(\frac{-\bar{\sigma}_i(t) - \sqrt{\bar{\sigma}_i(t)^2 - 4\bar{\mu}_i(t)V_i(t)}}{2\bar{\mu}_i(t)} \right)^2 \right\}.$$

ii. If $V_i(t) < 0$, $\bar{\mu}_i(t) \neq 0$, and $\bar{\sigma}_i(t)^2 - 4\bar{\mu}_i(t)V_i(t) \geq 0$ then

$$\Delta t = \min \left\{ \Delta t \ , \ \left(\frac{-\bar{\sigma}_i(t) + \sqrt{\bar{\sigma}_i(t)^2 - 4\bar{\mu}_i(t)V_i(t)}}{2\bar{\mu}_i(t)} \right)^2 \right\}.$$

iii. If $\bar{\mu}_i(t) = 0$ and $V_i(t)\bar{\sigma}_i(t) < 0$ then $\Delta t = \min \{ \Delta t \ , \ V_i(t)^2 / \bar{\sigma}_i(t)^2 \}$.

iv. If $\bar{\mu}_i(t)\bar{\sigma}_i(t) < 0$ then $\Delta t = \min \{ \Delta t \ , \ \bar{\sigma}_i(t)^2 / \bar{\mu}_i(t)^2 \}$.

(d) Compute $\Delta V(t) = \bar{\mu}(t)\Delta t + \bar{\sigma}(t)\sqrt{\Delta t}$.

(e) Define the matrix $\Lambda \in \{0,1\}^{(n+1)\times(n+1)}$ such that

$$\Lambda_{ij} = \begin{cases} 0 & \text{if } i = j \neq 0, \ V_i(t) > 0 \text{ or } [V_i(t) = 0, \ \Delta V_i(t) \geq 0] \\ 1 & \text{if } i = j \neq 0, \ V_i(t) < 0 \text{ or } [V_i(t) = 0, \ \Delta V_i(t) < 0] \\ 0 & \text{else} \end{cases}$$

(iv) Define the matrix $\bar{\Lambda} \in \{0,1\}^{(n+1)\times(n+1)}$ so that $\bar{\Lambda} = \begin{cases} 1 & \text{if } i = j \neq 0, \ V_i(t) < 0 \\ 0 & \text{else} \end{cases}$.

(v) Set

$$c(t + \Delta t) = c(t) + \mu(t, c(t))\Delta t + \sigma(t, c(t))\sqrt{\Delta t} Z$$
$$V(t + \Delta t) = V(t) + \Delta V(t)$$

$$A(t+\Delta t) = \bar{\Lambda}\left[A(t) + \operatorname{diag}(V(t)^-)^{-1}[\dot{L}(t) - A(t) * (\dot{L}(t)\mathbb{1})]\Delta t\right]$$
$$+ (I - \bar{\Lambda})\operatorname{diag}(\dot{L}(t)\vec{1})^{-1}\dot{L}(t).$$

where $\mathbb{1} = \{1\}^{(n+1)\times(n+1)}$ and $*$ denotes the element-wise multiplication operator.

(vi) Increment $t = t + \Delta t$.

If $t > T$ then set

$$c(T) = c(t-\Delta t) + \frac{c(t) - c(t-\Delta t)}{\Delta t}(T - [t - \Delta t])$$
$$V(T) = V(t-\Delta t) + \frac{V(t) - V(t-\Delta t)}{\Delta t}(T - [t - \Delta t])$$
$$A(T) = A(t-\Delta t) + \frac{A(t) - A(t-\Delta t)}{\Delta t}(T - [t - \Delta t]).$$

In the above event-finding algorithm for the continuous-time Eisenberg--Noe system, the main concern is that I do not increment time too far in any step so as to pass over an event (e.g., a solvent bank becoming a distressed bank). This is accomplished in the loop described in step (iiic). In particular, (iii(c)i)-(iii(c)iii) guarantee that $V_i(t) + \bar{\mu}_i(t)\Delta t + \bar{\sigma}_i(t)\sqrt{\Delta t}$ is nonnegative if $V_i(t) > 0$ and nonpositive if $V_i(t) < 0$. The additional condition in (iii(c)iv) guarantees that the direction of $\bar{\mu}_i(t)\Delta t + \bar{\sigma}_i(t)\sqrt{\Delta t}$ is maintained as Δt shrinks, i.e., if Δt is too large then the direction of the change in wealth could be impacted by choosing a smaller (and thus more accurate) step-size. While not strictly necessary, I include step (iii(c)iv) as it improves the accuracy of the algorithm.

Remark 4.4.9. As with the discrete-time setting discussed in Remark 4.3.7, I can introduce the concept of loans from a central bank to the continuous-time Eisenberg--Noe system. To do so I would need to introduce stopping times associated with each bank becoming insolvent. Notably, the receivership setting would act the same as the described continuous-time Eisenberg--Noe system after insolvencies occur. In contrast, a pure auction model would eliminate all need for continuous-time contagion. At the time of the auction a static system would be considered, e.g., the static Eisenberg--Noe clearing, based on the results of the auction; this would update the cash flow parameters for each firm going forward, but no dynamic contagion would need to be modeled.

4.5 Discussion

In this section, I will consider the implications of time on the clearing solutions in the Eisenberg--Noe setting. Specifically, I will focus on the continuous-time formulation, though all conclusions hold in the discrete-time setting as well. Notably, I deduce rules so as to recreate the static Eisenberg--Noe clearing solution via the continuous-time differential system, which (independently) replicates the results from [98]. Further, I consider the implications of time dynamics on the health of the financial system by determining bounds on how different the static clearing solution and a dynamic solution might be. This demonstrates the importance of time dynamics on accurately assessing the health and wealth of the financial system.

4.5.1 The static model as a differential system

Herein I will consider the case in which the relative liabilities are constant through time. That is, I consider the setting in which $dL_{ij}(s)/\sum_{k\in\mathcal{N}_0} dL_{ik}(s) = dL_{ij}(t)/\sum_{k\in\mathcal{N}_0} dL_{ik}(t)$ for all times $s,t \in \mathbb{T}$ and firms $i,j \in \mathcal{N}_0$ so long as $\sum_{k\in\mathcal{N}_0} dL_{ik}(s), \sum_{k\in\mathcal{N}_0} dL_{ik}(t) > 0$. The key implication of this assumption is that the relative exposures matrix in (4.13) can be found explicitly to equal the relative liabilities

$$a_{ij}(t) = \pi_{ij} := \begin{cases} \frac{dL_{ij}(s_i)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(s_i)} & \text{if } s_i < \sup\mathbb{T} \\ \frac{1}{n} & \text{if } s_i = \sup\mathbb{T},\; j \neq i \\ 0 & \text{if } s_i = \sup\mathbb{T},\; j = i \end{cases}$$

for all times t and banks $i,j \in \mathcal{N}_0$ where $s_i \in \{t \in \mathbb{T} \mid \sum_{k\in\mathcal{N}_0} dL_{ik}(t) > 0\}$ chosen arbitrarily strictly less than $\sup\mathbb{T}$ (and $s_i = \sup\mathbb{T}$ if the supremum is taken over the empty set).

Further, expanding and solving the differential system (4.12), I deduce that the continuous-time clearing wealths must satisfy the fixed point problem

$$V(t) = V(0) + \int_0^t dc(s) - \Pi^\top V(t)^- \tag{4.14}$$

at all time $t \in \mathbb{T}$. Therefore, if $\int_0^t dc(s) \geq \int_0^t dL(s)^\top \vec{1} - \int_0^t dL(s)\vec{1}$ at some time t, it follows that $V(t)$ are the *static* clearing wealths to the Eisenberg--Noe system with aggregated data with nominal liabilities matrix defined by $\int_0^t dL(s)$ and (incoming) external cash flow given by $\int_0^t dc(s) - \left(\int_0^t dL(s)^\top \vec{1} - \int_0^t dL(s)\vec{1}\right)$. Importantly, this means that, if the relative liabilities are kept constant over time, taking aggregated data and considering the static Eisenberg--Noe framework will produce the same *final* clearing wealths as the dynamic Eisenberg--Noe setting presented in this chapter. However, though the set of defaulting banks is the same as in the static setting, the order of defaults need not strictly follow the order given in the fictitious default algorithm of [45].

Definition 4.5.1. *A bank is called a* ***k*****th-order default** *in the static Eisenberg--Noe setting if it is determined to be in default in the kth iteration of the fictitious default algorithm (see, e.g., [45, Section 3.1] or the inner loop of Algorithm 4.3.6).*

I note that the first-order defaults are exactly those firms that have negative wealth even if it has no negative exposure to other firms (i.e., all other firms satisfy their obligations in full).

Proposition 4.5.2. *Let $(x, \bar{L}) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ denote the static incoming external cash flow and nominal liabilities. Define a dynamic system over the time period $\mathbb{T} = [0, T]$ such that $V(0) \in [0, x]$, $dL(t) = \frac{1}{T}\bar{L}dt$, and $dc(t) = \frac{1}{T}\left(x - V(0) + \bar{L}^\top \vec{1} - \bar{L}\vec{1}\right)dt$. The clearing wealths at the terminal time $V(T)$ are equal to those given in the static setting. Additionally, no firm will ever recover from distress in the dynamic setting. Finally, the first kth-order default will occur only after the first $(k-1)$th-order default in the static fictitious default algorithm; in particular, the first firm to become distressed will be a first-order default in the static fictitious default algorithm.*

Proof. The fact that the clearing wealths $V(T)$ are equal to the static Eisenberg--Noe clearing wealths follows from (4.14) and the logic given in the proof of Lemma **??**. Additionally, since $dc(t)$ is constant in time and firms are beginning in a solvent state, over time the unpaid liabilities may accumulate as a negative factor on bank balance sheets, but there is no outlet to allow for a firm to recover from distress. Finally, by definition, a kth-order default is only driven into distress through the failure of the $(k-1)$th-order defaults (and *not* solely by the $(k-2)$th-order defaults). Therefore, by way of contradiction, if a kth-order default were to occur before any $(k-1)$th-order default then such a firm must default without regard to what happens to the $(k-1)$th-order defaults, i.e., this firm must be a $(k-1)$th-order default. By this same logic, the first firm to become distressed must be a first-order default. □

The notion of real defaulting times differing from the order introduced by the fictitious default algorithm of [45] is unsurprising. Consider a financial system with two subgraphs that are only connected through their obligations to the societal node. By construction, the default of a firm in one subgraph will have no impact on the firms in the other subgraph. Thus I can construct a network so that all defaults in one subgraph (including higher order defaults as defined in Definition 4.5.1) occur before any first-order defaults in the other subgraph.

Notably, Proposition 4.5.2 states that, provided the aggregate data (until the terminal time) is kept constant, the clearing wealths at the terminal time will be path-independent in this setting. I will demonstrate this with an illustrative example demonstrating this setting in a small 4 bank (plus societal node) system. In particular, I will consider the cash flows c to be defined as a Brownian bridge so as to provide the appropriate aggregate data at the terminal time.

Example 4.5.3. Consider a financial system with four banks, each with an additional obligation to an external societal node. Consider the time interval $\mathbb{T} = [0,1]$ with aggregated data such that the initial wealths are given by $V(0) = (100, 1, 3, 2, 5)^\top$, cash flows dc are such that $\int_0^1 dc(s) = \bar{L}^\top \vec{1} - \bar{L}\vec{1}$, and where the nominal liabilities matrix $dL = \bar{L}dt$ is defined by

$$\bar{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 7 & 1 & 1 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 1 & 1 & 0 & 1 \\ 3 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

The *static* Eisenberg--Noe clearing wealths, with nominal liabilities $\bar{L}$ and external assets $V(0)$, are found to be $V(1) \approx (109.38, -6.81, -3.03, -0.32, 1.62)^\top$. Further, from the static fictitious default algorithm, I can determine that bank 1 is a first-order default, bank 2 is a second-order default, and bank 3 is a third-order default. Consider now three dynamic settings which are differentiated only by the choice of the cash flows dc:

(i) Consider the deterministic setting introduced in Proposition 4.5.2, i.e., $dc(t) = [\bar{L}^\top \vec{1} - \bar{L}\vec{1}]dt$ for all times $t \in \mathbb{T}$.

(ii) Consider a Brownian bridge with low volatility, i.e., $dc(t) = \frac{\bar{L}^\top \vec{1} - \bar{L}\vec{1} - c(t)}{1-t} dt + dW(t)$ for vector of independent Brownian motions W and with $c(0) = 0$.

(iii) Consider a Brownian bridge with high volatility, i.e., $dc(t) = \frac{\bar{L}^\top \vec{1} - \bar{L}\vec{1} - c(t)}{1-t} dt + 5dW(t)$ for vector of independent Brownian motions W and with $c(0) = 0$.

A single sample path for each dynamic setting is provided. In each plot I reduce the equity of the societal node by 100 so that it begins with an initial wealth of 0, but more importantly so that it can easily be displayed on the same plot as the other 4 institutions. First, I point out that, as indicated by the circles at the terminal time in each plot, the terminal wealths of the continuous-time setting match up with the clearing wealths in the static model. I further note that in the deterministic setting (Figure 4.2a) and the low volatility setting (Figure 4.2b) the order of defaults is maintained. However, in the high volatility setting (Figure 4.2c) the order of defaults given by the fictitious default algorithm no longer holds.

4.5.2 The implications of time dynamics

Now I will consider the case in which the relative liabilities change over time. As in the prior discussion, I will focus on the setting in which the aggregate cash flows and interbank liabilities correspond to a static Eisenberg--Noe model. As the liabilities are now changing over time there is an inherent prioritization in the obligations due to the rolling forward of unpaid debts. Any earlier obligations are more likely to be paid, and accumulate to be paid proportionally with any new obligations. As such, by altering only the rate at which the liabilities are due, the terminal wealths and also the set of defaulting firms can be modified. Proposition 4.5.5 provides analysis on which banks will always be solvent and which will always be in default at the terminal time. In particular, the results of Proposition 4.5.5 show that the *static* Eisenberg--Noe model applied to aggregate data can produce a viewpoint on the health of the financial system that is either incorrectly optimistic or pessimistic; without explicitly knowing the dynamics of the cash flows and liabilities, only rough estimates can be considered. This is in contrast to, e.g., [67] in which data from the European Banking

Authority's 2011 stress test was utilized to assess the health of the European financial system without time dynamics.

Definition 4.5.4. *In the static Eisenberg--Noe setting a bank is called a* **first-order solvency** *if it has positive wealth even under the maximum negative exposure (i.e., no other firms pay at all).*

Note that, by assumption, the societal node 0 will always be a first-order solvent institution.

Proposition 4.5.5. *Let $(x, \bar{L}) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ denote the static incoming external cash flow and nominal liabilities. Define a dynamic system over the time period $\mathbb{T} = [0, T]$ such that $V(0) \in [0, x]$, $\int_0^T dL(t) = \bar{L}$, and $\int_0^T dc(t) = x - V(0) + \bar{L}^\top \vec{1} - \bar{L}\vec{1}$. At time T, those banks that are first-order defaults in the static setting will be in default in the dynamic setting. Similarly, those banks that are first-order solvencies in the static setting will be solvent in the dynamic setting at the terminal time.*

Proof. This result follows from the definition of a first-order default or solvency as such firms allow me to disregard all interbank dynamics. □

To conclude this discussion, I will consider two examples with the same aggregate values as given in Example 4.5.3. The first example considers the case in which the nominal liabilities are shifted in time so as to have the maximum possible number of banks be solvent or, vice versa, the maximum number of banks be in default at the terminal time. The second example considers a fixed structure for the nominal liabilities in time (but non-constant relative liabilities), thus demonstrating the path-dependence of the clearing wealths on the cash flows.

Example 4.5.6. Consider the financial system described in Example 4.5.3 over the time interval $\mathbb{T} = [0, 1]$ with aggregated data such that the initial wealths $V(0) = (100, 1, 3, 2, 5)^\top$ and where the aggregate nominal liabilities matrix is defined by $\bar{L}$. Further, consider the cash flows $dc(t) = dL(t)^\top \vec{1} - dL(t)\vec{1}$ for all times $t \in \mathbb{T}$ where dL is either:

(i) prioritizing the defaulting firms: $dL(t) = 5\bar{L}\left(E_0 1_{\{t \in (0.8, 1]\}} + \sum_{i \in \mathcal{N}} E_i 1_{\{t \in (0.2(i-1), 0.2i]\}}\right)$, or

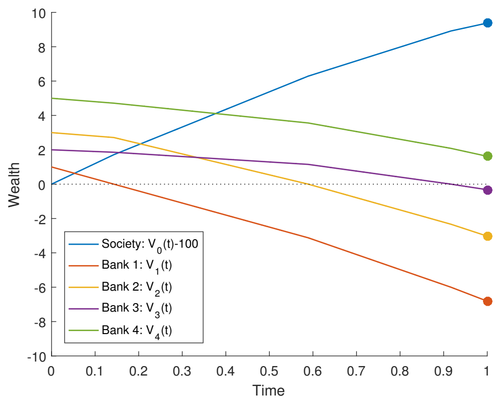
(ii) prioritizing society: $dL(t) = 5\bar{L}\left(E_0 1_{\{t\in[0,0.2)\}} + \sum_{i\in\mathcal{N}} E_i 1_{\{t\in(0.2i,0.2(i+1)]\}}\right)$

where the collection of matrices $E_i \in \{0,1\}^{(n+1)\times(n+1)}$ are such that $(E_i)_{ii} = 1$ and all other elements are set to 0. As in Figure 4.2, the circles at the terminal time in both plots denote the clearing wealths under the static Eisenberg--Noe setting. It is clear in both examples that the terminal dynamic clearing wealths now are *not* equal to the static wealths. Further, by choosing the liabilities to be introduced in the order provided I provide the settings so that only the first-order defaults, Bank 1, have negative terminal wealth (Figure 4.3a) or so that only the first-order solvencies, the societal firm, have positive terminal wealth (Figure 4.3b). In Figure 4.3a, I notice that firms 2 and 3 have a terminal wealth of 0, so although they are not defaulting, they do not have any positive equity either. Further, it is clear that though all financial firms have improved their wealth given this ordering of the nominal liabilities, the societal wealth is decreased (though to a lesser amount than the aggregate improvement for the banks) in comparison to the static results. In contrast, in the second scenario in which obligations to society are first (Figure 4.3b), the societal wealths are greater than those provided in the static setting but all banks have less wealth. Notice further that, even after the obligations to society have "ended" at time 0.2 the societal wealth still increases. This occurs as the banks in distress receive money as their incoming liabilities come due and thus they have cash to immediately transfer to cover the prior unpaid obligations to, e.g., society. Finally, this numerically verifies the results of Proposition 4.5.5 and demonstrates the importance of understanding the order of obligations for an accurate measure of the health of the financial system.

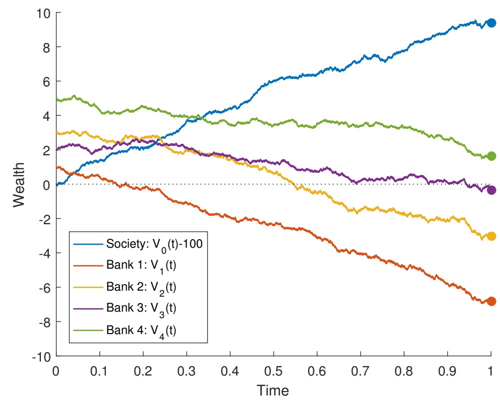
Example 4.5.7. Consider the financial system described in Example 4.5.3 over the time interval $\mathbb{T} = [0,1]$ with aggregated data such that the initial wealths $V(0) = (100,1,3,2,5)^\top$ and where the aggregate nominal liabilities matrix is defined by $\bar{L}$. Further, consider the nominal liabilities determined by

$$dL(t) = \bar{L}\begin{pmatrix} E_0 + \frac{1}{0.237}E_1 1_{\{t\in[0.145,0.382]\}} + \frac{1}{0.178}E_2 1_{\{t\in[0.331,0.509]\}} \\ +\frac{1}{0.439}E_3 1_{\{t\in[0.301,0.740]\}} + \frac{1}{0.105}E_4 1_{\{t\in[0.673,0.778]\}} \end{pmatrix}$$

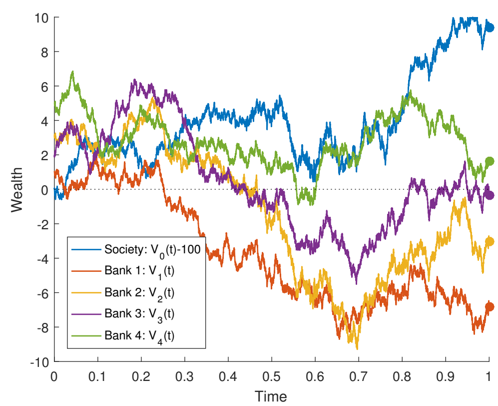
where the collection of matrices $E_i \in \{0,1\}^{(n+1)\times(n+1)}$ are such that $(E_i)_{ii} = 1$ and all other elements are set to 0. Finally, consider the cash flows determined by a Brownian bridge with volatility of 2, i.e., $dc(t) = \frac{\bar{L}^\top \vec{1} - \bar{L}\vec{1} - c(t)}{1-t}dt + 2dW(t)$ for vector of independent Brownian motions W and with $c(0) = 0$. Figure 4.4 depicts the empirical distribution of the terminal societal wealths under 10,000 samples of the Brownian bridge cash flows. The black curve depicts the kernel density for this empirical distribution. The $\times$ illustrates the societal wealth under the static Eisenberg--Noe framework considering the aggregated data (as provided in Example 4.5.3). The key takeaway of this figure is the payments to society range from 8.12 to 10.20 out of an obligated 12, i.e., society can experience anywhere from 16% to 32% shortfall in payments depending on the sample path. This also implies that society can experience anywhere from a 13.4% decrease to an 8.8% increase over the payments found under the static Eisenberg--Noe model. Similar results can be shown for the other firms in the system as well. Notably, firms 2, 3, and 4 all have simulations in which they are solvent at the terminal time and simulations in which they are defaulting on their obligations. Recall none of these three firms are first-order defaults or first-order solvencies. Empirically, firm 2 (a second-order default) is found to default in approximately 98% of the simulations; firm 3 (a third-order default) is found to default in approximately 3.6% of simulations; firm 4 (which does not default in the static setting) is found to default in just 0.03% of the provided simulations (i.e., 3 out of the 10,000 simulations). Therefore, if relative liabilities are not constant over time, the order of the cash flows can have a significant impact on the health of the system.



(a) Example 4.5.3: Clearing wealths over time under deterministic and constant cash flows.

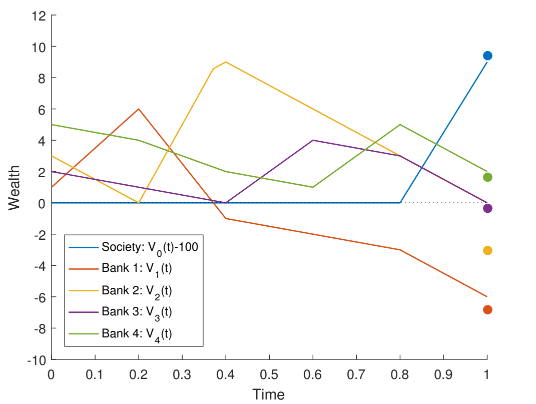


(b) Example 4.5.3: Clearing wealths over time under low volatility Brownian bridge cash flows.

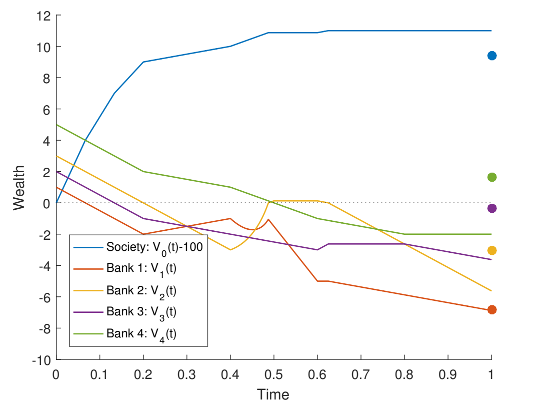


(c) Example 4.5.3: Clearing wealths over time under high volatility Brownian bridge cash flows.

Figure 4.2: Example 4.5.3: Comparison of clearing wealths under deterministic and random cash flows that aggregate to the same terminal values.



(a) Example 4.5.6: Clearing wealths over time under setting to have all but the first order-defaults solvent at the terminal time.



(b) Example 4.5.6: Clearing wealths over time under setting to have all but the first order-solvency defaulting at the terminal time.

Figure 4.3: Example 4.5.6: Comparison of clearing wealths under different ordering of the nominal liabilities in time that aggregate to the same terminal values.

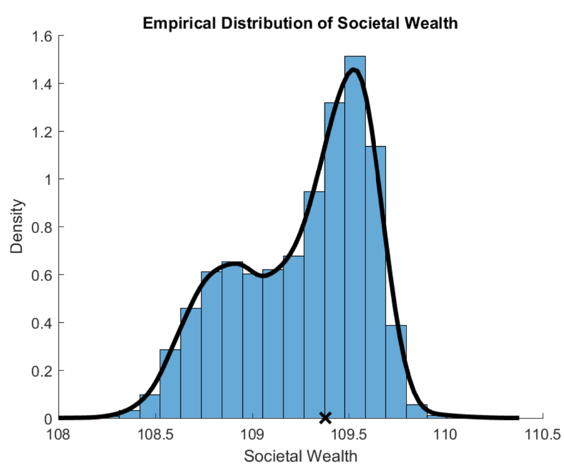


Figure 4.4: Example 4.5.7: Empirical distribution of the terminal societal wealths under random cash flows. The $\times$ marks the societal wealth under the static Eisenberg--Noe framework with aggregated data.

Chapter 5

Impact of contingent payments on systemic risk in financial networks

This chapter is based on [14] which is joint work with Zachary Feinstein.

5.1 Introduction

The 2008 financial crisis proved that linkages formed between banks and insurance companies can act as potential channels of financial contagion. These linkages are formed and resolved in a way that is different from normal bank loans. A typical example of such a linkage is a credit default swap [CDS]. A credit default swap is a contract in which a buyer pays a premium to a seller in order to protect itself against a potential loss due to the occurrence of a credit event that affects the value of the contract's underlying reference obligation, e.g., a corporate or sovereign bond. The contract specifies the credit events that will trigger payment from the seller to the buyer. Whereas such instruments can be used to hedge risks, they may also be used for speculative purposes to put a short position on the credit markets.

The important role that such contingent linkages play is demonstrated by the financial crisis of 2007-2009. As that crisis unfolded, AIG faced bankruptcy after the failure of Lehman Brothers due to the large payouts it was required to make on its CDS contracts referencing Lehman and mortgage-backed securities. When the crisis hit, the sudden calls to pay out the CDS contracts put great pressure on AIG, which traditionally had a thin capital base. Consequently AIG had to be rescued by the U.S. Department of Treasury so as to avoid jeopardizing the financial health of firms which bought CDSs from AIG. However, despite the importance of these linkages, current models are unable to account for the conditional payments that an insurance or credit default swap contract would require. I refer to [24] for a preliminary study of the insurance and reinsurance market.

As far as I am aware, theoretical work on contingent payments and CDS in relation to systemic risk has not been explored much. [19, 18] show that the clearing vector in the presence of generalized CDS contracts is not well-defined and need not exist. They further propose a static setting to model CDS payments and give sufficient conditions on the network topology for existence of a clearing solution. [81] considers such a model in a static framework and proposes a method to rewrite some classes of network topologies as an Eisenberg--Noe system. [17, 36] modeled CDS payments, but most of the reference entities are required to be external to the financial system. [78] modeled reinsurance networks and studied the implications of network topologies on existence and uniqueness of the liabilities and clearing payments. A different approach has been taken in [74] in which a stochastic setting is used to analyze contagion caused by credit default swaps. The role of credit default swaps in causing financial contagion has been captured in several empirical studies, see e.g. [92, 84].

This chapter aims to provide a generalized theoretical framework in which to study credit default swaps and other contingent payments in the Eisenberg--Noe setting. I focus on existence and uniqueness of the clearing payments under contingent payments without presupposing the nature of those payments or strong assumptions on the network topology. This is in contrast to the aforementioned literature on CDS network models in which there is no guarantee that the realized networks would obey the required conditions. Hence it is paramount to develop results for a general network, irrespective of the topology. I do this by first considering the problem in a static framework where all claims are settled simultaneously. In such a setting I find that uniqueness of the clearing solution follows so long as no firm is "speculating" on another firm's failure. However, with speculation the problem in a static setting no longer satisfies the sufficient mathematical properties for uniqueness. In order to overcome this issue, I introduce a dynamic framework. This setting ensures both existence and uniqueness of a clearing solution under the usual conditions from [45]. Additionally I demonstrate how this framework can be applied to problems that were ill-defined in the static framework through numerical examples.

This chapter is organized in the following way: First, in Section 5.2, I will introduce the mathematical and financial setting. In Section 5.3, I develop the static framework for incorporating contingent payments such as insurance and CDS, provide results on existence and develop conditions for uniqueness that are intimately related to considerations of insurance versus speculation. Further I demonstrate some shortcomings inherent to the static framework with contingent payments. In Section 5.4, I introduce a discrete time dynamic framework and discuss existence and uniqueness results.

5.2 Background

I begin this chapter by reviewing notation for the clearing system used in [45]. For a detailed discussion of this mathematical framework, I refer the reader to Chapter 1.3. The goal of this chapter is to extend this framework to include contingent payments.

Throughout this chapter, I will consider a network of n financial institutions. I will denote the set of all banks in the network by $\mathcal{N} := \{1, 2, \ldots, n\}$. I will consider an additional node 0, which encompasses the entirety of the financial system outside of the n banks; this node 0 will also be referred to as society or the societal node. The full set of institutions, including the societal node, is denoted by $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$.

I will be extending the model from [45] in this chapter. In that work, any bank $i \in \mathcal{N}$ may have obligations $L_{ij} \geq 0$ to any other firm or society $j \in \mathcal{N}_0$. I will assume that no firm has any obligations to itself, i.e., $L_{ii} = 0$ for all firms $i \in \mathcal{N}$, and the society node has no liabilities at all, i.e., $L_{0j} = 0$ for all firms $j \in \mathcal{N}_0$. Thus the *total liabilities* for bank $i \in \mathcal{N}$ is given by $\bar{p}_i := \sum_{j \in \mathcal{N}_0} L_{ij} \geq 0$ and relative liabilities $\pi_{ij} := \frac{L_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and arbitrary otherwise; for simplicity, in the case that $\bar{p}_i = 0$, I will let $\pi_{ij} = \frac{1}{n}$ for all $j \in \mathcal{N}_0 \backslash \{i\}$ and $\pi_{ii} = 0$ to retain the property that $\sum_{j \in \mathcal{N}_0} \pi_{ij} = 1$. On the other side of the balance sheet, all firms are assumed to begin with some amount of external assets $x_i \geq 0$ for all firms $i \in \mathcal{N}_0$. In particular, the societal node has $x_0 > 0$. The resultant *clearing payments*, under a no priority of payments assumption, satisfy the fixed point problem in payments $p \in [0, \bar{p}]$

$$p = \bar{p} \wedge \left(x + \Pi^\top p\right). \tag{5.1}$$

That is, each bank pays the minimum of what it owes ($\bar{p}_i$) and what it has ($x_i + \sum_{j \in \mathcal{N}} \pi_{ji} p_j$). The resultant vector of *wealths* for all firms is given by

$$V = x + \Pi^\top p - \bar{p}. \tag{5.2}$$

Due to the equivalence of the clearing payments and clearing wealths as discussed in Chapter 1.3.3, I am able to consider the Eisenberg--Noe system as a fixed point of clearing wealth rather than payments as given by the following equation. For a detailed discussion on this point, I refer the reader to Chapter 1.3.3.

$$V = x + \Pi^\top [\bar{p} - V^-]^+ - \bar{p}. \tag{5.3}$$

In Chapter 1.3, results for the existence and uniqueness of the clearing payments (and thus for the clearing wealths as well) are provided. In fact, it can be shown that there exists a unique clearing solution in the Eisenberg--Noe framework so long as $L_{i0} > 0$ for all firms $i \in \mathcal{N}$. I will take advantage of this result later in this paper. This is a reasonable assumption (as discussed in, e.g., [67]) as obligations to society include, e.g., deposits to the banks.

5.3 Simultaneous network clearing with contingent payments

5.3.1 General setting

Let me now consider the case when the nominal liabilities between financial institutions depend explicitly on the wealths of the firms. This is, for instance, the case with insurance, credit default swaps, reserve requirements with a central bank, or the default waterfall enacted by central counterparties; see Examples 5.3.2-5.3.5 for more details of those cases.

As a general setting, this corresponds to the situation in which the nominal liabilities $L_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}_+$ from bank $i \in \mathcal{N}$ to $j \in \mathcal{N}_0$ is a mapping from the vector of bank wealths into the obligations; as mentioned above, I will assume that $L_{0i} \equiv 0$ and $L_{ii} \equiv 0$ for all firms $i \in \mathcal{N}$. That is, dependent on the actualized wealths $V \in \mathbb{R}^{n+1}$ of all institutions in the system, the nominal liabilities will adjust to be $L(V) \in \mathbb{R}_+^{(n+1)\times(n+1)}$ a nonnegative matrix with 0 diagonal. In the case that the societal node is not desired, then this can be incorporated by setting $L_{i0} \equiv 0$ for all $i \in \mathcal{N}$. Thus I consider a static setting for these contingent payments, i.e., I assume all claims are resolved simultaneously and the nominal liabilities L account for all layers of contingent claims.

Assumption 5.3.1. *The nominal liabilities $L_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}_+$ are bounded with upper bound $\bar{L}_{ij} \geq 0$ for all institutions $i, j \in \mathcal{N}_0$.*

Example 5.3.2. Consider a static network model of external assets $x \in \mathbb{R}_+^{n+1}$ and liabilities $L^0 \in \mathbb{R}_+^{(n+1)\times(n+1)}$ (with corresponding total liabilities $\bar{p}^0$ and relative liabilities Π^0). Firm $j \in \mathcal{N}_0$ purchased an *insurance contract* from firm $i \in \mathcal{N}$ on the event that firm $k \in \mathcal{N}$ does not pay its obligations in full to firm j; this is encoded in the nominal liabilities function

$$L_{ij}(V) = L_{ij}^0 + \sum_{k \in \mathcal{N}} \eta_{ij}^k(V) \frac{L_{kj}(V)}{\sum_{l \in \mathcal{N}_0} L_{kl}(V)} V_k^-.$$

In the above equation I set the parameter $\eta_{ij}^k : \mathbb{R}^{n+1} \to [0,1]$ to denote the level of insurance offered by the contract. Logically I impose the condition that $\eta_{ij}^i \equiv 0$ for all firms $i \in \mathcal{N}$ and $j \in \mathcal{N}_0$ so as a firm is not insuring against itself. I further impose a tree structure on the insurance, that is insurers will not directly insure nonpayments from other insurers in a cyclical manner. This is codified in the condition that $\eta_{ij}^{k_1} \eta_{k_1 j}^{k_2} \dots \eta_{k_m j}^{i} = 0$ for all $i, j, k_1, \cdots, k_m \in \mathcal{N}_0$. This tree structure immediately implies the uniqueness of the nominal liabilities matrix $L : \mathbb{R}^{n+1} \to \mathbb{R}_+^{(n+1)\times(n+1)}$. These conditions are related to the “green core” system in [19]. In the case that $\eta_{ij}^k(V) > 1$, this is the situation of over-insurance which no longer is considered “insurance” in the strict legal sense; see Example 5.3.3 for this more general setting. More generally, over-insurance is implied by the condition $\sum_{i \in \mathcal{N}} \eta_{ij}^k(V) > 1$, i.e., the total amount of insurance on any payment should be bounded by 1. Though explained as a single insurance contract, multiple such contracts may be layered so that one financial institution may have insurance against the failures of multiple counterparties. The simplest insurance contracts are such that $\eta_{ij}^k \equiv \hat{\eta}_{ij}^k \in [0,1]$, though by considering the functional I allow for situations in which insurance only pays losses exceeding a threshold

τ_{ij}^k, e.g.,

$$\eta_{ij}^k(V) = \hat{\eta}_{ij}^k \frac{\left[L_{kj}(V) - \frac{L_{kj}(V)}{\sum_{l\in\mathcal{N}_0} L_{kl}(V)} V_k^- + \tau_{ij}^k\right]^+}{\left[L_{kj}(V) - \frac{L_{kj}(V)}{\sum_{l\in\mathcal{N}_0} L_{kl}(V)} V_k^-\right]^+}.$$

Also within this framework I allow for reinsurance contracts; that is, insurance contracts that pay out once payments from an insurer reach a certain threshold so as to contain the losses for the insurer itself.

Example 5.3.3. As in Example 5.3.2, consider an initial static network model with asset and liability parameters (x, L^0). Though similar to an insurance policy, a firm may purchase credit default swaps. Firm $j \in \mathcal{N}_0$ purchased a *credit default swap [CDS]* from firm $i \in \mathcal{N}$ on the failure of firm $k \in \mathcal{N}$ is encoded in the formula

$$L_{ij}(V) = L_{ij}^0 + \sum_{k\in\mathcal{N}} \eta_{ij}^k(V) V_k^-.$$

In this example I define $\eta_{ij}^k : \mathbb{R}^{n+1} \to \mathbb{R}_+$ without restriction on the number of swaps purchased or the existence of an insurable interest. In such a way I allow for so-called "naked" CDSs where the payments to firm j are not based on any insurable interest in firm k.

Example 5.3.4. As in Example 5.3.2, consider an initial static network model with asset and liability parameters (x, L^0). I will now consider a system in which all firms must pay towards a centralized stability fund. That is, prior to the start some amount $y \in [0, x]$ of the external assets are provided from each firm used in the stability fund. In the case of failures this fund would support the defaulting firms. Consider this centralized fund to be denoted as node B and let $\mathcal{N}_B = \mathcal{N}_0 \cup \{B\}$. This system can be described in which the bailout fund is capitalized prior to clearing or as part of clearing. If the bailout is collected prior to clearing than this system is described by external assets of $x_i - y_i \geq 0$ for all firms $i \in \mathcal{N}$ and $\sum_{i\in\mathcal{N}} y_i \geq 0$ for the stability fund node B and liabilities of

$$L_{ij}(V) = L_{ij}^0 \; \forall i,j \in \mathcal{N}_0, \qquad L_{Bi}(V) = V_i^- \; \forall i \in \mathcal{N}, \qquad L_{iB}(V) = 0 \; \forall i \in \mathcal{N}.$$

The payments to this stability fund can also be made as a part of clearing. In this case the external assets are x and liabilities are

$$L_{ij}(V) = L_{ij}^0 \; \forall i,j \in \mathcal{N}_0, \qquad L_{Bi}(V) = V_i^- \; \forall i \in \mathcal{N}, \qquad L_{iB}(V) = y_i \; \forall i \in \mathcal{N}.$$

This can be extended further by setting the payments to the stability fund y to itself be a function of the wealth of each institution. This allows for concepts such as pooled reserve requirements to be encoded into the general framework.

Example 5.3.5. The final general conceptual example I wish to present is the situation of introducing a *central counterparty [CCP]*. In this setting, the network topology follows a star shape, i.e., firms only have liabilities to and from some centralized CCP node. The true CCP rules, however, also include what is called a default waterfall. The default waterfall kicks in when the CCP is unable to pay out in full through the initial collected liabilities and margin payments. In such a case the remaining solvent firms are forced to provide more liquidity to the CCP node. In a broad sense, this fits within the general framework considered herein as the obligations to the CCP are directly dependent on the wealths of all firms in the system. CCPs are described in more detail in [2, 89, 34, 36].

As in the construction of the Eisenberg--Noe setting [45], the total and relative liabilities will implicitly be functions of the system wealths as well, i.e.,

$$\bar{p}_i(V) = \sum_{j \in \mathcal{N}_0} L_{ij}(V) \tag{5.4}$$

$$\pi_{ij}(V) = \begin{cases} \frac{L_{ij}(V)}{\bar{p}_i(V)} & \text{if } \bar{p}_i(V) > 0 \\ \frac{1}{n} & \text{if } \bar{p}_i(V) = 0, \; i \neq j \\ 0 & \text{if } \bar{p}_i(V) = 0, \; i = j \end{cases} \tag{5.5}$$

for firms $i,j \in \mathcal{N}_0$ and system equities $V \in \mathbb{R}^{n+1}$.

With this contingent setting I can define the extension of the Eisenberg--Noe framework as the fixed point problem

$$V = x + \Pi(V)^\top [\bar{p}(V) - V^-]^+ - \bar{p}(V). \tag{5.6}$$

That is, the wealths are the sum of external assets and payments from other banks minus the payments owed. This could equivalently be defined directly as the payments as is done in [45], I choose to consider the wealths directly in this work as it is easier to consider the examples, e.g., insurance payments. The realized payments can be defined (as discussed previously without contingent payments) by $p = [\bar{p}(V) - V^-]^+$.

Proposition 5.3.6. *Under Assumption 5.3.1, any fixed point wealth $V \in \mathbb{R}^{n+1}$ of (5.6) lies within the compact set $\prod_{i=1}^n [x_i - \sum_{j \in \mathcal{N}_0} \bar{L}_{ij}, x_i + \sum_{j \in \mathcal{N}} \bar{L}_{ji}]$.*

Proof. The result is immediate by the boundedness properties of Assumption 5.3.1. □

Corollary 5.3.7. *Under Assumption 5.3.1, there exists an equilibrium wealth of (5.6) if $L_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}_+$ is continuous as a function of wealths for all firms $i, j \in \mathcal{N}_0$.*

Proof. This follows from the compactness argument of Proposition 5.3.6 and the Brouwer fixed point theorem. □

Though in Corollary 5.3.7 I have proven the existence of an equilibrium solution to (5.6), this need not be a unique solution. The following example illustrates a simple network with multiple equilibria. Further, Corollary 5.3.7 and (5.6) implicitly assume that there are no bankruptcy costs. With such costs (as introduced in [96]), Corollary 5.3.7 will no longer apply. See also Remark 5.3.12 for a discussion on sufficient conditions to guarantee existence of a clearing wealths vector under bankruptcy costs.

Example 5.3.8. Consider the network with $n = 3$ banks, and *without* the societal node. This network is depicted in Figure 5.1. Banks 1 and 3 have $x_1 = x_3 = 0$ external assets and bank 2 begins with $x_2 = 3/16$ external assets. I consider the case in which $L_{23} = L_{32} \equiv 1$ are fixed obligations whereas the first bank has purchased a credit default swap on the third institution defaulting on its obligations from the second institution that pays out $L_{21}(V) = V_3^-$. No other exposures exist within this system. The system of wealths must therefore satisfy

$$V_1 = \frac{V_3^-}{1 + V_3^-}(1 + V_3^- - V_2^-)^+$$
$$V_2 = \frac{3}{16} + (1 - V_3^-)^+ - (1 + V_3^-)$$

$$V_3 = \frac{1}{1+V_3^-}(1+V_3^- - V_2^-)^+ - 1.$$

It can be shown that the following are both equilibrium wealths of the contingent network:

- $V = (0\ ,\ 3/16\ ,\ 0)^\top$, i.e., payments are given by $p = \bar{p}(V) - V^- = (0\ ,\ 1\ ,\ 1)^\top$, and
- $V = (3/16\ ,\ -21/16\ ,\ -3/4)^\top$, i.e., payments are given by $p = \bar{p}(V) - V^- = (0\ ,\ 7/16\ ,\ 1/4)^\top$.

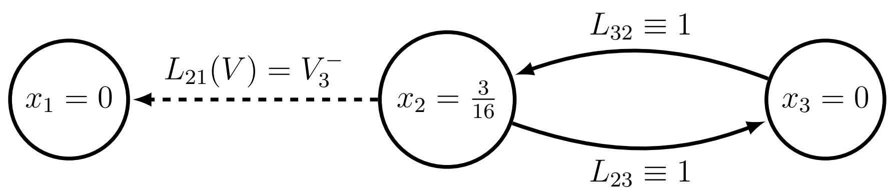


Figure 5.1: Example 5.3.8: A graphical representation of the network model with 3 banks which accepts more than one clearing solution.

5.3.2 A nonspeculative financial network

I will now impose additional properties upon the financial system to have stronger existence results, culminating in uniqueness of the clearing solutions. These results provide monotonicity of the wealth of the banks in the financial system. The first of such properties, defined as a nonspeculative property, is provided below in Definition 5.3.9.

Definition 5.3.9. *Firm $i \in \mathcal{N}_0$ is called* **nonspeculative** *if*

$$x_i + \sum_{j \in \mathcal{N}} \pi_{ji}(V)[\bar{p}_j(V) - V_j^-]^+ - \bar{p}_i(V)$$

is nondecreasing in $V \in \mathbb{R}^{n+1}$. The network $\mathcal{N}_0$ is called nonspeculative if all firms $i \in \mathcal{N}_0$ are nonspeculative.

I call the property in Definition 5.3.9 "nonspeculative" as it provides conditions so that firm $i \in \mathcal{N}_0$ does not benefit from (i.e., speculate on) the failure of another firm. I do, however, allow for firm i to *hedge* its exposure to other firms. This exposure can be either direct or indirect.

Remark 5.3.10. The nonspeculative framework considered herein is similar to, and can be considered as an extension of the properties considered in [19]. In that work, the monotonicity property, in the definition of nonspeculative systems that I consider, is specified for credit default swaps. Properties on solutions, which I will derive from this nonspeculative property, are considered as a function of the network topology in [19]; in fact, the topological features required in [19] guarantee that the "green core" system is inherently nonspeculative.

Lemma 5.3.11. *Under Assumption 5.3.1, any nonspeculative system has a greatest and least equilibrium wealth $V^\uparrow \geq V^\downarrow$ satisfying* (5.6) *existing within the compact space $\prod_{i\in\mathcal{N}_0}[x_i - \sum_{j\in\mathcal{N}_0}\bar{L}_{ij}, x_i + \sum_{j\in\mathcal{N}}\bar{L}_{ji}]$. Additionally, under all clearing vectors the value of the equity of each node of the financial system is the same, that is, if V and $\hat{V}$ are any two clearing wealths then $V^+ = \hat{V}^+$.*

Proof. By the nonspeculative property I can apply the Tarski fixed point theorem to get the existence of a maximal and minimal fixed point $V^\uparrow \geq V^\downarrow$. By Proposition 5.3.6 I have that such solutions must exist within the provided compact space.

Now I will show the uniqueness of the positive equities by proving that $(V^\uparrow)^+ = (V^\downarrow)^+$. By definition I know that $(V^\uparrow)^+ \geq (V^\downarrow)^+$, so as in [45, Theorem 1] I will prove that the total positive equity in the system remains constant. Let $V \in \mathbb{R}^{n+1}$ be some equilibrium wealth solution, then since $\bar{p}_0 \equiv 0$ by definition I recover that

$$\begin{aligned}
\sum_{i\in\mathcal{N}_0} V_i^+ &= \sum_{i\in\mathcal{N}_0}\left(x_i + \sum_{j\in\mathcal{N}}\pi_{ji}(V)[\bar{p}_j(V) - V_j^-]^+ - \bar{p}_i(V)\right)^+ \\
&= \sum_{i\in\mathcal{N}_0}\left(x_i + \sum_{j\in\mathcal{N}}\pi_{ji}(V)[\bar{p}_j(V) - V_j^-]^+ - [\bar{p}_i(V) - V_i^-]^+\right) \\
&= \sum_{i\in\mathcal{N}_0} x_i + \sum_{j\in\mathcal{N}}[\bar{p}_j(V) - V_j^-]^+ \sum_{i\in\mathcal{N}_0}\pi_{ji}(V) - \sum_{i\in\mathcal{N}_0}[\bar{p}_i(V) - V_i^-]^+ \\
&= \sum_{i\in\mathcal{N}_0} x_i + \sum_{j\in\mathcal{N}}[\bar{p}_j(V) - V_j^-]^+ - \sum_{i\in\mathcal{N}_0}[\bar{p}_i(V) - V_i^-]^+ \\
&= \sum_{i\in\mathcal{N}_0} x_i.
\end{aligned}$$

Therefore $\sum_{i\in\mathcal{N}_0}(V_i^\uparrow)^+ = \sum_{i\in\mathcal{N}_0}(V_i^\downarrow)^+$ and thus $(V^\uparrow)^+ = (V^\downarrow)^+$. □

Remark 5.3.12. The inclusion of bankruptcy costs to this setting, in much the same way as accomplished in [96, 19], would guarantee the existence of a maximal and minimal clearing wealths vector under the assumptions of Lemma 5.3.11. Much as in [19], without the nonspeculative assumption, a solution may not exist since Corollary 5.3.7 will no longer apply.

I will now give additional properties for the societal node 0 to satisfy.

Assumption 5.3.13. *All firms $i \in \mathcal{N}$ have strictly positive obligations to society, i.e., $L_{i0} : \mathbb{R}^{n+1} \to \mathbb{R}_{++}$. Additionally, the obligations to society depend only on the negative wealths of all firms, i.e., $L_{i0}(V) = L_{i0}(-V^-)$ for all $i \in \mathcal{N}$.*

Definition 5.3.14. *The societal node is called* **strictly nonspeculative** *if*

$$\sum_{j \in \mathcal{N}} \pi_{j0}(V)[\bar{p}_j(V) - V_j^-]^+$$

is strictly increasing in $V \in \mathbb{R}^{n+1}_-$. The network $\mathcal{N}_0$ is called strictly nonspeculative if the system is nonspeculative and the societal node is strictly nonspeculative.

I call this property strictly nonspeculative since it provides the condition that society does strictly worse as any firm defaults by any additional amount. This requires that society can never perfectly hedge its risk, and as a consequence is strictly not speculating on any firm's failure. This is a reasonable property as society should always be exposed to banking failures to some degree through, e.g., deposits and the payments necessary for deposit insurance.

Corollary 5.3.15. *Under Assumptions 5.3.1 and 5.3.13, any strictly nonspeculative system has a unique equilibrium wealth of* (5.6) *existing within the compact space $\prod_{i \in \mathcal{N}_0}[x_i - \sum_{j \in \mathcal{N}_0} \bar{L}_{ij}, x_i + \sum_{j \in \mathcal{N}} \bar{L}_{ji}]$.*

Proof. Using Lemma 5.3.11 I have the existence of greatest and least fixed points $V^\uparrow \geq V^\downarrow$. Assume there exists some firm $i \in \mathcal{N}$ such that $0 > V_i^\uparrow > V_i^\downarrow$ (otherwise uniqueness is guaranteed by nonexistence of such a firm as well as the uniqueness of the positive equities). By the definition of the equilibrium wealth of the society node

$$V_0^\uparrow = \sum_{j \in \mathcal{N}} \pi_{j0}(V^\uparrow)[\bar{p}_j(V^\uparrow) - (V_j^\uparrow)^-]^+$$

$$> \sum_{j \in \mathcal{N}} \pi_{j0}(V^{\downarrow})[\bar{p}_j(V^{\downarrow}) - (V_j^{\downarrow})^-]^+ = V_0^{\downarrow}.$$

However, immediately I know that the societal node has positive equity, therefore by Lemma 5.3.11 it must follow that $V_0^{\uparrow} = V_0^{\downarrow}$, which is a contradiction so uniqueness must follow. □

I will now provide a version of the *fictitious default algorithm* (as discussed in, e.g., [45, 96, 6, 51, 104]) for the contingent payments described in (5.6). Algorithm 5.3.16 provides the maximal fixed point $V^{\uparrow}$ under the conditions of Lemma 5.3.11, that is, for a network of nonspeculative banks. It can easily be modified to provide the minimal fixed point $V^{\downarrow}$ instead. Under the assumptions of Corollary 5.3.15 this algorithm results in the unique equilibrium wealths.

Algorithm 5.3.16. Consider the setting of Lemma 5.3.11 such that $L(V) = L(-V^-)$ for every $V \in \mathbb{R}^{n+1}$. The greatest clearing wealths $V^{\uparrow}$ can be found in at most n iterations of the following algorithm. Initialize $k = 0$, $D^0 = \emptyset$, and $V^0 = x + \Pi(0)^{\top}\bar{p}(0) - \bar{p}(0)$.

(i) Increment $k = k + 1$;

(ii) Denote the set of insolvent banks by $D^k = \{i \in \mathcal{N} \mid V_i^{k-1} < 0\}$;

(iii) If $D^k = D^{k-1}$ then terminate;

(iv) Define the matrix $\Lambda \in \{0, 1\}^{(n+1)\times(n+1)}$ so that

$$\Lambda_{ij} = \begin{cases} 1 & \text{if } i = j \in D^k \\ 0 & \text{else} \end{cases};$$

(v) $V^k = \hat{V}$ is the maximal solution of the following fixed point problem:

$$\hat{V} = x + \Pi(\Lambda\hat{V})^{\top}[\bar{p}(\Lambda\hat{V}) + \Lambda\hat{V}]^+ - \bar{p}(\Lambda\hat{V})$$

in the domain $\prod_{i \in \mathcal{N}_0}[x_i - \sum_{j \in \mathcal{N}_0} \bar{L}_{ij}, x_i + \sum_{j \in \mathcal{N}} \bar{L}_{ji}]$;

(vi) Go back to step i.

In Algorithm 5.3.16 at most n iterations are needed, as opposed to $n+1$ as would generally be stated for the fictitious default algorithm of [45]. This is due to the fact that, by definition, the societal node 0 has no obligations and therefore cannot default. The additional condition that $L(V) = L(-V^-)$ for Algorithm 5.3.16 corresponds to the case in which the nominal liabilities only depend on the set of solvent institutions and the shortfall of each insolvent institution. This is satisfied for, e.g., CDS as described in Example 5.3.3.

I now consider a simple sensitivity analysis of the clearing wealths V under uncertainty in the initial endowments $x \in \mathbb{R}^{n+1}_+$. In so doing I am able to consider similar comparative statics results as in Lemma 5 of [45].

Proposition 5.3.17. *Consider the setting of Corollary 5.3.15 such that the nominal liabilities $L_{ij} : \mathbb{R}^{n+1} \to \mathbb{R}_+$ are continuous for every pair of firms $i, j \in \mathcal{N}_0$. The unique clearing wealths $V : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}$ are continuous and nondecreasing as a function of the initial endowments $x \in \mathbb{R}^{n+1}_+$.*

Proof. The proof is presented in the appendix. □

5.3.3 Shortcomings

I wish to conclude my discussion of the system with contingent obligations under simultaneous claims by considering shortfalls to this approach. In the subsequent section I will consider a dynamic approach to overcome these shortcomings, as well the issues on existence and uniqueness considered above and by [19, 18].

Consider a setting in which a firm takes out an insurance contract on its own failure. This setting is of particular interest as it is inherently the type of contingent payment owed to a central counterparty as part of the default waterfall in the CCP framework considered in Example 5.3.5 or the stability fund discussed in Example 5.3.4. To further simplify this setting, and again to make it relevant with regards to the CCP framework, I will consider the case in which the firm(s) offering insurance have enough assets to make all payments in full. Allow the firm taking out the insurance contracts to be firm 1. If the sum total of all contingent payments are exactly enough to make firm 1 whole again, i.e., all contingent payments to firm 1 sum up to V_1^-, then in principle firm 1 will never default. However, in equilibrium, this is not what the insurance payments will be; in fact, if the insurance payments in a fixed point were to make firm 1 whole then no insurance payments would be made and the initial shortfall would be realized once more. Therefore, in equilibrium, it must be the case that firm 1 will default even if they are paid the insurance. As a simple demonstrative example, if firm 1 only has obligations to the societal node (allowing to ignore all feedback effects from firm 1 paying more and having a higher recovery rate through the network) then the insurance will add up to exactly *half* of firm 1's initial shortfall in the fixed point as the new shortfall for firm 1 will be equal to the contingent payments that are being made.

However, while this conceptual problem with a firm taking out an insurance contract on its own losses is important, it can conceivably be overcome by providing a sufficiently complicated structure to the contingent payments. A more subtle, but pernicious, flaw is that this contingent payment system is speculative by construction. Namely, if the wealth of firm 1 is lowered, no other firm does better (firm 1 will pay out less and the insurance companies will have higher claims to pay), but firm 1 itself improves its wealth. This is due to the nonspeculative property being constructed in which firm 1 does not directly get hit by its own lower wealth, but would only occur in network effects that would be on the second order, not in evidence in the single iteration of the definition. Thus, even though the network is constructed from the notion that no firm benefits in the case of defaults (and this would be evidenced in any equilibrium), the monotonicity of the nonspeculative property is a stronger construct that cannot be satisfied so easily from a conceptual standpoint.

The above described problems could, in specific circumstances (e.g., a single insurer and only one contingent payment contract or a "green core" system from [19]), be overcome by reformulating the payments appropriately. However, in the general case with each contract incorporating no speculation from a financial perspective, this system would have the aforementioned shortcomings. These challenges, along with the inability to deal with speculative systems in general, stem from structural issues in such a static framework. Specifically, insurance, and contingent payments more generally, are paid on specific claims, not simultaneous to the claim being made. This necessitates a dynamic approach to this problem, which I will discuss in the subsequent section.

5.4 Dynamic framework

As detailed above, the static, simultaneous claims, model presented has both mathematical and economic issues that I am not aware of any way to overcome in a general setting. These problems are associated with the presence of, potentially, infinite cycles. Much as with [80], these cycles could alternate between two states, particularly for speculative systems. That is, for instance, insurance is paid out because a bank is insolvent, but because of this insurance payment the firm is no longer insolvent and no payment would be necessary. As in [80], I will consider an algorithmic approach to this issue. I thus propose a simple dynamic framework. Additionally, I consider this setting to be more realistic than the static setting considered above and by [19] as the financial system does not include the payment of, e.g., a CDS on the obligation inherent in that contract.

5.4.1 General setting

I adapt the framework introduced in Chapter 4 for the purposes of constructing a simple dynamic framework for contingent payments. Consider a discrete set of clearing times $\mathbb{T}$, e.g., $\mathbb{T} = \{0, 1, \ldots, T\}$ for some (finite) terminal time $T < \infty$ or $\mathbb{T} = \mathbb{N}$. For processes I will use the notation from [35] such that the process $Z : \mathbb{T} \to \mathbb{R}^n$ has value of $Z(t)$ at time $t \in \mathbb{T}$ and history $Z_t := (Z(s))_{s=0}^t$. As an explicit extension to Chapter 4.3, I consider the external (incoming) cash flow $x : \mathbb{T} \times \mathbb{R}^{(n+1)\times|\mathbb{T}|} \to \mathbb{R}^{n+1}_+$ and nominal liabilities $L : \mathbb{T} \times \mathbb{R}^{(n+1)\times|\mathbb{T}|} \to \mathbb{R}^{(n+1)\times(n+1)}_+$ to be functions of the clearing time *and* prior wealths. For simplicity, I will consider $x(t, \cdot) := x(t)$ to be independent of the prior wealths, though it may still depend on time. The distinguishing feature of this model compared to the static Eisenberg--Noe model (or the static contingent payment model above and in [19]) is that the system parameters may depend on prior times. For example, if firm i has positive equity at time $t - 1$ (i.e., $V_i(t - 1) > 0$) then these surplus assets are available to firm i at time t in order to satisfy its obligations. In the contingent setting, the wealths of all banks at time $t - 1$ may affect the obligations due at time t as well.

I note that in Chapter 4.3 all unpaid obligations from a prior time are assumed to roll forward automatically. That is, if firm i has negative wealth at time $t - 1$ then the debts that the firm has not yet paid will roll forward in time and be due at the next time point. Under such an assumption, no firm is deemed to default on its obligations until the terminal time T. Herein, with the explicit consideration of the contingent payments, I may "zero out" a firm before the terminal date if it is deemed to default in much the same as in, e.g., [27]. While I can incorporate the notion of loans from [27] as well, I will restrict my analysis to debts rolling forward in time so as to simplify the discussion.

As noted above, in addition to the structure from Chapter 4.3, the nominal liabilities will explicitly depend on the clearing wealths of the prior time(s), i.e., $L : \mathbb{T} \times \mathbb{R}^{(n+1)\times|\mathbb{T}|} \to \mathbb{R}_+^{(n+1)\times(n+1)}$. Often, to make this difference explicit especially in examples, I consider the full nominal liabilities L to be a combination of two components: a non-contingent component $L^0 : \mathbb{T} \to \mathbb{R}_+^{(n+1)\times(n+1)}$ which is only a function of clearing times and a contingent component $L^c : \mathbb{T} \times \mathbb{R}^{(n+1)\times|\mathbb{T}|} \to \mathbb{R}_+^{(n+1)\times(n+1)}$ which is a function of both the clearing times and the past history (but only encodes the contingent payments based on the past history). That is, $L = L^0 + L^c$.

As a descriptive consideration of the contingent obligations L^c, consider the insurance-based (Example 5.3.2) or credit default swap (Example 5.3.3) scenarios of the previous section. For instance, a bank j may purchase a credit default swap from bank i on the failure of firm k as described in Example 5.3.3. As opposed to the simultaneous claims setting in Section 5.3, in this dynamic setting I consider an order of operations. That is, first firm k must fail at time $t-1$, and only then would the credit default swap be paid at time t. This delay in payments is a reflection of the real financial system in which there is a time between a claim being made by bank j to i and the payment on that claim. The payment due to this credit default swap would be incorporated in L^c_{ij} but not L^0_{ij}.

Even with this important distinction, I can use the same methodology as in Chapter 4.3 to prove existence and uniqueness of the clearing wealths in this setting. The following assumption, with the concept taken from Chapter 4.3 guarantees that all firms are solvent at the start of the system and that the system is a regular network as described by [45].

Assumption 5.4.1. *Before the time of interest, all firms are solvent and liquid. That is, $V_i(-1) \geq 0$ for all firms $i \in \mathcal{N}_0$. Additionally, all firms have positive external cash flow or obligations to society at all times $t \in \mathbb{T}$, i.e., $x_i(t) + L_{i0}(t) > 0$ for all firms $i \in \mathcal{N}$ and all times $t \in \mathbb{T}$.*

To incorporate the possibility of firms defaulting before the terminal time, let $\mathcal{N}_0^t(V_{t-1}) \subseteq \mathcal{N}_0$ denote the firms that are paying obligations at time $t \in \mathbb{T}$ based on the history of wealths up to time $t-1$. In particular, I will assume that $\mathcal{N}_0^0(V_{-1}) := \mathcal{N}_0$ and $\mathcal{N}_0^{t+1}(V_t) \subseteq \mathcal{N}_0^t(V_{t-1})$ for any time t and any wealths process V. That is, all firms are deemed solvent at time 0 as in Assumption 5.4.1 and no firm recovers from default. This notion allows for a consideration in much the same manner as [27]. Mathematically this does not require further consideration than in Chapter 4.3 as $\mathcal{N}_0^t$ only depends on the history up to time $t-1$. With this notation I can define $L_{ij}(t, V_{t-1}) = 0$ for all firms $j \in \mathcal{N}_0$ and $i \notin \mathcal{N}_0^t(V_{t-1})$. With the notion of an auction from [27] it will also follow that $L_{ji}(t, V_{t-1}) = 0$ for all firms $j \in \mathcal{N}_0$ and $i \notin \mathcal{N}_0^t(V_{t-1})$. I define the total liabilities and relative liabilities at time $t \in \mathbb{T}$ as

$$\bar{p}_i(t, V_{t-1}) := \sum_{j \in \mathcal{N}_0} L_{ij}(t, V_{t-1}) + V_i(t-1)^-$$

$$\pi_{ij}(t, V_{t-1}) := \begin{cases} \frac{L_{ij}(t,V_{t-1}) + \pi_{ij}(t-1,V_{t-2}) V_i(t-1)^-}{\bar{p}_i(t,V_{t-1})} & \text{if } \bar{p}_i(t, V_{t-1}) > 0 \\ \frac{1}{n} & \text{if } \bar{p}_i(t, V_{t-1}) = 0, \; j \neq i \\ 0 & \text{if } \bar{p}_i(t, V_{t-1}) = 0, \; j = i \end{cases} \quad \forall i, j \in \mathcal{N}_0.$$

Then the clearing wealths must satisfy the following fixed point problem in time t wealths:

$$V(t) = V(t-1)^+ + x(t) + \Pi(t, V_{t-1})^\top \operatorname{diag}(\mathbb{I}_{\{\}}\{i \in \mathcal{N}_0^t(V_{t-1})\}) \left[\bar{p}(t, V_{t-1}) - V(t)^-\right]^+ - \bar{p}(t, V_{t-1}). \tag{5.7}$$

I proceed to reformulate the problem as in Chapter 4.3. I consider a process of cash flows c and functional relative exposures A. These I define by

$$c(t, V_{t-1}) := x(t) + L(t, V_{t-1})^\top \vec{1} - L(t, V_{t-1}) \vec{1}$$

$$a_{ij}(t, V_t) := \begin{cases} \pi_{ij}(t, V_{t-1}) & \text{if } \bar{p}_i(t, V_{t-1}) \geq V_i(t)^-, \; i \in \mathcal{N}_0^t(V_{t-1}) \\ \frac{L_{ij}(t,V_{t-1}) + a_{ij}(t-1,V_{t-1}) V_i(t-1)^-}{V_i(t)^-} & \text{else} \end{cases} \quad \forall i, j \in \mathcal{N}_0. \tag{5.8}$$

That is, I consider $c(t, V_{t-1}) \in \mathbb{R}^{n+1}$ to be the vector of book capital levels at time t, i.e., the new wealth of each firm assuming all other firms pay in full. I can also consider $c_i(t, V_{t-1})$ to be the *net cash flow* for firm i at time t. I define the functional matrix $A : \mathbb{T} \times \mathbb{R}^{(n+1) \times |\mathbb{T}|} \to [0,1]^{(n+1) \times (n+1)}$ to be the relative exposure matrix. That is, $a_{ij}(t, V_t) V_i(t)^-$ provides the (negative) impact that firm i's losses have on firm j's wealth at time $t \in \mathbb{T}$. This is in contrast to Π, the relative liabilities, in that it endogenously imposes the limited exposures concept. This equivalent formulation provides mathematical simplicity to the analysis.

Thus the fixed point equation reduces to

$$V(t) = V(t-1) + c(t, V_{t-1}) - A(t, V_t)^\top V(t)^- + A(t-1, V_{t-1})^\top V(t-1)^-. \tag{5.9}$$

With this setup I now wish to extend the existence and uniqueness results of [45] to discrete time.

Corollary 5.4.2. *Let $(c, L) : \mathbb{T} \times \mathbb{R}^{(n+1)\times|\mathbb{T}|} \to \mathbb{R}^{n+1} \times \mathbb{R}_+^{(n+1)\times(n+1)}$ define a dynamic financial network such that every bank has cash flow at least at the level dictated by nominal interbank liabilities, i.e., $c_i(t, V_{t-1}) \geq \sum_{j\in\mathcal{N}} L_{ji}(t, V_{t-1}) - \sum_{j\in\mathcal{N}_0} L_{ij}(t, V_{t-1})$ for all times $t \in \mathbb{T}$ and all wealth processes V, and so that every bank owes to the societal node at all times $t \in \mathbb{T}$, i.e., $L_{i0}(t, V_{t-1}) > 0$ for all banks $i \in \mathcal{N}$, times $t \in \mathbb{T}$, and wealths V. Under Assumption 5.4.1, there exists a unique solution of clearing wealths $V : \mathbb{T} \to \mathbb{R}^{n+1}$ to (5.9).*

Proof. This follows directly from the proof of Theorem 4.3.3. □

With the construction of the existence and uniqueness of the solution, I now want to emphasize the *fictitious default algorithm* from [45] to construct this clearing wealths vector over time. I note that at each time t this algorithm takes at most n iterations as is the case for the fictitious default algorithm originally presented in [45]. Thus with a terminal time T, this algorithm will construct the full clearing solution over $\mathbb{T}$ in nT iterations.

Algorithm 5.4.3. Under the assumptions of Theorem 5.4.2, the clearing wealths process $V : \mathbb{T} \to \mathbb{R}^{n+1}$ can be found by the following algorithm. Initialize $t = -1$ and $V(-1) \geq 0$ as a given. Repeat until $t = \max \mathbb{T}$:

(i) Increment $t = t + 1$.

(ii) Initialize $k = 0$, $V^0 = V(t-1) + c(t, V_{t-1})$, and $D^0 = \emptyset$. Repeat until convergence:

  (a) Increment $k = k + 1$;

(b) Denote the set of illiquid banks by $D^k := \{i \in \mathcal{N}_0^t(V_{t-1}) \mid V_i^{k-1} < 0\}$.

(c) If $D^k = D^{k-1}$ then terminate and set $V(t) = V^{k-1}$.

(d) Define the matrix $\Lambda^k \in \{0,1\}^{n\times n}$ so that $\Lambda^k_{ij} = \begin{cases} 1 & \text{if } i = j \in D^k \\ 0 & \text{else} \end{cases}$.

(e) Define $V^k = (I - \Pi(t, V_{t-1})^\top \Lambda^k)^{-1} \left(V(t-1) + c(t, V_{t-1}) + A(t-1, V_{t-1})^\top V(t-1)^-\right)$.

In step (iie) of the fictitious default algorithm I am able to replace $A(t, V_t)$ with $\Pi(t, V_{t-1})$. This is beneficial as it allows to directly compute V^k without requiring a fixed point problem. I additionally note that the inclusion of defaulted banks only required the change that the fictitious set of illiquid banks is a subset of $\mathcal{N}_0^t(V_{t-1})$ at each time t.

Remark 5.4.4. The dynamic framework provides a flexible way to deal with contingent payments. In particular, I can have as many time steps as the number of contingent payment layers in the network. For example, to consider insurance I need to have two time points to incorporate the nominal claims and the insurance claims triggered by the clearing of these nominal claims. For reinsurance markets, I need three time steps, the third one to incorporate the reinsurance claims triggered by the clearing of the insurance claims. I feel this hierarchical resolution of the claims is widely observed in reality.

Remark 5.4.5. One of the advantages of the dynamic framework is that it provides a natural way to include bankruptcy costs. This is a deviation from the static framework where I might not have existence of solutions for bankruptcy costs. However in the dynamic framework I can always determine the time point when the equity of a bank reaches zero and include the bankruptcy costs for the successive time periods. Hence the solution will exist and be unique.

Remark 5.4.6. I can provide much stronger sensitivity results in this case, as compared to the static case. Since in this approach at every time step I get an Eisenberg--Noe system, the sensitivity results are a sequential application of Section 4 of [45]. Directional derivatives of the static Eisenberg--Noe approach have been considered in [82, 55].

I wish to finish this section by remarking on when the dynamic framework presented herein will provide a clearing solution from the simultaneous claim setting in the prior section.

Remark 5.4.7. In general, the clearing solutions of the simultaneous claims framework will not coincide with the terminal clearing wealths of the dynamic framework. These notions will, however, coincide if the relative liabilities are kept constant as a function of wealths and time. Other settings, as evidenced by the examples provided in the next section, may provide sufficient conditions for the dynamic framework to provide a clearing solution from the simultaneous clearing setting. In particular, this will occur if the contingent payments do not strongly feedback into the network itself, e.g. if insurance is owed to an already solvent firm. However, I want to emphasize that the conditions under which the static and dynamic solutions coincide are very restrictive and in general this will not be the case. This is appropriate given the shortcomings of the static setting as expressed in Section 5.3.3.

5.4.2 Examples

I now wish to provide three illustrative examples to demonstrate the value of the discrete time setting as a model over the static setting presented in Section 5.3. These three examples correspond to simple networks in which the static setting has no clearing wealths, has multiple clearing wealths, and has a poor interpretation of the clearing wealths respectively. I will show that in all three situations the discrete time model presented above provides a unique clearing wealth for which the interpretation of the results is as anticipated.

Example 5.4.8. I wish to consider a small network example in which the financial system does not admit a clearing solution in the static setting, but a unique and financially meaningful solution in the dynamic setting. In this case I will consider a digital CDS. That is, in the case the CDS is triggered, the payment is a fixed strictly positive value (herein set to be 1), otherwise it pays out nothing. Immediately I can see that this is not a continuous payout and therefore does not automatically provide a clearing solution in the static setting (see Corollary 5.3.7), however I will still need to prove that there does not exist any solution.

Consider the network with $n = 3$ banks, and *without* the societal node, depicted in Figure 5.2. That is, bank 1 begins with $x_1 = 1$, bank 2 with $x_2 = 0$, and bank 3 with $x_3 = 2$ in external assets. I consider the case in which $L_{12} \equiv 2$ and $L_{23} \equiv 1.5$ are fixed obligations whereas the first bank has purchased a digital credit default swap on the second institution defaulting on its obligations from the third institution, i.e. $L_{31}(V) = \mathbb{I}_{\{\}}\{V_2 < 0\}$. No other exposures exist within this system. The system of wealths must therefore satisfy

$$V_1 = 1 + (\mathbb{I}_{\{\}}\{V_2 < 0\} - V_3^-)^+ - 2$$
$$V_2 = (2 - V_1^-)^+ - 1.5$$
$$V_3 = 2 + (1.5 - V_2^-)^+ - \mathbb{I}_{\{\}}\{V_2 < 0\}.$$

To show that no clearing solution exists to this system, I will consider the two possible settings: bank 2 is solvent or bank 2 has negative wealth.

(i) Assume bank 2 is solvent, i.e. $\mathbb{I}_{\{\}}\{V_2 < 0\} = 0$. I can compute a unique solution to the clearing wealths $V = (-1, -0.5, 3)^\top$. However, since this violates my assumption that $V_2 \geq 0$, this cannot be a clearing solution to the full problem.

(ii) Assume bank 2 is insolvent, i.e. $\mathbb{I}_{\{\}}\{V_2 < 0\} = 1$. I can compute a unique solution to the clearing wealths $V = (0, 0.5, 2.5)^\top$. However, since this violates my assumption that $V_2 < 0$, this cannot be a clearing solution to the full problem.

As no other possible clearing solutions can exist, it must be the case that there does not exist a clearing solution to this *static* financial system.

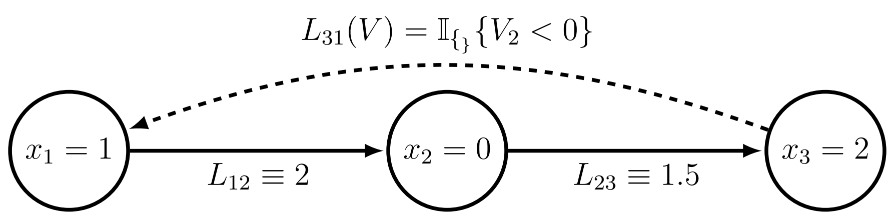


Figure 5.2: Example 5.4.8: A graphical representation of the network model with 3 banks which has no clearing in a static setting.

Now I wish to consider the same example but in the discrete time framework with $\mathbb{T} = \{0, 1\}$. Here I will consider all possible divisions of the external assets over the two time points. Formally, define $x^\epsilon(0) = (\epsilon, 0, 1)^\top$ and $x^\epsilon(1) = (1 - \epsilon, 0, 1)^\top$ for any $\epsilon \in [0, 1]$. Note that, by the topology of this network, it is a regular network (as defined in [45]) for any choice of $\epsilon \in [0, 1]$ as required by Assumption 5.4.1. In any scenario, define $L_{12}(0) = 2$ and $L_{23}(0) = 1.5$ with no other obligations at time 0. The only new obligation owed at time 1 is the contingent payment from bank 3 to 1, i.e., $L_{31}(1, V_0) = \mathbb{I}_{\{\}}\{V_2(0) < 0\}$ with no other new obligations at time 1. Further, all scenarios will be assumed to start from zero wealths (thus satisfying Assumption 5.4.1). I can easily compute the *unique* clearing wealths under x^ϵ (assuming no firms are removed from the system) as $V^\epsilon(0) = (\epsilon - 2, \epsilon - 1.5, \epsilon + 1)^\top$ and (noting that $V_2^\epsilon(0) < 0$ for any $\epsilon \in [0, 1]$) $V^\epsilon(1) = (0, 0.5, 2.5)^\top$. I note that this clearing solution is identical to the proposed *static* wealths under the assumption that bank 2 is insolvent. Additionally, the final wealths are independent of the choice of ϵ.

Example 5.4.9. Consider again Example 5.3.8 with three banks. In the static solution this was encoded by the parameters: external assets of $x = (0, 3/16, 0)^\top$ and sparse liabilities provided by $L_{23} = L_{32} \equiv 1$ and $L_{21}(V) = V_3^-$. Two clearing solutions existed, $V^* = (0, 3/16, 0)^\top$ and $V^* = (3/16, -21/16, -3/4)^\top$.

Now I wish to consider the same example but in the discrete time framework with $\mathbb{T} = \{0, 1\}$. Here I will consider all possible divisions of the external assets over the two time points. Formally, define $x^\epsilon(0) = (0, \epsilon, 0)^\top$ and $x^\epsilon(1) = (0, 3/16 - \epsilon, 0)^\top$ for any $\epsilon \in (0, 3/16]$ to guarantee the uniqueness of the clearing solutions as a regular network from [45] (and as required from Assumption 5.4.1 and since no societal node is included in this example). In any scenario, define $L_{23}(0) = L_{32}(0) \equiv 1$ with no other obligations at time 0. The only new obligation owed at time 1 is the contingent payment from bank 2 to 1, i.e., $L_{21}(1, V_0) = V_3(0)^-$ with no other new obligations at time 1. Further, all scenarios will be assumed to start from zero wealths (thus satisfying Assumption 5.4.1). I can easily compute the *unique* clearing wealths under x^ϵ as $V^\epsilon(0) = (0, \epsilon, 0)^\top$ and $V^\epsilon(1) = (0, 3/16, 0)^\top$. I note that this clearing solution is identical to the first clearing wealths solution of the static system and is independent of the choice of ϵ.

Example 5.4.10. Finally, I want to consider a simple financial system to demonstrate the issues discussed in Section 5.3.3 surrounding the static framework. I will then use this same network in the discrete time framework to find a unique, financially meaningful, clearing solution. To do so, consider a bank who takes out an insurance payment on its own losses. As discussed in Section 5.3.3, while the insured bank may, rightly, assume that their total losses will be made whole, in a static setting this will not happen. However, in the dynamic framework this does occur appropriately.

Consider the network with $n = 3$ banks, and *without* the societal node, depicted in Figure 5.3. That is, bank 1 begins with $x_1 = 1$, bank 2 with $x_2 = 0$, and bank 3 with $x_3 = 2$ in external assets. I consider the case in which $L_{12} \equiv 2$ and $L_{23} \equiv 1.5$ are fixed obligations whereas the first bank has purchased insurance on their own losses from the third institution, i.e. $L_{31}(V) = V_1^-$. No other exposures exist within this system. The system of wealths must therefore satisfy

$$\begin{aligned}
V_1 &= 1 + (V_1^- - V_3^-)^+ - 2 \\
V_2 &= (2 - V_1^-)^+ - 1.5 \\
V_3 &= 2 + (1.5 - V_2^-)^+ - V_1^-.
\end{aligned}$$

Without the insurance payment, the first bank will default with wealths of $V = (-1, -0.5, 3)^\top$. However, if the insurance is paid out in full then the first bank is made whole and the resultant wealths are $V = (0, 0.5, 2.5)$. In this case, the first bank does not default, which raises the question whether any insurance payment is to be made at all. Neither of these are clearing solutions as the system would infinitely cycle between needing insurance or not. The clearing wealths, instead, are given by $V = (-0.5, 0, 3)^\top$. That is, bank 1 will have a shortfall midway between its wealth with and without the insurance being paid. This, though, is *not* the notion that a firm purchasing insurance would expect as it cannot make them whole.

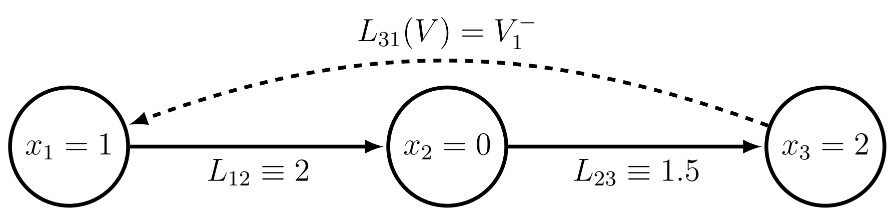


Figure 5.3: Example 5.4.10: A graphical representation of the network model with 3 banks which has poor interpretation in a static setting.

Now I wish to consider the same example but in the discrete time framework with $\mathbb{T} = \{0, 1\}$. Here I will consider all possible divisions of the external assets over the two time points. Formally, define $x^\epsilon(0) = (\epsilon, 0, 1)^\top$ and $x^\epsilon(1) = (1 - \epsilon, 0, 1)^\top$ for any $\epsilon \in [0, 1]$. Note that, by the topology of this network, it is a regular network (as defined in [45]) for any choice of $\epsilon \in [0, 1]$ as required by Assumption 5.4.1. In any scenario, define $L_{12}(0) = 2$ and $L_{23}(0) = 1.5$ with no other obligations at time 0. The only new obligation owed at time 1 is the contingent payment from bank 3 to 1, i.e., $L_{31}(1, V_0) = V_1(0)^-$ with no other new obligations at time 1. Further, all scenarios will be assumed to start from zero wealths (thus satisfying Assumption 5.4.1). I can easily compute the *unique* clearing wealths under x^ϵ (assuming no firms are removed from the system) as $V^\epsilon(0) = (\epsilon - 2, \epsilon - 1.5, \epsilon + 1)^\top$ and (noting that $V_1^\epsilon(0) < 0$ for any $\epsilon \in [0,1]$) $V^\epsilon(1) = (1 - \epsilon, 0.5, 1.5 + \epsilon)^\top$. I note that, in the case that $\epsilon = 1$, this clearing solution is identical to the proposed *static* wealths when the insurance is paid in full. As opposed to the prior examples, here the final wealths are a function of ϵ.

Chapter 6

Conclusion

In this dissertation, I have extended the Eisenberg--Noe framework to study channels of contagion such as fire sale and contingent payments and accounted for realistic financial situations viz. corporate debt pricing and dynamic clearing. These models are presented under general settings without presupposing the nature of the network or system parameters. Hence, these results can be used to augment stress tests, calibrate system parameters, improve data collection, and bolster current practices. Beyond their utility in systemic risk assessment, I have found these problems extremely interesting from a mathematical standpoint and have, in the process, opened up further avenues of research.

I have presented formulas for pricing of debt and equity of firms in a financial network under comonotonic endowments. In doing so, I have considered a problem that is analytically intractable under a general setting, developed conditions under which it can be solved analytically, and have proved that it provides a lower bound for the price of debt under the framework of [45]. This is particularly valuable as financial networks are of particular interest in performing stress tests and studying systemic risk. In particular, when the firms only invest in a risk-free bond and a risky asset following a geometric Brownian motion, I have deduced closed-form expressions for the price of debt and market capitalization for each firm in the system. Using empirical data, I have shown that the price of debt should be significantly higher under full network effects compared to the scenario in which it is assumed that the banks will be able to pay in full. Incorporating this risk properly in the valuation of assets will ensure that there are no pricing shocks to the system at a future date. An interesting extension of this work would be to consider the sensitivity analysis of pricing under misspecification of the network.

Next, I have developed a general mathematical and economic framework to study price-mediated contagion in a multi-asset setting where the firms liquidate assets during a crisis due to risk-weighted capital requirements. I have formulated the equilibrium pricing problem in a fixed point framework and developed conditions for existence and uniqueness of the solution under general settings of liquidation and inverse demand function. I have performed sensitivity analysis of the equilibrium prices with respect to the system parameters and used these results to evaluate the cost of regulation. Using a numerical example, I have studied the effect of diversification of bank portfolio in our framework and found that diversification does not uniformly lead to a more stable system, measured in this case by the total market capitalization. This model has several policy implications. First, it provides a framework to design stress tests beyond proportional liquidation and linear price impact. This is particularly important as I show through an example that the equilibrium price is very much dependent on the choice of the liquidation function. Secondly, the result on the bound of the risk-weight provides a method to calibrate the risk-weight depending on the liquidity of the asset. Thirdly, the sensitivity analysis can be used to give results on the robustness of the solution. Finally, the analysis of the cost of regulation would aid in the study of regulatory thresholds. An interesting extension would be the characterization of the joint pricing and liquidation equilibrium for the entire system.

I have considered an extension of the network model of financial contagion to allow for cash flows and obligations to be dynamic in time. I have presented this model in both discrete and continuous time, thus extending the frameworks of [27, 57, 80] which consider only discrete-time clearing. For the continuous time setting, I have determined conditions for existence and uniqueness of the clearing solutions under deterministic and Itô settings using an approach that does not require strong monotonicity properties as in the static or the discrete time setting. Additionally, I have found that the dynamical system for the Eisenberg--Noe contagion model may include an inherent prioritization scheme. Specifically, I have determined that if the relative liabilities are constant over time, then the dynamic Eisenberg--Noe model presented herein will reproduce the static system at the terminal time in a path-independent manner. Notably, in such a setting, I have been able to determine the true defaulting order rather than the fictitious order found in the fictitious default algorithm that is widely used in computing static clearing models. If, however, the relative liabilities are not constant over time, then the static Eisenberg--Noe model may report an incorrect picture of the financial system. This finding has two-fold policy implications. First, instead of collecting only aggregate data, further details should be sought on the time component of the cash flows and liabilities. Secondly, stress tests should be designed while taking the time dynamics into consideration. A natural extension, for which I believe the proposed dynamic model will be especially useful, is in considering strategic or dynamic actions by the market participants, e.g., incorporating bankruptcy costs and strategic decisions on the rolling forward of debt. I feel that the continuous-time framework will be particularly suitable for these extensions, as it allows us to construct unique clearing solutions without requiring strong monotonicity assumption.

Finally, I have proposed an extension of the network model of [45] to include contingent payments viz. insurance and CDSs with endogenous reference entities. I have studied these contingent payments in a static, simultaneous claims, framework, and developed conditions to provide existence and uniqueness of the clearing wealths. Further, sensitivity analysis and financial implications are considered in this setting. I have found that the static framework is suitable only for a certain class of networks and cannot guarantee the existence of a clearing solution beyond these systems. Indeed the problem often becomes ill-defined from a financial standpoint. Hence I have introduced the dynamic framework and shown that I can get existence and uniqueness under very mild assumptions. Lastly, I have shown that the problems which could not be solved in the simultaneous claims framework can be studied with this dynamic approach. This extension can be used to design stress tests that take into account the conditional nature of the payments under contingent claims. A clear extension of this model would be to include illiquid assets along with financial derivatives on these illiquid assets, i.e., options. These derivatives fall under the general class of contingent payments and can be used as a tool for either hedging (insurance) or speculation. Due to the possibility of speculation, in such a setting, a firm may have incentives to attempt to precipitate a fire sale and collect profit from the derivatives.

Appendix A

Expectations under random endowments

In this section I wish to consider a partition of the endowment space $\mathbb{R}^n_+$ into regions so that the defaulting set is constant in the 2^n subsets. This problem was considered in great detail in [69] for the setting without bankruptcy costs, i.e. $\beta = 1$, and with cross-ownership. Herein I will present a quick extension that allows for bankruptcy costs, i.e. for any $\beta \in [0, 1]$. Notably, when $\beta < 1$ the partitions need not be convex sets, while they are convex polyhedrons in a system without bankruptcy costs as given in [69]. In the below, if cross-ownership is desired then I refer to Remark 2.3.7 for the modifications necessary to the mappings Δ and $\bar{\delta}$.

To consider the partitions, fix $z \in \{0, 1\}^n$ to denote the defaulting banks. By construction, the resulting wealths given endowments $x \in \mathbb{R}^n_+$ are provided by $V(x) = \Delta(z)x - \bar{\delta}(z)$. For an endowment vector to be consistent with the defaulting set z, it would need to be such that $V_i(x) \geq 0$ if and only if $z_i = 0$. That is, the set of endowments that generate the defaulting set z is given by the system of inequalities:

$$\begin{aligned} e_i^\top \Delta(z)x &\geq e_i^\top \bar{\delta}(z) \quad \forall i: \; z_i = 0 \\ e_i^\top \Delta(z)x &< e_i^\top \bar{\delta}(z) \quad \forall i: \; z_i = 1. \end{aligned}$$

However, except in the special case that there are no bankruptcy costs ($\beta = 1$), these regions need not be disjoint. If an endowment x has two clearing wealth vectors $V^1 \neq V^2$, then it must be that $z^1 \neq z^2$ where $z^k = \mathbb{I}_{\{V^k<0\}}$. If $z^1 = z^2$ then, by construction of $\Delta(z^k), \bar{\delta}(z^k)$, it must follow that $V^1 = V^2$. In particular, I am interested in the maximal clearing wealth, thus I can construct the partition from the least to greatest number of defaults by considering the set of endowments that lead to z by $\mathcal{X}(z) \subseteq \mathbb{R}^n_+$ defined as:

$$\mathcal{X}(z) := \left\{ x \in \mathbb{R}^n_+ \;\middle|\; \begin{array}{l} e_i^\top \Delta(z) x \geq e_i^\top \bar{\delta}(z) \; \forall i : z_i = 0, \\ e_i^\top \Delta(z) x < e_i^\top \bar{\delta}(z) \; \forall i : z_i = 1 \end{array} \right\} \cap \bigcap_{\bar{z} \lneq z} \mathcal{X}(\bar{z})^c.$$

That is, $\mathcal{X}(z)$ is constructed as the intersection of a finite number of closed and open halfspaces as well as an additional condition that $x \notin \bigcup_{\bar{z} \lneq z} \mathcal{X}(\bar{z})$. This additional condition is the one that guarantees that $x \in \mathcal{X}(z)$ does not provide a "better" (i.e. fewer defaulting banks) clearing wealth vector than the maximal clearing solution $V(x)$. By the use of Tarski's fixed point theorem in the proof of Proposition 2.2.1, I am able to guarantee that this construction of $\mathcal{X}(z)$ is now, in fact, disjoint. In Figure A.1 I provide an image of the partitioning of the endowment space for a small network with 2 banks plus a societal node. I note that the societal node in this image can never default, this is due to it having no liabilities and thus always a nonnegative wealth.

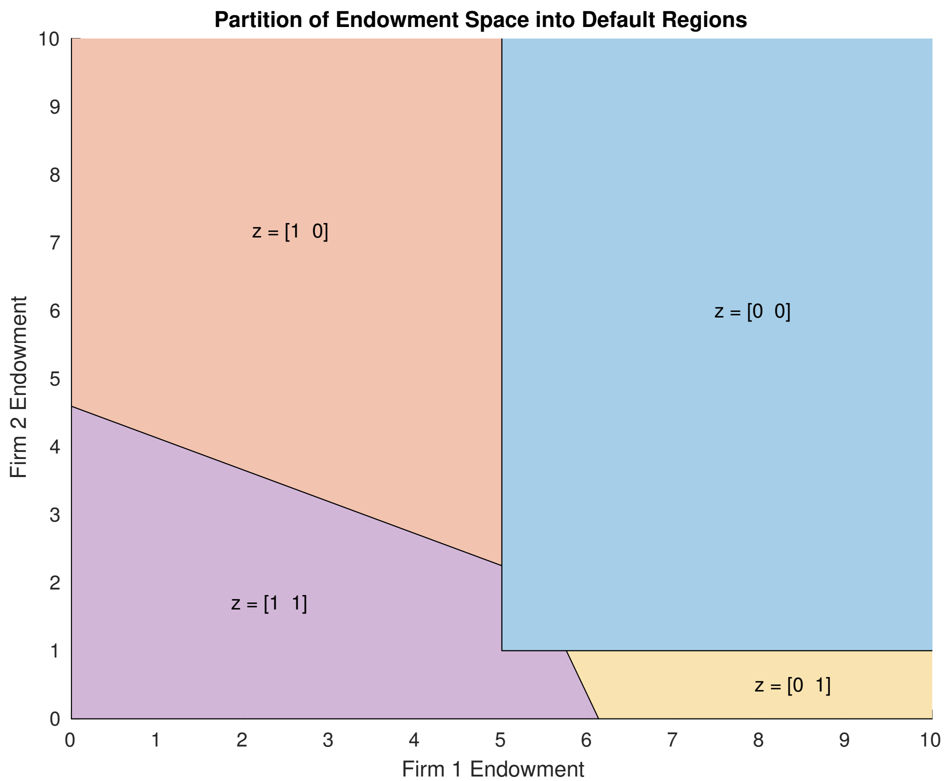


Figure A.1: Sample partition of the endowment space into default regions for 2 bank plus societal node network with bankruptcy costs.

Lastly, I wish to consider the particular case without bankruptcy costs ($\beta = 1$) as provided by [69]. Due to the uniqueness of the clearing wealths (Corollary 2.2.2) in this case I obtain only a single consistent default set $z \in \{0,1\}^n$ for every endowment $x \in \mathbb{R}^n_+$ and thus do not need to take a secondary intersection as in the case with bankruptcy costs. Further, in this case, those banks with 0 wealth can be considered equivalently both solvent and defaulting; thus the set of endowments that produce $z \in \{0,1\}^n$ can be considered as the finite intersection of *closed* halfspaces only, i.e. the convex polyhedron:

$$\mathcal{X}(z) := \left\{x \in \mathbb{R}^n_+ \mid (I - 2\,\mathrm{diag}(z))\Delta(z)x \geq (I - 2\,\mathrm{diag}(z))\bar{\delta}(z)\right\}.$$

I note that the partition $(\mathcal{X}(z))_{z\in\{0,1\}^n}$ is no longer disjoint as boundaries would be shared by the partitions. However, on this shared boundary the clearing wealths and payments will be equivalent, and this intersection has Lebesgue measure 0, therefore I (and prior authors) have discounted this situation for ease of constructing the sets $\mathcal{X}(z)$.

Appendix B

Proofs for Chapter 3

B.1 Proof of Proposition 3.4.1

Proof. A bank i can belong to any of the following three mutually exclusive and exhaustive sets:

- $S(q, \bar{q}) = \{i \in \mathcal{N} | h_i \leq q^\top [I - \theta_{min} A] s_i\}$.
- $L(q, \bar{q}) = \{i \in \mathcal{N} | q^\top [I - \theta_{min} A] s_i < h_i < \bar{q}^\top s_i\}$.
- $D(q, \bar{q}) = \{i \in \mathcal{N} | h_i \geq \bar{q}^\top s_i\}$.

With slight abuse in notation, I shall drop the dependence and refer to $L(q, \bar{q})$ as L throughout this proof. Let me define $\Gamma^L = \sum_{i \in L} \gamma_i$.

For $i \in S(q, \bar{q}) \cup D(q, \bar{q})$, $k = 1, 2, ...m$ and $j = 1, 2, ...m$

$$\frac{\partial \gamma_{ik}}{\partial q_j} = 0$$

Similarly, for $i \in S(q, \bar{q}) \cup D(q, \bar{q})$, $k = 1, 2, ...m$ and $j = 1, 2, ...m$

$$\frac{\partial \gamma_{ik}}{\partial \bar{q}_j} = 0$$

Hence for $k = 1, 2, ...m$ and $j = 1, 2, ...m$,

$$\frac{\partial \Gamma_k}{\partial q_j} = \frac{\partial \sum_{i \in L} \gamma_{ik}}{\partial q_j} = \frac{\partial \Gamma_k^L}{\partial q_j}$$

Similarly for $k = 1, 2, ...m$ and $j = 1, 2, ...m$,

$$\frac{\partial \Gamma_k}{\partial \bar{q}_j} = \frac{\partial \sum_{i \in L} \gamma_{ik}}{\partial \bar{q}_j} = \frac{\partial \Gamma_k^L}{\partial \bar{q}_j}$$

Thus, $I - W$ can be rewritten as

$$\begin{pmatrix} 1 - \bar{F}_1{}' \frac{\partial \Gamma_1^L}{\partial \bar{q}_1} & -\bar{F}_1{}' \frac{\partial \Gamma_1^L}{\partial \bar{q}_2} & ... & -\bar{F}_1{}' \frac{\partial \Gamma_1^L}{\partial q_1} & ... & -\bar{F}_1{}' \frac{\partial \Gamma_1^L}{\partial q_m} \\ -\bar{F}_2{}' \frac{\partial \Gamma_2^L}{\partial \bar{q}_1} & 1 - \bar{F}_2{}' \frac{\partial \Gamma_2^L}{\partial \bar{q}_2} & ... & -\bar{F}_2{}' \frac{\partial \Gamma_2^L}{\partial q_2} & ... & -\bar{F}_2{}' \frac{\partial \Gamma_1^L}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_1{}' \frac{\partial \Gamma_1^L}{\partial \bar{q}_1} & -F_2{}' \frac{\partial \Gamma_1^L}{\partial \bar{q}_2} & ... & 1 - F_1{}' \frac{\partial \Gamma_1^L}{\partial q_1} & ... & -F_1{}' \frac{\partial \Gamma_1^L}{\partial q_m} \\ ... & ... & ... & ... & ... & ... \\ -F_m{}' \frac{\partial \Gamma_m^L}{\partial \bar{q}_1} & -F_m{}' \frac{\partial \Gamma_m^L}{\partial \bar{q}_2} & ... & -F_m{}' \frac{\partial \Gamma_m^L}{\partial q_1} & ... & 1 - F_m{}' \frac{\partial \Gamma_m^L}{\partial q_m} \end{pmatrix}$$

By the liquidation condition (3.5),

$$\bar{q}^\top \Gamma^L + q^\top [I - \theta_{min} A](\sum_{i \in L} s_i - \Gamma^L) = \sum_{i \in L} h_i \tag{B.1}$$

Taking derivative of (B.1) with respect to $\bar{q}_k$ and rewriting, for $k = 1, 2, ..., m$, I have

$$\Gamma_k^L = -(\bar{q} - [I - \theta_{min} A]q)^\top (\nabla^{\bar{q}_k} \Gamma^L) \tag{B.2}$$

Similarly, taking derivative with respect to q_k, for $k = 1, 2, ..., m$, I have

$$(1 - \alpha_k \theta)(s_k - \Gamma_k^L) = -(\bar{q} - [I - \theta_{min} A]q)^\top (\nabla^{q_k} \Gamma^L) \tag{B.3}$$

From (iii) of Theorem 3.3.1,

$$\bar{F}(\Gamma)^\top \Gamma^L + F(\Gamma)^\top [I - \theta_{min} A](\sum_{i \in L} s_i - \Gamma^L) \text{ is increasing in } \Gamma.$$

Since I assume γ is strictly decreasing in $(q, \bar{q})$,

$$\bar{F}(\Gamma)^\top \Gamma^L + F(\Gamma)^\top [I - \theta_{min} A](\sum_{i \in L} s_i - \Gamma^L) \text{ is decreasing in}(q, \bar{q}).$$

Let $z_k \in \{q_k, \bar{q}_k\}$ for $k = 1, 2, ..m$.

Taking derivative of the above expression with respect to z_k and rearranging, for $k = 1, 2, ..., m$, I have

$$\begin{aligned}(\nabla^{z_k} \bar{F}(\Gamma))^\top \Gamma^L + (\nabla^{z_k} F(\Gamma))^\top [I - \theta_{min} A](\sum_{i \in L} s_i - \Gamma^L) \\ < -(\nabla^{z_k} \Gamma^L)^\top (\bar{F}(\Gamma) - [I - \theta_{min} A] F(\Gamma))\end{aligned}$$

Then at equilibrium,

$$(\nabla^{z_k} \bar{q})^\top \Gamma^L + (\nabla^{z_k} q)^\top [I - \theta_{min} A](\sum_{i \in L} s_i - \Gamma^L) < -(\nabla^{z_k} \Gamma^L)^\top (\bar{q} - [I - \theta_{min} A] q) \tag{B.4}$$

Now,

$$[(\Gamma^L)^\top \quad (\sum_{i \in L} s_i - \Gamma^L)^\top (I - \theta_{min} A)] \geq 0$$

Then applying (B.2),(B.3) and (B.4),

$$\begin{aligned}[(\Gamma^L)^\top \quad (\sum_{i \in L} s_i - \Gamma^L)^T (I - \theta_{min} A)] W &< -[(\frac{\partial \Gamma^L}{\partial \bar{q}_k})_{k=1}^m \quad (\frac{\partial \Gamma^L}{\partial q_k})_{k=1}^m]^\top (\bar{q} - (I - \theta_{min} A) q) \\ &= [(\Gamma^L)^\top \quad (\sum_{i \in L} s_i - \Gamma^L)^T (I - \theta_{min} A)]\end{aligned}$$

Then using Theorem 2.1 of [100], $(I - W)^{-1}$ exists and is given by $\sum_u W^u$.

$\square$

Appendix C

Proofs for Chapter 4

C.1 Proof of results in Section 4.3

Proof of Theorem 4.3.3. I will prove this result inductively. First consider time $t = 0$. Recall from Assumption 4.3.2 that $V(-1) \geq 0$. The clearing wealths at time 0 follow the fixed point equation

$$V(0) = \Phi(0, V(0)) := V(-1) + c(0) - A(0, V_0)^\top V(0)^-.$$

Note that, by construction, $A(0, V_0)^\top V(0)^- \leq L(0)^\top \vec{1}$. Therefore any clearing solution must fall within the compact range $[V(-1) + c(0) - L(0)^\top \vec{1}, V(-1) + c(0)] \subseteq \mathbb{R}^{n+1}$. It is clear from the definition that $\Phi(0, \cdot)$ is a monotonic operator, and thus there exists a greatest and least clearing solution $V^\uparrow(0) \geq V^\downarrow(0)$ by Tarski's fixed point theorem, both of which must fall within this domain. Further, $a_{ij}(0, V_0) = \frac{L_{ij}}{\sum_{k \in \mathcal{N}_0} L_{ik}}$ (for $i \in \mathcal{N}$ and $j \in \mathcal{N}_0$) for any wealth $V(0)$ in this domain since $V(-1) + c(0) - L(0)^\top \vec{1} \geq -L(0)\vec{1} = -\bar{p}(0, V_{-1})$. I will prove uniqueness as it is done in [45] by noting additionally that I can assume that the societal node will always have positive equity (i.e., $V^\downarrow(0) \geq 0$). First, I will show that the positive equities are the same for every firm no matter which clearing solution is chosen, i.e., $V_i^\uparrow(0)^+ = V_i^\downarrow(0)^+$ for every firm $i \in \mathcal{N}_0$. By definition $V^\uparrow(0) \geq V^\downarrow(0)$ and using $\sum_{j \in \mathcal{N}_0} a_{ij}(0) = 1$ for every firm $i \in \mathcal{N}_0$ I recover

$$\begin{aligned}
\sum_{i \in \mathcal{N}_0} V_i^\uparrow(0)^+ &= \sum_{i \in \mathcal{N}_0} \left[V_i^\uparrow(0) + V_i^\uparrow(0)^- \right] \\
&= \sum_{i \in \mathcal{N}_0} \left[V_i(-1) + c_i(0) - \sum_{j \in \mathcal{N}} a_{ji}(0, V_0^\uparrow) V_j^\uparrow(0)^- + V_i^\uparrow(0)^- \right]
\end{aligned}$$

$$= \sum_{i \in \mathcal{N}_0} [V_i(-1) + c_i(0)] - \sum_{j \in \mathcal{N}} V_j^{\uparrow}(0)^- \sum_{i \in \mathcal{N}_0} a_{ji}(0, V_0^{\uparrow}) + \sum_{i \in \mathcal{N}_0} V_i^{\uparrow}(0)^-$$
$$= \sum_{i \in \mathcal{N}_0} [V_i(-1) + c_i(0)] = \sum_{i \in \mathcal{N}_0} V_i^{\downarrow}(0)^+.$$

Therefore it must be the case that $V_i^{\uparrow}(0)^+ = V_i^{\downarrow}(0)^+$ for all firms $i \in \mathcal{N}_0$. Since I assume that the societal node will always have positive equity, it must be the case that $V_0^{\uparrow}(0) = V_0^{\downarrow}(0)$. Now since I assume that each node $i \in \mathcal{N}$ owes to the societal node, if any firm $i \in \mathcal{N}$ is such that $0 \geq V_i^{\uparrow}(0) > V_i^{\downarrow}(0)$ then it must be that $V_0^{\uparrow}(0) > V_0^{\downarrow}(0)$, which is a contraction.

Continuing with the inductive argument, assume that the history of clearing wealths V_{t-1} up to time $t-1$ is fixed and known. The clearing wealths at time t follow the fixed point equation

$$V(t) = \Phi(t, V(t)) := V(t-1) + c(t) - A(t, V_t)^\top V(t)^- + A(t-1, V_{t-1})^\top V(t-1)^-.$$

Note that, by construction, $A(t, V_t)^\top V(t)^- \leq L(t)^\top \vec{1} + A(t-1, V_{t-1})^\top V(t-1)^-$. Therefore any clearing solution must fall within the compact range $[V(t-1) + c(t) - L(t)^\top \vec{1}, V(t-1) + c(t) + A(t-1, V_{t-1})^\top V(t-1)^-] \subseteq \mathbb{R}^{n+1}$. Further, $a_{ij}(t, V_t) = \frac{L_{ij} + a_{ij}(t-1, V_{t-1}) V_i(t-1)^-}{\sum_{k \in \mathcal{N}_0} L_{ik} + V_i(t-1)^-}$ (for $i \in \mathcal{N}$ and $j \in \mathcal{N}_0$) for any wealth $V(t)$ in this domain since $V(t-1) + c(t) - L(t)^\top \vec{1} \geq -V(t-1)^- - L(t)\vec{1} = -\bar{p}(t, V_{t-1})$. Thus I can apply the same logic as in the time 0 case to recover existence and uniqueness of the clearing wealths $V(t)$ at time t. □

Proof of Corollary 4.3.5. This follows immediately from the proof of Theorem 4.3.3 using induction and noting that the lattice upper and lower bounds for the domain and range spaces of $\Phi(s, \cdot)$ are subsets of $\mathcal{L}_s^p(\mathbb{R}^{n+1})$. Therefore any clearing solution $V(t)$ is bounded above and below by an element of $\mathcal{L}_t^p(\mathbb{R}^{n+1})$ and the result is proven. □

C.2 Proof of results in Section 4.4

Proof of Corollary 4.4.2. Existence and uniqueness of the clearing solutions follows from Theorem 4.3.3. To prove continuity I will employ an induction argument. To do so, I will consider the reduced domain $V : \mathbb{T} \times [\epsilon, \infty) \to \mathbb{R}^{n+1}$ for some $\epsilon > 0$. That is, I restrict the step-size $\Delta t \geq \epsilon$. As I will demonstrate that the continuity argument holds for any $\epsilon > 0$ then the desired result must hold as well. Before continuing, consider an expanded version of the recursive formulation of (4.7), i.e.,

$$V(t, \Delta t) = V(-1) + \int_0^t dc(s) - A(t, \Delta t, V_t(\Delta t))^\top V(t, \Delta t)^- \tag{C.1}$$

for all times $t \in \mathbb{T}$. Fix the minimal step-size $\epsilon > 0$. Note that the relative exposures satisfy $a_{ij}(t, \Delta t, V_t(\Delta t)) := \frac{\int_0^t dL_{ij}(s)}{\sum_{k \in \mathcal{N}_0} \int_0^t dL_{ik}(s)}$ for any time $t \in [0, \epsilon)$ by the assumption that $V(-1) \geq 0$. Thus I can conclude $V : [0, \epsilon) \times [\epsilon, \infty) \to \mathbb{R}^{n+1}$ is continuous by an application of [56, Proposition A.2]. Now, by way of induction, assume that $V : [0, s) \times [\epsilon, \infty) \to \mathbb{R}^{n+1}$ is continuous for some $s > 0$. Again, by [56, Proposition A.2], I am able to immediately conclude that $V : [0, s + \epsilon) \cap \mathbb{T} \times [\epsilon, \infty) \to \mathbb{R}^{n+1}$ is continuous. As I am able to always extend the continuity result by $\epsilon > 0$ in time, the result is proven. □

Proof of Proposition 4.4.4. (i) Consider firm $i \in \mathcal{N}$. By assumption I have that $a_{ij}(t)$ for $t \nearrow \tau$ solves the first order differential equation:

$$\frac{da_{ij}(t)}{dt} + \frac{\sum_{k \in \mathcal{N}_0} dL_{ik}(t)/dt}{V_i(t)^-} a_{ij}(t) = \frac{dL_{ij}(t)/dt}{V_i(t)^-}.$$

For sake of simplicity, let this differential equation start at time 0 with $V_i(0) < 0$ and some initial value $a_{ij}(0)$. Then this differential equation can be solved via the integrating factor $\nu(t) := \int_0^t \frac{\sum_{k \in \mathcal{N}_0} dL_{ik}(s)}{V_i(s)^-} ds$. Thus for $t \nearrow \tau$ it follow that

$$a_{ij}(t) = e^{-\nu(t)} \left[\int_0^t e^{\nu(s)} \frac{dL_{ij}(s)}{V_i(s)^-} + a_{ij}(0) \right].$$

Therefore, utilizing L'Hôspital's rule,

$$\begin{aligned}
\lim_{t \nearrow \tau} a_{ij}(t) &= \lim_{t \nearrow \tau} e^{-\nu(t)} \left[\int_0^t e^{\nu(s)} \frac{dL_{ij}(s)}{V_i(s)^-} + a_{ij}(0) \right] \\
&= \lim_{t \nearrow \tau} \frac{e^{\nu(t)} \frac{dL_{ij}(t)}{V_i(t)^-}}{e^{\nu(t)} \frac{d}{dt} \nu(t)} = \lim_{t \nearrow \tau} \frac{dL_{ij}(t)/V_i(t)^-}{\sum_{k \in \mathcal{N}_0} dL_{ik}(t)/V_i(t)^-} = \frac{dL_{ij}(\tau)}{\sum_{k \in \mathcal{N}_0} dL_{ik}(\tau)}.
\end{aligned}$$

(ii) First, if $V_i(t) \geq 0$ then by construction (and the above result) it follows that $a_{ij}(t) = \frac{dL_{ij}(t)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(t)} \geq 0$ for any $i, j \in \mathcal{N}_0$ and $a_{i0}(t) \geq \delta$ by this construction. Consider now the case for $V_i(t) < 0$ and assume $a_{ij}(t) < 0$. Let $\tau = \sup\{s \leq t \mid V_i(s) = 0\}$. Since $a_{ij}(\tau) \in [0,1]$ by construction and the relative exposures are continuous, this implies there exists some time $s \in [\tau, t)$ such that $a_{ij}(s) = 0$. By the definition of the relative exposures, this must follow that $da_{ij}(s) \geq 0$ for any time $a_{ij}(s) \leq 0$ (with $da_{ij}(s) > 0$ if $a_{ij}(s) < 0$), thus $a_{ij}(t) < 0$ can never be reached. Further, assume $a_{i0}(t) < \delta$. By Assumption 4.4.1, if $a_{i0}(s) \leq \frac{dL_{i0}(s)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(s)}$ then $da_{i0}(s) \geq 0$. In particular, if $a_{i0}(s) \leq \delta$ then $da_{i0}(s) \geq 0$ (with $da_{i0}(s) > 0$ if $a_{i0}(s) < \delta$). Thus, by the same contradiction found in the case for $j \in \mathcal{N}$, I am able to bound $a_{i0}(t) \geq \delta$.

(iii) First, if $i = 0$ then $\sum_{j\in\mathcal{N}_0} a_{0j}(t) = 1$ by property that $a_{0j}(t) = \frac{1}{n}1_{\{j\neq 0\}}$ for all times t. Now consider $i \in \mathcal{N}$, if $V_i(t) \geq 0$ then by construction (and the above result) it follows that $\sum_{j\in\mathcal{N}_0} a_{ij}(t) = \sum_{j\in\mathcal{N}_0} \frac{dL_{ij}(t)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(t)} = 1$. Consider now the case for $V_i(t) < 0$ and let $\tau = \sup\{s \leq t \mid V_i(s) = 0\}$. Since $\sum_{j\in\mathcal{N}_0} a_{ij}(\tau) = 1$ by prior results, I will assume that $\sum_{j\in\mathcal{N}_0} a_{ij}(t) = 1$ to deduce

$$\sum_{j\in\mathcal{N}_0} da_{ij}(t) = \sum_{j\in\mathcal{N}_0} \frac{dL_{ij}(t) - a_{ij}(t)\sum_{k\in\mathcal{N}_0} dL_{ik}(t)}{V_i(t)^-}$$
$$= \frac{\sum_{j\in\mathcal{N}_0} dL_{ij}(t)}{V_i(t)^-} - \frac{\left(\sum_{j\in\mathcal{N}_0} a_{ij}(t)\right)\left(\sum_{k\in\mathcal{N}_0} dL_{ik}(t)\right)}{V_i(t)^-} = 0.$$

Therefore based on the initial conditions, $a_{ij}(t)$ must evolve so that it maintains the constant row sum of 1.

□

Proof of Theorem 4.4.5. Recall that the initial values to the Eisenberg-Noe differential system are $V_i(0) > 0$ and $a_{ij}(0) = \frac{dL_{ij}(0)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(0)} 1_{\{i\neq 0\}} + \frac{1}{n} 1_{\{i=0,\, j\neq 0\}}$ for all banks $i, j \in \mathcal{N}_0$. For ease of notation, consider $\tau_0 := 0$ and recursively define the stopping times

$$\tau_{m+1} := \inf\{t \in (\tau_m, T] \mid V_i(\tau_m)V_i(t) < 0 \text{ or } [V_i(\tau_m) = 0,\ dV_i(\tau_m)V_i(t) < 0]\}.$$

That is, $\tau_m \in \mathbb{T}$ is the time of the mth change in $\Lambda(V)$. Without loss of generality, I will assume that $\tau_m = T$ if the infimum is taken over an empty set. I note that the times τ_m are all stopping times with respect to the natural filtration.

With these times, note that in particular, on the interval $(\tau_m, \tau_{m+1}]$ I can consider the set of distressed banks to be constant; to simplify, and slightly abuse, notation I can thus consider a constant matrix of distressed firms $\Lambda(\tau_m)$ in the interval $(\tau_m, \tau_{m+1}]$. I will now construct the unique strong solution forward in time over these time intervals, noting that I update Λ and τ_{m+1} once the next event is found.

First, by construction, on $[0, \tau_1]$ there exists a unique solution to the differential system provided by $V(t) = V(0) + c(t)$ and $a_{ij}(t) = \frac{dL_{ij}(t)}{\sum_{k\in\mathcal{N}_0} dL_{ik}(t)} 1_{\{i\neq 0\}} + \frac{1}{n} 1_{\{i=0,\, j\neq 0\}}$ for all banks $i, j \in \mathcal{N}_0$. Assume there exists a strong solution in the time interval $[0, \tau_m]$ for $\tau_m < T$. Now I want to prove the existence and uniqueness for the clearing wealths and relative exposures on the interval $(\tau_m, \tau_{m+1}]$. Expanding $dc(t)$ based on its differential form allows me to consider (4.12) as

$$\begin{aligned} dV(t) &= [I - A(t)^\top \Lambda(\tau_m)]^{-1}(\mu(t, c(t)) - [\dot{L}(t)^\top - A(t)^\top \dot{L}(t)]\vec{1})dt \\ &\quad + [I - A(t)^\top \Lambda(\tau_m)]^{-1}\sigma(t, c(t))dW(t) \\ &= \bar{\mu}(t, c(t), A(t), V(t))dt + \bar{\sigma}(t, c(t), A(t), V(t))dW(t). \end{aligned}$$

Let me first consider the linear growth condition for dV. Utilizing the 1-norm and where $\|\cdot\|_1^{op}$ denotes the corresponding operator norm, let $A \in \mathbb{A}$ and $V \in \mathbb{R}^{n+1}$, then

$$\begin{aligned} &\|\bar{\mu}(t, c, A, V)\|_1 + \|\bar{\sigma}(t, c, A, V)\|_1^{op} \\ &\quad \leq \|(I - A^\top \Lambda(\tau_m))^{-1}\|_1^{op} \left(\|\mu(t, c)\|_1 + \|[\dot{L}(t)^\top - A^\top \dot{L}(t)]\vec{1}\|_1 + \|\sigma(t, c)\|_1^{op}\right) \\ &\quad \leq \sum_{k=0}^{\infty} \|[A^\top \Lambda(\tau_m)]^k\|_1^{op} \left(\|\mu(t, c)\|_1 + \|\dot{L}(t)^\top \vec{1}\|_1 + \|A^\top \dot{L}(t)\vec{1}\|_1 + \|\sigma(t, c)\|_1^{op}\right) \\ &\quad \leq \left(1 + \sum_{k=1}^{\infty} (1-\delta)^{k-1}\right) \left(\|\mu(t, c)\|_1 + \|\dot{L}(t)^\top \vec{1}\|_1 + \|A^\top\|_1^{op} \|\dot{L}(t)\vec{1}\|_1 + \|\sigma(t, c)\|_1^{op}\right) \\ &\quad \leq \left(1 + \frac{1}{\delta}\right) \left(\|\mu(t, c)\|_1 + \|[\dot{L}(t)^\top \vec{1}\|_1 + \|\dot{L}(t)\vec{1}\|_1 + \|\sigma(t, c)\|_1^{op}\right) \\ &\quad \leq \frac{1+\delta}{\delta} \sup_{s \in [\tau_m, \tau_{m+1}]} \left(\|\mu(s, c)\|_1 + \|\dot{L}(s)^\top \vec{1}\|_1 + \|\dot{L}(s)\vec{1}\|_1 + \|\sigma(s, c)\|_1^{op}\right) \end{aligned}$$

$$\leq \theta(1 + \|c\|_1)$$

The second line follows from the triangle inequality and definition of the operator norm. The third line is a result of Proposition C.2.1 and further use of the triangle inequality. The fourth line follows from Proposition 4.4.4 and noting that, by assumption, $\Lambda_{00} = 0$. The upper bound $\theta \geq 0$ can be determined by Assumption 4.4.1 and since all terms are continuous and being evaluated on a compact interval of time (since $\tau_{m+1} \leq T$ by definition). Further, I wish to prove $\bar{\mu} : \mathbb{T} \times \mathbb{R}^{n+1} \times \mathbb{A} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $\bar{\sigma} : \mathbb{T} \times \mathbb{R}^{n+1} \times \mathbb{A} \times \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)\times(n+1)}$ are jointly locally Lipschitz in (c, A, V). First $(c, A, V) \in \mathbb{R}^{n+1} \times \mathbb{A} \times \mathbb{R}^{n+1} \mapsto \mu(t, c) - [\dot{L}(t)^\top - A^\top \dot{L}(t)]\vec{1}$ and $(c, A, V) \in \mathbb{R}^{n+1} \times \mathbb{A} \times \mathbb{R}^{n+1} \mapsto \sigma(t, c)$ are Lipschitz continuous by their linear (or constant) forms with Lipschitz constants that can be taken independently of time (via continuity and the compact time domain) as well as the definitions of μ and σ. It remains to show that $(c, A, V) \in \mathbb{R}^{n+1} \times \mathbb{A} \times \mathbb{R}^{n+1} \mapsto (I - A^\top \Lambda(\tau_m))^{-1}$ is Lipschitz continuous. Let $A, B \in \mathbb{A}$, then by the same argument as above on the bounds of the norm of the matrix inverse,

$$\begin{aligned}
&\|(I - A^\top \Lambda(\tau_m))^{-1} - (I - B^\top \Lambda(\tau_m))^{-1}\|_1^{op} \\
&\qquad = \|(I - A^\top \Lambda(\tau_m))^{-1}[(I - B^\top \Lambda(\tau_m)) - (I - A^\top \Lambda(\tau_m))](I - B^\top \Lambda(\tau_m))^{-1}\|_1^{op} \\
&\qquad = \|(I - A^\top \Lambda(\tau_m))^{-1}[A - B]^\top \Lambda(\tau_m)(I - B^\top \Lambda(\tau_m))^{-1}\|_1^{op} \\
&\qquad \leq \|(I - A^\top \Lambda(\tau_m))^{-1}\|_1^{op} \|(I - B^\top \Lambda(\tau_m))^{-1}\|_1^{op} \|\Lambda(\tau_m)\|_1^{op} \|[A - B]^\top\|_1^{op} \\
&\qquad \leq \left(\frac{1 + \delta}{\delta}\right)^2 \|\Lambda(\tau_m)\|_1^{op} \|A - B\|_\infty^{op} \\
&\qquad \leq n \left(\frac{1 + \delta}{\delta}\right)^2 \|\Lambda(\tau_m)\|_1^{op} \|A - B\|_1^{op}.
\end{aligned}$$

Thus $\bar{\mu}$ and $\bar{\sigma}$ are appropriately locally Lipschitz continuous on $[\tau_m, \tau_{m+1}]$.

Now I wish to consider the differential form for the relative exposures matrix (4.13). First, if $\Lambda_{ii}(\tau_m) = 0$ (and in particular, $\Lambda_{00}(\tau_m) = 0$ by assumption of the societal node) then $a_{ij}(t) = \frac{dL_{ij}(t)}{\sum_{k \in \mathcal{N}_0} dL_{ik}(t)} 1_{\{i \neq 0\}} + \frac{1}{n} 1_{\{i=0,\, j \neq 0\}}$ is the unique solution for any firm $j \in \mathcal{N}_0$ over all times $t \in (\tau_m, \tau_{m+1}]$. In particular, this is independent of the evolution of the wealths V, so I need only consider the joint differential equation between the wealths V and the relative exposures a_{ij} where bank i is in distress between times τ_m and τ_{m+1}, i.e., $\Lambda_{ii}(\tau_m) = 1$. Consider bank $i \in \mathcal{N}$ with $\Lambda_{ii}(\tau_m) = 1$. Therefore by construction $V_i(t) < 0$ for all

$t \in (\tau_m, \tau_{m+1})$. If $V_i(\tau_{m+1}) = 0$ then from Proposition 4.4.4, it already follows that the unique solution $a_{ij}(\tau_{m+1}) = \frac{dL_{ij}(\tau_{m+1})}{\sum_{k \in \mathcal{N}_0} dL_{ik}(\tau_{m+1})}$ must hold, otherwise I can extend $V_i(t) < 0$ for $t \in (\tau_m, \tau_{m+1}]$. The differential form for all relative exposures (4.13) on the interval $(\tau_m, \tau_{m+1}]$ is provided by $da_{ij}(t) = \frac{dL_{ij}(t) - a_{ij}(t) \sum_{k \in \mathcal{N}_0} dL_{ik}(t)}{V_i(t)^-}$. By construction $(a_{ij}, V_i) \in [0,1] \times -\mathbb{R}_{++} \mapsto \frac{\dot{L}_{ij}(t) - a_{ij} \sum_{k \in \mathcal{N}_0} \dot{L}_{ik}(t)}{-V_i}$ is locally Lipschitz and satisfies a local linear growth condition (with constants bounded independent of time as above utilizing continuity of the parameters and the compact time domain).

Combining my results for the joint differential system for the cash flows c, clearing wealths V from (4.12), and relative exposures A from (4.13), I find that this system satisfies a joint local linear growth and local Lipschitz property on the interval $(\tau_m, \tau_{m+1}]$. Therefore, there exists some $\epsilon \in \mathcal{L}_T^\infty(\mathbb{R}_{++})$ (such that $\tau_m + \epsilon$ is a stopping time) for which a strong solution for $(c, V, A) : [\tau_m, \tau_m + \epsilon] \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{A}$ exists and is unique. Using the same logic with local properties, I can continue my unique strong solution sequentially. This can be continued until the stopping time τ_{m+1} is reached (found along the path of (c, V, A) as a stopping time) or this process reaches some maximal time $T^* < \tau_{m+1}$ for which a unique strong solution exists on the time interval $[\tau_m, T^*)$. First, as $c(t)$ can be calculated separately from the clearing wealths and relative exposures, I can immediately determine that $c(T^*) = \lim_{t \nearrow T^*} c(t)$ exists. Further, I note that any solution $V(t)$ must, almost surely, exist in the (almost surely) compact space

$$\left[V(\tau_m) - \left(I + \frac{1+\delta}{\delta}\mathbb{1}\right)\left(\int_{\tau_m}^t dc(s)^- + (L(t) - L(\tau_m))\,\vec{1}\right), V(\tau_m) + c(t) - c(\tau_m)\right] \subseteq \mathcal{L}_t^2(\mathbb{R}^{n+1})$$

where $\mathbb{1} = \{1\}^{(n+1)\times(n+1)}$. The lower bound is determined to be based on the bounding of the Leontief inverse; the upper bound follows from the continuous-time version of (C.1), i.e.,

$$V(t) = V(0) + c(t) - A(t)^\top V(t)^-.$$

Additionally, $a_{ij}(t)$ almost surely exists in the compact neighborhood $[0,1]$ by definition. Therefore $(V(T^*), A(T^*)) = \lim_{t \nearrow T^*}(V(t), A(t))$ exists by continuity of the solutions and compactness of the range space. Thus I can continue the differential equation from time T^* with values $(c(T^*), V(T^*), A(T^*))$ which contradicts the nature that T^* is the maximal time. Notably, if $V_i(T^*) = 0$ for some bank i then it is imperative to check if $\tau_{m+1} = T^*$ to update the set of distressed banks Λ.

Therefore, by induction, there exists a unique strong solution (V, A) to (4.12) and (4.13) on the domain $[0, \tau_m]$ for any index $m \in \mathbb{N}$ by use of [91, Theorem 5.2.1]. In particular this holds up to $\tau^* = \sup_{m\in\mathbb{N}} \tau_m$. If $\tau^* \geq T$ then the proof is complete. If $\tau^* < T$, then by the same argument as above I can find $(V(\tau^*), A(\tau^*))$ as I can bound both the wealths and relative exposures into an almost surely compact neighborhood (and a subset of $\mathcal{L}^2_{\tau^*}(\mathbb{R}^{n+1})$). Therefore, as before, I can start the process again at time τ^*, which contradicts the terminal nature of τ^*. This concludes the proof. □

Proposition C.2.1. *For any relative exposure matrix $A \in \mathbb{A}$ and any distress matrix $\Lambda \in \{0,1\}^{(n+1)\times(n+1)}$ such that $\Lambda_{00} = 0$ and $\Lambda_{ij} = 0$ for $i \neq j$, the matrix $I - A^\top\Lambda$ is invertible with Leontief form, i.e., $(I - A^\top\Lambda)^{-1} = \sum_{k=0}^{\infty}(A^\top\Lambda)^k$.*

Proof. By inspection, for any $A \in \mathbb{A}$, $(I - A^\top\Lambda)(I + A^\top(I - \Lambda A^\top)^{-1}\Lambda) = I$, i.e., the form of the inverse is provided by $I + A^\top(I - \Lambda A^\top)^{-1}\Lambda$. I refer to [55, Theorem 2.6] for a detailed proof that $(I - \Lambda A^\top)^{-1}$ is nonsingular and is provided by the Leontief inverse. Therefore, by construction

$$
\begin{aligned}
(I - A^\top\Lambda)^{-1} &= (I + A^\top(I - \Lambda A^\top)^{-1}\Lambda) = I + A^\top\left(\sum_{k=0}^{\infty}(\Lambda A^\top)^k\right)\Lambda \\
&= I + \sum_{k=0}^{\infty} A^\top\Lambda(A^\top)^k\Lambda^k = I + \sum_{k=0}^{\infty}(A^\top\Lambda)^{k+1} = \sum_{k=0}^{\infty}(A^\top\Lambda)^k.
\end{aligned}
$$

□

Appendix D

Proofs for Chapter 5

D.1 Proof of Proposition 5.3.17

Proof. First, as in (5.6), the clearing wealths as a function of initial endowments are defined by

$$V(x) = x + \Pi(V(x))^\top[\bar{p}(V(x)) - V(x)^-]^+ - \bar{p}(V(x)).$$

I will prove continuity by utilizing the closed graph theorem (see, e.g., [4, Theorem 2.58]) noting that Proposition 5.3.6 provides me with the condition that the clearing wealths map into a compact set. Theorem 4 of [87] immediately provides the monotonicity of the clearing wealths.

Fix $x \in \mathbb{R}^{n+1}_+$ and let $\mathcal{X} = x + [-1,1]^{n+1}$ be a closed compact neighborhood of x in the full Euclidean space $\mathbb{R}^{n+1}$. Then I can define $V^x : \mathcal{X} \to \mathbb{R}^{n+1}$ as the restriction (and possible expansion to negative terms) of the domain of V to $\mathcal{X}$. The graph of V^x is given by:

$$\operatorname{graph} V^x = \left\{(\hat{x}, \hat{V}) \in \mathcal{X} \times \prod_{i \in \mathcal{N}_0} [x_i - 1 - \sum_{j \in \mathcal{N}_0} \bar{L}_{ij}, x_i + 1 + \sum_{j \in \mathcal{N}} \bar{L}_{ji}] \mid \hat{V} = \hat{x} + \Pi(\hat{V})^\top[\bar{p}(\hat{V}) - \hat{V}^-]^+ - \bar{p}(\hat{V})\right\}$$

To see that $\operatorname{graph} V^x$ is closed let $(\hat{x}^k, \hat{V}^k)_{k \in \mathbb{N}} \subseteq \operatorname{graph} V^x \to (\hat{x}, \hat{V})$, then immediately

$$\hat{V} = \lim_{k\to\infty} \hat{V}^k = \lim_{k\to\infty} \left[\hat{x}^k + \Pi(\hat{V}^k)^\top[\bar{p}(\hat{V}^k) - (\hat{V}^k)^-]^+ - \bar{p}(\hat{V}^k)\right] = \hat{x} + \Pi(\hat{V})^\top[\bar{p}(\hat{V}) - \hat{V}^-]^+ - \bar{p}(\hat{V})$$

by continuity of the nominal liabilities matrix L. Therefore by the closed graph theorem I immediately recover that V^x is continuous for any $x \in \mathbb{R}_+^{n+1}$, which implies that V is continuous at any x as well and thus $V : \mathbb{R}_+^{n+1} \to \mathbb{R}^{n+1}$ is a continuous mapping. $\square$

References

[1] Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. Systemic risk and stability in financial networks. NBER Working Papers 18727, National Bureau of Economic Research, 2013.

[2] Viral Acharya and Alberto Bisin. Counterparty risk externality: Centralized versus over-the-counter markets. *Journal of Economic Theory*, 149:153–182, 2014.

[3] Tobias Adrian and Huyn Shin. Liquidity and leverage. *Federal Reserve Bank of New York Staff Reports*, 2010.

[4] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, 2007.

[5] Hamed Amini, Rama Cont, and Andreea Minca. Resilience to contagion in financial networks. *Mathematical Finance*, 26(2):329–365, 2016.

[6] Hamed Amini, Damir Filipović, and Andreea Minca. Uniqueness of equilibrium in a payment system with liquidation costs. *Operations Research Letters*, 44(1):1–5, 2016.

[7] Hamed Amini, Damir Filipović, and Andreea Minca. Systemic risk with central counterparty clearing. Swiss Finance Institute Research Paper No. 13-34, Swiss Finance Institute, 2017.

[8] Hamed Amini and Andreea Minca. Inhomogeneous financial networks and contagious links. *Operations Research*, 64(5):1109–1120, 2016.

[9] Kartik Anand, Guillaume Bédard-Pagé, and Virginie Traclet. Stress testing the Canadian banking system: A system-wide approach. *Bank of Canada Financial Stability Review*, 2014.

[10] Kartik Anand, Ben Craig, and Goetz Von Peter. Filling in the blanks: Network structure and interbank contagion. *Quantitative Finance*, 15(4):625–636, 2015.

[11] Yannick Armenti, Stephane Crepey, Samuel Drapeau, and Antonis Papapantoleon. Multivariate shortfall risk allocation and systemic risk. *SIAM Journal on Financial Mathematics*, 9(1):90–126, 2018.

[12] Tathagata Banerjee, Alex Bernstein, and Zachary Feinstein. Dynamic clearing and contagion in financial networks. 2018. Preprint. https://arxiv.org/abs/1801.02091.

[13] Tathagata Banerjee and Zachary Feinstein. Pricing of debt and equity in a financial network with comonotonic endowments. 2018. Preprint. https://arxiv.org/abs/1810.01372.

[14] Tathagata Banerjee and Zachary Feinstein. Impact of contingent payments on systemic risk in financial networks. *Mathematics and Financial Economics*, 2019. https://doi.org/10.1007/s11579-019-00239-9.

[15] Tathagata Banerjee and Zachary Feinstein. Price-mediated contagion through capital ratio requirements. 2019. Preprint.

[16] Paolo Barucca, Marco Bardoscia, Fabio Caccioli, Marco D'Errico, Gabriele Visentin, Stefano Battiston, and Guido Caldarelli. Network valuation in financial systems. 2016. Working paper.

[17] Stefano Battiston, Guido Caldarelli, and Michelangelo Puliga. Credit default swaps networks and systemic risk. *Scientific Reports*, 4, 2014.

[18] Stefano Battiston, Sven Seuken, and Steffen Schuldenzucker. Finding Clearing Payments in Financial Networks with Credit Default Swaps is PPAD-complete. In Christos H. Papadimitriou, editor, *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, volume 67 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 32:1–32:20, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[19] Stefano Battiston, Sven Seuken, and Steffen Schuldenzucker. Default ambiguity: Credit default swaps create new systemic risks in financial networks. 2019. Working paper.

[20] Francesca Biagini, Jean-Pierre Fouque, Marco Frittelli, and Thilo Meyer-Brandis. A unified approach to systemic risk measures via acceptance sets. *Mathematical Finance*, 2018. DOI: 10.1111/mafi.12170.

[21] Maxim Bichuch, Agostino Capponi, and Stephan Sturm. Arbitrage-free XVA. *Mathematical Finance*, 28(2):582–620, 2018.

[22] Maxim Bichuch, Agostino Capponi, and Stephan Sturm. Robust XVA. 2019. Working Paper.

[23] Maxim Bichuch and Zachary Feinstein. Optimization of fire sales and borrowing in systemic risk. *SIAM Journal on Financial Mathematics*, 10:68–88, 2019.

[24] Jose Blanchet and Yixi Shi. Stochastic risk networks: Modeling, analysis and efficient monte carlo. 2012. Working paper.

[25] Michael Boss, Helmut Elsinger, Martin Summer, and Stefan Thurner. Network topology of the interbank market. *Quantitative Finance*, 4(6):677–684, 2004.

[26] Yann Braouezec and Lakshithe Wagalath. Strategic fire-sales and price-mediated contagion in the banking system. *European Journal of Operational Research*, 274(3):1180–1197, 2019.

[27] Agostino Capponi and Peng-Chu Chen. Systemic risk mitigation in financial networks. *Journal of Economic Dynamics and Control*, 58:152–166, 2015.

[28] Agostino Capponi, Peng-Chu Chen, and David D. Yao. Liability concentration and systemic losses in financial networks. *Operations Research*, 64(5):1121–1134, 2016.

[29] René Carmona, Jean-Pierre Fouque, Seyyed Mostafa Mousavi, and Li-Hsien Sun. Systemic risk and stochastic games with delay. *Journal of Optimization Theory and Applications*, Mar 2018.

[30] René Carmona, Jean-Pierre Fouque, and Li-Hsien Sun. Mean field games and systemic risk. *Communications in Mathematical Sciences*, 13(4):911–933, 2015.

[31] Chen Chen, Garud Iyengar, and Ciamac C. Moallemi. An axiomatic approach to systemic risk. *Management Science*, 59(6):1373–1388, 2013.

[32] Nan Chen, Xin Liu, and David D. Yao. An optimization view of financial systemic risk modeling: The network effect and the market liquidity effect. *Operations Research*, 64(5), 2016.

[33] Rodrigo Cifuentes, Hyun Song Shin, and Gianluigi Ferrucci. Liquidity risk and contagion. *Journal of the European Economic Association*, 3(2-3):556–566, 2005.

[34] Rama Cont. The end of the waterfall: Default resources of central counterparties. *Journal of Risk Management in Financial Institutions*, 8(4):365–389, 2015.

[35] Rama Cont and David-Antoine Fournié. Functional Itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, 41(1):109–133, 2013.

[36] Rama Cont and Andreea Minca. Credit default swaps and systemic risk. *Annals of Operations Research*, 247(2):523–547, 2016.

[37] Rama Cont, Amal Moussa, and Edson Bastos e Santos. Network structure and systemic risk in banking systems. In *Handbook on Systemic Risk*, pages 327–368. Cambridge University Press, 2013.

[38] Rama Cont and Eric Schaanning. Fire sales, indirect contagion and systemic stress testing. 2017. Norges Bank Working paper.

[39] Rama Cont and Lakshithe Wagalath. Fire sale forensics: measuring endogenous risk. *Mathematical Finance*, 26(4):835–866, 2016.

[40] Hans Degryse and Nguyen Gregory. Interbank exposures: An empirical examination of contagion risk in the Belgian banking system. *International Journal of Central Banking*, 3(2):123–171, 2007.

[41] Nils Detering, Thilo Meyer-Brandis, Konstantinos Panagiotou, and Daniel Ritter. Managing default contagion in inhomogeneous financial networks. 2019. Working paper.

[42] Douglas Diamond and Raghuram Rajan. Fear of fire sales, illiquidity seeking, and credit freezes. *The Quarterly Journal of Economics*, 126:557â591, 2011.

[43] Fernando Duarte and Thomas Eisenbach. Fire-sale spillovers and systemic risk. *Federal Reserve Bank of New York Staff Reports*, 2018.

[44] Darrel Duffie and Kenneth Singleton. Simulating correlated defaults. 1999. Working paper.

[45] Larry Eisenberg and Thomas H. Noe. Systemic risk in financial systems. *Management Science*, 47(2):236–249, 2001.

[46] Matthew Elliott, Benjamin Golub, and Matthew O. Jackson. Financial networks and contagion. *American Economic Review*, 104(10):3115–3153, 2014.

[47] Helmut Elsinger. Financial networks, cross holdings, and limited liability. *Österreichische Nationalbank (Austrian Central Bank)*, 156, 2009.

[48] Helmut Elsinger, Alfred Lehar, and Martin Summer. Risk assessment for banking systems. *Management Science*, 52(9):1301–1314, 2006.

[49] Helmut Elsinger, Alfred Lehar, and Martin Summer. Using market information for banking system risk assessment. *International Journal of Central Banking*, 2(1):137–165, 2006.

[50] Helmut Elsinger, Alfred Lehar, and Martin Summer. Network models and systemic risk assessment. In *Handbook on Systemic Risk*, pages 287–305. Cambridge University Press, 2013.

[51] Zachary Feinstein. Financial contagion and asset liquidation strategies. *Operations Research Letters*, 45(2):109–114, 2017.

[52] Zachary Feinstein. Obligations with physical delivery in a multi-layered financial network. 2018. Working paper.

[53] Zachary Feinstein. Capital regulation under price impacts and dynamic financial contagion. 2019. Working paper.

[54] Zachary Feinstein and Fatena El-Masri. The effects of leverage requirements and fire sales on financial contagion via asset liquidation strategies in financial networks. *Statistics and Risk Modeling*, 34(3-4):113–139, 2017.

[55] Zachary Feinstein, Weijie Pang, Birgit Rudloff, Eric Schaanning, Stephan Sturm, and Mackenzie Wildman. Sensitivity of the Eisenberg and Noe clearing vector to individual interbank liabilities. *SIAM Journal on Financial Mathematics*, 9(4):1286–1325, 2018.

[56] Zachary Feinstein, Birgit Rudloff, and Stefan Weber. Measures of systemic risk. *SIAM Journal on Financial Mathematics*, 8(1):672–708, 2017.

[57] Gerardo Ferrara, Sam Langfield, Zijun Liu, and Tomohiro Ota. Systemic illiquidity in the interbank network. Staff Working Paper 586, Bank of England, 2016.

[58] Tom Fischer. No-arbitrage pricing under systemic risk: Accounting for cross-ownership. *Mathematical Finance*, 24(1):97–124, 2014.

[59] Jean-Pierre Fouque and Tomoyuki Ichiba. Stability in a model of interbank lending. *SIAM Journal on Financial Mathematics*, 4:784–803, 2013.

[60] Jean-Pierre Fouque and Li-Hsien Sun. Systemic risk illustrated. In *Handbook on Systemic Risk*, pages 444–452. Cambridge University Press, 2013.

[61] Prasanna Gai, Andrew Haldane, and Sujit Kapadia. Complexity, concentration and contagion. *Journal of Monetary Economics*, 58(5):453–470, 2011.

[62] Prasanna Gai and Sujit Kapadia. Contagion in financial networks. Bank of England Working Papers 383, Bank of England, 2010.

[63] Axel Gandy and Luitgard A.M. Veraart. A Bayesian methodology for systemic risk assessment in financial networks. *Management Science*, 63(12):4428–4446, 2016.

[64] Axel Gandy and Luitgard A.M. Veraart. Adjustable network reconstruction with applications to CDS exposures. *Journal of Multivariate Analysis*, 172:193–209, 2019.

[65] Josselin Garnier, George Papanicolaou, and Tzu-Wei Yang. Diversification in financial networks may increase systemic risk. In *Handbook on Systemic Risk*, pages 432–443. Cambridge University Press, 2013.

[66] Josselin Garnier, George Papanicolaou, and Tzu-Wei Yang. Large deviations for a mean field model of systemic risk. *SIAM Journal on Financial Mathematics*, 4:151–184, 2013.

[67] Paul Glasserman and H. Peyton Young. How likely is contagion in financial networks? *Journal of Banking and Finance*, 50:383–399, 2015.

[68] Paul Glasserman and Peyton Young. Contagion in financial networks. *Journal of Economic Literature*, 54(3):779–831, 2016.

[69] Christian Gouriéroux, Jean-Cyprian Héam, and Alain Monfort. Bilateral exposures and systemic solvency risk. *Canadian Journal of Economics*, 45(4):1273–1309, 2012.

[70] Christian Gouriéroux, Jean-Cyprian Héam, and Alain Monfort. Liquidation equilibrium with seniority and hidden CDO. *Journal of Banking and Finance*, 37(12):5261–5274, 2013.

[71] Robin Greenwood, Augustin Landier, and David Thesmar. Vulnerable banks. *Journal of Financial Economics*, 115(3):3–28, 2015.

[72] Grzegorz Hałaj and Christoffer Kok. Assessing interbank contagion using simulated networks. *Computational Management Science*, 10(2-3):157–186, 2013.

[73] Grzegorz Hałaj and Christoffer Kok. Modelling the emergence of the interbank networks. *Quantitative Finance*, 15(4):653–671, 2015.

[74] Andreas H. Hamel and Carola Schrage. Directional derivatives, subdifferentials and optimality conditions for set-valued convex functions. *Pacific Journal of Optimization*, 10(4):667–689, 2014.

[75] Andrey Itkin and Alexander Lipton. Structural default model with mutual obligations. *Review of Derivatives Research*, 20(1):15–46, 2017.

[76] George G. Kaufman and Kenneth E. Scott. What is systemic risk, and do bank regulators retard or contribute to it? *The Independent Review*, 7:371–391, 2003.

[77] Amir Khandani and Andrew Lo. What happened to the quants in August 2007? Evidence from factors and transactions data. *Journal of Financial Markets*, 14(1):1–46, 2011.

[78] Ariah Klages-Mundt and Andreea Minca. Cascading losses in reinsurance networks. 2018. Working paper.

[79] Eduard Kromer, Ludger Overbeck, and Katrin Zilch. Systemic risk measures on general probability spaces. *Mathematical Methods of Operations Research*, 84(2):323–357, 2016.

[80] Michael Kusnetsov and Luitgard A.M. Veraart. Interbank clearing in financial networks with multiple maturities. *SIAM Journal on Financial Mathematics*, 10(1):37–67, 2019.

[81] Matt Leduc, Sebastian Poledna, and Stefan Thurner. Systemic risk management in financial networks with credit default swaps. *Journal of Network Theory in Finance*, 3(3):19–39, 2017.

[82] Ming Liu and Jeremy Staum. Sensitivity analysis of the Eisenberg-Noe model of contagion. *Operations Research Letters*, 35(5):489–491, 2010.

[83] Alberto Manconi, Massimo Massa, and Ayako Yasuda. The role of institutional investors in propagating the crisis of 2007-2008. *Journal of Financial Economics*, 104(3):491–518, 2012.

[84] Sheri M Markose, Simone Giansante, Mateusz Gatkowski, and Ali Rais Shaghaghi. Too interconnected to fail: Financial contagion and systemic risk in network model of CDS and other credit enhancement obligations of us banks. Technical Report DP 683, Economics Department, University of Essex, 2010.

[85] Alexander J. McNeil, Rüdiger Frey, and Paul Embrechts. *Quantitative risk management: Concepts, techniques, and tools*. Princeton University Press, 2015.

[86] Robert C. Merton. On the pricing of corporate debt: the risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.

[87] Paul Milgrom and John Roberts. Comparing equilibria. *American Economic Review*, 84(3):441–459, 1994.

[88] Paolo Emilio Mistrulli. Assessing financial contagion in the interbank market: Maximum entropy versus observed interbank lending patterns. *Journal of Banking and Finance*, 35(5):1114–1127, 2011.

[89] David Murphy. The systemic risk of otc derivatives central clearing. *Journal of Risk Management in Financial Institutions*, 5(3):319–334, 2012.

[90] Sergey Nadtochiy and Mykhaylo Shkolnikov. Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *Annals of Applied Probability*, 29(1):89–129, 2019.

[91] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. 6 edition, 2007.

[92] Mark E. Paddrik, Sriram Rajan, and Peyton Young. Contagion in the CDS market. OFR Working Paper 16-12, Office of Financial Research, 2016.

[93] Bezalel Peleg and Menahem E. Yaari. A price characterization of efficient random variables. *Econometrica*, 43(2):283–292, 1975.

[94] Sebastian Poledna, José Luis Molina-Borboa, Serafín Martínez-Jaramillo, Marco van der Leij, and Stefan Thurner. The multi-layer network nature of systemic risk and its implications for the costs of financial crises. *Journal of Financial Stability*, 20:70–81, 2015.

[95] John K.-H. Quah. The comparative statics of constrained optimization problems. *Econometrica*, 75(2):401–431, 2007.

[96] Leonard C.G. Rogers and Luitgard A.M. Veraart. Failure and rescue in an interbank network. *Management Science*, 59(4):882–898, 2013.

[97] Moshe Shaked and J. George Shanthikumar. *Stochastic Orders*. Springer Series in Statistics. Springer, 2007.

[98] Isaac M. Sonin and Konstantin Sonin. Banks as tanks: A continuous-time model of financial clearing. 2017. Working paper.

[99] Teruyoshi Suzuki. Valuing corporate debt: the effect of cross-holdings of stock and debt. *Journal of Operations Research*, 45(2):123–144, 2002.

[100] Thijs ten Raa. *The Economics of Input-Output Analysis*. Cambridge University Press, 2006.

[101] Christian Upper. Simulation methods to assess the danger of contagion in interbank markets. *Journal of Financial Stability*, 7(3):111–125, 2011.

[102] Christian Upper and Andreas Worms. Estimating bilateral exposures in the German interbank market: Is there a danger of contagion? *European Economic Review*, 48(4):827–849, 2004.

[103] Luitgard A.M. Veraart. Distress and default contagion in financial networks. 2017. Working paper.

[104] Stefan Weber and Kerstin Weske. The joint impact of bankruptcy costs, fire sales and cross-holdings on systemic risk in financial networks. *Probability, Uncertainty and Quantitative Risk*, 2(1):9, 2017.

[105] Simon J. Wells. Financial interlinkages in the United Kingdom's interbank market and the risk of contagion. 2004. Bank of England working paper.