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Wavelet Factorization and Related Polynomials

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Dedicated to my parents.

CHAPTER 1

Introduction

Our goal is exploring and better understanding factorizations of polyphase matrices for finite impulse response (FIR) filters. In particular, we focus on nearest neighbor factorizations discussed by Wickerhauser and Zhu [**19**] that allow for efficient implementation of the discrete wavelet transform (DWT) for the algorithms of Daubechies and Sweldens [**8**] and Mallat [**10**]. Nearest neighbor lifting is a specific form of the general lifting scheme that improves the lifting algorithm by optimizing the number of efficient memory accesses. Nearest neighbor lifting factorizations are typically generated by implementing the Euclidean algorithm for Laurent polynomials, which introduces multiple choices of factorizations of a polyphase matrix associated with a filter, and are the main focus of this work.

1.1. Filters and the Euclidean Algorithm

A filter h is a linear map $h : \ell^2 \to \ell^2$ which is characterized completely by its impulse response, $\{h_{k\in Z}\}$. We only consider real-valued finite impulse response (FIR) filters which correspond to only finitely many nonzero filter coefficients (also referred to as "taps"). Since we work primarily with the lifting algorithm and polyphase matrices, it will be convenient to represent a filter by its z-transform, the Laurent polynomial with coefficients equal to the impulse response,

$$h\left(z\right) = \sum_{k\in\mathbb{Z}} h_k z^{-k}.$$

Definition 1. The support of a Laurent polynomial $h\left(z\right) = \sum_{k\in\mathbb{Z}} h_k z^{-k}$ is

$$\mathit{support}\left(h\left(z\right)\right) = \left\{k \in \mathbb{Z} \mid m_1 \leq k \leq m_2\right\} = \left[m_1, m_2\right],$$

where

$$m_1 = \inf\{k \in \mathbb{Z} : \; h_k \neq 0\}$$

$$m_2 = \sup\{k \in \mathbb{Z} : \; h_k \neq 0\}.$$

DEFINITION 2. The degree of the Laurent polynomial $h(z) = \sum_{k \in \mathbb{Z}} h_k z^{-k}$ is

$$|h| = m_2 - m_1,$$

where *support* $(h(z)) = [m_1, m_2] \subseteq \mathbb{Z}$.

This definition differs from the notion of degree for traditional polynomials. For example, $|z^2 + 3z| = 1$. With this Laurent definition of degree, we are able to implement the Euclidean algorithm on the ring of Laurent polynomials.

LEMMA 3. *Let $A(z)$ and $B(z)$ be nonzero Laurent polynomials satisfying $|A(z)| \geq |B(z)|$. Then there exists a quotient polynomial $Q(z)$, and a remainder polynomial $R(z)$ with degree strictly less than $B(z)$, satisfying*

$$A(z) = B(z)\,Q(z) + R(z).$$

In contrast to traditional polynomial division, there are choices for which terms to cancel in Laurent polynomial division which result in different quotient and remainder polynomials. We focus on examples involving quotients of degree at most one, since we wish to work primarily with nearest neighbor factorizations which require this condition.

EXAMPLE 4. Let $A(z) = 9z + 12 + 6z^{-1}$ and $B(z) = 3z + 2$. Since $|A| = |B| + 1$, there are three choices for the first division in the Euclidean algorithm according to which terms of

$A(z)$ are eliminated. If the lowest two terms are eliminated, denoted $\{right - right\}$, then

$$A(z) = B(z)\,Q(z) + R(z)$$

$$\left(9z + 12 + 6z^{-1}\right) = (3z + 2)\left(\frac{3}{2} + 3z^{-1}\right) + \left(\frac{9}{2}z\right).$$

We check the degree of the remainder is less than the divisor, $|B(z)| = 1 > 0 = |R(z)|$. Comparing the remainder with $A(z)$, we see that indeed the two rightmost terms, those with the two lowest powers, have been eliminated. The other two choices for the division are eliminating the highest and lowest power from $A(z)$, denoted $\{sym\}$, and eliminating the two highest power terms, denoted $\{left - left\}$,

$$\left(9z + 12 + 6z^{-1}\right) = (3z + 2)\left(3 + 3z^{-1}\right) + (-3) \quad \{sym\}$$

$$\left(9z + 12 + 6z^{-1}\right) = (3z + 2)\left(3 + 2z^{-1}\right) + \left(2z^{-1}\right) \quad \{left - left\}.$$

Since the degree of the remainder polynomial is reduced at each step, we can implement the Euclidean algorithm on the ring of Laurent polynomials. The choice of which terms to eliminate at each division, however, influences the result.

Definition 5. A division scheme is the sequence of choices for the divisions at each step in the Euclidean algorithm for two Laurent polynomials.

In general, the Euclidean algorithm is implemented on Laurent polynomials A, B satisfying $|A| \geq |B| \geq 0$ by defining $a_0 = A$ and $b_0 = B$ and performing the following for $k = 0, 1, 2, \ldots$:

$$a_{k+1} = b_k$$

$$b_{k+1} = a_k - q_k b_k.$$

In the above equations q_k is any of the possible quotients from division. The Euclidean algorithm terminates when $b_{k+1} = 0$ by finding a greatest common divisor (GCD) b_k, of the starting Laurent polynomials, similar to the traditional polynomial case, as shown by the following lemma from [**19**].

LEMMA 6. *Let n be the smallest positive integer for which $b_n = 0$. Then $a_n \in \gcd(A, B)$.*

The greatest common divisor for two Laurent polynomials is unique only up to multiplication by a unit, which, in the ring of Laurent polynomials, is any degree 0 polynomial (nonzero monomial). The choice of division scheme will be the key to controlling the factorizations resulting from the Euclidean algorithm in the lifting algorithm.

1.2. Polynomial Remainder Sequences

The sequence of remainder polynomials generated by the Euclidean algorithm is called the polynomial remainder sequence or PRS. For traditional polynomials, there is only one PRS associated to two polynomials as there are no choices in the Euclidean algorithm. We now extend the definition of polynomial remainder sequence to Laurent polynomials and define the property of normality.

DEFINITION 7. A Laurent PRS for a given division scheme is the set of remainder Laurent polynomials obtained by the Euclidean algorithm at each step. For polynomials A and B we denote it $PRS(A, B)$.

DEFINITION 8. A (Laurent) PRS is called normal if the (Laurent) degree decreases by exactly 1 at each step of the Euclidean algorithm. A (Laurent) PRS that is not normal is called abnormal.

DEFINITION 9. A single division in the Euclidean algorithm is called normal if the degree of the remainder polynomial decreases by exactly one, otherwise it is called abnormal.

Since the division algorithm reduces the remainder degree by at least one, normality characterizes when the maximal number of steps in the Euclidean Algorithm are needed. The following is an example of a normal PRS for the polynomials $A(x) = 2x^4 + 7x^3 + 8x^2 + 5x + 3$ and $B(x) = 3x^3 + 4x^2 + 2x + 5$.

EXAMPLE 10. The Euclidean Algorithm for the polynomials $A(x) = 2x^4 + 7x^3 + 8x^2 + 5x + 3$ and $B(x) = 3x^3 + 4x^2 + 2x + 5$ has only one division scheme as there are no choices for the divisions for traditional polynomials. If A and B are considered Laurent polynomials, then the division scheme corresponding to the Euclidean Algorithm for traditional polynomials is $\{left - left, left - left, \ldots, left - left\}$, shown below. We begin by computing the quotient q_1 and remainder r_1,

$$A(x) = q_1 B + r_1 = \left(\frac{13}{9} + \frac{2x}{3}\right) B + \left(\frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}\right).$$

Thus, the first polynomial in the PRS is $r_1 = \frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}$ and we continue with the Euclidean Algorithm to find q_2 and r_2,

$$\begin{aligned} B(x) &= q_2 \left(\frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}\right) + r_2 \\ &= \left(\frac{27x}{8} + \frac{585}{64}\right)\left(\frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}\right) + \left(\frac{1395x}{32} + \frac{1755}{64}\right). \end{aligned}$$

Thus, the second polynomial in the PRS is $r_2 = \frac{1395x}{32} + \frac{1755}{64}$. The final step in the Euclidean Algorithm yields the final polynomial in the PRS which is necessarily the GCD of A and B,

$$\begin{aligned} \left(\frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}\right) &= q_3 \left(\frac{1395x}{32} + \frac{1755}{64}\right) + r_3 \\ &= \left(\frac{512x}{15795} - \frac{11840}{123201}\right)\left(\frac{1395x}{32} + \frac{1755}{64}\right) - \left(\frac{448}{13689}\right). \end{aligned}$$

We conclude the final polynomial in the PRS is $r_3 = -\frac{448}{13689}$, and

$$PRS(A, B) = \left\{\frac{8x^2}{9} - \frac{11x}{9} - \frac{38}{9}, \frac{1395x}{32} + \frac{1755}{64}, -\frac{448}{13689}\right\}.$$

Since the degree of the polynomials of the PRS decreases by exactly one at each step, the PRS is normal.

1.3. Discrete Wavelet Transforms and Lifting

The discrete wavelet transform (DWT) takes a signal $u \in \ell^2$ and applies the analysis filters $\tilde{h}, \tilde{g}$ to decompose it into coefficients of the wavelet basis. The signal passes through the low-pass $\tilde{h}$ and high-pass $\tilde{g}$ filters and is then subsampled. The inverse transform (IDWT) reconstructs the signal by upsampling it and then applying the synthesis filters, h (low-pass) and g (high-pass). We only consider FIR filters in this dissertation, hence $h, g, \tilde{h}, \tilde{g}$ have finite support. A complete description of wavelet transforms can be found in [**2, 3, 7, 12, 16**].

A commonly desired property of filters is the perfect reconstruction property that allows the original signal to be exactly recovered by the synthesis filters after passing through the analysis filters.

Definition 11. The perfect reconstruction property in our z-transform notation is then

$$h(z)\,\tilde{h}\left(z^{-1}\right) + g(z)\,\tilde{g}\left(z^{-1}\right) = 2$$

$$h(z)\,\tilde{h}\left(-z^{-1}\right) + g(z)\,\tilde{g}\left(-z^{-1}\right) = 0.$$

The even and odd parts of a filter, defined below, are useful in representing the DWT and IDWT.

Definition 12. Let $h(z) = \sum h_k z^{-k}$ be a Laurent polynomial. Then the even part of h is

$$h_e(z) = \sum_k h_{2k} z^{-k},$$

and the odd part of h is

$$h_o(z) = \sum_k h_{2k+1} z^{-k}.$$

For synthesis filters, we define the polyphase matrix $P(z)$,

$$P(z) = \begin{bmatrix} h_e & g_e \\ h_o & g_o \end{bmatrix}$$

and similarly for $\tilde{P}(z)$ using $\tilde{h}$ and $\tilde{g}$. The perfect reconstruction property can be rewritten [**8**] as

$$P(z)\,\tilde{P}\left(z^{-1}\right)^T = \mathbf{Id}.$$

Since the entries of P and $\tilde{P}$ are all Laurent polynomials, their determinants are Laurent polynomials as well. Then,

$$\det(P(z))\det\left(\tilde{P}\left(z^{-1}\right)^T\right) = \det(\mathbf{Id}) = 1,$$

which can only occur when the determinants of P and $\tilde{P}$ are degree 0 (monomials). We can rescale g to ensure $\det(P(z)) = 1$. Suppose $\det(P(z)) = cz^m$ for some nonzero c, then,

$$\det\left(\begin{bmatrix} h_e & \frac{g_e}{cz^m} \\ h_o & \frac{g_o}{cz^m} \end{bmatrix}\right) = \frac{h_e g_o}{cz^m} - \frac{h_o g_e}{cz^m} = \frac{1}{cz^m}\det(P(z)) = 1.$$

Definition 13. A pair of filters h, g are called complementary if the associated polyphase matrix P satisfies $\det(P) = 1$.

Given an FIR filter h, a complementary filter can be found if and only if h_e and h_o are coprime [**7, 19**]. We can apply the Euclidean algorithm to h_e and h_o with any division scheme to obtain

$$\begin{bmatrix} h_e \\ h_o \end{bmatrix} = (-1)^N \prod_{k=0}^{N-1} \begin{bmatrix} q_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cz^m \\ 0 \end{bmatrix},$$

where $\{q_k\}$ are the quotients and $cz^m \in \gcd(h_e, h_o)$. The GCD is necessarily a monomial since h_e and h_o are coprime, and a complementary filter g is defined by

$$\begin{bmatrix} h_e & g_e \\ h_o & g_o \end{bmatrix} = (-1)^N \prod_{k=0}^{N-1} \begin{bmatrix} q_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} cz^m & 0 \\ 0 & \frac{1}{cz^m} \end{bmatrix}.$$

Given synthesis filters h, g, a pair of analysis filters can be found which satisfy the perfect reconstruction property by defining $\tilde{h}$ and $\tilde{g}$ by

$$\tilde{h}_e(z) = g_o\left(z^{-1}\right),$$

$$\tilde{h}_o(z) = -g_e\left(z^{-1}\right),$$

$$\tilde{g}_e(z) = -h_o\left(z^{-1}\right),$$

$$\tilde{g}_o(z) = -h_e\left(z^{-1}\right).$$

Thus, if an FIR filter h has comprime even and odd parts h_e and h_o, then we can always find $g, \tilde{h}, \tilde{g}$ with the perfect reconstruction property [**5**].

The lifting scheme is a way to build filters satisfying the perfect reconstruction property. The idea is to start with the lazy wavelet, which only downsamples the signal, and then multiply by matrices with unit determinant (lifting steps) to ensure the resulting filters h, g will be complementary. The choice of lifting steps leads to different properties of the resulting multiresolution analysis, and can be used to build any FIR wavelet. The two following theorems from [**8**] outline the lifting scheme.

THEOREM 14. *(Lifting) Let h, g be complementary filters. Then any other finite filter g^{new} complementary to h is of the form:*

$$g^{new}(z) = g(z) + h(z)\, s\left(z^2\right),$$

where $s(z)$ is a Laurent polynomial. Conversely, any filter of this form is complementary to h.

THEOREM 15. *(Dual Lifting) Let h, g be complementary filters. Then any other finite filter h^{new} complementary to g is of the form:*

$$h^{new}(z) = h(z) + g(z)\, t\left(z^2\right),$$

where $t(z)$ is a Laurent polynomial. Conversely, any filter of this form is complementary to g.

To build the desired FIR filter, start with the Lazy wavelet and alternate lifting and dual lifting steps, which correspond to multiplying the polyphase matrix by matrices of the form

$$\begin{bmatrix} 1 & s(z) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t(z) & 1 \end{bmatrix}$$

for lifting and dual lifting, respectively. Using the lifting scheme to construct wavelets with special properties is described in detail in [**4, 17**].

1.4. Overview of Results

Wickerhauser and Zhu [**19**] showed that every filter has a nearest neighbor factorization if additional matrices are added when the Euclidean algorithm does not directly produce a nearest neighbor factorization. In chapter 2, we show that for most popular filters, these additional matrices are needed exactly when the PRS generated from h_e and h_o is abnormal. The only candidates for a direct nearest neighbor factorizations are from the $\{left, sym, \ldots, sym\}$ or $\{right, sym, \ldots, sym\}$ division schemes, depending on the filter length, and exist when these PRS are normal. Uniqueness of direct nearest neighbor factorizations for filters with certain length restriction is shown, and an algorithm to compute the factorizations is given. The effects of an initial z-shift are characterized and related to normality of the PRS.

In chapter 3, the results about direct nearest neighbor factorizations are applied to Daubechies filters and the existence of a direct nearest neighbor factorization is numerically verified for filters with lengths up to 220. Asymptotics of Daubechies polynomial roots from [**14, 15**] are used to prove limiting behavior of Daubechies filter coefficients, and are related to properties of Daubechies filter factorizations.

In chapter 4, normality of PRS for traditional and Laurent polynomials is analyzed. Sturm sequences are introduced, and normality of the first division is related to the zeros of a higher order derivative of the starting polynomial. For quartic polynomials, this gives a geometric representation of normality for the first division. An example is constructed to show that convergence of the even and odd parts of a family of polynomials is not sufficient for normality. Sufficient conditions for normality are given for a particular family of polynomials.

CHAPTER 2

Nearest Neighbor Factorizations

2.1. Introduction

Recall that the Euclidean algorithm for a given division scheme of h_e and h_o for an FIR filter h results in a factorization of the polyphase matrix P, where the complementary filter g can be defined using the lifting steps [**8**]. Limiting the form of the lifting steps can result in fewer distant memory accesses. This motivates the nearest neighbor factorization definition from [**19**], repeated here:

Definition 16. Let P be the polyphase matrix of a filter bank. A lifting factorization of P,

$$P(z) = \prod_{k=0}^{N-1} \begin{bmatrix} 1 & s_k(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_k(z) & 1 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}$$

is called nearest neighbor if it satisfies the following conditions,

$$s_k(z) = \alpha_k + \beta_k z^{-1}$$

$$t_k(z) = \gamma_k z + \delta_k,$$

where $\alpha_k, \beta_k, \gamma_k, \delta_k, M \in \mathbb{C}$.

Wickerhauser and Zhu [**19**] showed that every FIR filter has a nearest neighbor factorization if additional matrices are added, often at the expense of the factorization having a higher condition number. With these additional matrices, every division scheme of an FIR filter results in a nearest neighbor factorization. We recall a lemma from [**19**] which includes the definition of the condition number of a matrix.

LEMMA 17. *If $P(z)$ is the polyphase matrix of a perfect reconstruction filter pair, then*

$$cond(P) := \frac{\sup\left\{\sqrt{\lambda_{max}(P^*P)} :\ |z| = 1\right\}}{\inf\left\{\sqrt{\lambda_{min}(P^*P)} :\ |z| = 1\right\}}$$

where $\lambda_{min}(M)$ and $\lambda_{max}(M)$ are eigenvalues of matrix M. *Furthermore, if $P = P_1 \cdots P_n$, then*

$$cond(P) \leq cond(P_1) \cdots cond(P_n).$$

EXAMPLE 18. Consider a polyphase matrix with the following lifting factorization

$$P(z) = \begin{bmatrix} 1 & z^{-3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This factorization is not nearest neighbor, but one can be found by decomposing the first matrix using additional matrices,

$$P(z) = \begin{bmatrix} 1 & z^{-3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1+z & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1-z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, we see that a single lifting step must be expanded into 12 matrices to satisfy the nearest neighbor form. Furthermore, 11 of the 12 additional matrices (all but the Id matrix) increase the condition number of the factorization.

For the Daubechies-4 filter with the $\{left, sym\}$ division scheme, the Euclidean algorithm results in a nearest neighbor factorization directly using the quotients as the lifting steps.

EXAMPLE 19. The Daubechies filter with four coefficients and shifted by z is

$$h(z) = \frac{1+\sqrt{3}}{4\sqrt{2}}z + \frac{3+\sqrt{3}}{4\sqrt{2}} + \frac{3-\sqrt{3}}{4\sqrt{2}}z^{-1} + \frac{1-\sqrt{3}}{4\sqrt{2}}z^{-2}.$$

The polyphase matrix factorization with the $\{left, sym\}$ division scheme is

$$P(z) = \begin{bmatrix} 1 & 0.57735 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.43301 + 2.79904z & 1 \end{bmatrix} \begin{bmatrix} 0.29886 & 0 \\ 0 & 3.34607 \end{bmatrix},$$

which results directly in a nearest neighbor factorization. Therefore, no additional matrices are needed.

DEFINITION 20. Given a filter h and a division scheme with Euclidean algorithm quotients $\{q_k\}$ such that

$$\begin{bmatrix} h_e \\ h_o \end{bmatrix} = (-1)^N \left(\prod_{k=0}^{N-1} \begin{bmatrix} q_k & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix},$$

and $\{q_k\}$ satisfy the nearest neighbor conditions, then the factorization is called a direct nearest neighbor factorization.

Note the equation in the direct nearest neighbor factorization definition can be written in nearest neighbor form using

$$\begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For factorizations with an even number of quotients from the Euclidean algorithm, the matrices can be paired and the flip matrices cancel for each pair, resulting in the desired nearest neighbor form,

$$\begin{bmatrix} q_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & q_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_1 & 1 \end{bmatrix}.$$

For factorizations with an odd number of quotients from the Euclidean algorithm, there is an additional matrix which cannot be paired and hence a flip matrix remains. This can corrected for by starting the Euclidean algorithm with the roles of h_e and h_o reversed. Then,

$$\begin{bmatrix} h_o \\ h_e \end{bmatrix} = \left(\prod_{k=1}^{N} \begin{bmatrix} q_k & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} h_o \\ h_e \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} \left(\prod_{k=2}^{N} \begin{bmatrix} q_k & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_o \\ h_e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} \left(\prod_{k=1}^{\frac{N-1}{2}} \begin{bmatrix} 1 & q_{k+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{k+2} & 1 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} h_e \\ h_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ q_1 & 1 \end{bmatrix} \left(\prod_{k=1}^{\frac{N-1}{2}} \begin{bmatrix} 1 & q_{2k} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{2k+1} & 1 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix},$$

and the factorization is nearest neighbor whenever the quotients satisfy the conditions of the nearest neighbor definition. Thus, whenever the $\{right, sym, \ldots, sym\}$ division scheme is given for a filter with an odd number of quotients, assume the Euclidean algorithm had input polynomials of $a_0 = h_o$ and $b_0 = h_e$, unless otherwise stated.

Wickerhauser and Zhu [**19**] showed that not all FIR filters have a direct nearest neighbor factorization for any z-shift and division scheme, demonstrated by the split Haar filter,

$$h = \frac{1}{\sqrt{2}} \left(1 + z^{-9}\right).$$

This is easily seen, as there is only one division scheme for the split Haar filter and it does not produce a direct nearest neighbor factorization.

2.2. Effects of z-Shifts

Since the GCD in a nearest neighbor factorization must be constant, it can be useful to multiply the z-transform of the filter by an initial shift before factoring into lifting steps. This corresponds to multiplying the z-transform of the filter by some power of z, which has no effect on the filter coefficients. We begin with two lemmas showing the effects of multiplying by even and odd powers of z.

LEMMA 21. *Multiplying the z-transform of a filter h by z^{2m}, $m \in \mathbb{Z}$, multiplies h_e and h_o by z^m.*

PROOF. The z-transform of the shifted filter is

$$h_{shift}(z) = z^{2m} h(z) = \left(z^{2m}\right) \sum_i h_i z^{-i} = \sum_i h_i z^{2m-i}.$$

Thus, the even part of $h_{shift}(z)$ is

$$h_{shift,even}(z) = \sum_i h_{2i} z^{m-i} = z^m \sum_i h_{2i} z^{-i} = z^m h_e(z).$$

Similarly for the odd part of $h_{shift}(z)$

$$h_{shift,odd}(z) = \sum_i h_{2i+1} z^{m-i} = z^m \sum_i h_{2i+1} z^{-i} = z^m h_o(z).$$

□

LEMMA 22. *Multiplying the z-transform of a filter h by z switches h_e and h_o, and multiplies h_e by z.*

PROOF. The z-transform of the shifted filter is

$$h_{shift}(z) = zh(z) = z\sum_i h_i z^{-i} = \sum_i h_i z^{1-i}.$$

Thus, the even part of $h_{shift}(z)$ is

$$h_{shift,even}(z) = \sum_i h_{2i} z^{1-i} = \sum_i h_{2i+1} z^{-i} = h_o(z),$$

and the odd part of $h_{shift}(z)$ is

$$h_{shift,odd}(z) = \sum_i h_{2i+1} z^{1-i} = z\sum_i h_{2i} z^{-i} = zh_e(z).$$

$\square$

An arbitrary integer power shift of a filter can be thought of as first an even power shift, and then a shift by z if the power is odd. Thus, multiplying a filter by z^{2m+1} shifts the even and odd parts of the filter by z^m according to lemma 21, and then swaps the even and odd parts and multiplies the even part by z as specified in lemma 22. In the context of lifting factorizations, shifts by an even power of z allow us to adjust the GCD to be constant without affecting the lifting steps, as shown in the next lemma.

Lemma 23. *Given a filter h factored into lifting steps $\{q_i\}$ with GCD Mz^j, the shifted filter $z^{-2j}h$ has the same lifting steps $\{q_i\}$, but with constant GCD M.*

Proof. A filter having lifting steps $\{q_i\}$ with a nonzero GCD Mz^j implies

$$\begin{bmatrix} h_e(z) \\ h_o(z) \end{bmatrix} = \left(\prod \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} Mz^j \\ 0 \end{bmatrix}.$$

Using lemma 21 with the shift z^{-2j}, $h_e(z)$ and $h_o(z)$ are multiplied by z^{-j}, hence the lifting factorization becomes

$$\begin{bmatrix} z^{-j}h_e(z) \\ z^{-j}h_o(z) \end{bmatrix} = z^{-j} \begin{bmatrix} h_e(z) \\ h_o(z) \end{bmatrix}$$

$$= z^{-j} \left(\prod \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} Mz^j \\ 0 \end{bmatrix} = \left(\prod \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix}.$$

□

This result allows us to record only the coefficients in the support of the remainder polynomials generated during the Euclidean algorithm on Laurent polynomials when finding nearest neighbor factorizations.

2.3. Number of Direct Nearest Neighbor Factorizations

We focus our attention on filters that satisfy $|h_e| = |h_o|$, as many popular wavelet filters satisfy this condition, including Daubechies filters which are the main topic of the next chapter.

REMARK 24. Let h be a filter of length $2N$ which satisfies $|h_e| = |h_o| = N - 1 \geq 1$. Then there are at most $4 \cdot 3^{N-2}$ division schemes with quotients of degree at most one.

The remark is shown in [**11**], and results from 4 choices for the first division depending upon which terms are canceled,

$$left, left - left, right, right - right,$$

no choices for the final division, and 3 choices for the remaining $N - 2$ divisions,

$$left - left, right - right, sym.$$

The $\{sym\}$ element corresponds to canceling the two extreme terms. The sequence of remainder polynomials resulting from a given division scheme will play a key role in nearest neighbor factorizations.

Remark 25. Given two Laurent polynomials, it is possible to have normal and abnormal PRS corresponding to different division schemes.

For a given division scheme to produce a direct nearest neighbor factorization, the quotients resulting from the Euclidean algorithm must be in the nearest neighbor form, otherwise additional matrices are required. We begin with a result about the first division for length $2N$ filters which satisfy $|h_e| = |h_o| = N-1$.

Theorem 26. *Given a filter h of length $2N$ which satisfies $|h_e| = |h_o| = N-1 \geq 1$, and a division scheme resulting in a direct nearest neighbor factorization, then the first element of the division scheme is either $\{left\}$ or $\{right\}$. This is equivalent to the first lifting step (quotient in the Euclidean algorithm) being a constant.*

Proof. Let $\{a_i\}_{i=0}^N$ and $\{b_i\}_{i=0}^N$ be the polynomials in the Euclidean algorithm, starting with $a_0 = h_e$ and $b_0 = h_o$, and let $\{q_i\}_{i=1}^N$ be the quotients. Since the division scheme results in a direct nearest neighbor factorization, the first lifting step is of the form $q_1 = c_1 z + d_1$ or $q_1 = c_1 z^{-1} + d_1$. For the case $q_1 = c_1 z + d_1$, assume toward contradiction that, $c_1 \neq 0$, which corresponds to having $\{right - right\}$ as the first element of the division scheme. Using lemma 23, assume h_e and h_o have constant lowest degree terms. Then

$$a_0 = \sum_{j=0}^{N-1} a_{0,j} z^j$$

$$b_0 = \sum_{j=0}^{N-1} b_{0,j} z^j$$

$$b_1 = a_0 - q_1 b_0 = \sum_{j=2}^{N} b_{1,j} z^j.$$

The (Laurent) degree of b_1 has been reduced by at least 1 as required for the Euclidean algorithm. For the next step in the algorithm, the quotient must be of the form $q_2 = c_2 z^{-1} + d_2$ for the factorization to be directly nearest neighbor. Then,

$$a_1 = b_0 = \sum_{j=0}^{N-1} a_{1,j} z^j$$

$$b_2 = a_1 - q_2 b_1 = \sum_{j=0}^{N-1} b_{0,j} z^j - \left(c_2 z^{-1} + d_2\right) \sum_{j=2}^{N} b_{1,j} z^j = \sum_{j=0}^{N} b_{2,j} z^j.$$

We note that $|b_2| = N$ when d_2 is nonzero, and $|b_2| = N - 1$ when $d_2 = 0$ since the extreme terms cannot cancel. But then the degree of the remainder has not been reduced in this step since $|b_1| = N - 1$, a contradiction. A similar argument leads to a contradiction for the case $q_1 = c_1 z^{-1} + d_1$ with $c_1 \neq 0$. Thus, q_1 must be constant, which corresponds to the first element of the division scheme being $\{left\}$ or $\{right\}$. $\square$

Thus, only two choices of the possible four $\{left, left - left, right, right - right\}$ for the first division can result in a direct nearest neighbor factorization. The next theorem shows that there is only one choice for the remaining steps in the Euclidean algorithm that can result in a direct nearest neighbor factorization.

THEOREM 27. *Given a length $2N$ filter h which satisfies $|h_e| = |h_o| = N - 1 \geq 1$, and a division scheme that results in a direct nearest neighbor factorization, then the division scheme must be either $\{left, sym, \ldots, sym\}$ or $\{right, sym, \ldots, sym\}$.*

PROOF. The first element being $\{left\}$ or $\{right\}$ is a result of theorem 26. Let $a_0 = h_e$ and $b_0 = h_o$. Assume the first division is normal, and hence $|a_1| = |b_1| + 1$. Without loss of generality, suppose that the lowest power of a_0 was eliminated, corresponding to $\{right\}$ as the first element of the division scheme. Then, using lemma 23, assume $a_1 = \sum_{j=0}^{N-1} a_{1,j} z^j$ and $b_1 = \sum_{j=1}^{N-1} b_{1,i} z^j$. There are then three possibilities for the division, eliminating the highest two, lowest two, or highest and lowest terms from a_1. The remainders corresponding to the division choices $\{left - left\}, \{right - right\}, \{sym\}$ are

$$b_2^L = \sum_{j=0}^{N-3} b_{2,j} z^j$$

$$b_2^R = \sum_{j=2}^{N-1} b_{2,j} z^j$$

$$b_2^{sym} = \sum_{j=1}^{N-2} b_{2,j} z^j,$$

respectively, with corresponding quotients of the form $q_2 = c_2 z^{-1} + d_2$. Note that $\left|b_2^R\right| = \left|b_2^L\right| = \left|b_2^{sym}\right| = N - 3$. With any choice, $a_2 = \sum_{j=1}^{N-1} a_{2,j} z^j$, and the corresponding q_3 must satisfy $|q_3| = 1$ in order to reduce the remainder degree in the division. Since the previous quotient was of the form $q_2 = c_2 z^{-1} + d_2$, the next step must have the form

$$q_3 = c_3 z + d_3.$$

The resulting b_3 polynomials corresponding to the above b_2 polynomials are:

$$b_3^L = a_2 - q_3 b_2^L = \sum_{j=1}^{N-1} a_{2,j} z^j - (c_3 z + d_3) \sum_{j=0}^{N-3} b_{2,j} z^j = \sum_{j=0}^{N-1} b_{3,j} z^j$$

$$b_3^R = a_2 - q_3 b_2^R = \sum_{j=1}^{N} a_{2,j} z^j - (c_3 z + d_3) \sum_{j=2}^{N-1} b_{2,j} z^j = \sum_{j=1}^{N} b_{3,j} z^j$$

$$b_3^{sym} = a_2 - q_3 b_2^{sym} = \sum_{j=1}^{N-1} a_{2,j} z^j - (c_3 z + d_3) \sum_{j=1}^{N-2} b_{2,j} z^j = \sum_{j=2}^{N-2} b_{3,j} z^j.$$

The extreme terms of b_3^R and b_3^L cannot be canceled by the subtraction, which can be seen by comparing the degrees of a_2 and $q_3 b_2$. Thus, $\left|b_3^R\right| = \left|b_3^L\right| = N - 1$ which cannot occur since that would imply the degree was not reduced. Then, $|b_3^{sym}| \leq N-4$, with equality exactly when the division is normal. Thus, the only possibility given a direct nearest neighbor factorization is for all elements in the division scheme after the first element to come from symmetric division.

Now suppose the first division is abnormal, then $|b_1| < |a_1| - 1$. The next step in the Euclidean algorithm would require a quotient of degree more than 1 to reduce the degree of the remainder sufficiently, and hence would not result in a nearest neighbor factorization. $\square$

Thus, out of the possible $4 \cdot 3^{N-2}$ division schemes for a filter satisfying $|h_e| = |h_o| = N-1$, there are only two candidate division schemes which can result in a nearest neighbor factorization. The next theorem reduces the number of candidate division schemes for such filters to one.

THEOREM 28. *Let h be a length $2N$ filter which satisfies $|h_e| = |h_o| = N-1 \geq 1$ with a direct nearest neighbor factorization. Then for even N, the division scheme must be $\{left, sym, \ldots sym\}$. For odd N, the division scheme must be $\{right, sym, \ldots sym\}$.*

PROOF. Let h be as above and suppose N is even. Then theorem 27 shows the only possible division schemes resulting in a direct nearest neighbor factorization are

$$\{left, sym, \ldots sym\}$$

and

$$\{right, sym, \ldots sym\}.$$

Suppose for contradiction the division scheme is

$$\{right, sym, \ldots sym\},$$

and let $\{q_i\}$ be the list of quotients from the Euclidean algorithm. Since the factorization is nearest neighbor, $\{q_i\}$ must contain only degree one Laurent polynomials. Since by assumption $|h_e| = |h_o| = N - 1$, there are at most N quotients. Thus,

$$\begin{bmatrix} h_e \\ h_o \end{bmatrix} = \left(\prod_{k=1}^{\frac{N}{2}} \begin{bmatrix} 1 & q_{2k-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_{2k} & 1 \end{bmatrix} \right) \begin{bmatrix} M \\ 0 \end{bmatrix} \tag{2.3.1}$$

where $M \in gcd\,(h_e, h_o)$. To satisfy nearest neighbor form, $q_2 = c_2 z + d_2$, but from theorem 27, the $\{right, sym, \ldots, sym\}$ has a constant q_1 and $q_2 = c_2 z^{-1} + d_2$. For q_2 to be of the correct form, $c_2 = 0$, hence q_2 must be a constant. But then the left hand side of equation (2.3.1) has polynomials of degree $N-1$ and the right hand side has degree at most $N-2$ since every other q_i is at most degree one, a contradiction. The same argument holds for odd N. $\square$

THEOREM 29. *Let h be a length $2N$ filter which satisfies $|h_e| = |h_o| = N - 1 \geq 1$. For odd N, the division scheme $\{right, sym, \ldots, sym\}$ having a normal polynomial remainder sequence $PRS\,(h_o, h_e)$ is equivalent to $\{right, sym, \ldots, sym\}$ resulting in a direct nearest neighbor factorization. For even N, the division scheme $\{left, sym, \ldots, sym\}$ having a normal polynomial remainder sequence $PRS\,(h_e, h_o)$ is equivalent to $\{left, sym, \ldots, sym\}$ resulting in a direct nearest neighbor factorization.*

PROOF. ($\Longrightarrow$) Let the filter be as above, N odd, and let $\{right, sym, \ldots, sym\}$ result in a normal PRS. Then since N is odd, we begin the Euclidean algorithm with $a_0 = h_o$ and $b_0 = h_e$. The first division cancels the lowest power term of b_0, and since the division scheme is normal, there are no additional terms canceled. Then the first lifting step q_1 is constant and, up to a shift by z,

$$a_1 = h_e = \sum_{j=0}^{N-1} a_{1,j} z^j$$

$$b_1 = \sum_{j=1}^{N-1} b_{1,j} z^j.$$

The extreme terms $b_{1,1}$ and $b_{1,N-1}$ must be nonzero for normality to hold, and the extreme powers are aligned for the highest powers and differ by one for the lowest powers. Continuing with $\{sym\}$ division, which again must be normal, then the next quotient, q_2, must be exactly degree one. Comparing the degrees of a_1 and b_1, the quotient must be of the form $q_2 = c_2 z^{-1} + d_2$. The next step of the Euclidean algorithm results in

$$a_2 = \sum_{j=1}^{N-1} a_{2,j} z^j$$

$$b_2 = a_1 - q_2 b_1 = \sum_{j=1}^{N-2} b_{2,j} z^j.$$

Since the division is normal, b_2 must have nonzero extreme terms, $b_{2,1}$ and $b_{2,N-2}$, and the quotients resulting from the division scheme thus far satisfy the nearest neighbor condition. The next division is similar the previous $\{sym\}$ step, but since the left powers align, the quotient will be exactly degree one, but of the form $q_3 = c_3 z + d_3 z$. The next step yields

$$a_3 = \sum_{j=1}^{N-2} a_{3,j} z^j$$

$$b_3 = a_2 - q_3 b_2 = \sum_{j=2}^{N-2} b_{3,j} z^j.$$

The extreme terms $b_{3,2}$ and $b_{3,N-2}$ again must be nonzero since the division is normal, and the highest powers are now aligned. The $\{sym\}$ divisions will hence alternate quotients in the necessary forms to satisfy the nearest neighbor condition. This pattern can only be disrupted if an extreme term of the remainder is zero, which cannot occur with the assumption of normality.

The proof for $\{left, sym, \ldots, sym\}$ follows the same arguments as $\{right, sym, \ldots, sym\}$, with the only change that the Euclidean algorithm starts with $a_0 = h_e$ and $b_0 = h_o$.

($\Longleftarrow$) Without loss of generality, assume the division scheme $\{left, sym, \ldots, sym\}$ is abnormal. Then for at least one division in the Euclidean algorithm, the degree of the remainder is reduced by more than 1. Then the total number of steps, and hence lifting steps, in the Euclidean algorithm is at most $N-2$. Then

$$\begin{bmatrix} h_e(z) \\ h_o(z) \end{bmatrix} = \left(\prod_{i=1}^{N-2} \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} gcd(h_e, h_o) \\ 0 \end{bmatrix},$$

but then the product on the right hand side of the equation must have degree strictly less than $N-1$ since $|q_i| \leq 1$. This is a contradiction as $|h_e| = |h_o| = N-1$ by assumption. Thus, no division in the division scheme can be abnormal. $\square$

The connection of normal PRS and direct nearest neighbor factorizations is promising as abnormal PRS form a measure 0 set considering $h = \sum_{j=0}^{N-1} h_j z^j$ as $(h_0, h_1, \ldots, h_{N-1}) \in \mathbb{R}^{\mathbb{N}}$. Unfortunately, there are few, if any, ways to check normality of a PRS in general without going through the entirety of the Euclidean algorithm and checking the degrees of the remainders. We will find families of polynomials in chapter 4 for which normality can be proven without computing the entire PRS.

These results show that with the restrictions of direct nearest neighbor factorizations of length $2N$ filters satisfying $|h_e| = |h_o| = N-1 \geq 1$, uniqueness is achieved exactly when the associated PRS is normal. In [**1**], Brislawn approaches the question of uniqueness in the lifting scheme with a group structure approach, very different from our direct computation approach. Our approach results in a uniqueness theorem for a smaller class of filters, but allows us to find an algorithm to find the factorizations whenever they exist.

2.4. Algorithm for Nearest Neighbor Factorization

Using the results from section §2.3, we outline an algorithm for checking whether an arbitrary filter of length $2N$ satisfying $|h_e| = |h_o| = N-1$ has a direct nearest neighbor factorization. Due to lemma 23, we need only track the coefficients of the polynomials in the Euclidean algorithm, and theorem 29 allows us only to check for normality of the $\{left, sym, \ldots, sym\}$ or $\{right, sym, \ldots, sym\}$ division scheme (depending on whether N is even or odd), instead of all $4 \cdot 3^{N-2}$ possibilities. We outline the algorithm for the $\{left, sym, \ldots, sym\}$ division scheme for a filter $h = \sum_{j=-2N-1}^{0} h_j z^j$ satisfying $|h_e| = |h_o| = N-1$. The algorithm for the $\{right, sym, \ldots, sym\}$ division scheme works similarly, the only alteration is starting the Euclidean algorithm with the roles of h_e and h_o reversed.

Step 1: Eliminate Constant Term of h_o. We denote the coefficients of h_e by $h_e = a_0 = (c_0, c_1, ..., c_{N-1})$ and the coefficients of h_o by $h_o = b_0 = (d_0, d_1, ..., d_{N-1})$. The first step in the Euclidean Algorithm eliminates the highest order term from h_o via the following:

$$q_1 = \frac{c_0}{d_0}$$

$$a_1 = b_0 = (d_0, d_1, ..., d_{N-1})$$

$$b_1 = a_0 - q_1 b_0 = (c_1 - q_1 d_1, c_2 - q_1 d_2, ..., , c_{N-1} - q_1 d_{N-1}) \, .$$

If the extreme terms in b_1 are 0, that is, the division is abnormal, then the algorithm terminates and there is no nearest neighbor factorization for the $\{left, sym, \ldots, sym\}$ division scheme. If the extreme powers do not equal 0, then set the first lifting step as q_1.

Step 2: Symmetric Division. We proceed with the Euclidean algorithm using symmetric division until the algorithm terminates after a total of N steps when $b_N = 0$ and

$a_N \in gcd\,(h_e, h_o)$. At each step, check that the extreme powers of b_i are nonzero, else the division scheme is abnormal and the algorithm terminates. We represent symmetric division in coefficient arrays with the following operations. For $a_i = \{c_0, \ldots, c_{m+1}\}$ and $b_i = \{d_0, \ldots, d_m\}$ which agree in the lowest power of z, symmetric division yields:

$$q_i = \frac{c_0}{d_0} + \frac{c_{m+1}}{d_m} z^{-1} = s_i + t_i z^{-1}$$

$$a_{i+1} = b_i = (d_0, d_1, ..., d_m)$$

$$b_{i+1} = (c_1 - s_i d_1 - t_i d_0, c_2 - s_i d_2 - t_i d_1, ..., c_m - s_i d_m - t_i d_{m-1})\,.$$

If a_i and b_i agree in the highest power of z, a_{i+1} and b_{i+1} are the same, but the quotient becomes

$$q_i = \frac{c_0}{d_0} z + \frac{c_{m+1}}{d_m} = s_i z + t_i.$$

At each step, set q_i as the nearest neighbor lifting step.

Step 3: Determine z-Shift. If the Euclidean algorithm terminates and results in a normal PRS, then let M be the coefficient of the GCD obtained from the last step of the Euclidean algorithm. Then the GCD is

$$Mz^{\lfloor -\frac{N}{2} \rfloor} \in \mathrm{gcd}\,(h_e, h_o)\,.$$

If the original filter, h, had a different z-shift, use lemma 23 to shift the filter so $h = \sum_{j=-2N-1}^{0} h_j z^j$ and then apply the lemma again to obtain the correct GCD via the appropriate z-shift.

EXAMPLE 30. The Daubechies filter with 8 coefficients is

$$\{0.23038, 0.71485, 0.63088, -0.027984, -0.18703, 0.030841, 0.032883, -0.010597\},$$

thus

$$h_e = \{0.23038, 0.63088, -0.1870, 0.032883\}$$

$$h_o = \{0.71485, -0.027984, 0.030841, -0.010597\}.$$

Step 1, left division, cancels the highest order term of the filter, which corresponds to the first element of h_e. After the first division, the coefficients of the remainder polynomial are

$$\{0.63990, -0.19697, 0.03630\},$$

and the corresponding quotient (lifting step) is

$$q_1 = 0.32228.$$

Applying Step 2 (sym division) yields remainder polynomial coefficients of

$$\{0.37888, -0.06722\}$$

$$\{0.12115\}$$

$$\{0\},$$

and corresponding quotients of

$$q_2 = -0.29195 + 1.1171z$$

$$q_3 = 1.6889 - \frac{0.5400}{z}$$

$$q_4 = -0.555 + 3.127z.$$

The coefficient of the GCD is the last nonzero remainder coefficient, $M = 0.12155$. Computing the GCD using Step 3,

$$0.12155z^{\lfloor -\frac{N}{2} \rfloor} = 0.12155z^{\lfloor -\frac{4}{2} \rfloor} = 0.12155z^{-2} \in \gcd(h_e, h_o).$$

Thus, the original filter requires a z-shift of z^2 to result in a constant GCD and a direct nearest neighbor factorization via $\{left, sym, sym, sym\}$.

2.5. Matrix Representation of the Reconstruction Algorithm

Given the lifting steps $\{q_i\}$ and $M \in gcd(h_e, h_o)$ of a filter, we can reconstruct the original filter using the following equation,

$$P(z) = \prod_{i=1}^{N/2} \begin{bmatrix} q_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & M^{-1} \end{bmatrix}.$$

If in addition to the filter, we want to recover all information in the associated Euclidean algorithm used to generate the lifting steps, we can use the following theorem.

THEOREM 31. *Suppose a given filter of length $2N$ with $|h_e| = |h_o| = N-1$ has a nearest neighbor factorization with lifting steps $\{q_i\}$ and $M \in gcd(h_e, h_o)$. Let A be any matrix, S be the zero matrix with 1's along the superdiagonal, and G be the zero matrix with $G_{N+1,1} = M$, each with dimension $(N+1) \times (N+1)$. Then define C as the diagonal matrix of coefficients on the highest power of $\{q_i\}$, with the first diagonal entry equal to 0*

$$C = \begin{bmatrix} 0 & 0 & \dots & & 0 \\ 0 & c_1 & 0 & & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & c_{N-1} & \vdots \\ 0 & 0 & \dots & 0 & c_N \end{bmatrix}.$$

Similarly, define D with diagonal entries equaling the coefficients of the lowest order terms of $\{q_i\}$.

$$D = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ 0 & d_1 & 0 & & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & d_{N-1} & \vdots \\ 0 & 0 & \cdots & 0 & d_N \end{bmatrix}.$$

Then $N+1$ iterations of the following matrix equation will converge to the matrix of coefficients obtained by the Euclidean algorithm producing the specified lifting steps:

$$A = SCA + ASDS + SSAS + G.$$

PROOF. The theorem is proved by writing the effects of the Euclidean algorithm on the remainder polynomials at each step in terms of matrix operations. $\square$

2.6. Filters with No Direct Nearest Neighbor Factorizations

The algorithm described in section §2.4 can be used to generate filters with no direct nearest neighbor factorizations. In [**19**], the split Haar filter,

$$h = \frac{1}{\sqrt{2}}\left(1 + z^{-9}\right),$$

was an example of a filter with no direct nearest neighbor factorization. Having multiple zeros in the support of the filter is often enough to guarantee no direct nearest neighbor factorizations, although we give examples to show it is not a necessary condition. We also show that common properties of filters such as orthogonality and vanishing moments are not enough to guarantee the existence of a direct nearest neighbor factorization. We begin by restating a lemma in [**11**] which shows the even and odd parts of a filter must have similar degrees or no direct nearest neighbor factorization can exist.

LEMMA 32. *Let $h = \{h_0, h_1, \ldots, h_{2N-1}\}$ be a filter such that $||h_e| - |h_o|| > 1$, then h has no direct nearest neighbor factorizations.*

PROOF. Let a_o and b_o be the starting polynomials in the Euclidean algorithm. Then for the factorization to be directly nearest neighbor, each quotient must have degree at most one. Suppose $|a_o| > |b_o| + 1$, then the first division can cancel at most two terms from a_o, hence

$$b_1 = a_0 - q_1 b_0$$

$$|b_1| \geq |a_o| - 2 > |b_o| - 1 \geq |b_o| \,.$$

But then the degree of the remainder has not been reduced, a contradiction. □

LEMMA 33. *Let $h = \{h_0, h_1, \ldots, h_{2N-1}\}$ be a filter satisfying $|h_e| = |h_o|$ and*

$$|support\,(h_e) \cap support\,(h_o)| + 1 < |support\,(h_e)| \,,$$

then h has no direct nearest neighbor factorizations.

PROOF. First, suppose N is even, and let $a_0 = h_e$ and $b_0 = h_o$ be the starting polynomials in the Euclidean algorithm. The first division must cancel at least one term from a_0 so the degree of b_1 is strictly less than b_0. The conditions $|h_e| = |h_o|$ and $|support\,(h_e) \cap support\,(h_o)| + 1 < |support\,(h_e)|$ imply the extreme terms of a_0 and b_0 differ by a monomial with traditional polynomial degree of at least two. Then,

$$b_1 = a_0 - q_1 b_0,$$

so the extreme terms of $q_1 b_0$ and a_0 must align, requiring q_1 to contain a term with traditional polynomial degree of at least two, hence the factorization cannot be directly nearest neighbor.

The same argument holds for odd N and $a_0 = h_o$ and $b_0 = h_e$. $\square$

Although filters with many zeros in the support often do not have a direct nearest neighbor factorization, it is not sufficient to ensure no direct nearest neighbor factorizations exist.

EXAMPLE 34. Consider the filter

$$h = \{36, 72, 72, 0, 48, 60, 60, 24\}.$$

Then, using the $\{left, sym, sym, sym\}$ division scheme yields a direct nearest neighbor factorization. The coefficient arrays of the PRS are

$$\{72, 0, 60, 24\}$$

$$\{72, 18, 48\}$$

$$\{-54, 3\}$$

$$\{886\}.$$

This PRS is easily seen to be normal as the length of the remainder coefficient array decreases by exactly 1 at each step.

A common property of filters is the perfect reconstruction property, which is also not sufficient to ensure a filter has a direct nearest neighbor factorization. A filter having the perfect reconstruction property is equivalent to coprimality of h_e and h_o [**19**]. This property is not enough to guarantee a normal PRS, as shown in the following proposition.

PROPOSITION 35. *There exists a filter h such that h_e and h_o are coprime, but $PRS\,(h_e, h_o)$ is abnormal in the $\{left, sym, \ldots, sym\}$ division scheme.*

Proof. Let $h_e = 9z^3 + 5z^2 + 4z + 2$ and $h_o = 10z^3 + 2z^2 + 6z + 3$. Then the $\{right, sym, sym, sym\}$ division scheme is abnormal, but h_e and h_o are coprime as they have no common roots. The polynomial remainder sequence is abnormal as the first remainder is

$$b_1 = \frac{7}{3}z^3 + \frac{11}{3}z^2.$$

Since the degree of the remainder decreased by more than one ($b_0 = h_o$ has degree 4, b_1 has degree 2), the division scheme results in an abnormal remainder sequence.

However, this filter has a normal remainder sequence for the $\{left, sym, \ldots, sym\}$ division scheme, which demonstrates normality is dependent on division scheme. $\square$

Many filters are designed to have orthogonality and vanishing moment conditions, and we investigate the effect of these properties on direct nearest neighbor factorizations. For FIR filters, these properties can be translated to conditions involving the filter coefficients [**7**].

Definition 36. Let $h = \sum_{i=0}^{2N-1} h_i z^i$ be the z-transform of an FIR filter of length $2N$. Then h is orthogonal if it satisfies the following double shift orthogonality equations,

$$\sum_{i=0}^{2N-1} h_i h_{i+2k} = \delta_k, \quad k = 0, 1, 2, \ldots, N-1.$$

A filter with l vanishing moments can also be classified using equations only involving the filter coefficients.

Definition 37. Let $h = \sum_{i=0}^{2N-1} h_i z^i$ be the z-transform of an FIR filter of length $2N$. Then h has l vanishing moments if it satisfies the following equations for $k = 0, \ldots, l$

$$\sum_{i=0}^{2N-1} (-1)^i \, i^k h_i = 0 \quad k = 0, 1, 2, \ldots, N-1.$$

These properties are not sufficient to guarantee a direct nearest neighbor factorization, even with no zeros in the support of the filter, as demonstrated in the following proposition.

PROPOSITION 38. *(Recluselet) There exist orthogonal filters with at least one vanishing moment and no zeros in the support with no direct nearest neighbor factorizations.*

PROOF. The filter

$$\{0.742661, -0.107110, 0.123776, 0.011555, 0.01, 0.06, -0.069335\}$$

is a length 8 orthogonal filter with one vanishing moment and no direct nearest neighbor factorizations. This filter solves the following system of equations for orthogonality and one vanishing moment, along with conditions to make both $\{left, sym, \ldots, sym\}$ and $\{right, sym, \ldots, sym\}$ division schemes be abnormal.

$$\sum_{i=0}^{7} h_i = \sqrt{2}$$

$$\sum_{i=0}^{7} h_i h_{i+2k} = \delta_k,\ k = 0, 1, 2, 3$$

$$h_0 - h_1 + h_2 - h_3 + h_4 - h_5 + h_6 - h_7 = 0$$

$$-\frac{h_1\left(h_4 - \frac{h_0 h_5}{h_1}\right)}{h_2 - \frac{h_0 h_3}{h_1}} - \frac{h_7\left(h_2 - \frac{h_0 h_3}{h_1}\right)}{h_6 - \frac{h_0 h_7}{h_1}} + h_3 = 0$$

$$-\frac{h_1\left(h_4 - \frac{h_5 h_6}{h_7}\right)}{h_0 - \frac{h_1 h_6}{h_7}} - \frac{h_7\left(h_2 - \frac{h_3 h_6}{h_7}\right)}{h_4 - \frac{h_5 h_6}{h_7}} + h_5 = 0.$$

The taps h_5 and h_6 were specified to ensure a real solution to the system of equations. The abnormality equations were generated by performing the nearest neighbor algorithm with arbitrary coefficients, and finding conditions such that the degree of a remainder was reduced by more than 1 by setting extreme terms equal to 0. Since at each step, the two extremal coefficients being zero results in an abnormal division, there are many abnormality equations that can be used which result in different filters. □

We call orthogonal filters with at least one vanishing moment "Recluselets" if they have no direct nearest neighbor factorization. The system of equations used in the previous theorem is closely related to the system of equations used to generate Daubechies filters. The Daubechies system has a maximal number of vanishing moment equations (N for a $2N$ length filter), whereas the Recluselet system has one vanishing moment condition but two abnormality equations are added. In general, we find that each vanishing moment equation and abnormality equation reduces the dimension of the solution set by one. This method can be used to generate longer Recluselet filters, although computation time becomes an obstacle around filter length 12.

CHAPTER 3

Nearest Neighbor Factorizations of Daubechies Filters

3.1. Introduction

In [**5, 7**], Daubechies constructs orthonormal, compactly supported wavelets with the maximum number of vanishing moments. We give a modified construction for Daubechies filters. Our goal is to use the asymptotic behavior of Daubechies polynomial roots along with Vieta's formulas to explore asymptotics of the nearest neighbor lifting factorizations. The algorithm described below generates a degree N (length $2N$) Daubechies filter, and is equivalent to the traditional construction presented in [**5**].

(1) Find the $N-1$ roots $\{Y_i\}^{N-1}$ of the polynomial:

$$B_N(y) = \sum_{i=0}^{N-1} \binom{N-1-i}{i} y^{i-1}.$$

(2) Transform the roots $\{Y_i\}^{N-1}$ into $2N-2$ roots, $\{Z_i\}^{2N-2}$, using

$$Z + Z^{-1} = 2 - 4Y.$$

(3) From $\{Z_i\}^{2N-2}$, select the $N-1$ roots which lie inside the unit circle, $\{r_i\}_{i=1}^{N-1}$.

(4) Form the polynomial $\tilde{H}(z)$ with $N-1$ roots $z = r_i$ and N roots at $z = -1$.

$$\tilde{H}(z) = \left(\prod_{i=1}^{N-1} (z - r_i)\right)(z+1)^N = \sum_{i=0}^{2N-1} \tilde{h}_{2N-1-i} z^i$$

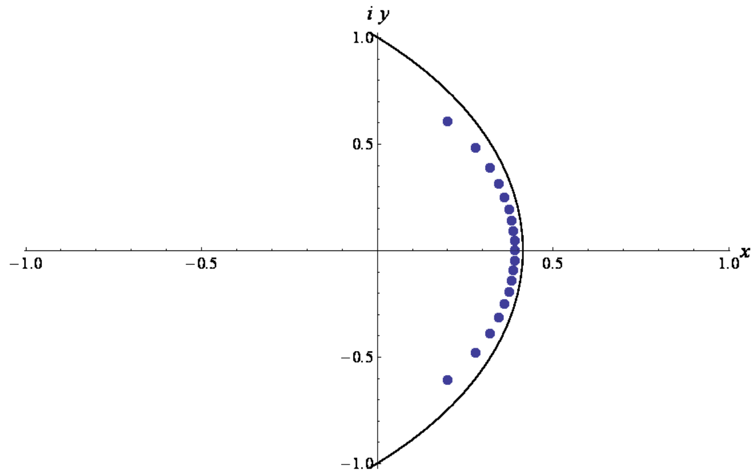


FIGURE 3.1.1. The transformed roots $\{r_i\}_{i=1}^{N-1}$ for $N = 20$ with the limiting curve $|z+1| = \sqrt{2}$

(5) Scale the filter so that $\sum |h_i| = 2$, by dividing $\tilde{H}(z)$ by the constant

$$C = 2^{N-\frac{1}{2}} \prod_{i=1}^{N-1} (1 - r_i).$$

Note that the resulting polynomial has the coefficients indexed in reverse order, so that the filter coefficients match the order found in Daubechies' original construction:

$$H(z) = \frac{\tilde{H}(z)}{C} = \frac{\left(\prod_{i=1}^{N-1} (z - r_i)\right)(z+1)^N}{2^{N-\frac{1}{2}} \prod_{i=1}^{N-1} (1 - r_i)} = \sum_{i=0}^{2N-1} h_{2N-1-i} z^i.$$

This construction has the advantage of not involving negative powers of z, thus allowing the use of Vieta's formulas. In section §3.4, we show $C = 2^{N-\frac{1}{2}} \prod_{i=1}^{N-1} (1 - r_i)$ is strictly positive, and thus results about signs of filter coefficients can typically be proven directly by looking at the unscaled filter $\tilde{H}(z)$.

We heavily use the results from Strang and Shen [**14**] with respect to the locations of the zeros $\{r_i\}_{i=1}^{N-1}$. The authors showed these zeros come in complex conjugate pairs, lie strictly in the right half plane, and converge on the circle $|z+1| = \sqrt{2}$ from the inside as $N \to \infty$.

3.2. Properties of Nearest Neighbor Lifting Step Roots

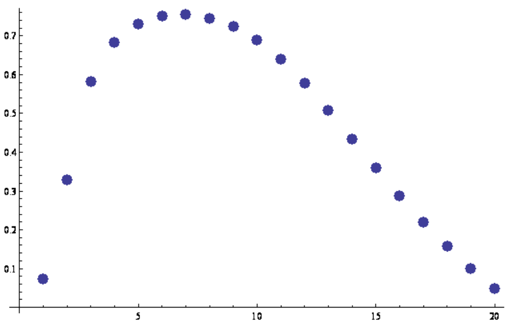
Wickerhauser and Zhu verified that Daubechies filters up to length 20 have a direct nearest neighbor factorization [**19**]. With the factorization algorithm described in section §2.4, there is only one possible division scheme that results in a direct nearest neighbor factorization. This allows for a more efficient algorithm for verifying higher order filters. With these considerations, we verify numerically that all Daubechies filters up to length 220 have exactly one direct nearest neighbor factorizations.

In contrast to general Laurent polynomials, nearest neighbor factorizations for Daubechies filters demonstrate remarkably stable characteristics. For nearest neighbor factorizations, the first lifting step is constant, and the others have the forms $q = cz + d$ or $q = c + dz^{-1}$. Plotting the roots of these lifting steps for the $\{left, sym, \ldots, sym\}$ division scheme for various degrees in figure 3.2.1 demonstrates the stability of the relative size of the lifting step coefficients.

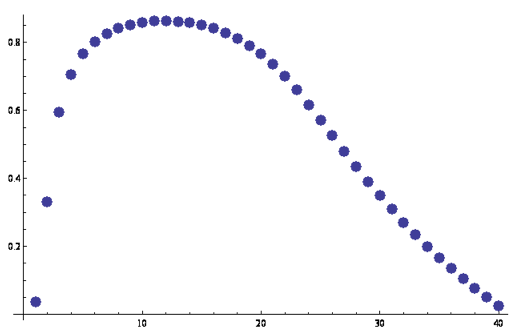
Similar behavior is also found for Daubechies filters with the $\{right, sym, \ldots, sym\}$ division scheme. Various degrees are shown in figure 3.2.2.

Since the lifting steps are degree one Laurent polynomials, the positivity of the lifting steps roots corresponds to opposite signs of the lifting step coefficients (c_i and d_i). This pattern has been observed up to $N = 110$ (length 220). All of the observed lifting step roots ($N = 2$ to $N = 110$) are bounded on $(0, 1)$ for the $\{left, sym, \ldots, sym\}$ division scheme. The existence of these points is enough to show a normal PRS and hence a direct nearest neighbor factorization exists for all N. Due to the predictability of this behavior, we make the following conjectures.

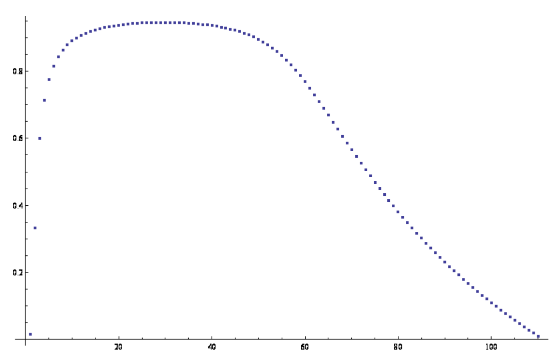
Conjecture 39. *All Daubechies filters of degree $N > 2$ have exactly one direct nearest neighbor factorizations corresponding to the $\{left, sym, \ldots, sym\}$ and $\{right, sym, \ldots, sym\}$ division schemes for even and odd N, respectively.*



(A) $N = 20$

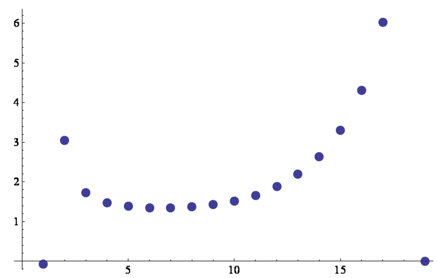


(B) $N = 40$

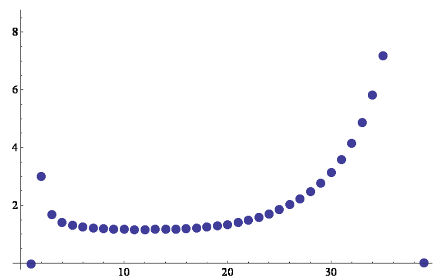


(C) $N = 110$

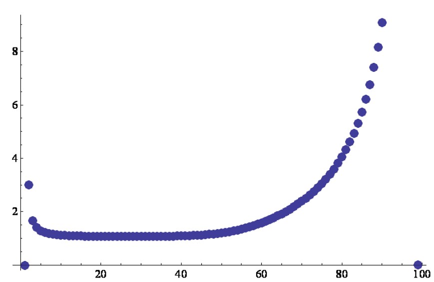
FIGURE 3.2.1. Nearest Neighbor Roots for degree N Daubechies Filter with $\{left, sym, \ldots, sym\}$ for (A) $N = 20$ (B) $N = 40$ (C) $N = 110$



(A) $N = 19$



(B) $N = 39$



(C) $N = 99$

FIGURE 3.2.2. Nearest Neighbor Roots for degree N Daubechies Filter with $\{right, sym, \ldots, sym\}$ for (A) $N = 19$ (B) $N = 39$ (C) $N = 99$

Numerical analysis of the lifting step roots for the $\{left, sym, \ldots, sym\}$ division scheme suggests asymptotic convergence among the first $0.2N$ lifting steps for a filter of length $2N$. This is summarized in the following conjecture.

CONJECTURE 40. *Given a degree N Daubechies filter, with N even, and the*

$$\{left, sym, \ldots, sym\}$$

division scheme, the zeros of the lifting steps $\{Z_i\}_{i=1}^{0.2N}$ are bounded by 1, and the first $0.2N$ lifting steps converge to

$$Z_i = \frac{2i-1}{2i+1}$$

as $N \to \infty$.

3.3. Bounds on Daubechies Filter Coefficients

Using the asymptotics of the Daubechies polynomial roots and their relationship to the Daubechies filter coefficients, we prove results involving the tails of the Daubechies filters and properties of their direct nearest neighbor factorizations. We use a different approach than in **[15]**, where Strang and Shen give global asymptotic behavior of Daubechies filter coefficients. The estimates from **[15]** do not have the accuracy necessary to analyze individual coefficients and their relative sizes.

LEMMA 41. *The scaling coefficient $C = 2^{N-\frac{1}{2}} \prod_{i=1}^{N-1} (1 - r_i)$ used in the above construction of Daubechies filters is strictly positive.*

PROOF. The roots $\{r_i\}_{i=1}^{N-1}$ come in complex conjugate pairs along with a single real positive root when N is even. Strang and Shen **[14]** proved the roots lie strictly in the right half plane and inside the circle $|z+1| = \sqrt{2}$ and $|r_i| < 1$ for all i. Let $M = max_i\left(|r_i|\right) < 1$.

Grouping the terms involving conjugate pairs for odd N and reindexing as needed gives:

$$\prod_{i=1}^{N-1}(1-r_i)=\prod_{i=1}^{\frac{N-1}{2}}(1-r_i)(1-\bar{r}_i)=\prod_{i=1}^{\frac{N-1}{2}}\left(1-r_i^2\right)$$

$$>\prod_{i=1}^{\frac{N-1}{2}}\left(1-M^2\right)>0.$$

For even N, there is a single real positive root, r_{N-1}, along with $\frac{N}{2}-1$ conjugate pairs of roots, so with reindexing we get:

$$\prod_{i=1}^{N-1}(1-r_i)=r_{N-1}\left(\prod_{i=1}^{\frac{N}{2}-1}(1-r_i)(1-\bar{r}_i)\right)=r_{N-1}\left(\prod_{i=1}^{\frac{N}{2}-1}\left(1-r_i^2\right)\right)$$

$$>r_{N-1}\left(\prod_{i=1}^{\frac{N}{2}-1}\left(1-M^2\right)\right)>0.$$

In either case, the product is strictly positive, hence $C>0$. □

Corollary 42. *Given a degree N Daubechies filter $h=\{h_0,h_1,\ldots,h_{2N-1}\}$, then*

$$h_0=\left(2^{N-\frac{1}{2}}\prod_{i=1}^{N-1}(1-r_i)\right)^{-1}=C^{-1}>0.$$

Proof. From the construction for Daubechies filters, we see

$$H(z)=\frac{\tilde{H}(z)}{C}=\frac{\left(\prod_{i=1}^{N-1}(z-r_i)\right)(z+1)^N}{2^{N-\frac{1}{2}}\prod_{i=1}^{N-1}(1-r_i)}=\sum_{i=0}^{2N-1}h_{2N-1-i}z^i$$

Matching coefficients in the above equation, we find h_0 is the leading coefficient. The numerator is a monic polynomial, hence:

$$h_0=C^{-1}=\left(2^{N-\frac{1}{2}}\prod_{i=1}^{N-1}(1-r_i)\right)^{-1}$$

Using lemma 41, $h_0=C^{-1}>0$. □

Using results from Strang and Shen [14], we bound the relative growth of h_0 and h_1.

LEMMA 43. *Given a degree $N > 1$ Daubechies filter $h = \{h_0, h_1, \ldots, h_{2N-1}\}$, then*

$$0 < h_0 \left(-\sqrt{2}N + 2N + \sqrt{2} - 1\right) < h_1 < Nh_0.$$

PROOF. Using the formula in the construction described above,

$$H(z) = \frac{\tilde{H}(z)}{C} = \frac{\left(\prod_{i=1}^{N-1}(z - r_i)\right)(z+1)^N}{2^{N-\frac{1}{2}}\prod_{i=1}^{N-1}(1 - r_i)} = \sum_{i=0}^{2N-1} h_{2N-1-i} z^i.$$

Using Vieta's formula for the h_1 coefficient

$$\sum_{i=1}^{N-1} r_i + \sum_{i=1}^{N}(-1) = \sum_{i=1}^{N-1} r_i - N = -\frac{h_1}{h_0}$$

$$h_1 = -h_0\left(\sum_{i=1}^{N-1} r_i - N\right).$$

The roots $\{r_i\}_{i=1}^{N-1}$ come in complex conjugate pairs and possibly a single real positive root when N is even. Strang and Shen [14] showed

$$0 < Re(r_i) < \sqrt{2} - 1$$

for all i.

Since the sum of the roots must be real, the imaginary parts cancel out, and the upper bound for h_1 is:

$$h_1 = -h_0\left(\sum_{i=1}^{N-1} Re(r_i) - N\right) < -h_0(0 - N) = h_0 N.$$

For the lower bound we use the positivity of the real part of the roots along with the previous result $h_0 > 0$. Again, the roots come in conjugate pairs so only the real parts contribute to the sum.

$$h_1 = -h_0 \left(\sum_{i=1}^{N-1} r_i - N \right) = -h_0 \left(\left(\sum_{i=1}^{N-1} Re\left(r_i\right) \right) - N \right)$$

$$> -h_0 \left((N-1) \left(\sqrt{2} - 1 \right) - N \right) = h_0 \left(-\sqrt{2}N + 2N + \sqrt{2} - 1 \right) > 0.$$

□

Corollary 44. *Given a degree $N > 2$ Daubechies filter $h = \{h_0, h_1, \ldots, h_{2N-1}\}$, then $0 < h_0 < h_1$.*

Proof. The corollary is obvious using $0 < h_0$ and lemma 43,

$$0 < h_0 < h_0 \left(-\sqrt{2}N + 2N + \sqrt{2} - 1 \right) < h_0 \left(0.6N + .5\right) < h_0 N < h_1$$

for all $N > 1$. □

We prove the following lemma which will be useful in upcoming theorems.

Lemma 45. *Given a degree N Daubechies filter $h = \{h_0, h_1, \ldots, h_{2N-1}\}$, h_{2N-1} is nonzero. For even N, $h_{2N-1} < 0$ and for odd N, $h_{2N-1} > 0$.*

Proof. Using the formula from the construction described above,

$$H\left(z\right) = \frac{\tilde{H}\left(z\right)}{C} = \frac{\left(\prod_{i=1}^{N-1} \left(z - r_i\right) \right) \left(z+1\right)^N}{2^{N-\frac{1}{2}} \prod_{i=1}^{N-1} \left(1 - r_i\right)} = \sum_{i=0}^{2N-1} h_{2N-1-i} z^i.$$

Computing the constant coefficient, h_{2N-1}

$$h_{2N-1} = \frac{\prod_{i=1}^{N-1} \left(-r_i\right)}{C} = \left(-1\right)^{N-1} \frac{\prod_{i=1}^{N-1} r_i}{C}.$$

From lemma 41, C is strictly positive, and $\{r_i\}_{i=1}^{N-1}$ are nonzero and strictly in the right half plane from [**14**], thus h_{2N-1} is nonzero. The sign change for even and odd N is apparent from the $\left(-1\right)^{N-1}$ term. □

Corollary 46. *Given a degree $N \geq 1$ Daubechies filter $h = \{h_0, h_1, \ldots, h_{2N-1}\}$, h_{2N-2} is nonzero.*

Proof. For $N = 1$, h is the Haar filter, and thus the corollary holds.

For $N > 1$, we have shown in previous results that h_0, h_1, h_{2N-1} are nonzero. All Daubechies filters satisfy double shift orthogonality conditions, in particular,

$$h_0 h_{2N-2} + h_1 h_{2N-1} = 0.$$

Thus, it is clear $h_{2N-2} = \frac{-h_1 h_{2N-1}}{h_0}$ must be nonzero. □

Lemma 47. *Given a degree $N > 1$ Daubechies filter $h = \{h_0, h_1, \ldots, h_{2N-1}\}$, $\frac{h_{2N-1}}{h_{2N-2}} < 0$. In particular, h_{2N-1} and h_{2N-2} have different signs.*

Proof. Since the filter satisfies double shift orthogonality, along with the previous results showing $h_0, h_1, h_{2N-2}, h_{2N-1}$ are nonzero,

$$\frac{h_{2N-1}}{h_{2N-2}} = \frac{-h_1}{h_0} < 0,$$

using the results that $h_0, h_1 > 0$. Since h_{2N-1} alternates sign as N increases, so does h_{2N-2}. □

3.4. Asymptotics of Daubechies Filter Coefficients

The asymptotics of the Daubechies polynomial roots from Theorem 5 in [**14**] allow us to obtain bounds on the Daubechies filter coefficients. Let $\{Y_k\}$ be the asymptotic estimates for the roots of $B_N(y)$, then:

$$Y_k = \frac{1 - \sqrt{1 - \exp\left(2\pi i \frac{k}{N-1}\right)}}{2}, \quad k = 0, ..., N-2.$$

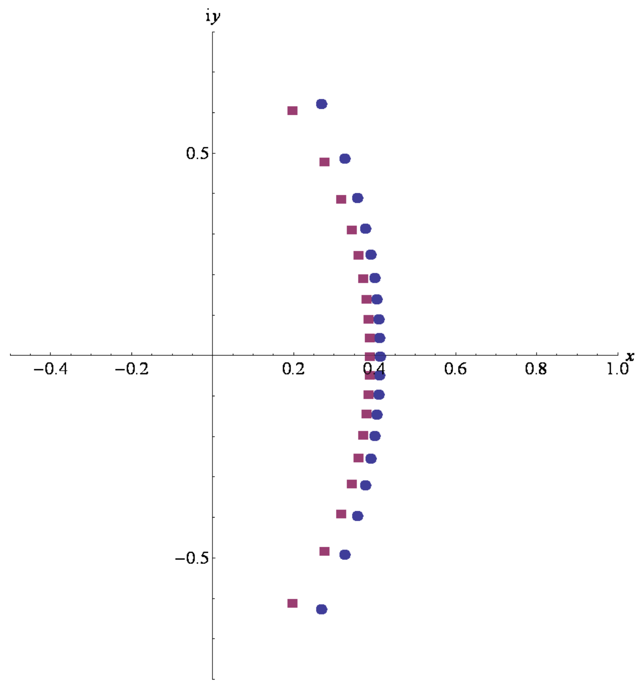


FIGURE 3.4.1. The Daubechies roots (squares) along with the Strang asymptotics (circles) for $N = 20$

Using the transformation $Z + Z^{-1} = 2 - 4Y$ to obtain asymptotic estimates Z_k for the $N-1$ roots $\{r_i\}^{N-1}$,

$$Z_k = \sqrt{1 - e^{\frac{2i\pi k}{N}}} - \sqrt{-e^{\frac{2i\pi k}{N}}}, \quad k = 1, ..., N-1.$$

A plot of the Daubechies polynomial roots and these asymptotic estimates is shown in figure 3.4.1.

We will use these regions to obtain asymptotic bounds for the filter coefficients since the Z_k estimates cannot be used directly. Let j be a positive integer, $\triangle$ be a small positive real value, and $\triangle_j = 2 + (j-1)\triangle$. Then denote the sets of asymptotic Daubechies polynomial roots in the upper half plane as

$$A_j^{\triangle} = \left\{ z \,\middle|\, |z+1| \leq \sqrt{2} \; and \; \arg\left(Z_{\frac{N}{\triangle_j}}\right) \leq \arg(z) \leq \arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) \right\}.$$

Let

$$\theta_j^{\triangle} = \left\{ \arg\left(z_k\right) \middle| z_k \in A_j^{\triangle} \right) \right\}$$

$$B_j^{\triangle} = \left\{ \left|z_k\right| \middle| z_k \in A_j^{\triangle} \right) \right\},$$

then,

$$\arg\left(Z_{\frac{N}{\triangle_j}}\right) \leq \min\left(\theta_j^{\triangle}\right) \leq \max\left(\theta_j^{\triangle}\right) \leq \arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)$$

$$\left|Z_{\frac{N}{\triangle_j}}\right| \leq \min\left(B_j^{\triangle}\right) \leq \max\left(B_j^{\triangle}\right) \leq \left|Z_{\frac{N}{\triangle_{j+1}}}\right|.$$

LEMMA 48. *Let* $\{r_i\}^{N-1}$ *and* $\{Z_i\}^{N-1}$ *be the Daubechies polynomial roots and associated asymptotic estimates defined as above. Then, for any* $A_j^{\triangle}$,

$$1 - o(1) < \frac{\#\left\{r_i \in A_j^{\triangle}\right\}}{\#\left\{Z_i \in A_j^{\triangle}\right\}} < 1 + o(1).$$

PROOF. Fix $A_j^{\triangle}$ and $\epsilon > 0$. In **[14]**, Strang and Shen show the global error for the Z_i approximations to each r_i is $O\left(N^{-\frac{1}{2}}\right)$. Then, for some of the $Z_i \in A_j^{\triangle}$, the associated r_i may not be in the $A_j^{\triangle}$ region. An example region is shown in figure 3.4.2, with the shaded region representing $A_j^{\triangle}$. The dashed region shows the possible locations of the r_i, obtained by drawing a circle around with Z_i with radius $O\left(N^{-\frac{1}{2}}\right)$. Strang proved that the r_i lie strictly inside the circle $|z+1| = \sqrt{2}$, hence we disregard the dashed region outside this circle. Denote the dashed region inside the circle $|z+1| = \sqrt{2}$ by C' and define $C = C' - A_j^{\triangle}$. Then C contains every $r_i \notin A_j^{\triangle}$ approximated by a $Z_i \in A_j^{\triangle}$.

The Z_i are asymptotically evenly distributed along the limiting circle $|z+1| = \sqrt{2}$, hence

$$\#\left\{Z_i \in A_j^{\triangle}\right\} \to \frac{N}{\triangle_j} - \frac{N}{\triangle_{j+1}}.$$

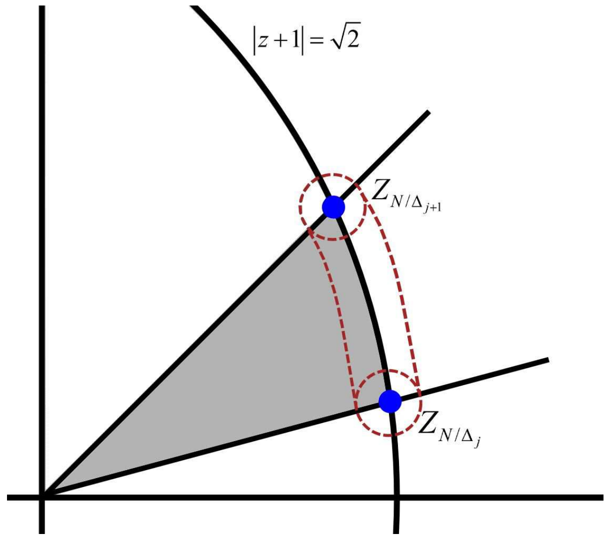


FIGURE 3.4.2. $A_j^{\triangle}$ region with error estimates shown

For a fixed N, there exist a constants K such that $|r_i - Z_i| < KN^{-\frac{1}{2}}$ for all i. Define the set

$$D = \left\{ Z_i \,|\, Z_i \in A_j^{\triangle} \; and \; \left| Z_i - Z_{\frac{N}{\triangle_j}} \right| > KN^{-\frac{1}{2}} \; and \; \left| Z_i - Z_{\frac{N}{\triangle_{j+1}}} \right| > cN^{-\frac{1}{2}} \right\}.$$

Then every $Z_i \in D$ has an associated $r_i \in A_j^{\triangle}$, and

$$\#\left\{ Z_i \in D \right\} = \frac{N}{\triangle_j} - \frac{N}{\triangle_{j+1}} - 2KN^{-\frac{1}{2}} \leq \#\left\{ r_i \in A_j^{\triangle} \right\}$$

Thus,

$$\frac{\#\left\{ r_i \in A_j^{\triangle} \right\}}{\#\left\{ Z_i \in A_j^{\triangle} \right\}} \geq \frac{\frac{N}{\triangle_j} - \frac{N}{\triangle_{j+1}} - 2KN^{-\frac{1}{2}}}{\frac{N}{\triangle_j} - \frac{N}{\triangle_{j+1}}} = 1 - o\left(1\right).$$

Next, consider the Z_i close to the endpoints of the $A_j^{\triangle}$ region,

$$\#\left\{ Z_i \in C \right\} \leq 2cN^{-\frac{1}{2}}.$$

Hence,

$$\frac{\#\left\{r_i \in A_j^{\triangle}\right\}}{\#\left\{Z_i \in A_j^{\triangle}\right\}} \leq \frac{\#\left\{Z_i \in A_j^{\triangle}\right\} + \#\left\{Z_i \in C\right\}}{\#\left\{Z_i \in A_j^{\triangle}\right\}} \leq 1 + \frac{2KN^{-\frac{1}{2}}}{\frac{N}{\triangle_j} - \frac{N}{\triangle_{j+1}}} = 1 + o\left(1\right).$$

□

This lemma shows that the expected number of Daubechies polynomial roots lie in each asymptotic region for sufficiently large N. For example, to find the percentage of Daubechies roots as $N \to \infty$ which have $\frac{\pi}{2} > arg\left(r_i\right) > \frac{\pi}{4}$, use the formula for Z_k with $k = \frac{N}{i}$, and find the i for which $arg\left(Z_{\frac{N}{i}}\right) = \frac{\pi}{4}$,

$$arg\left(Z_{\frac{N}{6}}\right) = arg\left(\sqrt{1 - e^{\frac{2i\pi}{6}}} - \sqrt{-e^{\frac{2i\pi}{6}}}\right) = \frac{\pi}{4}.$$

Thus, for sufficiently large N, $\frac{N-1}{6}$ of the Daubechies roots $\left\{r_i\right\}^{N-1}$ have $\frac{\pi}{2} > \arg\left(r_i\right) > \frac{\pi}{4}$.

We are now able to obtain much sharper bounds on the Daubechies polynomial roots.

LEMMA 49. *Let* $\left\{r_i\right\}^{N-1}$ be the roots of the N^{th} Daubechies polynomial inside the unit circle. Then for sufficiently large N,

$$\sum_{i=1}^{N-1} r_i < 0.36343N.$$

PROOF. We begin by rewriting the sum using the fact that the roots come in complex conjugate pairs with at most one real positive root when N is even. Assume that N is odd so there is no real root, and let $M = \frac{N-1}{2}$. Order and index the roots in the upper half plane $\left\{r_i\right\}^{M}$ by $0 < \arg\left(r_1\right) < ... < \arg\left(r_M\right)$. Then,

$$\sum_{i=1}^{N-1} r_i = \sum_{i=1}^{M}\left(r_i + \bar{r_i}\right) = 2\sum_{i=1}^{M} Re\left(r_i\right).$$

We use the asymptotic regions of the Daubechies polynomial roots to find an upper bound for the sum. Summing over the M roots in the upper half plane corresponds asymptotically to summing over $A_j^{\triangle}$ for $j \geq 1$ and a fixed $\triangle > 0$,

$$\sum_{i=1}^{M} Re\left(r_i\right) \leq \sum_{j=1}^{\infty} Re\left(Z_{\frac{N}{\triangle_j}}\right) \leq \sum_{j=1}^{L} Re\left(Z_{\frac{N}{\triangle_j}}\right) + \frac{N}{\triangle_{L+1}} Re\left(Z_{\frac{N}{\triangle_{L+1}}}\right).$$

These inequalities use lemma 48 to ensure that no significant portion of the r_i fall outside of the $A_j^{\triangle}$ regions. The upper bound decreases as L increases and $\triangle$ decreases. Selecting $L = 10^6$ and $\triangle = 0.01$,

$$\sum_{i=1}^{N-1} r_i \leq 2\left[\sum_{j=1}^{L} Re\left(Z_{\frac{N}{\triangle_j}}\right) + \frac{N}{\triangle_{L+1}} Re\left(Z_{\frac{N}{\triangle_{L+1}}}\right)\right] < 0.36343N.$$

For even N, the contribution of the additional real r_i is at most $\sqrt{2}-1$, which doesn't change the bound for sufficiently large N. $\square$

The key bound in lemma 49 is,

$$\sum_{i=1}^{M} Re\left(r_i\right) \leq \sum_{j=1}^{\infty} Re\left(Z_{\frac{N}{\triangle_j}}\right) + o\left(N\right)$$

as $N \to \infty$.

The 3.4.3 shows the $A_j^{\triangle}$ regions, with the A_2^1 region shaded in gray. The real part of any point in the gray region is bounded above by $Re\left(Z_{\frac{N}{\triangle_2}}\right)$.

A similar strategy allows for us to define a lower bound on the sum of the Daubechies polynomial roots.

Lemma 50. *Let* $\{r_i\}^{N-1}$ be the roots of the N^{th} Daubechies polynomial inside the unit circle. Then for sufficiently large N,

$$0.35581N < \sum_{i=1}^{N-1} r_i.$$

Proof. We begin as above by rewriting the sum using the fact that the roots come in complex conjugate pairs with at most one real positive root when N is even. Assume that

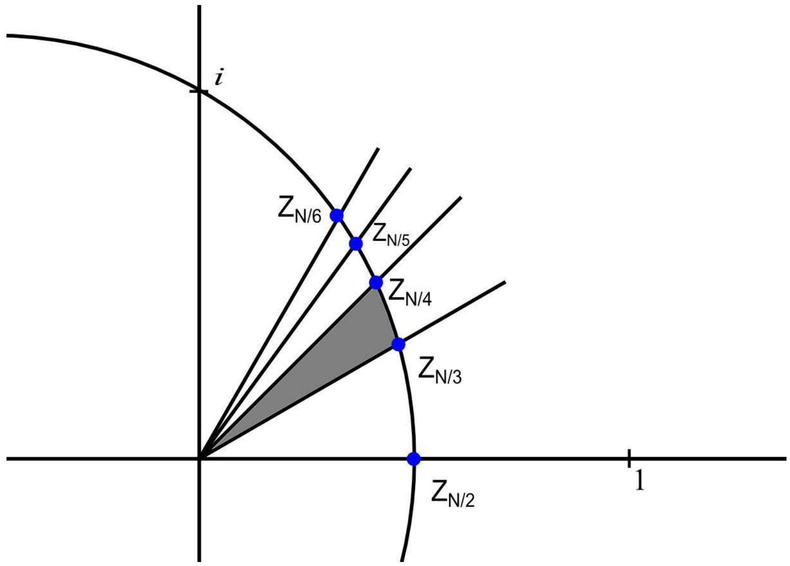


FIGURE 3.4.3. Examples of Asymptotic Regions for Daubechies Polynomial Roots

N is odd so there is no real root, and let $M = \frac{N-1}{2}$. Order and index the roots in the upper half plane $\{r_i\}^M$ by $0 < \arg(r_1) < ... < \arg(r_M)$. Then,

$$\sum_{i=1}^{N-1} r_i = \sum_{i=1}^{M} (r_i + \bar{r_i}) = 2\sum_{i=1}^{M} Re\,(r_i)$$

Summing over the M roots in the upper half plane corresponds asymptotically to summing over $A_j^{\triangle}$ for $j \geq 1$ and a fixed $\triangle > 0$,

$$\sum_{i=1}^{M} Re\,(r_i) \geq \sum_{j=1}^{\infty} Re\left(Z_{\frac{N}{\triangle j+1}}\right) - o\,(N) \geq \sum_{j=1}^{L} Re\left(Z_{\frac{N}{\triangle j+1}}\right) - o\,(N)\,,$$

as $N \to \infty$.

The lower bound increases as L increases and $\triangle$ decreases. Selecting $L = 10^6$ and $\triangle = 0.01$,

$$\sum_{i=1}^{N-1} r_i \geq 2\sum_{j=1}^{L} Re\left(Z_{\frac{N}{\triangle j+1}}\right) > 0.35581N,$$

for sufficiently large N.

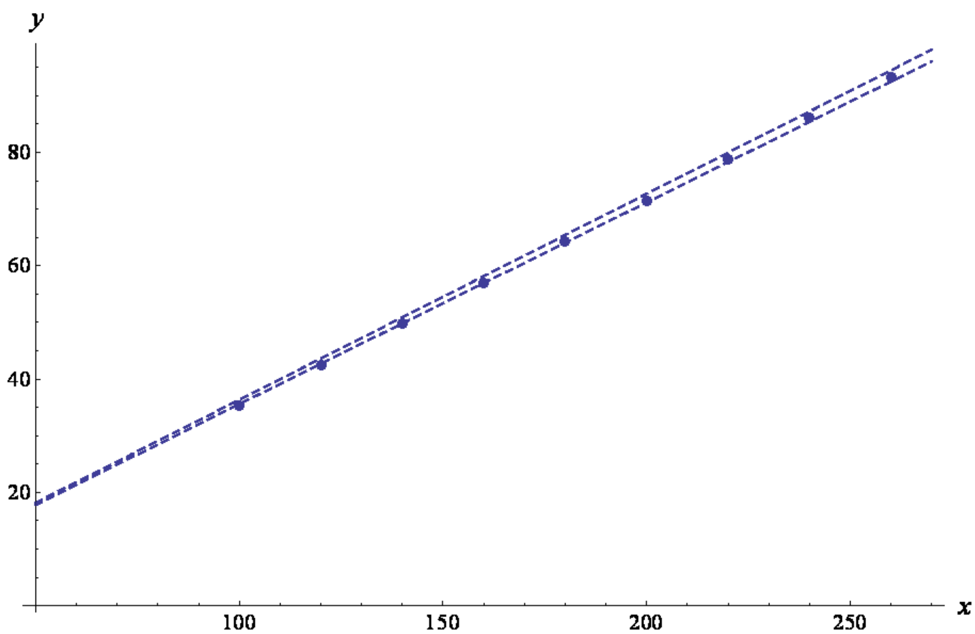


FIGURE 3.4.4. Actual values of $\sum_{i=1}^{N-1} r_i$ for values of $N = 100, 120, \ldots, 260$ along with asymptotic bounds from lemma 49 and lemma 50 (dashed lines).

For even N, the additional real root contributes at most $\sqrt{2} - 1$, and doesn't change the bound for sufficiently large N. $\square$

The behavior of the exact value of the summation along with the bounds given in lemma 49 and lemma 50 is shown in figure 3.4.4. We obtain the value of $\sum_{i=1}^{N-1} r_i$ from the filter coefficients using Vieta's formula for the first coefficient.

$$h_1 = -h_0 \left(\sum_{i=1}^{N-1} r_i - N \right)$$

$$\sum_{i=1}^{N-1} r_i = N - \frac{h_1}{h_0}$$

We use a similar strategy to obtain asymptotic behavior of more complicated symmetric polynomials of Daubechies polynomial roots.

N	$0.35581N$	$\sum_{i=1}^{N-1} r_i$	$0.36343N$
100	35.581	35.296	36.343
120	42.697	42.536	43.612
140	49.813	49.779	50.880
160	56.930	57.026	58.149
180	64.046	64.275	65.417
200	71.162	71.526	72.686
220	78.278	78.778	79.955
240	85.384	86.032	87.223
260	92.511	93.287	94.492

TABLE 1. Sum of Daubechies polynomial roots and asymptotic bounds for large values of N

LEMMA 51. *Let* $\{r_i\}^{N-1}$ be the roots of the N^{th} Daubechies polynomial inside the unit circle. Then for sufficiently large N,

$$\sum_{1\le i_1<i_2\le N-1} r_{i_1} r_{i_2} < 0.072753N^2.$$

PROOF. The roots $\{r_i\}^{N-1}$ come in complex conjugate pairs, with the possibility of a single real root. Assume that N is odd so there is no real root, and let $M = \frac{N-1}{2}$. Order and index the roots in the upper half plane $\{r_i\}^M$ by $0 < \arg(r_1) < ... < \arg(r_M)$. The sum can be rewritten as the sum of conjugate pairs:

$$\sum_{1\le i_1<i_2\le N-1} r_{i_1} r_{i_2} = \sum_{1\le i_1<i_2\le M} r_{i_1} r_{i_2} + r_{i_1}\bar{r_{i_2}} + \bar{r_{i_1}} r_{i_2} + \bar{r_{i_1}}\bar{r_{i_2}} + \sum_{i_1=1}^{M} r_{i_1}\bar{r_{i_1}}$$

$$= 2\sum_{1\le i_1<i_2\le M} Re\left(r_{i_1} r_{i_2}\right) + 2\sum_{1\le i_1<i_2\le M} Re\left(r_{i_1}\bar{r_{i_2}}\right) + \sum_{i_1=1}^{M} \left|r_{i_1}\right|^2.$$

The final summation is strictly positive and only contributes $O(N)$, so we need only bound the other terms. Summing over $\{r_i\}^M$ corresponds asymptotically to summing over $A_j^{\triangle}$ for $j \ge 1$ and a fixed $\triangle > 0$. Bounding the first sum involving products of roots, both of which are in the upper half plane,

$$\sum_{1\leq i_1<i_2\leq M} Re\left(r_{i_1}r_{i_2}\right) \leq \sum_{j=1}^{\infty}\binom{\left|A_j^{\triangle}\right|}{2} Re\left(\left|Z_{\frac{N}{\triangle_{j+1}}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_j}}\right)}\right)$$

$$+\sum_{j=1}^{\infty}\left(\left|A_j^{\triangle}\right|\right)\sum_{k=1}^{j}\left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_{j+1}}}\right|\left|Z_{\frac{N}{\triangle_{k+1}}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_j}}\right)+\arg\left(Z_{\frac{N}{\triangle_k}}\right)\right)}\right). \tag{3.4.1}$$

The first summation corresponds to terms with both roots coming from the same $A_j^{\triangle}$ region. Since for each section there are a total of $\left|A_j^{\triangle}\right|$ roots, and $\binom{\left|A_j^{\triangle}\right|}{2}$ total terms of this type. Given $r_{i_1}, r_{i_2} \in A_j^{\triangle}$, we bound the real part of these terms by

$$Re\left(r_{i_1}r_{i_2}\right) \leq Re\left(\left|Z_{\frac{N}{\triangle_{j+1}}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_j}}\right)}\right) + o\left(N^2\right).$$

This bound holds since,

$$\left|r_{i_1}\right| \leq \left|Z_{\frac{N}{\triangle_{j+1}}}\right|$$

$$\left|r_{i_2}\right| \leq \left|Z_{\frac{N}{\triangle_{j+1}}}\right|$$

$$0 < \arg\left(Z_{\frac{N}{\triangle_j}}\right) \leq \arg\left(r_{i_1}\right) < \frac{\pi}{2}$$

$$0 < \arg\left(Z_{\frac{N}{\triangle_j}}\right) \leq \arg\left(r_{i_2}\right) < \frac{\pi}{2}.$$

The bound consists of taking the extreme values for each $A_j^{\triangle}$, rather than a global estimate on the asymptotics, allowing for a much more accurate bound. The second summation in equation (3.4.1) corresponds to terms involving two roots in the upper half plane, one from $A_j^{\triangle}$ and the second from $A_k^{\triangle}$, where $k < j$.

To bound the remaining sum involving terms on opposite sides of the real axis, we use the same approach as above, with a modified bound for the real part. We again select the largest possible modulus and arguments resulting in the largest real part for the upper bound:

$$\sum_{1\leq i_1<i_2\leq M} Re\left(r_{i_1}\bar{r_{i_2}}\right) \leq \sum_{j=1}^{\infty}\left(\left|A_j^{\triangle}\right|\right)\sum_{k=1}^{j}\left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_{j+1}}}\right|\left|Z_{\frac{N}{\triangle_{k+1}}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_j}}\right)-\arg\left(Z_{\frac{N}{\triangle_{k+1}}}\right)\right)}\right).$$

The bound on the real part subtracts the arguments since this gives a lower bound on the real part of the product.

As L increases and $\triangle$ decreases, the lower bound increases. Selecting $L = 1000$ and $\triangle = 0.2$,

$$\frac{1}{2}\left(\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2}\right) = \sum_{1\leq i_1<i_2\leq M} Re\left(r_{i_1}r_{i_2}\right) + \sum_{1\leq i_1<i_2\leq M} Re\left(r_{i_1}\bar{r_{i_2}}\right)$$

$$\leq \sum_{j=1}^{L}\binom{\left|A_j^{\triangle}\right|}{2} Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)}\right)$$

$$+\sum_{j=1}^{L}\left(\left|A_j^{\triangle}\right|\right)\sum_{k=1}^{j}\left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|\left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)+\arg\left(Z_{\frac{N}{\triangle_{k+1}}}\right)\right)}\right)$$

$$+\sum_{j=1}^{L}\left(\left|A_j^{\triangle}\right|\right)\sum_{k=1}^{j}\left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|\left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)-\arg\left(Z_{\frac{N}{\triangle_k}}\right)\right)}\right)$$

$$+\frac{1}{2}\binom{\left(\frac{2N}{L}\right)}{2} Re\left(\left|Z_{\frac{N}{\triangle_L}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_L}}\right)}\right)$$

$$< \frac{1}{2}\left(0.072753N^2\right).$$

The final summation is an upper bound on the tails of the infinite sums, and increasing the L value greatly reduces the contribution of this term. If N is even, the additional real root contributes $O(N)$ terms, so the result is unchanged for sufficiently large N. $\square$

COROLLARY 52. *Let* $\{r_i\}^{N-1}$ be the roots of the N^{th} Daubechies polynomial inside the unit circle. Then for sufficiently large N,

$$\sum_{1\le i_1<i_2\le N-1} r_{i_1} r_{i_2} > 0.063902N^2.$$

PROOF. We find bounds for

$$\frac{1}{2}\left(\sum_{1\le i_1<i_2\le N-1} r_{i_1} r_{i_2}\right) = \sum_{1\le i_1<i_2\le M} Re\left(r_{i_1} r_{i_2}\right) + \sum_{1\le i_1<i_2\le M} Re\left(r_{i_1}\bar{r_{i_2}}\right) \tag{3.4.2}$$

following the same analysis as lemma 51 with the following changes to the bounds,

$$\sum_{1\le i_1<i_2\le M} Re\left(r_{i_1} r_{i_2}\right) \ge \sum_{j=1}^{\infty} \binom{\left|A_j^{\triangle}\right|}{2} Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)}\right)$$

$$+\sum_{j=1}^{\infty}\left(\left|A_j^{\triangle}\right|\right)\sum_{k=1}^{j}\left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|\left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)+\arg\left(Z_{\frac{N}{\triangle_{k+1}}}\right)\right)}\right).$$

These bounds are very similar to lemma 51, with the opposite endpoints of the asymptotic regions selected. In our notation, this corresponds to $j \to j+1$ and $j+1 \to j$. The key observation is

$$Re\left(r_{i_1} r_{i_2}\right) \ge Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)}\right) - o\left(N^2\right).$$

This bound holds as,

$$|r_{i_1}| \ge \left|Z_{\frac{N}{\triangle_j}}\right|$$

$$|r_{i_2}| \ge \left|Z_{\frac{N}{\triangle_j}}\right|$$

$$0 < \arg\left(r_{i_1}\right) \le \arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) < \frac{\pi}{2}$$

$$0 < \arg\left(r_{i_2}\right) \leq \arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) < \frac{\pi}{2}.$$

The alterations for the bound on the second summation in equation (3.4.2) is

$$\sum_{1 \leq i_1 < i_2 \leq M} Re\left(r_{i_1} \bar{r_{i_2}}\right) \geq \sum_{j=1}^{\infty} \left(\left|A_j^{\triangle}\right|\right) \sum_{k=1}^{j} \left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right| \left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) - \arg\left(Z_{\frac{N}{\triangle_k}}\right)\right)}\right),$$

which again switches the j and $j+1$ indices. As L increases and $\triangle$ decreases, the lower bound increases. Selecting $L = 50$ and $\triangle = 1$,

$$\begin{aligned}
\frac{1}{2}\left(\sum_{1 \leq i_1 < i_2 \leq N-1} r_{i_1} r_{i_2}\right) &= \sum_{1 \leq i_1 < i_2 \leq M} Re\left(r_{i_1} r_{i_2}\right) + \sum_{1 \leq i_1 < i_2 \leq M} Re\left(r_{i_1} \bar{r_{i_2}}\right) \\
&\geq \sum_{j=1}^{L} \binom{\left|A_j^{\triangle}\right|}{2} Re\left(\left|Z_{\frac{N}{\triangle_j}}\right|^2 e^{2i\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right)}\right) \\
&+ \sum_{j=1}^{L} \left(\left|A_j^{\triangle}\right|\right) \sum_{k=1}^{j} \left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right| \left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) + \arg\left(Z_{\frac{N}{\triangle_{k+1}}}\right)\right)}\right) \\
&+ \sum_{j=1}^{L} \left(\left|A_j^{\triangle}\right|\right) \sum_{k=1}^{j} \left|A_k^{\triangle}\right| Re\left(\left|Z_{\frac{N}{\triangle_j}}\right| \left|Z_{\frac{N}{\triangle_k}}\right| e^{i\left(\arg\left(Z_{\frac{N}{\triangle_{j+1}}}\right) - \arg\left(Z_{\frac{N}{\triangle_k}}\right)\right)}\right) \\
&> \frac{1}{2}\left(0.063902 N^2\right).
\end{aligned}$$

There is no need to bound the tails since we are bounding from below, and the case where N is even does not change the result for sufficiently large N as it only adds $O(N)$ terms to the original summation. $\square$

We compute the value of $\sum_{1 \leq i_1 < i_2 \leq N-1} r_{i_1} r_{i_2}$ for large N and compare to the results found in lemma 51 and 52. The value of $\sum_{1 \leq i_1 < i_2 \leq N-1} r_{i_1} r_{i_2}$ can be found from the filter

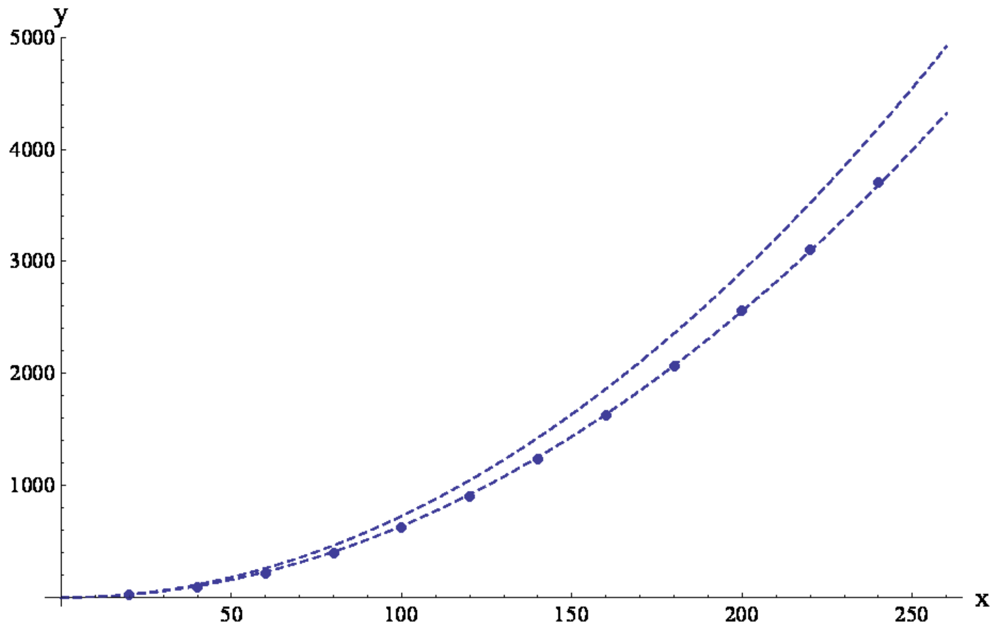


FIGURE 3.4.5. Values of $\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2}$ and asymptotic results from lemma 51 and 52 (dashed lines).

coefficients using Vieta's formulas,

$$\frac{h_2}{h_0} = \sum_{1\leq i_1<i_2\leq 2N-1} R_{i_1}R_{i_2} = \sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2} - N\left(\sum_{i=1}^{N-1} r_i\right) + \binom{N}{2}.$$

Using Vieta's formula to rewrite the sum of roots,

$$\frac{h_2}{h_0} = \sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2} - N\left(N - \frac{h_1}{h_0}\right) + \binom{N}{2}.$$

Thus, we can express $\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2}$ in terms of filter coefficients as

$$\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2} = N\left(N - \frac{h_1}{h_0}\right) - \binom{N}{2} - \frac{h_2}{h_0}.$$

The data for selected values of N are shown in figure 3.4.5 and table 2.

In addition to giving asymptotic behavior of the first few Daubechies filter coefficients, we will use these bounds to prove properties of nearest neighbor factorizations for Daubechies

N	$0.063902N^2$	$\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2}$	$0.072752N^2$
20	25.5608	20.8040	29.1008
40	102.243	92.7918	116.403
60	230.047	216.981	261.907
80	408.973	393.589	465.613
100	639.020	622.718	727.520
120	920.189	904.429	1047.63
140	1252.48	1238.76	1425.94
160	1635.89	1625.74	1862.45
180	2070.42	2065.40	2357.16
200	2556.08	2557.74	2910.08
220	3092.86	3102.79	3521.20
240	3680.76	3700.54	4190.52

TABLE 2. Values of $\sum_{1\leq i_1<i_2\leq N-1} r_{i_1}r_{i_2}$ and asymptotic results from lemma 51 and 52 for $20 \leq N \leq 240$.

filters in section §3.5. We estimate the asymptotic root behavior for the next symmetric polynomial, again providing an upper and lower bound.

LEMMA 53. *Let* $\{r_i\}^{N-1}$ be the roots of the N^{th} Daubechies polynomial inside the unit circle. Then for sufficiently large N,

$$\sum_{1\leq i_1<i_2<i_3\leq N-1} r_{i_1}r_{i_2}r_{i_3} < 0.04223N^3.$$

PROOF. We begin by rewriting the sum to take advantage of the root conjugate pairs. Without loss of generality, assume that N is odd and let $M = \frac{N-1}{2}$ so there is no single real root. Order and index the roots in the upper half plane $\{r_i\}^M$ by $0 < \arg(r_1) < ... < \arg(r_M)$, then

$$\sum_{1\leq i_1<i_2<i_3\leq N-1} r_{i_1}r_{i_2}r_{i_3}$$

$$= \sum_{i_3=1}^{M}(r_{i_3}+\bar{r_{i_3}}) \sum_{1\leq i_1<i_2<i_3}(r_{i_1}r_{i_2}+r_{i_1}\bar{r_{i_2}}+\bar{r_{i_1}}r_{i_2}+\bar{r_{i_1}}\bar{r_{i_2}}) + \sum_{i_1=1}^{M} r_{i_1}\bar{r_{i_1}}\left(\sum_{i_2\neq i_1} r_{i_2}+\bar{r_{i_2}}\right).$$

The last summation only contributes as $O(N^2)$ so we need only find bounds on the other terms:

$$\sum_{i_3=1}^{M}(r_{i_3}+\bar{r_{i_3}})\sum_{1\le i_1<i_2<i_3}(r_{i_1}r_{i_2}+r_{i_1}\bar{r_{i_2}}+\bar{r_{i_1}}r_{i_2}+\bar{r_{i_1}}\bar{r_{i_2}})$$

$$=2\sum_{i_3=1}^{M}Re(r_{i_3})\sum_{1\le i_1<i_2<i_3}(2Re(r_{i_1}r_{i_2})+2Re(r_{i_1}\bar{r_{i_2}}))$$

$$\le 2\sum_{i_3=1}^{M}|r_{i_3}|\sum_{1\le i_1<i_2<i_3}4|r_{i_1}r_{i_2}|\le 2\sum_{i_3=1}^{M}|r_{i_3}|\sum_{1\le i_1<i_2<i_3}4|r_{i_3}|^2=8\sum_{i_3=2}^{M}\binom{i_3}{2}|r_{i_3}|^3.$$

Summing over $\{r_i\}^M$ corresponds asymptotically to summing over $A_j^{\triangle}$ for $j\ge 1$ and a fixed $\triangle>0$. Thus,

$$\sum_{i_3=2}^{M}\binom{i_3}{2}|r_{i_3}|^3\le\sum_{j=1}^{\infty}\left|A_j^{\triangle}\right|\binom{\frac{N}{2}-\frac{N}{\triangle_j}}{2}\left|Z_{\frac{N}{\triangle_{j+1}}}\right|^3\le\sum_{j=1}^{L}\left|A_j^{\triangle}\right|\binom{\frac{N}{2}-\frac{N}{\triangle_j}}{2}\left|Z_{\frac{N}{\triangle_{j+1}}}\right|^3+2\frac{N}{\triangle_{L+1}}\binom{N}{2}.$$

Specifying L and $\triangle$ gives a bound on the symmetric polynomial. As L increases and $\triangle$ decreases, the upper bound decreases. Selecting $L=10^6$ and $\triangle=0.01$,

$$\sum_{1\le i_1<i_2<i_3\le N-1}r_{i_1}r_{i_2}r_{i_3}\le\sum_{j=1}^{L}\left|A_j^{\triangle}\right|\binom{\frac{N}{2}-\frac{N}{\triangle_j}}{2}\left|Z_{\frac{N}{\triangle_{j+1}}}\right|^3+2\frac{N}{\triangle_L}\binom{N}{2}<0.04223N^3.$$

As in previous results, if N is even, it only contributes an additional $O(N^2)$ terms and hence does not change the result. $\square$

The previous results allow us to prove basic facts about the asymptotic behavior of Daubechies filter coefficients. The strategy is to use Vieta's formulas to obtain symmetric polynomials of Daubechies polynomial roots, and then decompose the sum into a form where we can use the previous results.

LEMMA 54. *Given a degree N filter $(h_0,\ldots,h_{2N-1})$, $h_1<h_3$ for sufficiently large N.*

PROOF. From lemma 43, $h_1 < h_0 N$. Let $\{R_i\}^{2N-1}$ be the set of Daubechies polynomial roots $\{r_i\}^{N-1}$ along with the N roots at $z = -1$. Then from Vieta's formula

$$h_3 = h_0 \left(-1\right)^3 \sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} R_{i_1} R_{i_2} R_{i_3}.$$

This is the sum of all possible products of 3 roots from $\{R_i\}^{2N-1}$. Thus, we must show

$$N < \sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} -R_{i_1} R_{i_2} R_{i_3}.$$

There are $\binom{2N-1}{3}$ total terms in the symmetric polynomial which we classify into four types:

1. $\binom{N}{2}\left(N-1\right)$ terms with two roots at $z = -1$ and a single root from $\{r_i\}^{N-1}$: $\left(-1\right)\left(-1\right)\left(r_i\right)$
2. $\binom{N-1}{2} N$ terms with one root at $z = -1$ and two roots chosen from $\{r_i\}^{N-1}$: $\left(-1\right)\left(r_i\right)\left(r_j\right)$
3. $\binom{N-1}{3}$ terms with three roots chosen from $\{r_i\}^{N-1}$: $\left(r_i\right)\left(r_j\right)\left(r_k\right)$
4. $\binom{N}{3}$ terms with three roots at $z = -1$: $\left(-1\right)^3 = -1$

The Daubechies polynomial roots $\{r_i\}^{N-1}$ come in complex conjugate pairs, possibly with a single real positive root, all of which are strictly in the right half plane with $Re\left(r_i\right) < \sqrt{2} - 1$ and inside the unit circle. Since the Daubechies roots come in complex pairs, each term can be paired with its conjugate (of the same type), which results in a real valued filter coefficient, as expected. Using the previous results involving asymptotic bounds for each of the first three types of terms yields, for sufficiently large N,

$$\sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} -R_{i_1} R_{i_2} R_{i_3}$$
$$\ge \left(-0.36343N\right) \binom{N}{2} + \left(0.063902N^2\right) N - \left(0.04223N^3\right) + \binom{N}{3}$$

$$> 0.00662N^3 > N.$$

Using lemma 43, we conclude for sufficiently large N,

$$h_3 = h_0 \left(-1\right)^3 \sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} R_{i_1} R_{i_2} R_{i_3} > h_0 N > h_1.$$

Numerically verifying the lemma for values up to $N = 110$, we find the lemma holds for $7 \le N \le 110$, and numerical results suggest the lemma holds for all $N \ge 7$. □

COROLLARY 55. *Given a degree N filter $h = \{h_0, \ldots, h_{2N-1}\}$, $h_2 < h_3$ for sufficiently large N.*

PROOF. Using Vieta's formulas for the filter coefficients and the previous results

$$h_3 = h_0 \left(-1\right)^3 \sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} R_{i_1} R_{i_2} R_{i_3} > h_0 \left(0.00662 N^3\right).$$

Bounding h_2 using that all terms are comprised of complex numbers on or inside the unit circle yields

$$\begin{aligned} h_2 = h_0 \sum_{1 \le i_1 < i_2 \le 2N-1} R_{i_1} R_{i_2} &\le h_0 \sum_{1 \le i_1 < i_2 \le 2N-1} \left|R_{i_1} R_{i_2}\right| \\ \le h_0 \sum_{1 \le i_1 < i_2 \le 2N-1} 1 &\le h_0 \binom{2N-1}{2} = h_0 O\left(2N^2\right). \end{aligned}$$

□

We can generalize the bound from the proof of 55 in the following lemma.

LEMMA 56. *Given a degree N filter $h = \{h_0, \ldots, h_{2N-1}\}$, $|h_i| \le h_0 \binom{2N-1}{i}$.*

PROOF. Using Vieta's formula along results from [**14**] that Daubechies polynomial roots are inside the unit circle

$$|h_i| = |h_0| \left| \sum_{1 \le j_1 < \ldots < j_i \le 2N-1} R_{j_1} \ldots R_{j_i} \right| \le h_0 \sum_{1 \le j_1 < \ldots < j_i \le 2N-1} \left|R_{j_1} \ldots R_{j_i}\right|$$

$$\leq h_0 \sum_{1 \leq j_1 < \ldots < j_i \leq 2N-1} 1 \leq h_0 \binom{2N-1}{i}.$$

$\square$

3.5. Asymptotics of Nearest Neighbor Factorizations

Using the results of section §3.4, we are able to prove results about the asymptotics of direct nearest neighbor factorizations of Daubechies filters.

THEOREM 57. *Given a degree N Daubechies filter $h = \{h_0, \ldots, h_{2N-1}\}$ with polyphase matrix factored with the $\{left, sym, ..., sym\}$ division scheme, the first division is normal for sufficiently large N.*

PROOF. The first division being normal is equivalent to the extreme terms of the first remainder polynomial being nonzero,

$$h_{2N-2} - \frac{h_0 h_{2N-1}}{h_1} \neq 0 \tag{3.5.1}$$

$$h_2 - \frac{h_0 h_3}{h_1} \neq 0.$$

By the previous lemma, h_1 and h_{2N-2} are nonzero, hence equation (3.5.1) is equivalent to

$$h_1 h_{2N-2} \neq h_0 h_{2N-1}.$$

By the previous lemma, $h_0, h_1 > 0$ and

$$\frac{h_{2N-1}}{h_{2N-2}} < 0.$$

Thus, the right and left hand sides have different signs, so inequality must hold for the first normality condition.

For the second normality condition,

$$h_2 - \frac{h_0 h_3}{h_1} \neq 0$$

$$h_1 h_2 \neq h_0 h_3.$$

Let $\{R_i\}^{2N-1}$ be the set of Daubechies polynomial roots $\{r_i\}^{N-1}$ along with the N roots at $z = -1$. Using Vieta's formulas,

$$h_1 h_2 = \left(h_0 (-1) \sum_{i=1}^{2N-1} R_i\right) \left(h_0 \sum_{1 \leq i_1 < i_2 \leq 2N-1} R_{i_1} R_{i_2}\right)$$

$$h_0 h_3 = (h_0) \left(h_0 (-1) \sum_{1 \leq i_1 < i_2 < i_3 \leq 2N-1} R_{i_1} R_{i_2} R_{i_3}\right).$$

Thus, it is enough to show

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq 2N-1} -R_{i_1} R_{i_2} R_{i_3} < \left(\sum_{i=1}^{2N-1} -R_i\right) \left(\sum_{1 \leq i_1 < i_2 \leq 2N-1} R_{i_1} R_{i_2}\right).$$

Expanding each of the symmetric polynomials into types of terms involving R_i and r_i and using bounds from previous results, for sufficiently large N,

$$-\sum_{i=1}^{2N-1} R_i = -\sum_{i=1}^{N-1} r_i - \sum_{i=1}^{N} (-1) = N - \sum_{i=1}^{N-1} r_i \geq N - 0.36333N = 0.63666N$$

and

$$\begin{aligned} \sum_{1 \leq i_1 < i_2 \leq 2N-1} R_{i_1} R_{i_2} &= \sum_{1 \leq i_1 < i_2 \leq N-1} r_{i_1} r_{i_2} - N \sum_{i=1}^{N-1} r_i + \binom{N}{2} (-1)^2 \\ &\geq \left(0.063902N^2\right) - (0.36333N)\, N + \binom{N}{2} \geq 0.20057N^2 \end{aligned}$$

and

$$\sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} -R_{i_1} R_{i_2} R_{i_3}$$

$$= - \sum_{1 \le i_1 < i_2 < i_3 \le N-1} r_{i_1} r_{i_2} r_{i_3} + N \sum_{1 \le i_1 < i_2 \le N-1} r_{i_1} r_{i_2} - \binom{N}{2} \sum_{i=1}^{N-1} r_i + \binom{N}{3}$$

$$\le \left(-0.042234N^3\right) + N\left(0.07276N^2\right) - \binom{N}{2}(0.36333N) + \binom{N}{3} \le 0.01553N^3$$

Thus, for sufficiently large N

$$\sum_{1 \le i_1 < i_2 < i_3 \le 2N-1} -R_{i_1} R_{i_2} R_{i_3} \le 0.01553N^3$$

$$< \left(0.20057N^2\right)(0.63666N) \le \left(\sum_{i=1}^{2N-1} -R_i\right)\left(\sum_{1 \le i_1 < i_2 \le 2N-1} R_{i_1} R_{i_2}\right).$$

Hence,

$$h_1 h_2 \ne h_0 h_3.$$

Thus, the second normality condition must be satisfied for sufficiently large N

$$h_2 - \frac{h_0 h_3}{h_1} \ne 0.$$

The theorem was verified numerically for $2 \le N \le 110$, and numerical evidence suggests the theorem is true for all $N \ge 2$. $\square$

COROLLARY 58. *Given a degree N Daubechies filter $h = \{h_0, \ldots, h_{2N-1}\}$ with polyphase matrix factored with the $\{right, sym, ..., sym\}$ division scheme, the first division is normal for sufficiently large N.*

PROOF. Given a polyphase matrix with factorization coming from the

$$\{right, sym, ..., sym\}$$

division scheme, the starting polynomials of the Euclidean algorithm are h_o and h_e. Then the normality conditions for the first division are

$$h_3 - \frac{h_1 h_2}{h_0} \neq 0$$

and

$$h_{2N-1} - \frac{h_1 h_{2N-2}}{h_0} \neq 0.$$

Then since $h_0 \neq 0$, these conditions are equivalent to

$$h_0 h_3 \neq h_1 h_2$$

and

$$h_0 h_{2N-1} \neq h_1 h_{2N-2}$$

which are exactly the conditions shown in theorem 57. $\square$

CHAPTER 4

Normality of Polynomial Remainder Sequences

4.1. Introduction

In this chapter we focus on conditions for determining normality of PRS from the starting polynomials, without having to compute the entire PRS. We continue with the notation for Laurent PRS using a single starting Laurent polynomial and splitting it into its even and odd parts as inputs for the Euclidean algorithm. We keep this notation as a convenience, and note that we can define PRS for any two starting Laurent polynomials.

In general, the problem of determining normality of a PRS from the starting polynomials requires computing the entire PRS via the Euclidean algorithm. The PRS can fail to be normal if any extreme coefficient of a remainder polynomial is 0, causing the degree to go down by more than one in a given step. For a starting polynomial with N coefficients, this results in $O(N)$ terms which must be nonzero for the PRS to be normal. If even a single coefficient can take an arbitrary value, it is often possible to make the PRS abnormal. This is the reasoning for the conjectures in the previous chapters as degree N Daubechies filters have a finite solution set [**18**], and hence there are no additional degrees of freedom which can be used to make the PRS abnormal (in contrast with Recluselets). When seeking normality results, it is often useful to reduce the degrees of freedom in the coefficients of the starting polynomial. This is done in the next section with Sturm sequences, which are PRS generated from a polynomial and its derivative. For Sturm sequences, traditional polynomials are used rather than Laurent polynomials, reducing the degrees of freedom in the Euclidean algorithm since there is only one division scheme.

4.2. Sturm Sequences

Sturm sequences are commonly used to find locations of roots of polynomials. The PRS is computed for a polynomial and its derivative, and the sign changes are found at various points. Normality conditions are not related to this root finding method, and are thus largely ignored. For a detailed description of Sturm sequences and Sturm's Theorem see [**13**].

Definition 59. Let A be a square-free polynomial. The Sturm sequence for A is $PRS\left(A, A'\right)$ where A' is the derivative of A.

While the definition can be extended to Laurent polynomials, we only consider Sturm sequences involving traditional polynomials.

Example 60. Let $A\left(x\right) = x^4 + 4x^3 + 6x^2 + 7x + 2$. Then the Sturm sequence for A is

$$\begin{aligned} PRS\left(A, A'\right) &= PRS\left(x^4 + 4x^3 + 6x^2 + 7x + 2, 4x^3 + 12x^2 + 7\right) \\ &= \left\{ \frac{9x}{4} + \frac{1}{4}, \frac{4235}{729} \right\}. \end{aligned}$$

The Sturm sequence is abnormal since the degree decreases by two from A' to the first element of the PRS.

As mentioned in section §4.1, Sturm sequences have fewer degrees of freedom than the general two polynomial case, and much fewer than the general two Laurent polynomial PRS case. The goal is to find algebraic or analytic conditions on the starting polynomial which relate to normality. We begin with a few basic results.

Lemma 61. *Let $A\left(x\right) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n \geq 3$ with real coefficients. The first division of the Sturm sequence of A is abnormal if and only if*

$$a_{n-2} = \frac{\left(a_{n-1}\right)^2 \left(n-1\right)}{2a_n \cdot n}.$$

Proof. Let A be defined as above. Then

$$A'(x) = \sum_{i=0}^{n} i \cdot a_i x^{i-1}$$

Performing the first division of the Sturm sequence yields the leading coefficient (LC) of the remainder

$$LC(r_1) = \frac{2}{n} a_{n-2} - \frac{(n-1)(a_{n-1})^2}{a_n}.$$

The division in the first step is abnormal exactly when this leading coefficient equals 0. Thus, we set the expression equal to zero and solve for a_{n-2}.

$$\frac{2}{n} a_{n-2} - \frac{(n-1)(a_{n-1})^2}{a_n} = 0$$

$$a_{n-2} = \frac{(a_{n-1})^2 (n-1)}{2a_n \cdot n}.$$

□

Unsurprisingly, the normality of the Sturm sequence for a given step only involves a subset of the coefficients on the higher powers of the polynomial. We can make this statement precise by inspecting the Euclidean algorithm in the following lemma.

Lemma 62. *Let $A(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n \geq 3$ with real coefficients. Then the leading coefficient of the j^{th} Sturm sequence element, and hence normality at step j, depends only on $\{a_n, a_{n-1}, \ldots, a_{n-2j}\}$.*

This lemma shows that normality at a given step is tied to only a subset of the roots. This further demonstrates the difficulty in determining normality from the starting polynomials. Every coefficient plays a role in at least one division and hence even a single degree of freedom for the coefficients can often be manipulated to cause the PRS to be abnormal.

The next result links the normality of the first division with the $n-2$ derivative for a degree n polynomial.

THEOREM 63. *Let* $A(x) = \sum_{i=0}^{n} a_i x^i$ *be a polynomial of degree* n *with real coefficients. If* $A^{(n-2)}$*, the* $(n-2)^{nd}$ *derivative of* A*, has a repeated root, then the Sturm sequence of* A *is abnormal. If* $A^{(n-2)}$ *does not have a repeated root, then the first division of the Sturm sequence is normal.*

PROOF. We first compute the $(n-2)^{nd}$ derivative of A,

$$A^{(n-2)}(x) = \frac{n!}{2} a_n x^2 + (n-1)! a_{n-1} x + (n-2)! a_{n-2}.$$

Set this polynomial equal to 0 and solve for x,

$$\frac{n!}{2} a_n x^2 + (n-1)! a_{n-1} x + (n-2)! a_{n-2} = 0$$

$$x = \frac{-a_{n-1}(n-1)! \pm \sqrt{(a_{n-1})^2 (n-1)! - 2 a_n a_{n-2} (n-2)! n!}}{an!}.$$

A$^{(n-2)}$ will have repeated roots exactly when the discriminant is 0. Setting the expression equal to 0 and solving for a_{n-2},

$$(a_{n-1})^2 (n-1)! - 2 a_n a_{n-2} (n-2)! n! = 0$$

$$a_{n-2} = \frac{(a_{n-1})^2 (n-1)}{2 a_n \cdot n}.$$

Comparing with lemma 61, we find this is exactly the condition for the first division to be abnormal. $\square$

This theorem suggests much more algebraic structure to the normality property of Sturm sequences and PRS in general than is currently known, a similar observation also made in [**9**]. Only the first derivative is computed for the Sturm sequence of a polynomial, so a

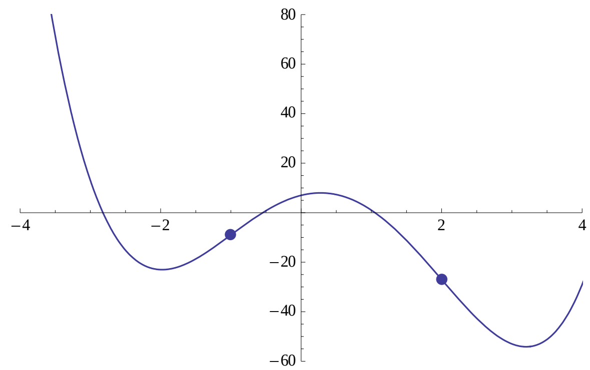


FIGURE 4.2.1. $A(x) = x^4 - 2x^3 - 12x^2 + 7x + 7$ with inflection points shown.

relationship to the roots of a much higher order derivative is surprising. When $deg(A) = 4$, this theorem relates the graph of a polynomial with the normality of its Sturm sequence.

COROLLARY 64. *Let A be a polynomial with real variables and $deg(A) = 4$. If A has two distinct points of inflection, then the first division of the Sturm sequence of A is normal.*

PROOF. The proof follows directly from theorem 63, noting that the $(n-2)^{nd}$ derivative of A in this case is A'', the second derivative. Since the roots of A are the inflection points of A, the corollary follows. $\square$

EXAMPLE 65. Using the corollary, we can now observe the graph of the following quartic polynomial and determine normality of the first division in the Sturm sequence.

The inflection points of the polynomial at $x = -1, 2$ are shown in figure 4.2.1. Since they are distinct, the first division of the Sturm sequence is normal.

An exact condition for normality in terms of the polynomial coefficients can be found by performing the Euclidean algorithm on general coefficients.

COROLLARY 66. *The Sturm sequence for any quadratic polynomial, $A(x) = a_0 + a_1 x + a_2 x^2$, is normal.*

Proof. There is only one division in the Sturm sequence which yields the GCD. The normality condition is then

$$a_0 - \frac{a_1^2}{4a_2} \neq 0.$$

For $a_2 \neq 0$, this is equivalent to the discriminant being nonzero, so $A(x) = a_0 + a_1 x + a_2 x^2$ need only be square-free. Since every Sturm sequence has a square-free polynomial as the starting polynomial by definition, the corollary holds. □

We can generalize the fact observation in the proof of corollary 66.

Fact 67. *Let* $A(x)$ be the polynomial input for a Sturm sequence. Then by definition, $A(x)$ is square-free and hence coprime with its derivative. Then $\gcd(A, A') = c$ for some nonzero constant c.

To completely classify normality for Sturm sequences, perform the Euclidean algorithm on general coefficients and extract the leading coefficients of each element of the PRS. Since every leading coefficient must be nonzero for the Sturm sequence to be normal, the product of the leading coefficients is a normality condition, although performing the Euclidean algorithm on general coefficients results in an exponential growth of expression lengths and computation time.

4.3. Abnormality Conditions and Examples

As previously mentioned, there are many degrees of freedom available to find examples of abnormal PRS. The following example demonstrates that for a given Laurent polynomial h, h_e and h_o can be coprime, have interlaced and strictly monotonic coefficients and still result in an abnormal PRS.

Example 68. The Laurent polynomial

$$A(z) = \frac{170}{9} z^8 + 17z^7 + 16z^6 + 15z^5 + 14z^4 + 13z^3 + 12z^2 + 11z$$

with the $\{left, sym, \ldots, sym\}$ division scheme applied to h_e and h_o is abnormal.

Examples of abnormal PRS with even a single degree of freedom in one of the coefficients are easily constructed by performing the division on the general coefficients, and then setting an extreme coefficient of a remainder polynomial equal to zero and solving. In some cases, however, the expressions for the extreme terms have no solution, which leads to families of polynomials with no abnormal PRS for specified division schemes.

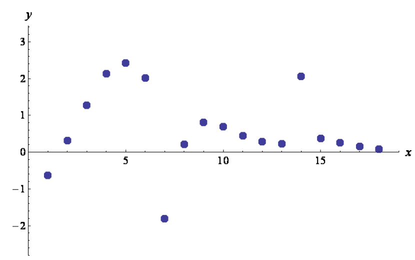
We investigate the plots leading to the conjectures in chapter 2, where the lifting step roots appear to be samples of a continuous limiting curve. While Daubechies filter coefficients are not samples of a single continuous function, the even and odd parts are converging. Coiflets are another family of orthogonal wavelets, discovered by Daubechies in [**6**], and similar behavior is seen in the lifting step roots as shown in figure 4.3.1. Just as in the Daubechies filter case, Coiflets have convergent even and odd parts. Unfortunately, this convergence is not enough to ensure a normal PRS, even for arbitrarily fine samples, as demonstrated with the following example.

EXAMPLE 69. We want to construct a Lipschitz continuous function F on $[0, 1]$ such that any polynomial formed by taking coefficients equal to samples of F at dyadic points results in an abnormal PRS for the $\{left, sym, \ldots, sym\}$ division scheme on the even and odd parts. Since F will be Lipschitz continuous, the even and odd parts of the polynomial formed by sampling will converge, demonstrating this is not a sufficient condition for a normal PRS.

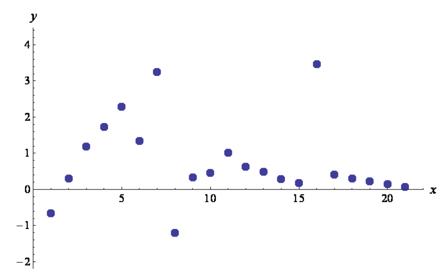
Let

$$F\left(\frac{j}{2^{i+1}}\right) = c_j^i$$

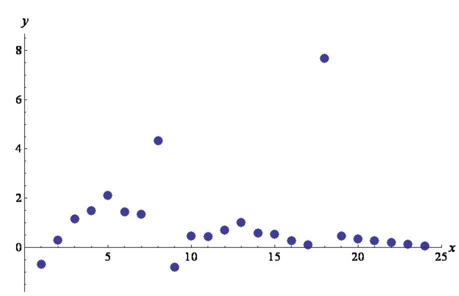
represent the value of the limiting function at dyadic level i and position j. The idea is to manipulate the first sample at each level so the PRS is abnormal, and interpolate the other



(A) Filter Length 36



(B) Filter Length 42



(C) Filter Length 48

FIGURE 4.3.1. Nearest Neighbor Roots for Coiflet filters of various lengths with $\{left, sym, \ldots, sym\}$ division scheme

samples. For the first level,

$$c_1^1 = \frac{1}{4},\ c_2^1 = \frac{1}{2},\ c_3^1 = \frac{3}{4},\ c_4^1 = 1$$

which are samples of $f(x) = x$. Then for $i, j > 1$,

$$c_j^i = c_{\left(\frac{j}{2}\right)}^{i-1}\ for\ even\ j$$

$$c_j^i = \frac{c_{\left(\frac{j-1}{2}\right)}^{i-1} + c_{\left(\frac{j+1}{2}\right)}^{i-1}}{2}\ for\ odd\ j$$

Then the first sample ($j = 1$) for levels $i > 1$ is defined by the following sequence,

$$c_1^i = \frac{c_1^{i-1}\left(c_1^{i-2} + c_1^{i-1}\right)}{2c_1^{i-2}}$$

$$c_1^0 = \frac{1}{2}$$

$$c_1^1 = \frac{1}{4}.$$

The first few values of the sequence are

$$c_1^i = \left\{\frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{21}{128}, \frac{315}{2048}, \ldots\right\}.$$

Then at each level $i > 1$, the abnormality condition

$$c_1^i c_4^i - c_2^i c_3^i = 0$$

holds.

The first few levels are given,

$$\left\{c_j^2\right\}_{j=1}^{8} = \left\{\frac{3}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1\right\}$$

$$\left\{c_j^3\right\}_{j=1}^{16} = \left\{\frac{21}{128}, \frac{3}{16}, \frac{7}{32}, \frac{1}{4}, \frac{5}{16}, \frac{3}{8}, \frac{7}{16}, \frac{1}{2}, \frac{9}{16}, \frac{5}{8}, \frac{11}{16}, \frac{3}{4}, \frac{13}{16}, \frac{7}{8}, \frac{15}{16}, 1\right\}.$$

Define F on $[0,1]$ to be the limit of the dyadic points as $i \to \infty$.

LEMMA 70. *The sequence*

$$\left\{c_1^i\right\}_{i=1}^{\infty}$$

in the construction of F in example 69 is monotonically decreasing and $F(0) > 0$.

PROOF. Recall from example 69,

$$c_1^0 = \frac{1}{2}$$

$$c_1^1 = \frac{1}{4}$$

$$c_1^i = \frac{c_1^{i-1}\left(c_1^{i-2} + c_1^{i-1}\right)}{2c_1^{i-2}}.$$

By induction, we see the sequence is monotonically decreasing as

$$c_1^i = \frac{c_1^{i-1}\left(c_1^{i-2} + c_1^{i-1}\right)}{2c_1^{i-2}} = c_1^{i-1}\left(\frac{c_1^{i-2} + c_1^{i-1}}{2c_1^{i-2}}\right) < c_1^{i-1}.$$

From the construction it is clear

$$F(0) = \lim_{i \to \infty} c_1^i.$$

This limit has a strictly positive value, as:

$$0 < \left(c_1^i - c_1^{i-1}\right)^2$$

$$2c_1^i c_1^{i-1} < \left(c_1^{i-1}\right)^2 + \left(c_1^i\right)^2$$

$$3c_1^i c_1^{i-1} - \left(c_1^{i-1}\right)^2 < c_1^i c_1^{i-1} + \left(c_1^i\right)^2$$

$$\frac{3c_1^i c_1^{i-1} - c_1^{i-1}}{2} < \frac{c_1^i c_1^{i-1} + \left(c_1^i\right)^2}{2c_1^{i-1}}$$

$$2c_1^i - \frac{c_1^i + c_1^{i-1}}{2} < c_1^{i+1}$$

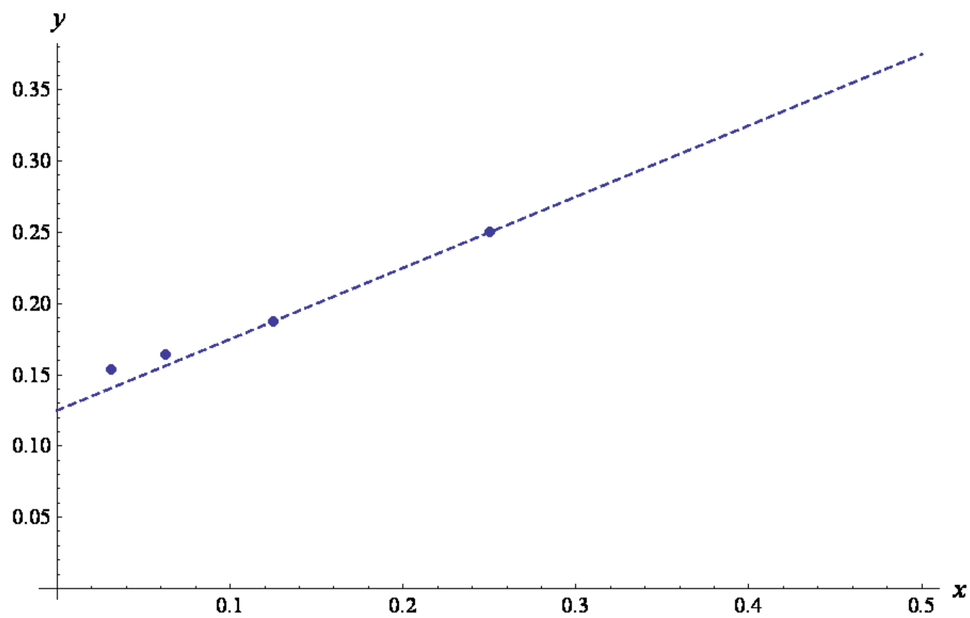


FIGURE 4.3.2. The points from right to left are $c_1^1, c_1^2, c_1^3, c_1^4$ shown with the line through c_1^1 and c_1^2.

The last inequality shows that c_1^{i+1} is strictly above the line through c_1^i and c_1^{i-1}. Hence, $\lim_{i\to\infty} c_1^i$ must lie above the y-intercept of every line segment joining c_1^i and c_1^{i-1} for any i. Since the y-intercept of the segment joining c_1^1 and c_1^2 is 1/8 as shown in figure 4.3.2, the limit is strictly positive.

□

LEMMA 71. *The function F defined in example 69 is Lipschitz continuous.*

PROOF. The construction interpolates every point except the leftmost sample at each level i. Thus, F is a continuous piecewise linear function on the interval

$$\left[\frac{1}{2^{i+1}}, 1\right]$$

for $i \to \infty$, where the sequence $\{c_1^i\}_{i=1}^{\infty}$ are the endpoints of the linear segments. Using lemma 70 shows this sequence is monotonically decreasing and converges to a positive value, thus the slopes of each linear segment are

$$\left\{c_1^i - c_1^{i+1}\right\}_{i=1}^{\infty},$$

which is a strictly decreasing sequence converging to 0. Thus, for any $x, y \in [0,1]$

$$|F(x) - F(y)| \leq |x - y|,$$

hence F is Lipschitz continuous. $\square$

We now find a family of polynomials with normal PRS for the $\{left, sym, \ldots, sym\}$ and $\{right, sym, \ldots, sym\}$ division schemes. These polynomials are generated in a similar manner as the previous example by sampling a given function to generate the coefficients. A nonconstant linear function sampled at equal intervals always has a normal PRS whenever the even and odd parts have full degree. We begin with some technical lemmas.

LEMMA 72. *Let $c, d \in \mathbb{R}$ be nonzero constants and let n be an integer with $n > 1$. Suppose the coefficients of two Laurent polynomials are arithmetic progressions of the forms:*

$$\{c, 2c, \ldots, nc\}$$

and

$$\{d, 2d, \ldots, (n-1)\,d\}.$$

Then symmetric division on the above Laurent polynomials results in a remainder polynomial with coefficients in an arithmetic progression of the form

$$\left\{-\frac{nc}{n-1}, -\frac{2nc}{n-1}, \ldots, -\frac{(n-2)\,nc}{n-1}\right\}.$$

In addition, the symmetric division is normal.

Proof. The proof is a straightforward computation. Symmetric division is equivalent to the following operations on the coefficient arrays,

$$\left\{2c-\frac{2d}{d}c-\frac{1}{n-1}nc,3c-\frac{3d}{d}c-\frac{2}{n-1}nc,\ldots,(n-1)\,c-\frac{(n-1)\,d}{d}c-\frac{n-2}{n-1}nc\right\}=$$

$$\left\{\frac{nc}{n-1},\frac{2nc}{n-1},\ldots,\frac{(n-2)\,nc}{n-1}\right\}.$$

The division is normal since $c\neq 0$ and $n>1$ by assumption. $\square$

We now show that polynomials with coefficients generated by sampling nonconstant linear functions have normal PRS for the $\{left, sym, \ldots, sym\}$ and $\{right, sym, \ldots, sym\}$ division schemes.

Theorem 73. *Let $P(x)$ be any Laurent polynomial with coefficients of the form*

$$\{A, A+k, A+2k, A+3k, \ldots, A+(2n-1)\,k\},$$

where $k\neq 0$ and the even and odd parts of $P(x)$ have degree $n-1$. Then the

$$\{left, sym, \ldots, sym\} \text{ and } \{right, sym, \ldots, sym\}$$

division schemes starting with the even and odd parts of $P(x)$ are normal.

Proof. Using lemma 72 it is enough to show that two consecutive remainder polynomials have coefficients satisfying the conditions in lemma 72 for normality to hold. We start with the $\{left, sym, \ldots, sym\}$ division scheme. The coefficients of the starting polynomials are

$$\{A, A+2k, A+4k, \ldots, A+(2n-2)\,k\}$$

and

$$\{A+k, A+3k, A+5k, \ldots, A+(2n-1)\,k\}.$$

Performing the $\{left\}$ division results in a remainder with coefficients

$$\left\{(A+2jk)-\frac{A}{A+k}\left(A+(2j+1)\,k\right)\right\}_{j=1}^{n-1}$$

$$\left\{\frac{2k^2}{A+k},\frac{4k^2}{A+k},\ldots,\frac{(2n-2)\,k^2}{A+k}\right\}.$$

Note that $k \neq 0$ and since the even and odd parts of A have degree $n-1$ by assumption, $A+k \neq 0$, so the first division is normal. In addition, the remainder polynomial has coefficients in arithmetic progression of the form given in lemma 72.

Continuing with symmetric division yields a remainder polynomial with coefficients of the form

$$\left\{A+(2j-1)\,k-\frac{A+k}{\left(\frac{2k^2}{A+k}\right)}\left(\frac{2jk^2}{A+k}\right)-\frac{A+(2n-1)\,k}{\left(\frac{(2n-2)k^2}{A+k}\right)}\left(\frac{2\,(j-1)\,k^2}{A+k}\right)\right\}_{j=2}^{n-1}=$$

$$\left\{-\frac{n}{n-1}\left(A+k\right),-\frac{2n}{n-1}\left(A+k\right),\ldots,-\frac{(n-2)\,n}{n-1}\left(A+k\right)\right\}.$$

Thus, the second division is normal, and the coefficients of the remainder polynomial are in arithmetic progression of the form given in lemma 72. Since the rest of the divisions are symmetric division and the inputs both have coefficients in arithmetic progression, the theorem holds.

For the $\{right, sym, \ldots, sym\}$ division scheme, the same argument holds with the array manipulations associated with $\{right\}$ division. □

Corollary 74. *Any two Laurent polynomials with coefficients of the following forms are coprime,*

$$\{A, A+2k, A+4k, \ldots, A+(2n-2)\,k\}$$

and

$$\{A+k, A+3k, A+5k, \ldots, A+(2n-1)\,k\},$$

given that the extreme coefficients for both polynomials are nonzero and $n > 1$.

PROOF. We need only observe that the extreme coefficients for any of the remainder polynomials in the Euclidean algorithm are nonzero as shown in theorem 73. Then the GCD is a monomial and hence the polynomials are coprime. $\square$

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