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Regularity of the Bergman Projection on Variants of the Hartogs Triangle

by

Liwei Chen

A dissertation presented to the
Graduate School of Arts & Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
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Table of Contents

	Page
Acknowledgments	iv
ABSTRACT OF THE DISSERTATION	vi
1 Introduction	1
1.1 Setup and Background History	1
1.2 Problems and Results	3
1.3 Outline and Organization	13
1.4 Future Directions	15
1.4.1 The two-weight inequality	15
1.4.2 Non-smooth domains	15
1.4.3 Smooth domains	16
1.4.4 Biholomorphic mappings	16
2 The Notion of Inflation	18
2.1 Preliminaries and Basic Definitions	18
2.2 The Inflation Theorem	19
3 The L^p Regularity	24
3.1 The Punctured Disk	24
3.2 The Hartogs Triangle	34
3.3 The n -dimensional Hartogs Triangle	35
4 The Two-weight Inequality	39
4.1 Preliminaries and Basic Definitions	39
4.2 The Properties of the Operator E	41
4.3 Analysis on Variants of A_p Class	44
4.4 Proof of Theorem 6.	48
5 A Wider Class of Weights	53
5.1 Alternative Proof of Theorem 2 and Its Extension	53
5.2 A Two-weight Estimate on the Hartogs Triangle	61
6 More Analysis on the $A_p^+(\mathbb{R}_+^2)$ Class	63
6.1 Preliminaries and Basic Definitions	63
6.2 Properties of the Special Maximal Function Operator	65
7 Twisted-Weighted Projections	74

	Page
7.1 Preliminaries and Basic Definitions	74
7.2 The L^p Regularity	76
8 The L^p Sobolev Regularity	79
8.1 Preliminaries and Basic Definitions	79
8.2 Transfer to the Product Model	81
8.3 Convert the Differential Operators on \mathbb{D}^*	83
8.4 Proof of Theorem 13	86
8.5 Generalization to n -dimensional Hartogs Triangle	91
References	94

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Dedicated to My Family.

ABSTRACT OF THE DISSERTATION

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by

Liwei Chen

Doctor of Philosophy in Mathematics,
Washington University in St. Louis, 2015.

Professor Steven G. Krantz, Chair

The Bergman projection is the orthogonal projection from the space of square integrable functions onto the space of square integrable holomorphic functions on a domain. Initially, the projection is defined on the L^2 space, but its behavior on other function spaces, e.g. L^p , Sobolev and Hölder spaces, is of considerable interest.

In this dissertation, we focus on the Hartogs triangle which is a classical source of counterexamples in several complex variables, and generalize it to higher dimensions. We investigate the L^p mapping properties of the weighted Bergman projections on these Hartogs domains. As applications, we obtain the L^p regularity of the twisted-weighted Bergman projections and the weighted L^p Sobolev regularity of the ordinary Bergman projection on the corresponding domains.

1. Introduction

1.1 Setup and Background History

The Bergman theory has been a seminal part of geometric analysis and partial differential equations since its invention by Stefan Bergman in 1922. It is based on a very simple idea: suppose Ω is a domain in \mathbb{C}^n ; the set of square integrable holomorphic functions on Ω , denoted by $A^2(\Omega)$, forms a closed subspace of the Hilbert space $L^2(\Omega)$. The *Bergman projection* associated to Ω , is the orthogonal projection

$$\mathcal{B} : L^2(\Omega) \rightarrow A^2(\Omega),$$

which has an integral representation

$$\mathcal{B}(f)(z) = \int_{\Omega} B(z, \zeta) f(\zeta) dV(\zeta),$$

for all $f \in L^2(\Omega)$ and $z \in \Omega$. Here the function $B(z, \zeta)$ defined on $\Omega \times \Omega$ is the *Bergman kernel*.

Different types of regularity of the Bergman projection are of particular interest. When Ω is bounded, smooth, and strongly pseudoconvex (or weakly pseudoconvex with additional regularity properties on the boundary, e.g. finite type, property (P), and etc.), the regularity of \mathcal{B} in $W^k(\Omega)$ and hence in $C^\infty(\overline{\Omega})$ have been intensively studied through the literature.¹ See, for example, [Str10] and references therein for details.

¹Here the Sobolev space $W^k(\Omega) = L_k^2(\Omega)$, and for the definition of $L_k^p(\Omega)$, see Chapter 8.

As well as the regularity in $W^k(\Omega)$, the regularity of \mathcal{B} in $L_k^p(\Omega)$ and the Hölder estimates also have been considerably studied for many years. We mention some important results here. In [PS77], Phong and Stein dealt with bounded smooth strongly pseudoconvex domains by applying the estimates of the Bergman kernel in [Fef74]. In [NRSW89], [MS94] and [CD06], the corresponding authors studied smoothly bounded pseudoconvex domains of finite type under additional assumptions. In [KR14], Khanh and Raich considered smoothly bounded pseudoconvex domains satisfying f -property, hence obtained the regularity for the finite type case and a class of domains of infinite type.

There are also results for irregularity of \mathcal{B} in $L_k^p(\Omega)$ when the underlying domains are smooth and bounded, see [Bar84, Bar92, Chr96, BŞ12], and regularity in $L^p(\Omega)$ when the underlying domains are only assumed to be C^2 smooth and bounded, see [LS12]. From these, we see besides the smoothness and boundedness of the underlying domain, we need additional regular assumption on the boundary.

When dealing with non-smooth domains, in [KP07] and [KP08], Krantz and Peloso show that the Bergman projection for the two dimensional non-smooth worm domain is bounded only when p is in a range depending on the winding of the domain. Even in a planar domain, we cannot expect that the L^p regularity holds for all $p \in (1, \infty)$, and the A_p^+ class in some sense interprets the regularity condition on the boundary, see [LS04], [Zey13], and Chapter 4 for details.²

²For the definition of A_p^+ , see Chapter 4.

1.2 Problems and Results

In this dissertation, we study a class of bounded Hartogs domains which are variants of the Hartogs triangle³

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

To be precise, for $j = 1, \dots, l$, let Ω_j be a smooth bounded domain in \mathbb{C}^{m_j} with a biholomorphic mapping $\phi_j : \Omega_j \rightarrow \mathbb{B}^{m_j}$ between Ω_j and the unit ball \mathbb{B}^{m_j} in \mathbb{C}^{m_j} . We use the notation \tilde{z}_j to denote the j th m_j -tuple in $z' \in \mathbb{C}^{m_1 + \dots + m_l}$, that is $z' = (\tilde{z}_1, \dots, \tilde{z}_l)$. Let $m = m_1 + \dots + m_l \geq 1$, $d = n - m \geq 1$, and $z = (z', z'')$ where $z'' \in \mathbb{C}^d$, we define the n -dimensional Hartogs triangle by

$$\mathbb{H}_{\phi_j}^n = \{z \in \mathbb{C}^n : \max_{1 \leq j \leq l} |\phi_j(\tilde{z}_j)| < |z_1''| < |z_2''| < \dots < |z_d''| < 1\}. \quad (1.1)$$

When $m = 1$, $n = 2$, and ϕ_1 is the identity map, we obtain the classical Hartogs triangle.

Example 1 For nontrivial examples, we can take ϕ_j to be nonsingular linear mappings.

When $n = 4$, $l = 2$, $m_1 = 1$, $m_2 = 2$, $\phi_1(z_1) = 2z_1 - 1$, and $\phi_2(z_2, z_3) = (z_2 + \frac{1}{2}z_3, z_3)$, we obtain a bounded domain which is the intersection of two unbounded domains,

$$\left\{ z \in \mathbb{C}^4 : |2z_1 - 1|^2 < |z_4|^2 < 1, \left| z_2 + \frac{1}{2}z_3 \right|^2 + |z_3|^2 < |z_4|^2 < 1 \right\}.$$

We can also take ϕ_j to be nonlinear; then we may obtain other types of domains.

When $n = 3$, $l = 1$, $m_1 = 2$, and $\phi_1(z_1, z_2) = (\frac{z_1}{z_2 - 10}, 3z_2 + 1)$, the domain becomes

$$\left\{ z \in \mathbb{C}^3 : \left| \frac{z_1}{z_2 - 10} \right|^2 + |3z_2 + 1|^2 < |z_3|^2 < 1 \right\}.$$

By this consideration, the domains $\mathbb{H}_{\phi_j}^n$ can be a large class of bounded Hartogs domains.

³It should be mentioned that, there are several people who recently care about the regularity of the Bergman projection on the Hartogs triangle, see [CS13] and [CZ14] for related results.

It is well known that the topological closure of \mathbb{H} does not possess a Stein neighborhood basis, and the solution to the $\bar{\partial}$ -equation on \mathbb{H} is not globally regular. We cannot expect the L^p regularity of Bergman projection on these variants of the Hartogs triangle to behave very well. A first question is the following.

Question 1 *Determine the exact range for p , so that the Bergman projection on $\mathbb{H}_{\phi_j}^n$ is L^p bounded.*

To answer this question, we have the following theorem.

Theorem 1 *The Bergman projection $\mathcal{B}_{\mathbb{H}_{\phi_j}^n}$ on $\mathbb{H}_{\phi_j}^n$ is bounded on $L^p(\mathbb{H}_{\phi_j}^n)$ if and only if p is in the range $(\frac{2n}{n+1}, \frac{2n}{n-1})$.*

It is quite interesting that the boundedness range for p does not depend on m and $\{\phi_j\}$, but only on the dimension n . For the classical Hartogs triangle, taking $n = 2$, we see the Bergman projection on the Hartogs triangle is L^p bounded if and only if $p \in (\frac{4}{3}, 4)$. Then we may ask the following natural question.

Question 2 *When we look at some weighted space of the (n -dimensional) Hartogs triangle, can we obtain the L^p regularity of the weighted Bergman projection for a larger range of p ?*

The answer is affirmative. Note that, for the classical Hartogs triangle, the boundary at $(0, 0)$ is not even Lipschitz, and this singularity may blow things up. A natural way to control boundary behavior of the singularity is the use of weights which measure the distance from the points near the boundary to the singularity at the boundary. Since on the Hartogs triangle we have $|z_2| < |z| < \sqrt{2}|z_2|$ where $z = (z_1, z_2) \in \mathbb{H}$, it is reasonable to consider a weight of the form $|z_2|^{s'}$, for some $s' \in \mathbb{R}$.⁴

⁴See Chapter 2 for detailed definitions of weighted space and the weighted Bergman projection.

Before stating the results, we first consider the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, with weight $\mu(z) = |z|^{s'}$, where $z \in \mathbb{D}^*$ and $s' \in \mathbb{R}$.

Theorem 2 For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\mathcal{B}_{s'}$ be the weighted Bergman projection on the space (\mathbb{D}^*, μ) , where $\mu(z) = |z|^{s'}$.

- (a) For $s' \in (0, \infty)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1}\right)$.
- (b) For $s' \in [-3, 0]$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded for $p \in (1, \infty)$.
- (c) For $s' \in (-4, -3)$, with $k = -2$ and $s \in (0, 1)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(2 - s, \frac{2-s}{1-s}\right)$.
- (d) When $s' = -4$, \mathcal{B}_{-4} is $L^p(\mu)$ bounded for $p \in (1, \infty)$.
- (e) For $s' \in (-\infty, -4)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1}\right)$.

By the notion of inflation,⁵ one obtains the following result which improves the classical case in Theorem 1.

Theorem 3 For $s' \in [0, \infty)$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\lambda(z) = |z_2|^{s'}$, where $z \in \mathbb{H}$. The weighted Bergman projection \mathcal{B}_λ on the weighted space (\mathbb{H}, λ) is $L^p(\mathbb{H}, \lambda)$ bounded if and only if $p \in \left(\frac{s+2k+4}{s+k+2}, \frac{s+2k+4}{k+2}\right)$.

If we allow the weight λ on \mathbb{H} to be unbounded, then we can obtain another result.

Theorem 4 Given any $p_0 \in [1, 2)$ with its conjugate exponent p'_0 , let $\lambda(z) = |z_2|^{-(p_0+4)}$, where $z \in \mathbb{H}$. Then the weighted Bergman projection \mathcal{B}_λ on the weighted space (\mathbb{H}, λ) is $L^p(\mathbb{H}, \lambda)$ bounded if and only if $p \in (p_0, p'_0)$.

⁵See Chapter 2 for details.

We can apply the same technique several times to obtain the result of the n -dimensional Hartogs triangle, which give a full answer to the second question.

Theorem 5 *Using the notation as in (1.1), let $\lambda(z) = |z_1''|^{s_1} \cdots |z_d''|^{s_d}$, where $z \in \mathbb{H}_{\phi_j}^n$ and $s_1, \dots, s_d \in \mathbb{R}$. Let $\mathcal{B}_{s'}$ be as in Theorem 2. Then the weighted Bergman projection on $(\mathbb{H}_{\phi_j}^n, \lambda)$ is $L^p(\lambda)$ bounded if and only if each of the following projections*

$$\mathcal{B}_{2m+s_1}, \mathcal{B}_{2m+s_1+s_2+2}, \dots, \mathcal{B}_{2m+s_1+\dots+s_d+2(d-1)}$$

is L^p bounded on the corresponding weighted space.

In other words, assume that $p > 1$ and for $j = 1, 2, \dots, d$ we let I_j be one of the intervals for p in Theorem 2, so that the j th projection above is L^p bounded if and only if $p \in I_j$. Then the weighted Bergman projection on $(\mathbb{H}_{\phi_j}^n, \lambda)$ is $L^p(\lambda)$ bounded if and only if $p \in \cap I_j$.

After considering some particular weights on the underlying domains, one may ask another natural question.

Question 3 *Can we obtain the L^p regularity of the weighted Bergman projection on the Hartogs triangle for a wider class of weights rather than some power of the norm of the variable?*

The answer is partly affirmative. Follow Zeytuncu's idea in [Zey13], and using the singular integral approach due to Lanzani and Stein in [LS04], we can consider the weighted space (\mathbb{D}^*, μ) , where $\mu(z) = |z|^{s'} |g(z)|^2$, $s' \in \mathbb{R}$, and g is a non-vanishing holomorphic function on the unit disk. Note that, by applying a Möbius transform, the isolated pole or zero indeed can be any point in the unit disk.

The key observation is that the weighted Bergman kernel $B_{s'}(z, \zeta)$ associated to $(\mathbb{D}^*, |z|^{s'})$, can be expressed as a "homotopy" between two weighted Bergman kernels

$$B_{s'}(z, \zeta) = \frac{s}{2} B_{2k+2}(z, \zeta) + \left(1 - \frac{s}{2}\right) B_{2k}(z, \zeta)$$

for $(z, \zeta) \in \mathbb{D}^* \times \mathbb{D}^*$, where $s' = s + 2k$, $k \in \mathbb{Z}$ and $s \in (0, 2]$. Then, after applying the Cayley transform $\varphi : \mathbb{R}_+^2 \rightarrow \mathbb{D}$, where $\varphi(z) = \frac{i-z}{i+z}$, one arrives at different types of the following two-weight inequality on the upper half plane

$$\int_{\mathbb{R}_+^2} \left| \mathcal{B}_{\mathbb{R}_+^2}(f)(z) \right|^p \mu_1(z) dV(z) \leq C \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) dV(z), \quad (1.2)$$

where μ_1 and μ_2 are two weights on \mathbb{R}_+^2 and

$$\mathcal{B}_{\mathbb{R}_+^2}(f)(z) = \frac{1}{\pi} \int_{\mathbb{R}_+^2} -\frac{f(w) dV(w)}{(z - \bar{w})^2}$$

is the Bergman projection on the upper half plane.⁶

This observation leads to an extremely interesting and elegant theory in harmonic analysis. Note that the operator $\mathcal{B}_{\mathbb{R}_+^2}$ has a similar expression as the so-called Hilbert integral

$$\mathcal{H}(f)(x) = \int_0^\infty \frac{f(y) dy}{x + y},$$

where $x > 0$, see [PS86a, PS86b] for further details. Both $\mathcal{B}_{\mathbb{R}_+^2}$ and \mathcal{H} , although not singular at the diagonal line, have a very close relation with the general Calderón-Zygmund singular integrals, see [LS04]. In potential theory and classical harmonic analysis, it is of particular interest to obtain a weighted L^p estimate of a singular integral for some weight on the underlying domain. In the 1970s, Muckenhoupt introduced the A_p class to show

⁶We should point out that, initially the underlying domain is $\mathbb{R}_+^2 \setminus \{i\}$. However, once we extend the corresponding weights on $\mathbb{R}_+^2 \setminus \{i\}$ to \mathbb{R}_+^2 , then the validity of (1.2) over $\mathbb{R}_+^2 \setminus \{i\}$ and \mathbb{R}_+^2 are equivalent. See arguments in Chapter 5.

that a necessary and sufficient condition for a general singular integral being weighted L^p bounded is the weight satisfying the A_p condition.⁷

In view of the one-weight case in [LS04], we have the following conjecture.

Conjecture 1 *For $p > 1$, if the two weights μ_1 and μ_2 satisfy $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, then (1.2) holds for some $C > 0$.*

By considering the results in [Neu83], the "power-bump" condition is also of interest.

Conjecture 2 *For $p > 1$, if the two weights μ_1 and μ_2 satisfy $(\mu_1^r, \mu_2^r) \in A_p^+(\mathbb{R}_+^2)$ for some $r > 1$, then (1.2) holds for some $C > 0$.*

Although the validity of Conjecture 1 and Conjecture 2 will make the whole theory neater and more elegant, we have the following partial result which is sufficient for our application.

Theorem 6 *For $p > 1$, suppose that μ_1 and μ_2 are two weights such that $c\mu_1 \geq \mu_2$ for some $c > 0$. Then (1.2) holds for some $C > 0$ if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.*

By using different weights in (1.2) and applying Theorem 6, we extend Theorem 2 to a wider class of weights.

Theorem 7 *Assume that $p > 1$. Let $\mu(z) = |z|^{s'} |g(z)|^2$, where g is a non-vanishing holomorphic function on \mathbb{D} and $s' \in \mathbb{R}$. Suppose the weighted Bergman projection $\mathcal{B}_{|g|^2}$ on $(\mathbb{D}, |g|^2)$ is $L^p(|g|^2)$ bounded, and suppose the weighted Bergman projection $\mathcal{B}_{s'}$ on $(\mathbb{D}^*, |z|^{s'})$ is $L^p(|z|^{s'})$ bounded. Then the weighted Bergman projection \mathcal{B}_μ on (\mathbb{D}^*, μ) is $L^p(\mu)$ bounded.*

⁷See Chapter 4 for detailed definitions of variants of A_p condition.

Moreover, suppose $\mathcal{B}_{|g|^2}$ is $L^p(|g|^2)$ bounded if and only if $p \in (p_0, p'_0)$ for some $p_0 \geq 1$ and suppose $\mathcal{B}_{s'}$ is $L^p(|z|^{s'})$ bounded if and only if $p \in (p_1, p'_1)$ for some $p_1 \geq 1$ as in Theorem 2. If $(p_1, p'_1) \subset (p_0, p'_0)$ properly, then \mathcal{B}_μ is $L^p(\mu)$ bounded if and only if $p \in (p_1, p'_1)$.

Example 2 As in [Zey13], if we take $g(z) = (z-1)^t$ for some $t > 0$, then we see $(p_0, p'_0) = (\frac{2t+2}{t+2}, \frac{2t+2}{t})$. By Theorem 2, when $s' \in (0, \infty)$, we have $(p_1, p'_1) = (\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1})$. So \mathcal{B}_μ is $L^p(\mu)$ bounded if $p \in (\frac{2t+2}{t+2}, \frac{2t+2}{t}) \cap (\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1})$.

To answer the third question, for $z \in \mathbb{H}$ we take $\lambda(z) = |z_2|^{s'-2} |g(z_2)|^2$, where g is any non-vanishing holomorphic function on \mathbb{D} , then we have the following result.

Theorem 8 Assume that $p > 1$. Let λ be as above, and let p_0, p_1 be as in Theorem 7. Then the weighted Bergman projection \mathcal{B}_λ on (\mathbb{H}, λ) is $L^p(\lambda)$ bounded if $p \in (p_0, p'_0) \cap (p_1, p'_1)$. In addition, if $(p_1, p'_1) \subset (p_0, p'_0)$ properly, then \mathcal{B}_λ is $L^p(\lambda)$ bounded if and only if $p \in (p_1, p'_1)$.

Again by inflation, we can apply the same technique several times to obtain a similar result for the n -dimensional Hartogs triangle. But we will skip this part here, since there are no new ideas.

In (1.2), we deal with two weights. Then one may ask whether we can do the same thing for the Bergman projection on the Hartogs triangle.

Question 4 Can we also obtain the L^p regularity of the (weighted) Bergman projection mapping from one (weighted) space to the other?

The answer is, of course, yes. We can consider the L^p regularity of the weighted Bergman projection $\mathcal{B}_{\mathbb{H}, s'}$ on $(\mathbb{H}, |z_2|^{s'})$ mapping from $L^p(\mathbb{H}, |z_2|^{s'})$ to $L^p(\mathbb{H}, |z_2|^t)$ for some $t \in \mathbb{R}$. For simplicity, we focus on the case $s' = 2k + 2 \geq 0$, for $k \in \mathbb{Z}$.

Theorem 9 *Suppose that $\mathcal{B}_{\mathbb{H},2k+2}$ is the weighted Bergman projection on the weighted space $(\mathbb{H}, |z_2|^{2k+2})$. Assume $p > 1$ and $k \geq -1$. Then, for $t \leq 2k + 2$, $\mathcal{B}_{\mathbb{H},2k+2}$ is L^p bounded from $L^p(\mathbb{H}, |z_2|^{2k+2})$ to $L^p(\mathbb{H}, |z_2|^t)$ if and only if $p \in (\frac{2k+6}{k+4}, \frac{t+4}{k+2})$. For $t > 2k+2$, $\mathcal{B}_{\mathbb{H},2k+2}$ is bounded from $L^p(\mathbb{H}, |z_2|^{2k+2})$ to $L^p(\mathbb{H}, |z_2|^t)$ if $p \in (\frac{2k+6}{k+4}, \frac{2k+6}{k+2}) \cup (\frac{t+4}{k+4}, \frac{t+4}{k+2})$.*

When applying the two-weight inequality (1.2), we can also consider a wider class of weights $|z_2|^{s'} |g(z_2)|^2$ rather than $|z_2|^{s'}$, where $z = (z_1, z_2) \in \mathbb{H}$ and g is a non-vanishing holomorphic function on \mathbb{D} . Indeed, if we restrict ourself to the weighted Bergman projection $\mathcal{B}_{\mathbb{H},s'}$ on $(\mathbb{H}, |z_2|^{s'})$, we can consider the L^p regularity of the $\mathcal{B}_{\mathbb{H},s'}$ mapping from $L^p(\mathbb{H}, |z_2|^{s'})$ to $L^p(\mathbb{H}, \mu)$ for a more general weight μ satisfying the corresponding A_p^+ conditions. On the other hand, one may also consider a weight of the form $\mu(z) = |z_1|^{s''} |z_2|^{s'}$. In this case, we have to consider once more the weighted space $(\mathbb{D}, |z|^{s''})$. The statements of these variants are almost the same, and the arguments involve no new ideas, so we will skip them.

As we have already seen, the two-weight inequality (1.2) has many interesting applications. In order to prove Conjecture 1 and Conjecture 2, we need a better understanding of the A_p^+ class. Based on the results in [Muc72, Neu83] and [Ste93, Chapter 5], we may ask the following question.

Question 5 *Since the classical A_p condition is closely related to the maximal function theory, can we formulate and prove some analogues for the A_p^+ class?*

The answer is partly affirmative, and it seems to provide a promising yet technical approach to attack Conjecture 1 and Conjecture 2. Let \widetilde{M}^+ denote the special maximal function operator.⁸ We have the following results.

⁸See Chapter 6 for detailed definitions of the special maximal function operator.

Theorem 10 *Let f be a measurable function on \mathbb{R}_+^2 . Then, for any $0 < q < 1$, the function $(\widetilde{M}^+(f))^q$ is in $A_1^+(\mathbb{R}_+^2)$.*

Theorem 11 *Assume that $p \geq 1$. Suppose μ_1 and μ_2 are two weights on \mathbb{R}_+^2 . Then we have a weak-type (p, p) inequality: namely, there is a constant $c > 0$ so that*

$$\mu_1\left(\{z \in \mathbb{R}_+^2 : \widetilde{M}^+(f)(z) > \alpha\}\right) \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) dV(z) \quad (1.3)$$

for all $\alpha > 0$, if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

Consider the following operator

$$\widetilde{\mathcal{B}}_{\mathbb{R}_+^2}(f)(z) = \frac{1}{\pi} \int_{\mathbb{R}_+^2} \frac{f(w) dV(w)}{|z - \bar{w}|^2},$$

which is the "absolute value" of the Bergman projection on the upper half plane. It is easy to see, for $f \geq 0$ and $z, z' \in \mathbb{R}_+^2$, that

$$\widetilde{\mathcal{B}}_{\mathbb{R}_+^2}(f)(z') \geq \widetilde{\mathcal{B}}_{\mathbb{R}_+^2}(f)(z),$$

whenever $\Re(z') = \Re(z)$ and $\Im(z') \leq \Im(z)$. Note that the special maximal function operator \widetilde{M}^+ also enjoys the same property above as $\widetilde{\mathcal{B}}_{\mathbb{R}_+^2}$.⁹ In view of this, we hope Theorem 10 and Theorem 11 could provide some clues to prove Conjecture 1 and Conjecture 2.

In [McN12], McNeal introduced a family of twist-weight factors (τ, μ) to generalize Bell's idea about Condition R, see [Bel81, McN12] for further details. Following this idea, we can consider the twisted-weighted Bergman projection and investigate its L^p regularity.¹⁰

Question 6 *Can we obtain the L^p regularity of the twisted-weighted Bergman projection as we did for the weighted Bergman projection?*

⁹See Chapter 6 for details.

¹⁰See Chapter 7 for a detailed definition of the twisted-weighted Bergman projection. See also [McN12].

The answer is partly affirmative. Under some restriction on the twist-weight factors, namely, assuming the holomorphicity of the ratio between the twist-weight factors, the L^p regularity of the twisted-weighted Bergman projection can be derived from the L^p regularity of a corresponding weighted Bergman projection. In particular, if we focus on planar domains, by applying Theorem 6, we have the following result.

Theorem 12 *Let Ω be a proper simply connected domain in \mathbb{C} , and let $\phi : \mathbb{R}_+^2 \rightarrow \Omega$ be a biholomorphism. For a weight $\mu \in C^1(\Omega)$ and for any non-vanishing $g \in \mathcal{O}(\Omega)$, define $\tau = \mu^{\frac{1}{2}}/|g|$. Then the twisted-weighted Bergman projection $\mathcal{B}_{\tau,\mu}$ is $L^p(\Omega, \mu)$ bounded if and only if $\mu(\phi(z))^{1-\frac{p}{2}} |\phi'(z)|^{2-p} \in A_p^+(\mathbb{R}_+^2)$.*

We point out that, for a different function g , we have a different twist factor τ , hence different coset \mathcal{O}_τ .¹¹ However, it is interesting to see that the L^p regularities of these different non-holomorphic projections are the same, and they depend only on the weight factor μ .

Again, we can apply the idea of inflation to formulate and prove similar results of twisted-weighted Bergman projections on the (n -dimensional) Hartogs triangle, and we will skip these for the same reason as before.

As the last application, we consider the L^p Sobolev regularity of the Bergman projection on the Hartogs triangle \mathbb{H} . We have already mentioned that the topological closure of \mathbb{H} does not possess a Stein neighborhood basis, the solution to the $\bar{\partial}$ -equation on \mathbb{H} is not globally regular, and the boundary at $(0, 0)$ is not even Lipschitz. So we cannot expect to obtain the regularity in the ordinary L^p Sobolev spaces, nor for all $p \in (1, \infty)$. In view of the result in [CS13] and the previous results of the weighted Bergman projections, we may ask the following question.

¹¹See Chapter 7 for definition of \mathcal{O}_τ .

Question 7 *Can we obtain a weighted L^p Sobolev regularity of the Bergman projection on the Hartogs triangle, where the weight measures the distance from points near the boundary to the singularity at the boundary?*

The answer is affirmative. Recall that on the Hartogs triangle we have $|z_2| < |z| < \sqrt{2}|z_2|$, where $z = (z_1, z_2) \in \mathbb{H}$. Therefore, we can consider the weighted Sobolev space¹² $L_k^p(\mathbb{H}, |z_2|^{pk})$, and prove the following theorem.

Theorem 13 *The Bergman projection $\mathcal{B}_{\mathbb{H}}$ on the Hartogs triangle \mathbb{H} maps continuously from $L_k^p(\mathbb{H})$ to $L_k^p(\mathbb{H}, |z_2|^{pk})$ for $p \in (\frac{4}{3}, 4)$.*

If we let $p = 2$ and replace k by $2k$, then Theorem 13 will imply the result in [CS13]. Note that, in our result, there is no loss of smoothness¹³ of $\mathcal{B}_{\mathbb{H}}(f)$.

Moreover, the method here also applies to the n -dimensional Hartogs triangle. By adopting the notation in (1.1), we have the following generalization of Theorem 13.

Theorem 14 *The Bergman projection on the n -dimensional Hartogs triangle $\mathbb{H}_{\phi_j}^n$ maps continuously from $L_k^p(\mathbb{H}_{\phi_j}^n)$ to $L_k^p(\mathbb{H}_{\phi_j}^n, |z_1''|^{pk})$ for $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$.*

We should point out that the idea of the proof remains the same, however, the weight $|z_1''|^{pk}$ is no longer comparable to $|z|^{pk}$, some power of the distance from points near the boundary to the singularity at the boundary.

1.3 Outline and Organization

The original proof of Theorem 1 follows from the idea in [CS13], where the Hartogs triangle is transferred to a product domain by a biholomorphism.¹⁴ However, by in-

¹²See Chapter 8 for definition of weighted Sobolev spaces.

¹³Compare with [CS13, Theorem 2.3]

¹⁴See also Chapter 8.

roducing the notion of inflation in Chapter 2, the arguments become neat and simple. Moreover, it can be applied to the weighted cases to prove Theorem 3, Theorem 4, and Theorem 5. So, in Chapter 3, we first prove Theorem 2 by Schur’s test, then we present the new proof of the weighted cases by using inflation arguments. As a consequence, we obtain Theorem 1 from Theorem 5.

The second part of this dissertation is to investigate the two-weight inequality (1.2) and extend the previous results to a wider class of weights. In Chapter 4, we introduce the A_p^+ class and follow the idea from [LS04] to prove Theorem 6. In Chapter 5, we apply the results in Chapter 4 to give an alternative proof of Theorem 2 and extend it to prove Theorem 7. By applying the inflation arguments, it is easy to see that Theorem 8 follows from Theorem 7, and Theorem 9 follows from Theorem 6.¹⁵ To get a better understanding of the A_p^+ class, we introduce the special maximal function operator in Chapter 6 and prove Theorem 10 and Theorem 11 by following the ideas from the classical results in [Muc72] and [Ste93, Chapter 5].

In the last part of this dissertation, we present two applications of the previous theory. In Chapter 7, we study the L^p regularity of twisted-weighted Bergman projections. In particular, by applying Theorem 6, we obtain an interesting result—Theorem 12. In Chapter 8, we study the L^p Sobolev regularity of the ordinary Bergman projection on the (n -dimensional) Hartogs triangle. To prove Theorem 13, we start with the idea from [CS13] to transfer \mathbb{H} to $\mathbb{D} \times \mathbb{D}^*$. We then use a result in [Str86] and integration by parts to convert the differential operators. Finally, we apply the weighted L^p estimates (Theorem 2) to complete the proof. It is not difficult to see the same argument also applies to Theorem 14.

¹⁵Theorem 9 can also be proved by using a variant of Schur’s test.

1.4 Future Directions

We conclude this chapter by mentioning several possible directions for future work.

1.4.1 The two-weight inequality

We have already seen many applications of the two-weight inequality (1.2), and we also have a partial result for the conjectures we mentioned in §1.2. It would be very interesting if we can prove Conjecture 1 and Conjecture 2 for general weights and give a full understanding of the A_p^+ class. Based on the special features of the special maximal function operator, we need to develop additional analysis on this operator, especially on the behavior of the operator along the boundary (i.e. the real line). Moreover, it is reasonable to extend this theory to higher dimension. We believe that, under some regularity condition on the boundary of a higher dimensional domain, the Bergman projection can still be expressed in a fashion similar to the integral in (1.2).

1.4.2 Non-smooth domains

We have already obtained some results on a class of Hartogs domains. However, there should be other variants of Hartogs triangle which are also of interest. It would be also interesting to study the regularity of Bergman projection on other types of non-smooth domains, especially in higher dimension. Up until now, there are relatively few results in this direction. As we mentioned previously, on the n -dimensional Hartogs triangle, $|z_1''|$ is not comparable to $|z|$ as long as $n > 2$. So other types of weighted L^p Sobolev regularity of the Bergman projection are also of interest. It would be very nice if we can develop a

method to control the behavior of the singularity at the boundary by some function in $|z|$ in the higher-dimensional cases.

1.4.3 Smooth domains

There are lots of classical results about the regularity of Bergman projection on smooth domains as we mentioned in §1.1. However, the story is still far from reaching the end, since there are several big problems still being studied. For example, Kohn conjectures that for the Bergman projection on a smooth bounded pseudoconvex domain, the exact regularity is equivalent to the global regularity. When the boundary of a smoothly bounded pseudoconvex domain possesses a set of points of infinite type, we in general (except for some special cases) do not know the L^p (or even L^2) Sobolev regularity of the Bergman projection. It would be also interesting to ask whether we can obtain the regularity of the Bergman projection by only assuming minimal smoothness of the boundary, for example, a bounded pseudoconvex domain satisfying property (P) with C^2 boundary. On the other hand, the worm domains provide a class of smooth bounded pseudoconvex domains that have irregular Bergman projections. It would be also interesting to summarize the geometric condition on the boundary of the worm domains to explain why the regularity fails.

1.4.4 Biholomorphic mappings

The Sobolev regularity of the Bergman projection also plays an important role in the mapping properties of biholomorphisms. Inspired by Fefferman's famous work on the boundary behavior of biholomorphic mappings in [Fef74], Bell formulated his Condition R in terms of regularity properties of Bergman projection in [Bel81]. As we mentioned pre-

viously, in [McN12] McNeal introduced a family of twist-weight factors to generalize Bell's idea. Now we have obtained the L^p regularity of the twisted-weighted Bergman projection under a certain restriction on the twist-weight factor. It would be very interesting to explore this idea further in higher dimension, weaken the restriction on the twist-weight factors, and investigate the Sobolev regularity of the corresponding twisted-weighted Bergman projections. Therefore we can study the boundary behavior of biholomorphic mappings by using this new idea in [McN12].

2. The Notion of Inflation

2.1 Preliminaries and Basic Definitions

Let us temporarily consider the general setting for a moment, and suppose Ω is a domain in \mathbb{C}^n .

Definition 2.1.1 *A measurable function μ on Ω is a weight if it is locally integrable and positive almost everywhere.*

As long as we have a weight μ , we can define the weighted $L^p(\mu)$ norm for a measurable function f on Ω by

$$\|f\|_{L^p(\Omega, \mu)} = \left(\int_{\Omega} |f(z)|^p \mu(z) dV(z) \right)^{\frac{1}{p}},$$

and the weighted $L^p(\mu)$ space by $\{f \text{ measurable on } \Omega : \|f\|_{L^p(\Omega, \mu)} < \infty\}$, for $p \geq 1$.

Sometimes, we will use a different notation $L^p(\Omega, \mu)$ to emphasize the underlying domain.

Supposing the set of all holomorphic function on Ω is denoted by $\mathcal{O}(\Omega)$, then we consider

the analytic subspace of $L^p(\mu)$ which is denoted by $A_\mu^p(\Omega) = L^p(\mu) \cap \mathcal{O}(\Omega)$. We focus on

$p = 2$ in this section, since the closedness of $A_\mu^2(\Omega)$ in $L^2(\mu)$ is of principal interest.

Definition 2.1.2 *A weight μ is admissible on Ω , if for any compact subset K of Ω , there exists $C_K > 0$ such that*

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_{L^2(\Omega, \mu)}$$

for all $f \in A_\mu^2(\Omega)$. For instance, if μ is continuous and non-vanishing, then it is admissible.

It is easy to see, if μ is admissible on Ω , then $A_\mu^2(\Omega)$ is closed in $L^2(\mu)$.

Definition 2.1.3 For an admissible weight μ on Ω , we define the weighted Bergman projection $\mathcal{B}_{\Omega,\mu}$ to be the orthogonal projection from $L^2(\mu)$ to $A_\mu^2(\Omega)$. The weighted Bergman projection is an integral operator

$$\mathcal{B}_{\Omega,\mu}(f)(z) = \int_{\Omega} B_{\Omega,\mu}(z, \zeta) f(\zeta) \mu(\zeta) dV(\zeta),$$

where $B_{\Omega,\mu}(z, \zeta)$ is the weighted Bergman kernel with $(z, \zeta) \in \Omega \times \Omega$.

It is not hard to see that every basic property of the ordinary Bergman theory can be moved to the weighted setting.

Definition 2.1.4 If μ is a non-vanishing weight on Ω , we define the inflation $\tilde{\Omega}$ of Ω by

$$\tilde{\Omega} = \{(z, w) \in \mathbb{C}^{m+n} : |z|^2 < \mu(w), w \in \Omega\}.$$

Note that $\tilde{\Omega}$ is a Hartogs domain.

Suppose μ is a non-vanishing weight on Ω , and suppose $\lambda > 0$ is a function on Ω such that it is admissible on $\tilde{\Omega}$. Then it is easy to see that $\mu^m \lambda$ is admissible on Ω . Throughout this dissertation, as long as we deal with the weighted Bergman theory, the weight is assumed to be admissible on the corresponding domain.

2.2 The Inflation Theorem

Before going further, we first give two useful lemmas and the corresponding corollaries.

Lemma 2.2.1 *Let $F : X_1 \rightarrow X_2$ be an isometry between two Banach spaces X_1 and X_2 . Then it induces an isometry $F^* : \mathfrak{B}(X_1) \rightarrow \mathfrak{B}(X_2)$ between the spaces of the bounded operators by $F^*(T) = F \circ T \circ F^{-1}$, for any $T \in \mathfrak{B}(X_1)$.*

In particular, suppose that $X_j = H_j$ is a Hilbert space, $j = 1, 2$. Let S be a closed subspace of H_1 , and let $P : H_1 \rightarrow S$ be the orthogonal projection. Then F induces an orthogonal decomposition $H_2 = F(S) \oplus F(S^\perp)$, that is, $F(S)$ is closed in H_2 and $F(S)^\perp = F(S^\perp)$. Hence, $F^(P) : H_2 \rightarrow F(S)$ is the orthogonal projection.*

Proof The first part of the lemma is straightforward, we only prove the second part. Since S is closed in H_1 , and since F is an isometry, it is easy to see that $F(S)$ is closed in H_2 .

To prove the equality $F(S)^\perp = F(S^\perp)$, we consider the following. For any $x \in F(S)^\perp$, we have $F^{-1}(x) \in S^\perp$, then $F^{-1}(x) \perp S$. Therefore $x \perp F(S)$, which implies $x \in F(S)^\perp$. This shows $F(S^\perp) \subset F(S)^\perp$. The other direction follows from the same argument in the reverse direction.

In the last statement, for any $x \in H_2$, we have a decomposition $x = y + z$, where $y \in F(S)$ and $z \in F(S^\perp)$. Then $F^{-1}(x) = F^{-1}(y) + F^{-1}(z)$, with $F^{-1}(y) \in S$ and $F^{-1}(z) \in S^\perp$. So $F^*(P)(x) = F(P(F^{-1}(x))) = F(F^{-1}(y)) = y$ is orthogonal. ■

Corollary 2.2.2 *Let $\Phi : \Omega_1 \rightarrow \Omega_2$ be a biholomorphism between two domains in \mathbb{C}^n . Suppose Ω_j is equipped with the weight μ_j , $j = 1, 2$, and $\mu_2 = \mu_1 \circ \Phi^{-1}$. Then we have the transformation formula for the weighted Bergman kernels*

$$B_{\Omega_1, \mu_1}(z, \zeta) = \det J_{\mathbb{C}}\Phi(z) B_{\Omega_2, \mu_2}(\Phi(z), \Phi(\zeta)) \det \overline{J_{\mathbb{C}}\Phi(\zeta)},$$

where $(z, \zeta) \in \Omega_1 \times \Omega_1$.

Proof Let $F : L^2(\Omega_1, \mu_1) \rightarrow L^2(\Omega_2, \mu_2)$ be the isometry, by $F(f) = \det J_{\mathbb{C}}(\Phi^{-1})f \circ \Phi^{-1}$, for any $f \in L^2(\Omega_1, \mu_1)$. Then, by Lemma 2.2.1, we have $F^*(\mathcal{B}_{\Omega_2, \mu_2}) = \mathcal{B}_{\Omega_1, \mu_1}$. By the uniqueness of the weighted Bergman kernel, we obtain the transformation formula above.

■

Corollary 2.2.3 *Let $\Phi : \Omega_1 \rightarrow \Omega_2$ be a biholomorphism between two domains in \mathbb{C}^n . Suppose \mathcal{B}_j is the weighted Bergman projection for (Ω_j, μ_j) , $j = 1, 2$, and $\mu_2 = |\det J_{\mathbb{C}}(\Phi^{-1})|^2 \mu_1 \circ \Phi^{-1}$. Then, for $p \geq 1$, \mathcal{B}_1 is $L^p(\mu_1)$ bounded if and only if \mathcal{B}_2 is $L^p(\mu_2)$ bounded.*

Proof Let $F : L^p(\Omega_1, \mu_1) \rightarrow L^p(\Omega_2, \mu_2)$ be the isometry given by $F(f) = f \circ \Phi^{-1}$, for any $f \in L^p(\Omega_1, \mu_1)$. In particular, when $p = 2$, we have $F^*(\mathcal{B}_1) = \mathcal{B}_2$. Since this F is an isometry for all $p \geq 1$, we see from the first part of Lemma 2.2.1 that \mathcal{B}_1 is L^p -bounded if and only if \mathcal{B}_2 is L^p -bounded. ■

Lemma 2.2.4 *Suppose we have a weight $\mu_1 > 0$ on Ω_1 and a weight $\mu_2 > 0$ on Ω_2 , both non-vanishing.¹ Let \mathcal{T}_1 and \mathcal{T}_2 be the integral operators with kernels $T_1(w_1, \eta_1)$ on $\Omega_1 \times \Omega_1$ and $T_2(w_2, \eta_2)$ on $\Omega_2 \times \Omega_2$, respectively. That is,*

$$\mathcal{T}_1(f)(w_1) = \int_{\Omega_1} T_1(w_1, \eta_1) f(\eta_1) \mu_1(\eta_1) dV(\eta_1),$$

$$\mathcal{T}_2(g)(w_2) = \int_{\Omega_2} T_2(w_2, \eta_2) g(\eta_2) \mu_2(\eta_2) dV(\eta_2).$$

Given any $p \in [1, \infty)$, if \mathcal{T}_1 is bounded on $L^p(\Omega_1, \mu_1)$ and \mathcal{T}_2 is bounded on $L^p(\Omega_2, \mu_2)$, then their product operator $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$ with kernel $T_1 \otimes T_2$, is bounded on $L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$.

Conversely, assuming \mathcal{T}_1 and \mathcal{T}_2 both are non-trivial, if one of these two operator is unbounded, then \mathcal{T} is unbounded.

¹The weights can be assumed to be non-negative, i.e. $\mu_1, \mu_2 \geq 0$, if we adopt the convention $0 \cdot \infty = 0$.

Proof By definition, we have

$$\begin{aligned}
\mathcal{T}(f)(w_1, w_2) &= \int_{\Omega_1 \times \Omega_2} T_1(w_1, \eta_1) T_2(w_2, \eta_2) f(\eta_1, \eta_2) \mu_1(\eta_1) \mu_2(\eta_2) dV(\eta_1, \eta_2) \\
&= \int_{\Omega_1} T_1(w_1, \eta_1) f(\eta_1, \eta_2) \mu_1(\eta_1) dV(\eta_1) \int_{\Omega_2} T_2(w_2, \eta_2) \mu_2(\eta_2) dV(\eta_2) \\
&= \int_{\Omega_2} \mathcal{T}_{1, \eta_1}(f(\eta_1, \eta_2))(w_1) T_2(w_2, \eta_2) \mu_2(\eta_2) dV(\eta_2) \\
&= \mathcal{T}_{2, \eta_2}(\mathcal{T}_{1, \eta_1}(f(\eta_1, \eta_2))(w_1))(w_2),
\end{aligned}$$

where \mathcal{T}_{1, η_1} and \mathcal{T}_{2, η_2} are operators \mathcal{T}_1 and \mathcal{T}_2 acting on η_1 and η_2 respectively. If \mathcal{T}_1 and \mathcal{T}_2 are both bounded, then

$$\begin{aligned}
\|\mathcal{T}(f)\|_{L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p &= \int_{\Omega_1 \times \Omega_2} |\mathcal{T}_{2, \eta_2}(\mathcal{T}_{1, \eta_1}(f(\eta_1, \eta_2))(w_1))(w_2)|^p \mu_1(w_1) \mu_2(w_2) dV(w_1, w_2) \\
&= \int_{\Omega_2} |\mathcal{T}_{2, \eta_2}(\mathcal{T}_{1, \eta_1}(f(\eta_1, \eta_2))(w_1))(w_2)|^p \mu_2(w_2) dV(w_2) \int_{\Omega_1} \mu_1(w_1) dV(w_1) \\
&\leq c \int_{\Omega_2} |\mathcal{T}_{1, \eta_1}(f(\eta_1, w_2))(w_1)|^p \mu_2(w_2) dV(w_2) \int_{\Omega_1} \mu_1(w_1) dV(w_1) \\
&= \int_{\Omega_1} |\mathcal{T}_{1, \eta_1}(f(\eta_1, w_2))(w_1)|^p \mu_1(w_1) dV(w_1) \int_{\Omega_2} \mu_2(w_2) dV(w_2) \\
&\leq c \int_{\Omega_1} |f(w_1, w_2)|^p \mu_1(w_1) dV(w_1) \int_{\Omega_2} \mu_2(w_2) dV(w_2) \\
&= \|f\|_{L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p.
\end{aligned}$$

Conversely, without loss of generality, if we assume \mathcal{T}_1 is unbounded, then there is a sequence $\{f_n\} \subset L^p(\Omega_1, \mu_1)$ such that $\|f_n\|_{L^p(\Omega_1, \mu_1)} \leq c < \infty$ for some $c > 0$ and $\|\mathcal{T}_1(f_n)\|_{L^p(\Omega_1, \mu_1)} \rightarrow \infty$ as $n \rightarrow \infty$.

Since \mathcal{T}_2 is non-trivial, there is a function $g \in L^p(\Omega_2, \mu_2)$ such that $g \neq 0$ and $\mathcal{T}_2(g) \neq 0$.

If we consider the sequence $\{g \otimes f_n\}$, we have

$$\begin{aligned}
\|g \otimes f_n\|_{L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p &= \int_{\Omega_1 \times \Omega_2} |f_n(w_1) g(w_2)|^p \mu_1(w_1) \mu_2(w_2) dV(w_1, w_2) \\
&= \int_{\Omega_1} |f_n(w_1)|^p \mu_1(w_1) dV(w_1) \int_{\Omega_2} |g(w_2)|^p \mu_2(w_2) dV(w_2) \\
&\leq c,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|\mathcal{T}(g \otimes f_n)\|_{L^p(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)}^p &= \lim_{n \rightarrow \infty} \int_{\Omega_1 \times \Omega_2} |\mathcal{T}_1(f_n)(w_1) \mathcal{T}_2(g)(w_2)|^p \mu_1(w_1) \mu_2(w_2) dV(w_1, w_2) \\
&= \lim_{n \rightarrow \infty} \|\mathcal{T}_1(f_n)\|_{L^p(\Omega_1, \mu_1)}^p \|\mathcal{T}_2(g)\|_{L^p(\Omega_2, \mu_2)}^p \\
&= \infty.
\end{aligned}$$

■

Remark 2.2.5 *Lemma 2.2.4 typically applies to the Bergman projection on product space, since it is easy to see that $\mathcal{B}_{\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2} = \mathcal{B}_{\Omega_1, \mu_1} \otimes \mathcal{B}_{\Omega_2, \mu_2}$.*

Now we are ready to prove the inflation theorem, which generalizes the result [Zey13, Corollary 4.6].

Proposition 2.2.6 (Inflation Theorem) *Let $\Omega \subset \mathbb{C}^n$ be a domain, and $\mu = |g|^2$ for some non-vanishing holomorphic function on Ω . Suppose that $\tilde{\Omega} \subset \mathbb{C}^{m+n}$ is the inflation of Ω via μ , and suppose that $\lambda > 0$ is a function on Ω such that it is admissible on $\tilde{\Omega}$. Then, for $p \geq 1$, $\mathcal{B}_{\tilde{\Omega}, \lambda}$ is $L^p(\lambda)$ bounded if and only if $\mathcal{B}_{\Omega, \mu^m \lambda}$ is $L^p(\mu^m \lambda)$ bounded.*

Proof Since g is holomorphic and non-vanishing, using the notation in Definition 2.1.4, we have the biholomorphism $\Phi : \tilde{\Omega} \rightarrow \mathbb{B}^m \times \Omega$ via $\Phi(z, w) = (z/g(w), w)$, where \mathbb{B}^m is the unit ball in \mathbb{C}^m .

A direct computation shows that $|\det J_{\mathbb{C}}(\Phi^{-1})|^2 = \mu^m$. So, by Corollary 2.2.3, we see that $\mathcal{B}_{\tilde{\Omega}, \lambda}$ is L^p -bounded if and only if $\mathcal{B}_{\mathbb{B}^m \times \Omega, \lambda \mu^m}$ is L^p -bounded.

But we know that $\mathcal{B}_{\mathbb{B}^m \times \Omega, \lambda \mu^m} = \mathcal{B}_{\mathbb{B}^m} \otimes \mathcal{B}_{\Omega, \lambda \mu^m}$, and $\mathcal{B}_{\mathbb{B}^m}$ is L^p -bounded for all $p \in (1, \infty)$. By Lemma 2.2.4, we see that $\mathcal{B}_{\tilde{\Omega}, \lambda}$ is L^p -bounded if and only if $\mathcal{B}_{\Omega, \mu^m \lambda}$ is L^p -bounded. ■

3. The L^p Regularity

If we take $\Omega = \mathbb{D}^*$ and $\mu(w) = |w|^2$ in Proposition 2.2.6,¹ then $\tilde{\Omega}$ is the classical Hartogs triangle \mathbb{H} . So in this chapter, we first prove Theorem 2, then prove Theorem 3, Theorem 4, and Theorem 5 by using inflation arguments. As a result, we obtain Theorem 1 from Theorem 5.

3.1 The Punctured Disk

We first look at the weighted space $(\mathbb{D}^*, |z|^{s'})$, for any $s' \in \mathbb{R}$.

Lemma 3.1.1 *For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, the weighted Bergman kernel $B_{s'}(z, \zeta)$ on $(\mathbb{D}^*, |z|^{s'})$ has a "homotopic" expression*

$$\begin{aligned} B_{s'}(z, \zeta) &= \frac{s}{2} B_{2k+2}(z, \zeta) + \left(1 - \frac{s}{2}\right) B_{2k}(z, \zeta) \\ &= \frac{s}{2} (z\bar{\zeta})^{-(k+1)} B_0(z, \zeta) + \left(1 - \frac{s}{2}\right) (z\bar{\zeta})^{-k} B_0(z, \zeta), \end{aligned} \tag{3.1}$$

where $B_0(z, \zeta)$ is the ordinary Bergman kernel on the unit disk and $(z, \zeta) \in \mathbb{D}^* \times \mathbb{D}^*$.

Proof We first determine an orthonormal basis for the space $A^2(\mathbb{D}^*, |z|^{s'})$. Suppose $m, n \in \mathbb{Z}$; a direct computation shows,² for $m + n + s' + 2 > 0$,

$$\int_{\mathbb{D}^*} z^n \bar{z}^m |z|^{s'} dV(z) = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{2}{2m+2+s'}, & \text{if } n = m. \end{cases}$$

¹Here $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is the punctured disk.

²We have normalized the area of \mathbb{D} by setting $\text{Area}(\mathbb{D}) = 1$.

Therefore $\left\{ \sqrt{\frac{2m+2+s'}{2}} z^m \right\}_{m > -(1+\frac{s'}{2})}$ is an orthonormal basis. So the weighted Bergman kernel for the space $(\mathbb{D}^*, |z|^{s'})$ is

$$\begin{aligned} B_{s'}(z, \zeta) &= \sum_{m > -(1+\frac{s'}{2})} \frac{2m+2+s'}{2} z^m \bar{\zeta}^m \\ &= \frac{(t+\frac{s'}{2})(z\bar{\zeta})^{t-1} - (t-1+\frac{s'}{2})(z\bar{\zeta})^t}{(1-z\bar{\zeta})^2}, \end{aligned} \quad (3.2)$$

where t is the smallest integer satisfying $t > -\frac{s'}{2}$.

Suppose $s'' = s' + 2$ and t_1 is the smallest integer such that $t_1 > -\frac{s''}{2}$. Then $t_1 = t - 1$.

In this case, from (3.2), we see that

$$\begin{aligned} B_{s''}(z, \zeta) &= \frac{(t_1+\frac{s''}{2})(z\bar{\zeta})^{t_1-1} - (t_1-1+\frac{s''}{2})(z\bar{\zeta})^{t_1}}{(1-z\bar{\zeta})^2} \\ &= \frac{(t-1+\frac{s'+2}{2})(z\bar{\zeta})^{t-2} - (t-2+\frac{s'+2}{2})(z\bar{\zeta})^{t-1}}{(1-z\bar{\zeta})^2} \\ &= (z\bar{\zeta})^{-1} \cdot \frac{(t+\frac{s'}{2})(z\bar{\zeta})^{t-1} - (t-1+\frac{s'}{2})(z\bar{\zeta})^t}{(1-z\bar{\zeta})^2} \\ &= (z\bar{\zeta})^{-1} B_{s'}(z, \zeta). \end{aligned} \quad (3.3)$$

Hence 2 is a "period" of s' for the weighted Bergman kernel $B_{s'}(z, \zeta)$. Let $s' = s \in (0, 2]$.

Then $t = 0$, and from (3.2) we have

$$\begin{aligned} B_s(z, \zeta) &= \frac{\frac{s}{2}(z\bar{\zeta})^{-1} + (1-\frac{s}{2})}{(1-z\bar{\zeta})^2} \\ &= \frac{s}{2}(z\bar{\zeta})^{-1} B_0(z, \zeta) + \left(1 - \frac{s}{2}\right) B_0(z, \zeta). \end{aligned} \quad (3.4)$$

Therefore, combining (3.3) and (3.4), we obtain (3.1). ■

Following the idea in [HKZ00], to prove Theorem 2 we need three lemmas.

Lemma 3.1.2 (Schur's Test) *Suppose X is a measure space with a positive measure μ . Let $T(x, y)$ be a positive measurable function on $X \times X$, and let \mathcal{T} be the integral operator associated to the kernel function $T(x, y)$.*

Given $p \in (1, \infty)$ with its conjugate exponent p' , if there exists a strictly positive function h a.e. on X and a constant $M > 0$, such that

1. $\int_X T(x, y)h(y)^{p'} d\mu(y) \leq Mh(x)^{p'}$, for a.e. $x \in X$, and

2. $\int_X T(x, y)h(x)^p d\mu(x) \leq Mh(y)^p$, for a.e. $y \in X$.

Then \mathcal{T} is bounded on $L^p(X, d\mu)$ with $\|\mathcal{T}\| \leq M$.

Proof Let $f \in L^p(X, d\mu)$. By Hölder's inequality and Condition 1 in the assumption, we have

$$\begin{aligned} |\mathcal{T}(f)(x)| &\leq \int_X T(x, y) |f(y)| d\mu(y) \\ &\leq \left(\int_X T(x, y)h(y)^{p'} d\mu(y) \right)^{\frac{1}{p'}} \left(\int_X T(x, y)h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\ &\leq M^{\frac{1}{p'}} h(x) \left(\int_X T(x, y)h(y)^{-p} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \end{aligned}$$

for a.e. $x \in X$. So, by Fubini's theorem and Condition 2 in the assumption, we have

$$\begin{aligned} \int_X |\mathcal{T}(f)(x)|^p d\mu(x) &\leq M^{\frac{p}{p'}} \int_X h(x)^p d\mu(x) \int_X T(x, y)h(y)^{-p} |f(y)|^p d\mu(y) \\ &= M^{\frac{p}{p'}} \int_X T(x, y)h(x)^p d\mu(x) \int_X h(y)^{-p} |f(y)|^p d\mu(y) \\ &\leq M^{\frac{p}{p'}+1} \int_X |f(y)|^p d\mu(y) \\ &= M^p \|f\|^p. \end{aligned}$$

This completes the proof. ■

Lemma 3.1.3 For $-1 < \alpha < 0$ and $\beta > -2$, define

$$I_{\alpha, \beta}(z) = \int_{\mathbb{D}^*} \frac{(1 - |\zeta|^2)^\alpha |\zeta|^\beta dV(\zeta)}{|1 - z\bar{\zeta}|^2},$$

where $z \in \mathbb{D}^*$.³ Then we have $I_{\alpha, \beta}(z) \sim (1 - |z|^2)^\alpha$,⁴ for any $z \in \mathbb{D}^*$.

³The restrictions $\alpha > -1$ and $\beta > -2$ make the integral $I_{\alpha, \beta}(z)$ convergent.

⁴The notation $A \sim B$ means there is a constant $c > 0$ so that $c^{-1}B \leq A \leq cB$.

Proof For $z, \zeta \in \mathbb{D}^*$, we expand the kernel function

$$\frac{1}{|1 - z\bar{\zeta}|^2} = \sum_{n=0}^{\infty} (z\bar{\zeta})^n \sum_{m=0}^{\infty} (\bar{z}\zeta)^m.$$

Substitute the expansion back to the integral, and integrate term by term. By the rotational symmetry on \mathbb{D}^* , we obtain

$$\begin{aligned} I_{\alpha, \beta}(z) &= \sum_{n=0}^{\infty} \int_{\mathbb{D}^*} |z\zeta|^{2n} (1 - |\zeta|^2)^\alpha |\zeta|^\beta dV(\zeta) \\ &= \sum_{n=0}^{\infty} |z|^{2n} \int_0^1 2r^{2n+\beta+1} (1 - r^2)^\alpha dr \\ &= \sum_{n=0}^{\infty} |z|^{2n} \int_0^1 t^{n+\frac{\beta}{2}} (1 - t)^\alpha dt \\ &= \sum_{n=0}^{\infty} |z|^{2n} B\left(\alpha + 1, n + \frac{\beta}{2} + 1\right) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1)\Gamma(n + \frac{\beta}{2} + 1)}{\Gamma(n + \frac{\beta}{2} + \alpha + 2)} |z|^{2n} \\ &\sim \sum_{n=0}^{\infty} \frac{\Gamma(n - \alpha)}{\Gamma(n + 1)\Gamma(-\alpha)} |z|^{2n} \\ &= (1 - |z|^2)^\alpha. \end{aligned}$$

Here we first use the polar coordinate $\zeta = re^{i\theta}$, then apply the substitution $t = r^2$. By basic properties of the Beta function⁵ and Stirling's formula, we obtain the asymptotic behavior for the Gamma functions as $n \rightarrow \infty$. The last equality holds since $\alpha < 0$. \blacksquare

Lemma 3.1.4 Let $a_j = \left(\frac{1}{j}\right)^j$, $j = 1, 2, 3, \dots$. For $p \geq 1$, the sum $A_{n,p} = \sum_{j=1}^n j \left(a_j^{p/j} - a_{j+1}^{p/j}\right)$ diverges when $p = 1$ and converges when $p > 1$, as $n \rightarrow \infty$. More precisely, we have

$$\lim_{n \rightarrow \infty} A_{n,1} = \infty$$

and

$$\lim_{n \rightarrow \infty} A_{n,p} \leq c \sum_{j=1}^{\infty} \frac{1}{j^{1+\epsilon}} < \infty$$

⁵We temporarily denote the Beta function by B . Except for the argument in this lemma, a function B is understood as the Bergman kernel on some domain.

for all $p > 1$ with some $c > 0$ and sufficiently small $\epsilon > 0$.

Proof We first prove the following statement,

$$\left(\frac{1}{j}\right)^2 \lesssim \frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}} \lesssim \left(\frac{1}{j}\right)^{2-\epsilon'} \quad (3.5)$$

for any $\epsilon' > 0$, as $j \rightarrow \infty$.⁶

We obtain the first inequality in (3.5) by looking at the limit (with L'Hôpital's rule applied)

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}}{\left(\frac{1}{j}\right)^2} &= \lim_{j \rightarrow \infty} \frac{-\frac{1}{j^2} + \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}} \left(-\frac{1}{j^2} \log(j+1) + \frac{1}{j}\right)}{-2 \left(\frac{1}{j}\right)^3} \\ &= \frac{1}{2} \lim_{j \rightarrow \infty} \frac{1 + \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \left(\frac{\log(j+1)}{j+1} - \frac{j}{j+1}\right)}{\frac{1}{j}} \\ &= \frac{1}{2} \left[\lim_{j \rightarrow \infty} \frac{1 - \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{j}{j+1}}{\frac{1}{j}} + \lim_{j \rightarrow \infty} \frac{\left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{\log(j+1)}{j+1}}{\frac{1}{j}} \right] \\ &= \frac{1}{2} \left[\lim_{j \rightarrow \infty} \frac{j}{j+1} \cdot \lim_{j \rightarrow \infty} \frac{1 + \frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{1}{j}}}{\frac{1}{j}} + \lim_{j \rightarrow \infty} \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{j}{j+1} \log(j+1) \right] \\ &= \frac{1}{2} \left[\lim_{j \rightarrow \infty} \frac{1 - \left(\frac{1}{j+1}\right)^{\frac{1}{j}}}{\frac{1}{j}} + 1 + \lim_{j \rightarrow \infty} \log(j+1) \right] \\ \text{(L'Hôpital's rule)} &= \frac{1}{2} \left[\lim_{j \rightarrow \infty} \frac{\left(\frac{1}{j+1}\right)^{\frac{1}{j}} \left(-\frac{1}{j^2} \log(j+1) + \frac{1}{j(j+1)}\right)}{-\frac{1}{j^2}} + 1 + \lim_{j \rightarrow \infty} \log(j+1) \right] \\ &= \lim_{j \rightarrow \infty} \log(j+1) \\ &= \infty. \end{aligned}$$

⁶The notation $A \lesssim B$ means there is a constant $c > 0$ so that $A \leq cB$.

Similarly, for the second inequality in (3.5), we look at the limit (with L'Hôpital's rule applied)

$$\begin{aligned}
\lim_{j \rightarrow \infty} \frac{\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}}{\left(\frac{1}{j}\right)^{2-\epsilon'}} &= \lim_{j \rightarrow \infty} \frac{-\frac{1}{j^2} + \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}} \left(-\frac{1}{j^2} \log(j+1) + \frac{1}{j}\right)}{(\epsilon' - 2) \left(\frac{1}{j}\right)^{3-\epsilon'}} \\
&= \frac{1}{2 - \epsilon'} \lim_{j \rightarrow \infty} \frac{1 + \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \left(\frac{\log(j+1)}{j+1} - \frac{j}{j+1}\right)}{\left(\frac{1}{j}\right)^{1-\epsilon'}} \\
&= \frac{1}{2 - \epsilon'} \left[\lim_{j \rightarrow \infty} \frac{1 - \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{j}{j+1}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \lim_{j \rightarrow \infty} \frac{\left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{\log(j+1)}{j+1}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} \right] \\
&= \frac{1}{2 - \epsilon'} \left[\lim_{j \rightarrow \infty} \frac{j}{j+1} \cdot \lim_{j \rightarrow \infty} \frac{1 + \frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{1}{j}}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \lim_{j \rightarrow \infty} \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \frac{j}{j+1} \frac{\log(j+1)}{j^{\epsilon'}} \right] \\
&= \frac{1}{2 - \epsilon'} \left[\lim_{j \rightarrow \infty} \frac{1 - \left(\frac{1}{j+1}\right)^{\frac{1}{j}}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + \lim_{j \rightarrow \infty} \frac{\frac{1}{j}}{\left(\frac{1}{j}\right)^{1-\epsilon'}} + 0 \right] \\
\text{(L'Hôpital's rule)} &= \frac{1}{2 - \epsilon'} \left[\lim_{j \rightarrow \infty} \frac{\left(\frac{1}{j+1}\right)^{\frac{1}{j}} \left(-\frac{1}{j^2} \log(j+1) + \frac{1}{j(j+1)}\right)}{(\epsilon' - 1) \left(\frac{1}{j}\right)^{2-\epsilon'}} + 0 + 0 \right] \\
&= \frac{1}{(2 - \epsilon')(1 - \epsilon')} \lim_{j \rightarrow \infty} \left(\frac{1}{j+1}\right)^{\frac{1}{j}} \left(\frac{\log(j+1)}{j^{\epsilon'}} - \frac{j}{(j+1)j^{\epsilon'}}\right) \\
&= 0.
\end{aligned}$$

This shows the second inequality in (3.5).

Now, for $p = 1$, we have

$$\begin{aligned}
A_{n,1} &= \sum_{j=1}^n j \left(a_j^{\frac{1}{j}} - a_{j+1}^{\frac{1}{j+1}} \right) \\
&= \sum_{j=1}^n j \left[\left(\frac{1}{j}\right)^{j \cdot \frac{1}{j}} - \left(\frac{1}{j+1}\right)^{(j+1) \cdot \frac{1}{j+1}} \right] \\
&= \sum_{j=1}^n j \left[\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}} \right]
\end{aligned}$$

So, from (3.5), we have

$$\lim_{n \rightarrow \infty} A_{n,1} \gtrsim \sum_{j=1}^{\infty} j \cdot \left(\frac{1}{j}\right)^2 = \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

For $p > 1$, we consider the function $\phi(x) = x^p$, $x \in (0, 1]$. By the mean-value theorem,

for each j , we have

$$\phi\left(\frac{1}{j}\right) - \phi\left(\left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right) = \left[\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right] \phi'(x_j),$$

where $\phi'(x) = px^{p-1}$ and $\left(\frac{1}{j+1}\right)^{\frac{j+1}{j}} \leq x_j \leq \frac{1}{j}$. Since $p-1 > 0$, we have

$$x_j^{p-1} \leq \left(\frac{1}{j}\right)^{p-1}.$$

So we obtain

$$\phi\left(\frac{1}{j}\right) - \phi\left(\left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right) \leq \left[\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right] p \left(\frac{1}{j}\right)^{p-1}.$$

Therefore, from (3.5), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} A_{n,p} &= \sum_{j=1}^{\infty} j \left(a_j^{\frac{p}{j}} - a_{j+1}^{\frac{p}{j+1}}\right) \\ &= \sum_{j=1}^{\infty} j \left[\phi\left(\frac{1}{j}\right) - \phi\left(\left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right)\right] \\ &\leq \sum_{j=1}^{\infty} j \left[\frac{1}{j} - \left(\frac{1}{j+1}\right)^{\frac{j+1}{j}}\right] p \left(\frac{1}{j}\right)^{p-1} \\ &\lesssim \sum_{j=1}^{\infty} j \left(\frac{1}{j}\right)^{2-\epsilon'} \left(\frac{1}{j}\right)^{p-1} \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{j}\right)^{p-\epsilon'} \\ &< \infty, \end{aligned}$$

for sufficiently small $\epsilon' > 0$, such that $p - \epsilon' = 1 + \epsilon$ for some $\epsilon > 0$. ■

Now we are ready to prove Theorem 2.

Theorem 2 For $s' \in \mathbb{R}$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\mathcal{B}_{s'}$ be the weighted Bergman projection on the space (\mathbb{D}^*, μ) , where $\mu(z) = |z|^{s'}$.

(a) For $s' \in (0, \infty)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1}\right)$.

(b) For $s' \in [-3, 0]$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded for $p \in (1, \infty)$.

(c) For $s' \in (-4, -3)$, with $k = -2$ and $s \in (0, 1)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(2 - s, \frac{2-s}{1-s}\right)$.

(d) When $s' = -4$, \mathcal{B}_{-4} is $L^p(\mu)$ bounded for $p \in (1, \infty)$.

(e) For $s' \in (-\infty, -4)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1}\right)$.

Proof For the boundedness part, by (3.1), we have

$$|B_{s'}(z, \zeta)| \leq |z\bar{\zeta}|^{-(k+1)} |B_0(z, \zeta)|$$

since $(z, \zeta) \in \mathbb{D}^* \times \mathbb{D}^*$. So it suffices to apply Lemma 3.1.2 to the kernel

$$T(z, \zeta) = |z\bar{\zeta}|^{-(k+1)} \cdot \frac{1}{|1 - z\bar{\zeta}|^2}$$

on $\mathbb{D}^* \times \mathbb{D}^*$, with the positive function

$$h(z) = (1 - |z|^2)^\delta |z|^\sigma$$

on \mathbb{D}^* for some $\delta, \sigma \in \mathbb{R}$ and the measure $d\mu(z) = |z|^{s'} dV(z)$.

Now we check the first condition in Lemma 3.1.2. By Lemma 3.1.3, we see that

$$\begin{aligned} \mathcal{T}(h^{p'}) &= \int_{\mathbb{D}^*} \frac{(1 - |\zeta|^2)^{\delta p'} |\zeta|^{\sigma p' + s' - (k+1)} dV(\zeta)}{|z|^{k+1} |1 - z\bar{\zeta}|^2} \\ &= I_{\delta p', \sigma p' + s + k - 1}(z) \cdot |z|^{-(k+1)} \\ &\lesssim (1 - |z|^2)^{\delta p'} |z|^{-(k+1)} \\ &\leq h(z)^{p'}, \end{aligned}$$

provided $-1 < \delta p' < 0$, $-2 < \sigma p' + s + k - 1$, and $\sigma p' \leq -(k + 1)$, i.e., $\delta \in (-\frac{1}{p'}, 0)$ and $\sigma \in (-\frac{s+k+1}{p'}, -\frac{k+1}{p'}]$. Similarly, the second condition in Lemma 3.1.2

$$\mathcal{T}(h^p)(\zeta) \lesssim h(\zeta)^p$$

holds if $\delta \in (-\frac{1}{p}, 0)$ and $\sigma \in (-\frac{s+k+1}{p}, -\frac{k+1}{p}]$.

Therefore such δ and σ exist if $(-\frac{1}{p'}, 0) \cap (-\frac{1}{p}, 0) \neq \emptyset$ and $(-\frac{s+k+1}{p'}, -\frac{k+1}{p'}] \cap (-\frac{s+k+1}{p}, -\frac{k+1}{p}] \neq \emptyset$. By Lemma 3.1.2, the existence of h will imply the boundedness of $\mathcal{B}_{s'}$. By a direct computation, we see that $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded, when

1. $k \geq 0$, $p \in (\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1})$;
2. $k = -1$, $p \in (1, \infty)$;
3. $k = -2$ and $1 \leq s \leq 2$, $p \in (1, \infty)$;
4. $k = -2$ and $0 < s < 1$, $p \in (2 - s, \frac{2-s}{1-s})$;
5. $k = -3$ and $s = 2$, $p \in (1, \infty)$;
6. $k = -3$ and $0 < s < 2$, $p \in (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$;
7. $k < -3$, $p \in (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$.

It is easy to see, the conditions above is equivalent to the conditions in Theorem 2.

To show the unboundedness part, since $\mathcal{B}_{s'}$ is self-adjoint, by the interpolation theorem, we only need to look at the endpoint $p = \frac{s+2k+2}{s+k+1}$. Let $a_j = (\frac{1}{j})^j$, $j = 1, 2, 3, \dots$, and define a function g on $(0, 1]$ such that $g(r) = r^{\frac{1}{j} - (s+k+1)}$, $r \in (a_{j+1}, a_j]$. We consider the sequence

$$f_n(z) = \begin{cases} g(|z|) \left(\frac{\bar{z}}{|z|}\right)^{k+1}, & |z| \in (a_{n+1}, 1], \\ 0, & |z| \in [0, a_{n+1}]. \end{cases}$$

Since each f_n is supported away from the origin and bounded above, it is easy to see that

$\{f_n\} \subset L^2(\mathbb{D}^*, |z|^{s'})$. Since $p = \frac{s+2k+2}{s+k+1} > 1$, by Lemma 3.1.4, we have

$$\begin{aligned}
\|f_n\|_{L^p(\mu)}^p &= \int_{a_{n+1} < |z| < 1} g(|z|)^p |z|^{s'} dV(z) \\
&= \sum_{j=1}^n \int_{a_{j+1}}^{a_j} 2r \left[\frac{1}{j} - (s+k+1) \right]^{p+s+2k+1} dr \\
&= 2 \sum_{j=1}^n \frac{j}{p} \left(a_j^{\frac{p}{j}} - a_{j+1}^{\frac{p}{j}} \right) \\
&= \frac{2}{p} A_{n,p} \\
&\leq c \sum_{j=1}^{\infty} \frac{1}{j^{1+\epsilon}},
\end{aligned}$$

for some $\epsilon > 0$ and some $c > 0$.

On the other hand, by (3.1), we see that

$$\begin{aligned}
\mathcal{B}_{s'}(f_n)(z) &= \frac{s}{2} \int_{\mathbb{D}^*} \frac{(z\bar{\zeta})^{-(k+1)} f_n(\zeta) |\zeta|^{s'} dV(\zeta)}{(1 - z\bar{\zeta})^2} + \left(1 - \frac{s}{2}\right) \int_{\mathbb{D}^*} \frac{(z\bar{\zeta})^{-k} f_n(\zeta) |\zeta|^{s'} dV(\zeta)}{(1 - z\bar{\zeta})^2} \\
&= I + II.
\end{aligned}$$

For the second integral,

$$\begin{aligned}
II &= \left(1 - \frac{s}{2}\right) z^{-k} \int_{a_{n+1} < |\zeta| < 1} \frac{g(|\zeta|) \bar{\zeta} |\zeta|^{s+k-1} dV(\zeta)}{(1 - z\bar{\zeta})^2} \\
&= 0.
\end{aligned}$$

In the last line, we expand the kernel $(1 - z\bar{\zeta})^{-2}$ and use the rotational symmetries. For

the first integral, a similar computation shows that

$$\begin{aligned}
I &= \frac{s}{2} z^{-(k+1)} \int_{a_{n+1} < |\zeta| < 1} \frac{g(|\zeta|) |\zeta|^{s+k-1} dV(\zeta)}{(1 - z\bar{\zeta})^2} \\
&= \frac{s}{2} z^{-(k+1)} \int_{a_{n+1} < |\zeta| < 1} g(|\zeta|) |\zeta|^{s+k-1} dV(\zeta) \\
&= s z^{-(k+1)} \sum_{j=1}^n \int_{a_{j+1}}^{a_j} r^{\frac{1}{j} - (s+k+1) + s+k} dr \\
&= s z^{-(k+1)} \sum_{j=1}^n j \left(a_j^{\frac{1}{j}} - a_{j+1}^{\frac{1}{j}} \right).
\end{aligned}$$

Therefore we have

$$\mathcal{B}_{s'}(f_n)(z) = sz^{-(k+1)}A_{n,1}.$$

It is easy to see that

$$s + 2k - (k + 1)p = -2 + \nu,$$

for some $\nu > 0$. So we obtain

$$\|\mathcal{B}_{s'}(f_n)\|_{L^p(\mu)} = s \left(\frac{2}{\nu}\right)^{\frac{1}{p}} A_{n,1}.$$

Hence, from Lemma 3.1.4, we see that

$$\lim_{n \rightarrow \infty} \|\mathcal{B}_{s'}(f_n)\|_{L^p(\mu)} = \infty.$$

This completes the proof. ■

Remark 3.1.5 *The range for p does not change continuously as s' varies. In fact, there are jumps around the integers $s' = 0, 2, 4, \dots$, and the integers $s' = -4, -6, -8, \dots$. The range changes continuously only when s' lies between the consecutive integers above or $s' \in (-4, 0]$ (there is no jump around $s' = -2$).*

3.2 The Hartogs Triangle

Now we are ready to prove Theorem 3 and Theorem 4 by using inflation arguments.

Theorem 3 *For $s' \in [0, \infty)$ with the unique expression $s' = s + 2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\lambda(z) = |z_2|^{s'}$, where $z \in \mathbb{H}$. The weighted Bergman projection \mathcal{B}_λ on the weighted space (\mathbb{H}, λ) is $L^p(\mathbb{H}, \lambda)$ bounded if and only if $p \in \left(\frac{s+2k+4}{s+k+2}, \frac{s+2k+4}{k+2}\right)$.*

Proof Using the notation in Proposition 2.2.6, if we take $\Omega = \mathbb{D}^*$ and $\mu(w) = |w|^2$, then we see that $\mathcal{B}_{\mathbb{H}, \lambda}$ is $L^p(\mathbb{H}, \lambda)$ bounded if and only if $\mathcal{B}_{\mathbb{D}^*, \mu\lambda}$ is $L^p(\mathbb{D}^*, \mu\lambda)$ bounded.

By Theorem 2 (a), the latter is true if and only if $p \in \left(\frac{s+2k+4}{s+k+2}, \frac{s+2k+4}{k+2}\right)$ for $s' \in [0, \infty)$. ■

Theorem 4 *Given any $p_0 \in [1, 2)$ with its conjugate exponent p'_0 , let $\lambda(z) = |z_2|^{-(p_0+4)}$, where $z \in \mathbb{H}$. Then the weighted Bergman projection \mathcal{B}_λ on the weighted space (\mathbb{H}, λ) is $L^p(\mathbb{H}, \lambda)$ bounded if and only if $p \in (p_0, p'_0)$.*

Proof Similarly, this is a direct consequence by combining Proposition 2.2.6 with $\Omega = \mathbb{D}^*$ and $\mu(w) = |w|^2$, and Theorem 2 (b) (c) with $s' = -(2 + p_0)$. ■

3.3 The n -dimensional Hartogs Triangle

To prove Theorem 5, we need a lemma.

Lemma 3.3.1 *Let $\mathbb{H}^{d*} = \{z \in \mathbb{C}^d \mid 0 < |z_1| < \dots < |z_d| < 1\}$ be the punctured d -dimensional standard Hartogs triangle. Suppose we have a weight $\lambda(z) = |z_1|^{s_1} \dots |z_d|^{s_d}$ on \mathbb{H}^{d*} , where $s_1, \dots, s_d \in \mathbb{R}$. Then the weighted Bergman projection $\mathcal{B}_{\mathbb{H}^{d*}, \lambda}$ is $L^p(\lambda)$ bounded if and only if each of the following projections*

$$\mathcal{B}_{s_1}, \mathcal{B}_{s_1+s_2+2}, \dots, \mathcal{B}_{s_1+\dots+s_d+2(d-1)}$$

is L^p bounded on the corresponding weighted space.

Proof As in the proof of Proposition 2.2.6, we have the biholomorphism $\Phi : \mathbb{H}^{d*} \rightarrow (\mathbb{D}^*)^{\times d}$ via $\Phi(z) = \left(\frac{z_1}{z_2}, \dots, \frac{z_{d-1}}{z_d}, z_d\right)$. By Corollary 2.2.3, we see that $\mathcal{B}_{\mathbb{H}^{d*}, \lambda}$ is $L^p(\lambda)$ bounded if and only if $\mathcal{B}_{(\mathbb{D}^*)^{\times d}, \tilde{\lambda}}$ is $L^p(\tilde{\lambda})$ -bounded, where $\tilde{\lambda} = |\det J_{\mathbb{C}} \Phi^{-1}|^2 \lambda(\Phi^{-1})$. A direct computation shows that

$$|\det J_{\mathbb{C}} \Phi^{-1}(w)|^2 = |w_2 w_3^2 \dots w_d^{d-1}|^2,$$

where $w \in (\mathbb{D}^*)^{\times d}$. So we see that

$$\mathcal{B}_{(\mathbb{D}^*)^{\times d}, \tilde{\lambda}} = \mathcal{B}_{s_1} \otimes \mathcal{B}_{s_1+s_2+2} \otimes \dots \otimes \mathcal{B}_{s_1+\dots+s_d+2(d-1)},$$

which implies the conclusion in view of Lemma 2.2.4. ■

Now we are ready to prove Theorem 5.

Theorem 5 *Using the notation as in (1.1), let $\lambda(z) = |z_1''|^{s_1} \cdots |z_d''|^{s_d}$, where $z \in \mathbb{H}_{\phi_j}^n$ and $s_1, \dots, s_d \in \mathbb{R}$. Let $\mathcal{B}_{s'}$ be as in Theorem 2. Then the weighted Bergman projection on $(\mathbb{H}_{\phi_j}^n, \lambda)$ is $L^p(\lambda)$ bounded if and only if each of the following projections*

$$\mathcal{B}_{2m+s_1}, \mathcal{B}_{2m+s_1+s_2+2}, \dots, \mathcal{B}_{2m+s_1+\dots+s_d+2(d-1)}$$

is L^p bounded on the corresponding weighted space.

In other words, assume that $p > 1$ and for $j = 1, 2, \dots, d$ we let I_j be one of the intervals for p in Theorem 2, so that the j th projection above is L^p bounded if and only if $p \in I_j$. Then the weighted Bergman projection on $(\mathbb{H}_{\phi_j}^n, \lambda)$ is $L^p(\lambda)$ bounded if and only if $p \in \cap I_j$.

Proof Iteratively apply Proposition 2.2.6 l times to $\Omega = \mathbb{H}^{d*} = \{z'' \in \mathbb{C}^d \mid 0 < |z_1''| < \cdots < |z_d''| < 1\}$ with the same weight $|z_1''|^2$. Then we will arrive at the space

$$\mathbb{H}^n = \{(z', z'') \in \mathbb{C}^{m_1+\dots+m_l+d} : \max_{1 \leq j \leq l} |\tilde{z}_j| < |z_1''| < |z_2''| < \cdots < |z_d''| < 1\}.$$

So the weighted Bergman projection $\mathcal{B}_{\mathbb{H}^n, \lambda}$ is $L^p(\lambda)$ bounded if and only if $\mathcal{B}_{\mathbb{H}^{d*}, \tilde{\lambda}}$ is $L^p(\tilde{\lambda})$ bounded, where $\lambda(z) = |z_1''|^{s_1} \cdots |z_d''|^{s_d}$ and $\tilde{\lambda}(z) = |z_1''|^{2m} \lambda(z)$. Applying Lemma 3.3.1 to $\mathcal{B}_{\mathbb{H}^{d*}, \tilde{\lambda}}$, we obtain a special case of Theorem 5, namely, the L^p boundedness of the weighted Bergman projection on (\mathbb{H}^n, λ) .

On the other hand, we look at the biholomorphism $\Phi : \mathbb{H}_{\phi_j}^n \rightarrow \mathbb{H}^n$ via

$$\Phi(z) = (\phi_1(\tilde{z}_1), \dots, \phi_l(\tilde{z}_l), z_1'', \dots, z_d'').$$

A direct computation shows, for $z \in \mathbb{H}_{\phi_j}^n$, that

$$\det J_{\mathbb{C}} \Phi(z) = \prod_{j=1}^l \det J_{\mathbb{C}} \phi_j(\tilde{z}_j).$$

For each j , we have the biholomorphism $\phi_j : \Omega_j \rightarrow \mathbb{B}^{m_j}$, with both Ω_j and \mathbb{B}^{m_j} being smooth and bounded. Since \mathbb{B}^{m_j} is strongly pseudoconvex, it satisfies condition R. By Bell's extension theorem in [Bel81], both ϕ_j and ϕ_j^{-1} extend smoothly to the boundaries. Therefore we can find two positive real numbers c_j and d_j , so that for any $\tilde{z}_j \in \Omega_j$,

$$0 < c_j \leq |\det J_{\mathbb{C}}\phi_j(\tilde{z}_j)| \leq d_j.$$

Hence, if we let $c = \prod_{j=1}^l c_j$ and $d = \prod_{j=1}^l d_j$, then for $z \in \mathbb{H}_{\phi_j}^n$ we have

$$0 < c \leq |\det J_{\mathbb{C}}\Phi(z)| \leq d.$$

Now suppose for some $p \in (1, \infty)$ that the weighted Bergman projection $\mathcal{B}_{\mathbb{H}_{\phi_j}^n, \lambda}$ is bounded on $L^p(\mathbb{H}_{\phi_j}^n, \lambda)$. Then, by the transformation formula in Corollary 2.2.2, for $f \in L^p(\mathbb{H}^n, \lambda)$ we have

$$\begin{aligned} \|\mathcal{B}_{\mathbb{H}^n, \lambda}(f)\|_{L^p(\mathbb{H}^n, \lambda)}^p &= \int_{\mathbb{H}^n} |\mathcal{B}_{\mathbb{H}^n, \lambda}(f)(z)|^p \lambda(z) dV(z) \\ &= \int_{\mathbb{H}_{\phi_j}^n} \left| \mathcal{B}_{\mathbb{H}_{\phi_j}^n, \lambda}(\det J_{\mathbb{C}}\Phi \cdot f \circ \Phi)(z) \right|^p |\det J_{\mathbb{C}}\Phi(z)|^{2-p} \lambda(z) dV(z) \\ &\leq \max\{c^{2-p}, d^{2-p}\} \left\| \mathcal{B}_{\mathbb{H}_{\phi_j}^n}(\det J_{\mathbb{C}}\Phi \cdot f \circ \Phi) \right\|_{L^p(\mathbb{H}_{\phi_j}^n, \lambda)}^p \\ &\leq C \max\{c^{2-p}, d^{2-p}\} \|\det J_{\mathbb{C}}\Phi \cdot f \circ \Phi\|_{L^p(\mathbb{H}_{\phi_j}^n, \lambda)}^p \\ &\leq C c^{-|p-2|} d^{|p-2|} \|f\|_{L^p(\mathbb{H}^n, \lambda)}^p \end{aligned}$$

for some $C > 0$. So the weighted Bergman projection $\mathcal{B}_{\mathbb{H}^n, \lambda}$ is bounded on $L^p(\mathbb{H}^n, \lambda)$.

Conversely, if we apply the same argument to $\Phi^{-1} : \mathbb{H}^n \rightarrow \mathbb{H}_{\phi_j}^n$, we see the weighted L^p boundedness of $\mathcal{B}_{\mathbb{H}^n, \lambda}$ will imply the weighted L^p boundedness of $\mathcal{B}_{\mathbb{H}_{\phi_j}^n, \lambda}$. Therefore, we see that $\mathcal{B}_{\mathbb{H}_{\phi_j}^n, \lambda}$ is $L^p(\lambda)$ bounded if and only if $\mathcal{B}_{\mathbb{H}^n, \lambda}$ is $L^p(\lambda)$ bounded. This completes the proof. ■

As a consequence, we obtain Theorem 1 from Theorem 5.

Theorem 1 *The Bergman projection $\mathcal{B}_{\mathbb{H}_{\phi_j}^n}$ on $\mathbb{H}_{\phi_j}^n$ is bounded on $L^p(\mathbb{H}_{\phi_j}^n)$ if and only if p is in the range $(\frac{2n}{n+1}, \frac{2n}{n-1})$.*

Proof Letting $s_1 = s_2 = \cdots = s_d = 0$ in Theorem 5, we see that $\lambda(z) = 1$. Therefore, the Bergman projection $\mathcal{B}_{\mathbb{H}_{\phi_j}^n}$ on $\mathbb{H}_{\phi_j}^n$ is bounded on $L^p(\mathbb{H}_{\phi_j}^n)$ if and only if each of the following projections

$$\mathcal{B}_{2m}, \mathcal{B}_{2m+2}, \cdots, \mathcal{B}_{2m+2(d-1)}$$

is L^p bounded on the corresponding weighted space. Note that, for $j = 1, 2, \dots, d$, by Theorem 2, $\mathcal{B}_{2(m+j-1)}$ is L^p bounded if and only if $p \in (\frac{2(m+j)}{m+j+1}, \frac{2(m+j)}{m+j-1})$. Hence, $\mathcal{B}_{\mathbb{H}_{\phi_j}^n}$ is L^p bounded if and only if $p \in \bigcap_{j=1}^d (\frac{2(m+j)}{m+j+1}, \frac{2(m+j)}{m+j-1}) = (\frac{2n}{n+1}, \frac{2n}{n-1})$. ■

4. The Two-weight Inequality

In this chapter, we mainly focus on the two-weight inequality (1.2) introduced in §1.2, and prove Theorem 6. In the next chapter, we will see as an application, that Theorem 6 gives an alternative proof of Theorem 2 and extends it to a wider class of weights.

4.1 Preliminaries and Basic Definitions

Throughout this chapter, z and w will denote complex variables in \mathbb{R}_+^2 (or $\mathbb{C} = \mathbb{R}^2$), any weight μ will be considered to be locally integrable on \mathbb{R}_+^2 (or \mathbb{R}^2), \mathcal{B} will denote the *Bergman projection on \mathbb{R}_+^2* , and we have

$$\mathcal{B}(f)(z) = \int_{\mathbb{R}_+^2} -\frac{1}{(z - \bar{w})^2} f(w) dV(w),$$

for all measurable f on \mathbb{R}_+^2 whenever it is well-defined.¹ We also consider the "*absolute value*" operator $\tilde{\mathcal{B}}$ of \mathcal{B} , which is defined as

$$\tilde{\mathcal{B}}(f)(z) = \int_{\mathbb{R}_+^2} \frac{1}{|z - \bar{w}|^2} f(w) dV(w),$$

where we replace the kernel $-\frac{1}{(z - \bar{w})^2}$ by its absolute value.

For a weight μ , a measurable function f , and any measurable set W with its Lebesgue measure $|W|$, we may use the notation

$$\mu(W) = \int_W \mu(z) dV(z)$$

¹One may consider $f \in C_c^\infty(\mathbb{R}_+^2)$, and the conclusion of Theorem 6 will follow from the passage from $C_c^\infty(\mathbb{R}_+^2)$ to $L^p(\mathbb{R}_+^2, \mu_2)$. We omit the coefficient $\frac{1}{\pi}$ in front of the integral in order to coincide with the normalization $\text{Area}(\mathbb{D}) = 1$ in the previous chapter.

and

$$\int_W f(z) dV(z) = \frac{1}{|W|} \int_W f(z) dV(z).$$

In this chapter, the symbol c will denote some positive constant independent of the variables and the functions in the context. So two c 's in the same equation could be different, but it does not matter at all.

In our notation, the inequality (1.2) becomes

$$\int_{\mathbb{R}_+^2} |\mathcal{B}(f)(z)|^p \mu_1(z) dV(z) \leq c \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) dV(z).$$

In order to prove Theorem 6, following Lanzani and Stein's idea in [LS04], we first give the definitions of several variants of the A_p condition, which was introduced by Muckenhoupt.²

Definition 4.1.1 For $p > 1$, let p' denote the conjugate exponent of p . We say the two weights μ_1 and μ_2 are in the $A_p(\mathbb{R}_+^2)$ class denoted by $(\mu_1, \mu_2) \in A_p(\mathbb{R}_+^2)$ if there is a positive constant c so that

$$\int_{D \cap \mathbb{R}_+^2} \mu_1(z) dV(z) \left(\int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) \right)^{\frac{p}{p'}} \leq c,$$

for all disks D centered at $z \in \overline{\mathbb{R}_+^2}$. A disk is said to be a special disk if it is centered at $x \in \mathbb{R}$. We say $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ if the above inequality holds for all special disks. The class $A_p(\mathbb{R}^2)$ is defined in the same way as $A_p(\mathbb{R}_+^2)$ except replacing \mathbb{R}_+^2 by \mathbb{R}^2 . For some weight μ , if $(\mu, \mu) \in A_p^+(\mathbb{R}_+^2)$ (resp. $A_p(\mathbb{R}_+^2)$ and $A_p(\mathbb{R}^2)$), we may simply adopt the notation $\mu \in A_p^+(\mathbb{R}_+^2)$ (resp. $A_p(\mathbb{R}_+^2)$ and $A_p(\mathbb{R}^2)$).

Remark 4.1.1 The class $A_p^+(\mathbb{R}_+^2)$ is strictly wider than the class $A_p(\mathbb{R}_+^2)$. For the one-weight case see the comments following the definition of A_p^+ in [LS04]. For two-weight case see Proposition 5.1.1 in Chapter 5.

²See also [Muc72] and [Ste93, Chapter 5].

As in [LS04], we now introduce a standard “tiling” of \mathbb{R}_+^2 and the associated averaging operator (the “conditional expectation”).

Definition 4.1.2 *The standard “tiling” of \mathbb{R}_+^2 are the squares $\{S_{j,k}\}$ of form*

$$S_{j,k} = \{z = x + iy \in \mathbb{C} : 2^k \leq y \leq 2^{k+1} \text{ and } j \cdot 2^k \leq x \leq (j+1) \cdot 2^k\}$$

for all $j, k \in \mathbb{Z}$. Note that each $S_{j,k}$ has side-length 2^k , the interiors of $S_{j,k}$ ’s are disjoint, and $\mathbb{R}_+^2 = \bigcup_{j,k \in \mathbb{Z}} S_{j,k}$.

Define the associated averaging operator E by

$$E(f)(z) = \int_{S_{j,k}} f(z) dV(z), \quad \text{if } z \in S_{j,k},$$

for any nonnegative measurable function f on \mathbb{R}_+^2 .

4.2 The Properties of the Operator E

In this section, we give two propositions about the operator E . The proofs can also be found in [LS04], but we will give the details here, in order to be self-contained.

Proposition 4.2.1 *We have the following basic properties of E . For any nonnegative measurable functions f and g , letting p' be the conjugate exponent of p , we have*

$$(a) \int_{\mathbb{R}_+^2} E(f)(z)g(z) dV(z) = \int_{\mathbb{R}_+^2} E(f)(z)E(g)(z) dV(z),$$

$$(b) \int_{\mathbb{R}_+^2} (E(f)(z))^p g(z) dV(z) \leq \int_{\mathbb{R}_+^2} E(f^p)(z)g(z) dV(z),$$

$$(c) E(fg)(z) \leq \left(E(f^p)(z)\right)^{\frac{1}{p}} \left(E(g^{p'})(z)\right)^{\frac{1}{p'}} \quad \text{for all } z \in \mathbb{R}_+^2.$$

Proof For (a), the left hand side of the equality is

$$\begin{aligned} \sum_{j,k} \int_{S_{j,k}} E(f)(z)g(z) dV(z) &= \sum_{j,k} E(f)(z) \chi_{S_{j,k}}(z) \int_{S_{j,k}} g(z) dV(z) \\ &= \sum_{j,k} E(f)(z) \chi_{S_{j,k}}(z) E(g)(z) |S_{j,k}|. \end{aligned}$$

On the other hand, the right hand side of the equality is

$$\sum_{j,k} \int_{S_{j,k}} E(f)(z)E(g)(z) dV(z) = \sum_{j,k} E(f)(z)E(g)(z)\chi_{S_{j,k}}(z) |S_{j,k}|,$$

which equals the left hand side.

For (b), we apply Jensen's inequality to the integral $\int_{S_{j,k}} f(z) dV(z)$, via the convex function x^p . We see that

$$(E(f)(z))^p g(z) \leq E(f^p)(z)g(z),$$

for all $z \in S_{j,k}$. Integrate over $S_{j,k}$ and sum over all $j, k \in \mathbb{Z}$, then we get the desired inequality.

For (c), it is a direct consequence of Hölder's inequality. ■

Proposition 4.2.2 *For any $f \geq 0$, we have*

$$\tilde{\mathcal{B}}(f) \leq cE\tilde{\mathcal{B}}E(f). \quad (4.1)$$

Proof We first show, for any $z, w \in \mathbb{R}_+^2$, that

$$E_z E_w \left(\frac{1}{|z - \bar{w}|^2} \right) \geq \frac{c}{|z - \bar{w}|^2}, \quad (4.2)$$

where E_z and E_w are operators E acting on z and w respectively. Suppose that $z \in S_{j,k}$ and $w \in S_{j',k'}$, with $k \geq k'$. We separate our arguments into two cases.

Case (I), $|\Re(z - \bar{w})| \leq 2^k$.

Since $z \in S_{j,k}$ and $w \in S_{j',k'}$, we see $|\Im(z - \bar{w})| \geq 2^k + 2^{k'} \geq 2^k$. Then

$$\frac{1}{|z - \bar{w}|^2} \leq \frac{1}{|\Im(z - \bar{w})|^2} \leq \frac{1}{2^{2k}}.$$

On the other hand, for any $\zeta \in S_{j,k}$ and $\eta \in S_{j',k'}$, we have

$$|\Re(\zeta - \bar{\eta})| \leq |\Re(\zeta - z)| + |\Re(z - \bar{w})| + |\Re(\bar{w} - \bar{\eta})| \leq 2^k + 2^k + 2^{k'} \leq 3 \cdot 2^k,$$

and

$$|\Im(\zeta - \bar{\eta})| \leq 2^{k+1} + 2^{k'+1} \leq 2^{k+2}.$$

So we obtain

$$\frac{1}{|\zeta - \bar{\eta}|^2} \geq \frac{1}{(3 \cdot 2^k)^2 + (2^{k+2})^2} = \frac{1}{2^{2k}} \cdot \frac{1}{25}.$$

Therefore we have

$$\begin{aligned} E_z E_w \left(\frac{1}{|z - \bar{w}|} \right) &= \int_{S_{j,k}} \int_{S_{j',k'}} \frac{1}{|\zeta - \bar{\eta}|^2} dV(\eta) dV(\zeta) \\ &\geq \frac{1}{25} \cdot \frac{1}{2^{2k}} \\ &\geq \frac{1}{25} \cdot \frac{1}{|z - \bar{w}|^2}. \end{aligned}$$

Case (II), $|\Re(z - \bar{w})| > 2^k$.

Similarly, we have

$$\frac{1}{|z - \bar{w}|^2} \leq \frac{1}{|\Re(z - \bar{w})|^2}.$$

For any $\zeta \in S_{j,k}$ and $\eta \in S_{j',k'}$, again we see

$$|\Re(\zeta - \bar{\eta})| \leq 2^k + |\Re(z - \bar{w})| + 2^{k'} \leq 3 |\Re(z - \bar{w})|,$$

and

$$|\Im(\zeta - \bar{\eta})| \leq 2^{k+1} + 2^{k'+1} \leq 4 |\Re(z - \bar{w})|.$$

So we obtain

$$\frac{1}{|\zeta - \bar{\eta}|^2} \geq \frac{1}{(3 |\Re(z - \bar{w})|)^2 + (4 |\Re(z - \bar{w})|)^2} = \frac{1}{25} \cdot \frac{1}{|\Re(z - \bar{w})|^2}.$$

Therefore we have

$$\begin{aligned} E_z E_w \left(\frac{1}{|z - \bar{w}|} \right) &= \int_{S_{j,k}} \int_{S_{j',k'}} \frac{1}{|\zeta - \bar{\eta}|^2} dV(\eta) dV(\zeta) \\ &\geq \frac{1}{25} \cdot \frac{1}{|\Re(z - \bar{w})|^2} \\ &\geq \frac{1}{25} \cdot \frac{1}{|z - \bar{w}|^2}. \end{aligned}$$

For $k' \geq k$, all arguments remain the same except switching k' with k . Thus we have proved (4.2). Now (4.1) follows from the argument below,

$$\begin{aligned}
E\tilde{\mathcal{B}}E(f)(z) &= E_z \left(\int_{\mathbb{R}_+^2} \frac{1}{|z - \bar{w}|^2} E(f)(w) dV(w) \right) \\
&= E_z \left(\int_{\mathbb{R}_+^2} E_w \left(\frac{1}{|z - \bar{w}|^2} \right) f(w) dV(w) \right) \\
&= \int_{\mathbb{R}_+^2} E_z E_w \left(\frac{1}{|z - \bar{w}|^2} \right) f(w) dV(w) \\
&\geq \int_{\mathbb{R}_+^2} \frac{c}{|z - \bar{w}|^2} f(w) dV(w) \\
&= c\tilde{\mathcal{B}}(f)(z),
\end{aligned}$$

where we have applied Proposition 4.2.1 (a) in the second line. ■

4.3 Analysis on Variants of A_p Class

In this section, we do some analysis on the variants of the A_p class. One important observation is the following proposition.

Proposition 4.3.1 *If $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, then $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}_+^2)$.*

The proof is essentially the same as that of the one weight case [LS04, Proposition 4.6]. Again, to be self-contained, we will give all the details here.

Definition 4.3.1 *Let W_1, \dots, W_N be N measurable subsets in \mathbb{R}_+^2 , we say they are comparable to each other if there is a special disk $D_R(x_0)$ centered at $x_0 \in \mathbb{R}$, so that $W_l \subset D_R(x_0) \cap \mathbb{R}_+^2$, while $|D_R(x_0) \cap \mathbb{R}_+^2| \leq a |W_l|$, for some $a \geq 1$, $l = 1, 2, \dots, N$.*

Lemma 4.3.2 *If $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and if W_1 and W_2 are comparable, then there is a $C = C(a) \geq 1$, such that $\mu_1(W_1) \leq C\mu_2(W_2)$.*

Proof Since W_1 and W_2 are comparable, we can find a special disk $D_R(x_0)$ so that $W_l \subset D_R(x_0) \cap \mathbb{R}_+^2$ and $|D_R(x_0) \cap \mathbb{R}_+^2| \leq a|W_l|$, $l = 1, 2$. Then it is easy to see

$$\int_{W_1} \mu_1(z) dV(z) \leq a \int_{D_R(x_0) \cap \mathbb{R}_+^2} \mu_1(z) dV(z). \quad (4.3)$$

Apply Jensen's inequality to the integral $\int_{W_2} \mu_2(z) dV(z)$, via the convex function $x^{-\frac{p'}{p}}$, we have

$$\left(\int_{W_2} \mu_2(z) dV(z) \right)^{-\frac{p'}{p}} \leq \int_{W_2} \mu_2(z)^{-\frac{p'}{p}} dV(z) \leq a \int_{D_R(x_0) \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z). \quad (4.4)$$

Combine (4.3) and (4.4), since $|W_1| \leq |D_R(x_0) \cap \mathbb{R}_+^2| \leq a|W_2|$ and since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, we see

$$\mu_1(W_1) \leq ca^{2+\frac{p}{p'}} \mu_2(W_2).$$

■

Proof [Proof of Proposition 4.3.1.]

For any disk $D_R(z_0)$ centered at $z_0 \in \overline{\mathbb{R}_+^2}$, let $x_0 = \Re(z_0)$ and $y_0 = \Im(z_0)$, denote $\{W_l\} = \{S_{j,k} : D_R(z_0) \cap \mathbb{R}_+^2 \cap S_{j,k} \neq \emptyset\}$. We separate our arguments into two cases.

Case (I), $y_0 \geq 2R$.

Suppose $2^{k_0} \leq y_0 - R \leq 2^{k_0+1}$, for some $k_0 \in \mathbb{Z}$. Since $y_0 \geq 2R$, we see $R \leq 2^{k_0+1}$, $y_0 \leq 2^{k_0+2}$, and $y_0 + R \leq 2^{k_0+3}$. Therefore, $D_R(z_0)$ must be covered by the union of 2 squares of side-length 2^{k_0+2} , 4 squares of side-length 2^{k_0+1} , and 8 squares of side-length 2^{k_0} . So the cardinality of the collection $\{W_l\}$ is at most $2 + 4 + 8 = 14$.

Now, if $x+iy \in D_R(z_0) \cap S_{j,k}$, for some $j, k \in \mathbb{Z}$. Then we have $y_0 - R \leq y \leq y_0 + R$ and $2^k \leq y \leq 2^{k+1}$. So it is easy to see that $\frac{y_0}{4} \leq 2^k \leq \frac{3}{2}y_0$. For such $S_{j,k}$, any $x' + iy' \in S_{j,k}$, we have $y' \leq 2^{k+1} \leq 3y_0$, and $|x' - x_0| \leq R + 2^k \leq 2y_0$. Then $x' + iy' \in D_{4y_0}(x_0) \cap \mathbb{R}_+^2$, and hence $S_{j,k} \subset D_{4y_0}(x_0) \cap \mathbb{R}_+^2$. But $|D_{4y_0}(x_0) \cap \mathbb{R}_+^2| = 8\pi y_0^2$ and $|S_{j,k}| = (2^k)^2 \geq \frac{y_0^2}{16}$, so we

see that $|D_{4y_0}(x_0) \cap \mathbb{R}_+^2| \leq a |S_{j,k}|$ for $a = 128\pi$. Therefore we obtain a finite collection $\{W_l\}$ whose elements are comparable to each other.

Let $M = \frac{\mu_1(W_1)}{|W_1|} = \max \frac{\mu_1(W_l)}{|W_l|}$ and $m = \frac{\mu_2(W_2)}{|W_2|} = \min \frac{\mu_2(W_l)}{|W_l|}$. When $z \in D_R(z_0)$, we must have $z \in W_l$ for some l , then $E(\mu_1)(z) = \frac{\mu_1(W_l)}{|W_l|} \leq M$ and $E(\mu_2)(z) = \frac{\mu_2(W_l)}{|W_l|} \geq m$. Since W_1 and W_2 are comparable, by Lemma 4.3.2, we see that

$$\mu_1(W_1) \leq C(a)\mu_2(W_2).$$

Note that $|W_2| \leq a |W_1|$, we obtain $M \leq C(a)m$, for some $C(a) > 1$ independent of $D_R(z_0)$. Therefore

$$\int_{D_R(z_0)} E(\mu_1)(z) dV(z) \left(\int_{D_R(z_0)} E(\mu_2)(z)^{-\frac{p'}{p}} dV(z) \right)^{\frac{p}{p'}} \leq Mm^{-1} \leq c.$$

Case (II), $y_0 < 2R$.

Let $S^* = \cup W_l$ be the union of the collection $\{W_l\}$, then $D_R(z_0) \cap \mathbb{R}_+^2 \subset S^* \subset D_{8R}(x_0) \cap \mathbb{R}_+^2$. To see the second inclusion, we follow a similar argument as case (I). If $x + iy \in D_R(z_0) \cap \mathbb{R}_+^2 \cap S_{j,k}$, for some $j, k \in \mathbb{Z}$, then $y \leq y_0 + R$ and $2^k \leq y$. Since $y_0 < 2R$, so we see $2^k < 3R$. For such $S_{j,k}$, any $x' + iy' \in S_{j,k}$, we have $y' \leq 2^{k+1} \leq 6R$, and $|x' - x_0| \leq R + 2^k < 4R$. So $x' + iy' \in D_{8R}(x_0)\mathbb{R}_+^2$, and hence $S_{j,k} \subset D_{8R}(x_0)\mathbb{R}_+^2$ as desired.

It is easy to see $\int_{S_{j,k}} E(\mu_1)(z) dV(z) = \int_{S_{j,k}} \mu_1(z) dV(z)$ for every $S_{j,k}$, so we have

$$\begin{aligned} \int_{D_R(z_0) \cap \mathbb{R}_+^2} E(\mu_1)(z) dV(z) &\leq \int_{S^*} E(\mu_1)(z) dV(z) \\ &= \int_{S^*} \mu_1(z) dV(z) \\ &\leq \int_{D_{8R}(x_0) \cap \mathbb{R}_+^2} \mu_1(z) dV(z). \end{aligned}$$

Note that $|D_{8R}(x_0) \cap \mathbb{R}_+^2| = 32R^2\pi$ and $|D_R(z_0) \cap \mathbb{R}_+^2| \geq \frac{R^2}{2}\pi$, so we obtain

$$\int_{D_R(z_0) \cap \mathbb{R}_+^2} E(\mu_1)(z) dV(z) \leq 64 \int_{D_{8R}(x_0) \cap \mathbb{R}_+^2} \mu_1(z) dV(z). \quad (4.5)$$

Next, for every $S_{j,k}$, when $z \in S_{j,k}$, we have $E(\mu_2)(z)^{-\frac{p'}{p}} \leq \int_{S_{j,k}} \mu_2(z)^{-\frac{p'}{p}} dV(z)$ by Jensen's inequality applied to the convex function $x^{-\frac{p'}{p}}$. So we see that $\int_{S_{j,k}} E(\mu_2)(z)^{-\frac{p'}{p}} dV(z) \leq \int_{S_{j,k}} \mu_2(z)^{-\frac{p'}{p}} dV(z)$, and hence

$$\begin{aligned} \int_{D_R(z_0) \cap \mathbb{R}_+^2} E(\mu_2)(z)^{-\frac{p'}{p}} dV(z) &\leq \int_{S^*} E(\mu_2)(z)^{-\frac{p'}{p}} dV(z) \\ &\leq \int_{S^*} \mu_2(z)^{-\frac{p'}{p}} dV(z) \\ &\leq \int_{D_{8R}(x_0) \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z). \end{aligned}$$

Again, by taking the average, we obtain

$$\int_{D_R(z_0) \cap \mathbb{R}_+^2} E(\mu_2)(z)^{-\frac{p'}{p}} dV(z) \leq 64 \int_{D_{8R}(x_0) \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z). \quad (4.6)$$

Since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, by (4.5) and (4.6), we obtain $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}_+^2)$. \blacksquare

Now we turn to another observation on the two-weight A_p condition.

Lemma 4.3.3 *If $(\mu_1, \mu_2) \in A_p(\mathbb{R}^2)$, then $\mu_1 \leq c\mu_2$ for some $c > 0$.*

Proof It is easy to see that the A_p condition is equivalent to the following

$$\mu_1(Q) \left(\int_Q f(z) dV(z) \right)^p \leq c \int_Q f(z)^p \mu_2(z) dV(z), \quad (4.7)$$

for all $f \geq 0$ and all squares Q in \mathbb{R}^2 .³

Letting $f = \chi_Q$ in (4.7), we see that $\mu_1(Q) \leq c\mu_2(Q)$ for all squares Q in \mathbb{R}^2 . But the σ -algebra can be generated from the set of squares, we obtain $\mu_1 \leq c\mu_2$ almost everywhere. \blacksquare

³See an analogue for the one weight case in [Ste93, Chapter 5]. See also the proof of the necessary part of Theorem 6.

4.4 Proof of Theorem 6.

Before proving Theorem 6, we first prove the following proposition. Then Theorem 6 can be derived from it very easily.

Proposition 4.4.1 *For $p > 1$, if the two weights μ_1 and μ_2 satisfy $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and either $\mu_1 \in A_p^+(\mathbb{R}_+^2)$ or $\mu_2 \in A_p^+(\mathbb{R}_+^2)$, then (1.2) holds for some $C > 0$.*

Proof We first assume $\mu_2 \in A_p^+(\mathbb{R}_+^2)$. It suffices to prove the boundedness of $\tilde{\mathcal{B}}$ for $f \geq 0$. By Proposition 4.3.1, we have $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}_+^2)$ and $E(\mu_2) \in A_p(\mathbb{R}_+^2)$. We extend $E(\mu_1)$ and $E(\mu_2)$ to \mathbb{R}^2 by reflection about the x -axis, that is, $E(\mu_j)(z) = E(\mu_j)(\bar{z})$ for $z \in \mathbb{R}_-^2$, $j = 1, 2$. Then it is easy to see that $(E(\mu_1), E(\mu_2)) \in A_p(\mathbb{R}^2)$ and $E(\mu_2) \in A_p(\mathbb{R}^2)$. Hence, by Lemma 4.3.3, $E(\mu_1) \leq cE(\mu_2)$ almost everywhere, and in particular, it is true on \mathbb{R}_+^2 . We also extend f to \mathbb{R}^2 by setting $f(z) = 0$ for $z \in \mathbb{R}_-^2$.

Let $K(z) = \kappa(\theta)/r^2$ for $z \in \mathbb{R}^2 \setminus \{0\}$, where $z = re^{i\theta}$, κ is smooth with $\kappa(\theta) = 1$ if $\theta \in [0, \pi]$, and $\int_0^{2\pi} \kappa(\theta) d\theta = 0$. Then $T(f) = K * f$ defines a singular integral operator with the cancellation property $\int_{|z|=1} K(z) d\sigma(z) = 0$ and the radial decreasing property $\left| \left(\frac{\partial}{\partial x} \right)^{\alpha_1} \left(\frac{\partial}{\partial y} \right)^{\alpha_2} K(z) \right| \leq c|z|^{-2-|\alpha|}$, for all $z \neq 0$ and $\alpha_1 + \alpha_2 \leq 1$, where $z = x + iy$. Since $E(\mu_2) \in A_p(\mathbb{R}^2)$ and it is easy to see $E(\mu_2)$ is locally integrable on \mathbb{R}^2 , we have

$$\int_{\mathbb{R}^2} T(f)(z)^p E(\mu_2)(z) dV(z) \leq c \int_{\mathbb{R}^2} f(z)^p E(\mu_2)(z) dV(z)$$

for all⁴ $f \in L^p(\mathbb{R}^2, E(\mu_2))$. Letting $\mathcal{J}f(z) = f(\bar{z})$, since $E(\mu_2)(\bar{z}) = E(\mu_2)(z)$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} T(\mathcal{J}f)(z)^p E(\mu_2)(z) dV(z) &\leq c \int_{\mathbb{R}^2} \mathcal{J}f(z)^p E(\mu_2)(z) dV(z) \\ &= c \int_{\mathbb{R}^2} f(z)^p E(\mu_2)(z) dV(z) \\ &= c \int_{\mathbb{R}_+^2} f(z)^p E(\mu_2)(z) dV(z) \end{aligned}$$

⁴This follows from the classical result of weighted L^p boundedness of singular integrals. For a general consideration of this topic, see [Ste93, Chapter 5, Chapter 6 §4.5, and Chapter 7] for details.

for our extended f . Noting that $T(\mathcal{J}f)(z) = K * \mathcal{J}f(z) = \tilde{\mathcal{B}}(f)(z)$ for $z \in \mathbb{R}_+^2$, we see that

$$\int_{\mathbb{R}_+^2} \tilde{\mathcal{B}}(f)(z)^p E(\mu_2)(z) dV(z) \leq c \int_{\mathbb{R}_+^2} f(z)^p E(\mu_2)(z) dV(z).$$

Together with the relation $E(\mu_1) \leq cE(\mu_2)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}_+^2} \tilde{\mathcal{B}}(f)(z)^p E(\mu_1)(z) dV(z) &\leq c \int_{\mathbb{R}_+^2} \tilde{\mathcal{B}}(f)(z)^p E(\mu_2)(z) dV(z) \\ &\leq c \int_{\mathbb{R}_+^2} f(z)^p E(\mu_2)(z) dV(z). \end{aligned} \quad (4.8)$$

Now, for $f \geq 0$, as in [LS04, Proposition 4.5], we have

$$\begin{aligned} \int_{\mathbb{R}_+^2} \left(\tilde{\mathcal{B}}(f)(z) \right)^p \mu_1(z) dV(z) &\leq c \int_{\mathbb{R}_+^2} \left(E\tilde{\mathcal{B}}E(f)(z) \right)^p \mu_1(z) dV(z) \\ &\leq c \int_{\mathbb{R}_+^2} E \left(\tilde{\mathcal{B}}E(f)(z) \right)^p \mu_1(z) dV(z) \\ &= c \int_{\mathbb{R}_+^2} \left(\tilde{\mathcal{B}}E(f)(z) \right)^p E(\mu_1)(z) dV(z) \\ &\leq c \int_{\mathbb{R}_+^2} (E(f)(z))^p E(\mu_2)(z) dV(z), \end{aligned} \quad (4.9)$$

where the first line follows from (4.1), the second line follows from Proposition 4.2.1 (b), the third line follows from Proposition 4.2.1 (a), and the last line follows from (4.8).

On the other hand, by Proposition 4.2.1 (c), we see

$$E(f)(z) = E(f\mu_2^{\frac{1}{p}} \cdot \mu_2^{-\frac{1}{p}})(z) \leq \left(E(f^p \mu_2)(z) \right)^{\frac{1}{p}} \left(E(\mu_2^{-\frac{p'}{p}})(z) \right)^{\frac{1}{p'}}. \quad (4.10)$$

For any $z = x + iy \in S_{j,k}$, let x_0 be the real part of the center of $S_{j,k}$, then $|z - x_0| \leq 2^k + 2^{k+1} \leq 2^{k+2}$. Let $D = D_{2^{k+2}}(x_0)$ be the special disk of radius 2^{k+2} centered at

x_0 , so we have $z \in D \cap \mathbb{R}_+^2$, and hence $S_{j,k} \subset D \cap \mathbb{R}_+^2$. Note that $|S_{j,k}| = 2^{2k}$ and $|D \cap \mathbb{R}_+^2| = 2^{2k+3}\pi$, we have $\frac{1}{|S_{j,k}|} = \frac{8\pi}{|D \cap \mathbb{R}_+^2|}$. Since $\mu_2 \in A_p^+(\mathbb{R}_+^2)$, for $z \in S_{j,k}$ we see that

$$\begin{aligned} E(\mu_2)(z) \left(E(\mu_2^{-\frac{p'}{p}})(z) \right)^{\frac{p}{p'}} &= \int_{S_{j,k}} \mu_2(z) dV(z) \left(\int_{S_{j,k}} \mu_2(z)^{-\frac{p'}{p}} dV(z) \right)^{\frac{p}{p'}} \\ &\leq c \int_{D \cap \mathbb{R}_+^2} \mu_2(z) dV(z) \left(\int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) \right)^{\frac{p}{p'}} \quad (4.11) \\ &\leq c, \end{aligned}$$

for some $c > 0$ independent of $S_{j,k}$. Therefore combining (4.9), (4.10) and (4.11), we see that

$$\begin{aligned} \int_{\mathbb{R}_+^2} \left(\tilde{\mathcal{B}}(f)(z) \right)^p \mu_1(z) dV(z) &\leq c \int_{\mathbb{R}_+^2} E(f^p \mu_2)(z) \left(E(\mu_2^{-\frac{p'}{p}})(z) \right)^{\frac{p}{p'}} E(\mu_2)(z) dV(z) \\ &\leq c \sum_{j,k} \int_{S_{j,k}} E(f^p \mu_2)(z) dV(z) \\ &= c \int_{\mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z) \end{aligned}$$

for $f \in L^p(\mathbb{R}_+^2, \mu_2)$, which completes the proof of the case $\mu_2 \in A_p^+(\mathbb{R}_+^2)$.

Now assume $\mu_1 \in A_p^+(\mathbb{R}_+^2)$. Then we have $E(\mu_1) \in A_p(\mathbb{R}_+^2)$. Almost the same argument shows that (4.9) becomes

$$\int_{\mathbb{R}_+^2} \left(\tilde{\mathcal{B}}(f)(z) \right)^p \mu_1(z) dV(z) \leq c \int_{\mathbb{R}_+^2} (E(f)(z))^p E(\mu_1)(z) dV(z),$$

and (4.11) becomes

$$E(\mu_1)(z) \left(E(\mu_2^{-\frac{p'}{p}})(z) \right)^{\frac{p}{p'}} \leq c,$$

since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$. This completes the proof. \blacksquare

Now we are ready to prove Theorem 6.

Theorem 6 *For $p > 1$, suppose that μ_1 and μ_2 are two weights such that $c\mu_1 \geq \mu_2$ for some $c > 0$. Then (1.2) holds for some $C > 0$ if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.*

Proof The sufficiency is immediate. Since $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and $c\mu_1 \geq \mu_2$, it is easy to see that $\mu_2 \in A_p^+(\mathbb{R}_+^2)$, then the boundedness of \mathcal{B} follows from Proposition 4.4.1.

Now we show the necessary part. For any special disk $D = D_R(x_0)$ with $x_0 \in \mathbb{R}$ and $R > 0$, let $D' = D_R(x_0 + 10Ri)$ be the disk⁵ with the same radius but centered at $(x_0, 10R) \in \mathbb{R}_+^2$. For this pair D and D' , we see that if $z \in D'$ and $w \in D$, or if $z \in D$ and $w \in D'$, we have $\Re\left(-\frac{1}{(z-\bar{w})^2}\right) \geq cR^{-2}$ for some $c > 0$. If \mathcal{B} is bounded, we see that

$$\int_{D \cap \mathbb{R}_+^2} |\mathcal{B}(f)(z)|^p \mu_1(z) dV(z) \leq c \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) dV(z). \quad (4.12)$$

Taking $f = \chi_{D'}$ in (4.12), we obtain

$$\mu_1(D \cap \mathbb{R}_+^2) \leq c\mu_2(D') \leq c\mu_1(D'). \quad (4.13)$$

On the other hand, for any $f \geq 0$ on \mathbb{R}_+^2 , we apply (4.12) to the function $f\chi_{D \cap \mathbb{R}_+^2}$, provided $f\chi_{D \cap \mathbb{R}_+^2} \in L^p(\mathbb{R}_+^2, \mu_2)$. We see that

$$\int_{D'} |\mathcal{B}(f)(z)|^p \mu_1(z) dV(z) \leq \int_{\mathbb{R}_+^2} |\mathcal{B}(f)(z)|^p \mu_1(z) dV(z) \leq c \int_{D \cap \mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z)$$

and, for $z \in D'$,

$$|\mathcal{B}(f)(z)|^p \geq \left(cR^{-2} \int_{D \cap \mathbb{R}_+^2} f(w) dV(w) \right)^p \geq c \left(\int_{D \cap \mathbb{R}_+^2} f(z) dV(z) \right)^p.$$

So we obtain

$$\mu_1(D') \left(\int_{D \cap \mathbb{R}_+^2} f(z) dV(z) \right)^p \leq c \int_{D \cap \mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z), \quad (4.14)$$

provided $f\chi_{D \cap \mathbb{R}_+^2} \in L^p(\mathbb{R}_+^2, \mu_2)$. Therefore, combining (4.13) and (4.14), we have

$$\mu_1(D \cap \mathbb{R}_+^2) \left(\int_{D \cap \mathbb{R}_+^2} f(z) dV(z) \right)^p \leq c \int_{D \cap \mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z), \quad (4.15)$$

for some $c > 0$, provided $f\chi_{D \cap \mathbb{R}_+^2} \in L^p(\mathbb{R}_+^2, \mu_2)$.

⁵Here i denotes $\sqrt{-1}$ as usual.

To show (4.15) is indeed the $A_p^+(\mathbb{R}_+^2)$ condition, we argue as in the proof of the necessary part of [Muc72, Theorem 1]. Suppose that $\int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) = \infty$. Then, by duality of the space $L^p(D \cap \mathbb{R}_+^2)$, there is a $g \in L^p(D \cap \mathbb{R}_+^2)$, so that $\int_{D \cap \mathbb{R}_+^2} g(z) \mu_2(z)^{-\frac{1}{p}} dV(z) = \infty$. Take $f = g \mu_2^{-\frac{1}{p}} \chi_{D \cap \mathbb{R}_+^2}$ in (4.15). Then $\int_{D \cap \mathbb{R}_+^2} f(z) dV(z) = \infty$ and $\int_{D \cap \mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z) < \infty$. So (4.15) gives $\mu_1(D \cap \mathbb{R}_+^2) = 0$, which contradicts the assumption $\mu_1 > 0$ almost everywhere. So we see that $\int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) < \infty$.

Now take $f = (\mu_2)^{-\frac{p'}{p}} \chi_{D \cap \mathbb{R}_+^2}$ in (4.15) and note that

$$\int_{D \cap \mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z) = \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) < \infty,$$

since $\frac{p'}{p} = p' - 1$. We see (4.15) implies the $A_p^+(\mathbb{R}_+^2)$ condition.

In fact, (4.15) is equivalent to the $A_p^+(\mathbb{R}_+^2)$ condition. The other direction, the $A_p^+(\mathbb{R}_+^2)$ condition implies (4.15), follows easily from the Hölder's inequality applied to the integral $\int_{D \cap \mathbb{R}_+^2} f(z) \mu_2(z)^{\frac{1}{p}} \mu_2(z)^{-\frac{1}{p}} dV(z)$. ■

5. A Wider Class of Weights

In this chapter, we apply Theorem 6 to extend the previous results to a wider class of weights.

5.1 Alternative Proof of Theorem 2 and Its Extension

Before giving an alternative proof of Theorem 2, we first have the following observation.

Proposition 5.1.1 *For $z \in \mathbb{R}_+^2$, $k \in \mathbb{Z}$, $s \in (0, 2]$ and $p > 1$, suppose*

$$\mu_1(z) = \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k}$$

and

$$\mu_2(z) = \left| \frac{i-z}{i+z} \right|^{(1-s-k)p+s+2k},$$

then $(\mu_1, \mu_2) \notin A_p(\mathbb{R}_+^2)$ for $s \neq 2$. But we have $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ if and only if $s + 2k + 2 > (k + 1)p$ and $p(s + k + 1) > s + 2k + 2$.

Proof To show $(\mu_1, \mu_2) \notin A_p(\mathbb{R}_+^2)$, we consider any disk $D_\epsilon(i)$ centered at i with radius $\epsilon < \frac{1}{2}$. For $z \in D_\epsilon(i)$, since $|i - z| < \epsilon < \frac{1}{2}$, we see that $\frac{3}{2} \leq |i + z| \leq \frac{5}{2}$. So, by Definition 4.1.1, we only need to look at

$$\int_{D_\epsilon(i)} |i-z|^{-(k+1)p+s+2k} dV(z) \left(\int_{D_\epsilon(i)} |i-z|^{-\frac{p'}{p}[(1-s-k)p+s+2k]} dV(z) \right)^{p/p'}.$$

Assuming both integrands are integrable, we obtain $\epsilon^{(s-2)p}$. But $s \in (0, 2)$ and $p > 1$, so we see the quantity above tends to ∞ as $\epsilon \rightarrow 0$.

To show $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, we consider two integrals

$$I_1 = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k} dV(z),$$

and

$$I_2 = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \left| \frac{i-z}{i+z} \right|^{-\frac{p'}{p}[(1-s-k)p+s+2k]} dV(z),$$

where $D = D_R(x_0)$ is any special disk with radius R centered at $x_0 \in \mathbb{R}$. Let $D_0 = D_{\frac{1}{2}}(i)$ be the disk with radius $\frac{1}{2}$ centered at i . We separate our arguments into two cases.

Case (I), $R < \frac{1}{2}$.

It is easy to see that $D \cap D_0 = \emptyset$ hence $|i-z| > \frac{1}{2}$. Note that, as $|z| \rightarrow \infty$, $\left| \frac{i-z}{i+z} \right| \rightarrow 1$, so there is an M such that, when $|z| > M$, $1 \geq \left| \frac{i-z}{i+z} \right| \geq \frac{1}{2}$. But when $|z| \leq M$, $|i+z| \leq M+1$, so $1 \geq \left| \frac{i-z}{i+z} \right| \geq \frac{1}{2(M+1)}$. Therefore, the integrands in I_1 and I_2 are bounded above by some constants that are independent of the special disk D . Then $I_1 I_2^{\frac{p}{p'}} \leq c$, for some $c > 0$.

Case (II), $R \geq \frac{1}{2}$.

We split both I_1 and I_2 into two integrals respectively, one integrates over $D \cap \mathbb{R}_+^2 \setminus D_0$ and the other integrates over $D \cap \mathbb{R}_+^2 \cap D_0$. For the same reasoning as in case (I), the parts integrated over $D \cap \mathbb{R}_+^2 \setminus D_0$ is bounded. The parts integrated over $D \cap \mathbb{R}_+^2 \cap D_0$ are bounded respectively by

$$\frac{8}{\pi} \int_{D_0} \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k} dV(z),$$

and

$$\frac{8}{\pi} \int_{D_0} \left| \frac{i-z}{i+z} \right|^{-\frac{p'}{p}[(1-s-k)p+s+2k]} dV(z).$$

Since $|i - z| \leq \frac{1}{2}$ for $z \in D_0$, we see that $\frac{3}{2} \leq |i + z| \leq \frac{5}{2}$, so the two integrals above are bounded respectively by

$$c \int_{D_0} |i - z|^{-(k+1)p+s+2k} dV(z)$$

and

$$c \int_{D_0} |i - z|^{-\frac{p'}{p}[(1-s-k)p+s+2k]} dV(z).$$

The first integral above is bounded by a constant if and only if $-(k+1)p+s+2k+2 > 0$, and the second is bounded if and only if $-\frac{p'}{p}[(1-s-k)p+s+2k]+2 > 0$. Solving the two inequalities, we see that $s+2k+2 > (k+1)p$ and $p(s+k+1) > s+2k+2$. ■

Remark 5.1.2 *From the proof we see that if two weights $c\mu_1 \geq \mu_2$ only have zeros or poles away from the x -axis and bounded above and below at ∞ , then $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ if and only if both μ_1 and $\mu_2^{-\frac{p'}{p}}$ are locally integrable.*

Remark 5.1.3 *Combining the fact that $c\mu_1 \geq \mu_2$ with Theorem 6 we see that, for the Bergman projection on \mathbb{R}_+^2 , the two-weight A_p condition is not a necessary condition for (1.2) (compare to the general Calderón-Zygmund type singular integral). The reason is that the Bergman kernel is not singular at all on \mathbb{R}_+^2 , and it should be a two dimensional analogue of the so-called Hilbert integral.¹*

Now we are ready to give an alternative proof of Theorem 2.

Theorem 2 *For $s' \in \mathbb{R}$ with the unique expression $s' = s+2k$, where $k \in \mathbb{Z}$ and $s \in (0, 2]$, let $\mathcal{B}_{s'}$ be the weighted Bergman projection on the space (\mathbb{D}^*, μ) , where $\mu(z) = |z|^{s'}$.*

(a) *For $s' \in (0, \infty)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in \left(\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1}\right)$.*

(b) *For $s' \in [-3, 0]$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded for $p \in (1, \infty)$.*

¹See also [PS86a, PS86b].

(c) For $s' \in (-4, -3)$, with $k = -2$ and $s \in (0, 1)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in (2 - s, \frac{2-s}{1-s})$.

(d) When $s' = -4$, \mathcal{B}_{-4} is $L^p(\mu)$ bounded for $p \in (1, \infty)$.

(e) For $s' \in (-\infty, -4)$, $\mathcal{B}_{s'}$ is $L^p(\mu)$ bounded if and only if $p \in (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$.

Proof For $s' = s + 2k$, from (3.1) we know that

$$B_{s'}(z, \zeta) = \frac{s}{2}(z\bar{\zeta})^{-(k+1)}B_0(z, \zeta) + \left(1 - \frac{s}{2}\right)(z\bar{\zeta})^{-k}B_0(z, \zeta)$$

for $z, \zeta \in \mathbb{D}^*$. So we may write $\mathcal{B}_{s'} = \frac{s}{2}\mathcal{T}_1 + (1 - \frac{s}{2})\mathcal{T}_2$, where \mathcal{T}_1 is the operator associated to the kernel $(z\bar{\zeta})^{-(k+1)}B_0(z, \zeta)$ with weight $|\zeta|^{s'}$ and \mathcal{T}_2 is the operator associated to the kernel $(z\bar{\zeta})^{-k}B_0(z, \zeta)$ with weight $|\zeta|^{s'}$.

For the operator \mathcal{T}_1 , showing its boundedness is the same as showing

$$\int_{\mathbb{D}^*} \left| \int_{\mathbb{D}^*} \frac{(z\bar{\zeta})^{-(k+1)}f(\zeta)}{(1 - z\bar{\zeta})^2} |\zeta|^{s'} dV(\zeta) \right|^p |z|^{s'} dV(z) \leq C \int_{\mathbb{D}^*} |f(z)|^p |z|^{s'} dV(z).$$

By the Cayley transform $\varphi : \mathbb{R}_+^2 \rightarrow \mathbb{D}$, where $\varphi(z) = \frac{i-z}{i+z}$, and using the notations in Chapter 4, we see the above inequality is equivalent to

$$\int_{\mathbb{R}_+^2 - \{i\}} \left| \int_{\mathbb{R}_+^2 - \{i\}} \frac{\tilde{f}(w) dV(w)}{(z - \bar{w})^2} \right|^p \mu_1(z) dV(z) \leq C \int_{\mathbb{R}_+^2 - \{i\}} |\tilde{f}(z)|^p \mu_2(z) dV(z),$$

where $\tilde{f}(z) = f(\varphi(z))\varphi(z)^{k+1}|\varphi(z)|^{s-2} \cdot \frac{1}{(i+1)^2}$, $\mu_1(z) = 4 \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k} \cdot \left| \frac{1}{(i+z)^2} \right|^{2-p}$ and $\mu_2(z) = 4 \left| \frac{i-z}{i+z} \right|^{(1-s-k)p+s+2k} \cdot \left| \frac{1}{(i+z)^2} \right|^{2-p}$. This is exactly (1.2), except that μ_1 and μ_2 are only locally integrable on $\mathbb{R}_+^2 \setminus \{i\}$ —not on \mathbb{R}_+^2 . However, since $\left| \frac{i-z}{i+z} \right| \leq 1$ and $s \leq 2$, we have $\mu_1(z)/\mu_2(z) = |(i-z)/(i+z)|^{(s-2)p} \geq 1$, so the local integrability of μ_2 will be guaranteed by the local integrability of μ_1 at i as we will see below. Moreover, by the relation $\mu_1 \geq \mu_2$, Theorem 6 applies,² that is, \mathcal{T}_1 is bounded if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

²From the proof of the necessity, we see the boundedness of \mathcal{T}_1 will imply the local integrability of μ_1 and μ_2 at i .

For the condition $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$, we first note that $\sigma(z) = \left| \frac{1}{(i+z)^2} \right|^{2-p} \in A_p^+(\mathbb{R}_+^2)$ for all $p > 1$. To see this, from the classical result \mathcal{B}_0 is L^p bounded for all $p > 1$, where \mathcal{B}_0 is the ordinary Bergman projection on \mathbb{D}^* which is the same as the ordinary Bergman projection on the unit disk. Then (1.2) holds with both μ_1 and μ_2 replaced by σ , and hence $\sigma \in A_p^+(\mathbb{R}_+^2)$ for all $p > 1$ by Theorem 6.³

As in Proposition 5.1.1, we consider two integrals

$$I_1 = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} 4 \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k} \cdot \left| \frac{1}{(i+z)^2} \right|^{2-p} dV(z)$$

and

$$I_2 = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \left(4 \left| \frac{i-z}{i+z} \right|^{(1-s-k)p+s+2k} \cdot \left| \frac{1}{(i+z)^2} \right|^{2-p} \right)^{-\frac{p'}{p}} dV(z),$$

where $D = D_R(x_0)$ is again any special disk with radius R centered at $x_0 \in \mathbb{R}$.

For $R < 1/2$, the same argument as in the proof of Proposition 5.1.1 case (I), shows that the integrands in I_1 and I_2 are bounded above by $c\sigma$ and $c\sigma^{-\frac{p'}{p}}$ respectively. So this case follows from the fact that $\sigma \in A_p^+(\mathbb{R}_+^2)$ for all $p > 1$.

For $R \geq 1/2$, the same argument as in the proof of Proposition 5.1.1 case (II) and Remark 5.1.2 show that we only need to consider whether

$$|i-z|^{-(k+1)p+s+2k} \quad \text{and} \quad |i-z|^{-\frac{p'}{p}[(1-s-k)p+s+2k]}$$

are locally integrable away from the x -axis. This shows the local integrabilities of μ_1 and μ_2 and, by Proposition 5.1.1, we see this is true if and only if $s + 2k + 2 > (k + 1)p$ and $p(s + k + 1) > s + 2k + 2$.

Denoting by $U_1 = \{p \in (1, \infty) \mid s + 2k + 2 > (k + 1)p \text{ and } p(s + k + 1) > s + 2k + 2\}$ the range for p , it is not difficult to see that U_1 is an open interval. So we obtain $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$ and hence \mathcal{T}_1 is bounded if and only if $p \in U_1$.

³Indeed, the fact that $\sigma \in A_p^+(\mathbb{R}_+^2)$ for all $p > 1$ can be derived by a direct computation.

Similarly, \mathcal{T}_2 is bounded if and only if (1.2) holds for $\mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-kp+s+2k}$ and $\mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(s+k)p+s+2k}$. Again, μ_1 and μ_2 may not be locally integrable. Fortunately, we have $\mu_1(z)/\mu_2(z) = |(i-z)/(i+z)|^{sp} \leq 1$, since $|(i-z)(i+z)| \leq 1$ and $s > 0$. So we do not need both of μ_1 and μ_2 to be locally integrable, indeed, they will not be in some cases. We instead apply Theorem 6 to a single weight, either to μ_1 or to μ_2 , then by $\mu_1 \leq \mu_2$ we get the desired inequality. That is, \mathcal{T}_2 is bounded if $\mu_1 \in A_p^+(\mathbb{R}_+^2) \cap L_{\text{loc}}^1(\mathbb{R}_+^2)$ or $\mu_2 \in A_p^+(\mathbb{R}_+^2) \cap L_{\text{loc}}^1(\mathbb{R}_+^2)$.

Following a similar argument, $\mu_j \in A_p^+(\mathbb{R}_+^2)$ will guarantee the local integrability of μ_j , for $j = 1, 2$. Then we see, by listing all possibilities of $k \in \mathbb{Z}$, $\mu_1 \in A_p^+(\mathbb{R}_+^2)$ or $\mu_2 \in A_p^+(\mathbb{R}_+^2)$ if and only if $p \in U_2 = \{p \in (1, \infty) \mid s+2k+2 > pk \text{ and } (s+k+2)p > s+2k+2\}$ for $s \neq 2$. But for $s = 2$ we do not need to worry about \mathcal{T}_2 , since $\mathcal{B}_{s'} = \frac{s}{2}\mathcal{T}_1 + (1 - \frac{s}{2})\mathcal{T}_2$. It is not hard to see that U_2 is also an open interval, and we have \mathcal{T}_2 is bounded if $p \in U_2$.

Now, if both \mathcal{T}_1 and \mathcal{T}_2 are bounded, then $\mathcal{B}_{s'}$ is bounded. Since a simple argument shows that $U_1 \subset U_2$ properly, we see that $\mathcal{B}_{s'}$ is bounded if $p \in U_1$. Conversely, if we look at the endpoint p of U_1 , then $p \notin U_1$ but $p \in U_2$. In this case, we see that \mathcal{T}_1 is unbounded, \mathcal{T}_2 is bounded, and hence $\mathcal{B}_{s'}$ is unbounded. So by interpolation we see that $\mathcal{B}_{s'}$ is unbounded for all $p \notin U_1$.

Therefore, for $p > 1$, $\mathcal{B}_{s'}$ is bounded if and only if $p \in U_1$. When $s' \in (0, \infty)$, $U_1 = (\frac{s+2k+2}{s+k+1}, \frac{s+2k+2}{k+1})$. When $s' \in [-3, 0]$, $U_1 = (1, \infty)$. When $s' \in (-4, -3)$, $U_1 = (2-s, \frac{2-s}{1-s})$. When $s' = -4$, $U_1 = (1, \infty)$. When $s' \in (-\infty, -4)$, $U_1 = (\frac{s+2k+2}{k+1}, \frac{s+2k+2}{s+k+1})$. ■

Remark 5.1.4 Besides Theorem 6, the analysis for \mathcal{T}_2 here also supports our Conjecture 1, since the ‘‘effective’’ bound for p is obtained by checking that $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$. Note that, in this case, we may extend our Conjecture 1 to a more general situation, that is, without the assumption of local integrabilities of μ_1 and μ_2 .

Now we extend the arguments above to prove Theorem 7.

Theorem 7 *Assume that $p > 1$. Let $\mu(z) = |z|^{s'} |g(z)|^2$, where g is a non-vanishing holomorphic function on \mathbb{D} and $s' \in \mathbb{R}$. Suppose the weighted Bergman projection $\mathcal{B}_{|g|^2}$ on $(\mathbb{D}, |g|^2)$ is $L^p(|g|^2)$ bounded, and suppose the weighted Bergman projection $\mathcal{B}_{s'}$ on $(\mathbb{D}^*, |z|^{s'})$ is $L^p(|z|^{s'})$ bounded. Then the weighted Bergman projection \mathcal{B}_μ on (\mathbb{D}^*, μ) is $L^p(\mu)$ bounded.*

Moreover, suppose $\mathcal{B}_{|g|^2}$ is $L^p(|g|^2)$ bounded if and only if $p \in (p_0, p'_0)$ for some $p_0 \geq 1$ and suppose $\mathcal{B}_{s'}$ is $L^p(|z|^{s'})$ bounded if and only if $p \in (p_1, p'_1)$ for some $p_1 \geq 1$ as in Theorem 2. If $(p_1, p'_1) \subset (p_0, p'_0)$ properly, then \mathcal{B}_μ is $L^p(\mu)$ bounded if and only if $p \in (p_1, p'_1)$.

Proof For the Bergman kernel on the weighted space $(\mathbb{D}, |g|^2)$, we have

$$B_{|g|^2}(z, \zeta) = \frac{1}{g(z)g(\zeta)} \frac{1}{(1 - z\bar{\zeta})^2}.$$

This follows from applying Lemma 2.2.1 with the isometry,

$$F : L^2(\mathbb{D}, |g|^2) \rightarrow L^2(\mathbb{D})$$

where⁴ $F(f) = fg$ for $f \in L^2(\mathbb{D}, |g|^2)$. Similar arguments as in the alternative proof of Theorem 2 show that $\mathcal{B}_{|g|^2}$ is L^p bounded if and only if (1.2) holds with both μ_1 and μ_2 replaced by $\sigma(z) = \left| g(\varphi(z)) \cdot \frac{1}{(i+z)^2} \right|^{2-p}$. From the statement of Theorem 7, $\mathcal{B}_{|g|^2}$ is $L^p(|g|^2)$ bounded if and only if $p \in (p_0, p'_0)$ for some $p_0 \geq 1$. So, by Theorem 6, we see that $\sigma \in A_p^+(\mathbb{R}_+^2)$ if and only if $p \in (p_0, p'_0)$.

Note that $\mu(z) = |z|^{s'} |g(z)|^2$ on \mathbb{D}^* . The same argument above applies to the weighted Bergman projection \mathcal{B}_μ , whose associated weighted Bergman kernel is

$$B_\mu(z, \zeta) = \frac{1}{g(z)g(\zeta)} B_{s'}(z, \zeta).$$

⁴See also [Zey13, Theorem 3.4].

Then by the relation $\mathcal{B}_{s'} = \frac{s}{2}\mathcal{T}_1 + (1 - \frac{s}{2})\mathcal{T}_2$ in the alternative proof of Theorem 2 above, \mathcal{B}_μ is $L^p(\mu)$ bounded if (1.2) holds both for the first pair

$$\mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(k+1)p+s+2k}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{(1-s-k)p+s+2k}$$

and for the second pair

$$\mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-kp+s+2k}, \quad \mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(s+k)p+s+2k}.$$

Follow the same argument as in the alternative proof of Theorem 2 above, and noting that g is bounded above and below on the disk $D_{\frac{1}{2}}(i)$ with radius $\frac{1}{2}$ centered at i , we see (1.2) holds for the first pair if and only if $p \in (p_0, p'_0) \cap (p_1, p'_1)$. Similarly, (1.2) holds for the second pair if $p \in (p_0, p'_0) \cap U$, for some larger open interval U which contains p_1 and p'_1 .⁵ Therefore \mathcal{B}_μ is $L^p(\mu)$ bounded if $p \in (p_0, p'_0) \cap (p_1, p'_1)$.

If, in addition, $(p_1, p'_1) \subset (p_0, p'_0)$ properly then, for $p = p_1$ and $p = p'_1$, (1.2) fails for the first pair, but holds for the second. So $\mathcal{B}_{s', |g|^2}$ is unbounded for these ps . Hence \mathcal{B}_μ is bounded if and only if $p \in (p_1, p'_1)$. ■

By inflation, we can extend the previous result to a wider class of weights on the Hartogs triangle. Namely, we consider a weight of the form $\lambda(z) = |z_2|^{s'-2} |g(z_2)|^2$ for $z \in \mathbb{H}$, where g is some non-vanishing holomorphic function on \mathbb{D} .

Theorem 8 *Assume that $p > 1$. Let λ be as above, and let p_0, p_1 be as in Theorem 7. Then the weighted Bergman projection \mathcal{B}_λ on (\mathbb{H}, λ) is $L^p(\lambda)$ bounded if $p \in (p_0, p'_0) \cap (p_1, p'_1)$. In addition, if $(p_1, p'_1) \subset (p_0, p'_0)$ properly, then \mathcal{B}_λ is $L^p(\lambda)$ bounded if and only if $p \in (p_1, p'_1)$.*

Proof This is a direct consequence of Proposition 2.2.6 and Theorem 7. ■

⁵If $(p_1, p'_1) = (1, \infty)$, then the conclusion is trivial.

5.2 A Two-weight Estimate on the Hartogs Triangle

As a last application in this chapter, we use the two-weight inequality to prove the L^p regularity of the weighted Bergman projection $\mathcal{B}_{\mathbb{H}, 2k+2}$ on $(\mathbb{H}, |z_2|^{2k+2})$ mapping from $L^p(\mathbb{H}, |z_2|^{2k+2})$ to $L^p(\mathbb{H}, |z_2|^t)$ for some $t \in \mathbb{R}$. For simplicity, we focus on $k \geq -1$ and $k \in \mathbb{Z}$.

Theorem 9 *Suppose that $\mathcal{B}_{\mathbb{H}, 2k+2}$ is the weighted Bergman projection on the weighted space $(\mathbb{H}, |z_2|^{2k+2})$. Assume $p > 1$ and $k \geq -1$. Then, for $t \leq 2k + 2$, $\mathcal{B}_{\mathbb{H}, 2k+2}$ is L^p bounded from $L^p(\mathbb{H}, |z_2|^{2k+2})$ to $L^p(\mathbb{H}, |z_2|^t)$ if and only if $p \in (\frac{2k+6}{k+4}, \frac{t+4}{k+2})$. For $t > 2k+2$, $\mathcal{B}_{\mathbb{H}, 2k+2}$ is bounded from $L^p(\mathbb{H}, |z_2|^{2k+2})$ to $L^p(\mathbb{H}, |z_2|^t)$ if $p \in (\frac{2k+6}{k+4}, \frac{2k+6}{k+2}) \cup (\frac{t+4}{k+4}, \frac{t+4}{k+2})$.*

Proof The boundedness of the mapping is equivalent to

$$\int_{\mathbb{H}} |\mathcal{B}_{\mathbb{H}, 2k+2}(f)(z)|^p |z_2|^t dV(z) \leq C \int_{\mathbb{H}} |f(z)|^p |z_2|^{2k+2} dV(z).$$

By Corollary 2.2.2, and considering the biholomorphism $\Phi : \mathbb{H} \rightarrow \mathbb{D} \times \mathbb{D}^*$ via $\Phi(z) = (\frac{z_1}{z_2}, z_2)$, the above inequality is equivalent to

$$\begin{aligned} & \int_{\mathbb{D} \times \mathbb{D}^*} \left| \int_{\mathbb{D} \times \mathbb{D}^*} B_0 \otimes B_{2k+2}(z, \zeta) f(\zeta) |\zeta_2|^{2k+2} dV(\zeta) \right|^p |z_2|^{t+2-p} dV(z) \\ & \leq C \int_{\mathbb{D} \times \mathbb{D}^*} |f(z)|^p |z_2|^{2k+4-p} dV(z), \end{aligned}$$

where B_{2k+2} is the weighted Bergman kernel on $(\mathbb{D}^*, |z|^{2k+2})$ and B_0 is the ordinary Bergman kernel on \mathbb{D} . By Lemma 2.2.4, and the fact that the ordinary Bergman projection on the unit disk is L^p bounded for all $p > 1$, we see that the above inequality is equivalent to

$$\int_{\mathbb{D}^*} \left| \int_{\mathbb{D}^*} B_{2k+2}(z, \zeta) f(\zeta) |\zeta|^{2k+2} dV(\zeta) \right|^p |z|^{t+2-p} dV(z) \leq C \int_{\mathbb{D}^*} |f(z)|^p |z|^{2k+4-p} dV(z).$$

Applying the same argument as in the alternative proof of Theore 2, since $s = 2$, we see that $\mathcal{B}_{\mathbb{H}, 2k+2}$ is bounded if (1.2) holds for

$$\mu_1(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(k+2)p+t+2} \quad \text{and} \quad \mu_2(z) = 4\sigma(z) \left| \frac{i-z}{i+z} \right|^{-(k+2)p+2k+4},$$

where $\sigma(z) = \left| \frac{1}{(i+z)^2} \right|^{2-p} \in A_p^+(\mathbb{R}_+^2)$ for all $p > 1$.

When $t \leq 2k + 2$, Theorem 6 tells us that $\mathcal{B}_{\mathbb{H}, 2k+2}$ is bounded if and only if $p \in \left(\frac{2k+6}{k+4}, \frac{t+4}{k+2} \right)$. When $t > 2k + 2$, Theorem 6 again tells us that $\mathcal{B}_{\mathbb{H}, 2k+2}$ is bounded if $p \in \left(\frac{2k+6}{k+4}, \frac{2k+6}{k+2} \right) \cup \left(\frac{t+4}{k+4}, \frac{t+4}{k+2} \right)$. This completes the proof. \blacksquare

Remark 5.2.1 *If we take $k = -1$ and $t = p - 2$, then Theorem 9 will imply [CZ14, Theorem 1.1].*

6. More Analysis on the $A_p^+(\mathbb{R}_+^2)$ Class

To prove Conjecture 1 and Conjecture 2 introduced in Chapter 1, we need a better understanding of the $A_p^+(\mathbb{R}_+^2)$ class. In this chapter, we introduce the special maximal function operator and prove Theorem 10 and Theorem 11.

6.1 Preliminaries and Basic Definitions

In Chapter 4, we introduced the $A_p^+(\mathbb{R}_+^2)$ class for $p > 1$. Now we extend this class to the case $p = 1$.

Definition 6.1.1 *Two weights μ_1 and μ_2 on \mathbb{R}_+^2 are in the class $A_1^+(\mathbb{R}_+^2)$, denoted by the ordered pair $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}_+^2)$, if there is a $c > 0$ so that, for all special disks D , we have*

$$\frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu_1(\zeta) dV(\zeta) \leq c\mu_2(z)$$

for any $z \in D \cap \mathbb{R}_+^2$. For some weight μ , if it satisfies $(\mu, \mu) \in A_1^+(\mathbb{R}_+^2)$, we simply adopt the notation $\mu \in A_1^+(\mathbb{R}_+^2)$.

According to the classical results, the A_p class is closely related to the maximal function operator.¹ So it is reasonable to introduce a suitable maximal function operator associated to the $A_p^+(\mathbb{R}_+^2)$ class.

¹See [Ste93, Chapter 5] for a general consideration of this topic.

Definition 6.1.2 For any measurable function f on \mathbb{R}_+^2 , we define the special maximal function operator \widetilde{M}^+ by

$$\widetilde{M}^+(f)(z) = \sup_{z \in D} \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta)$$

for $z \in \mathbb{R}_+^2$, where the supremum is taken over all special disks D containing z . It is clear that $\widetilde{M}^+(f)$ is lower semi-continuous.

Remark 6.1.1 It is easy to see $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}_+^2)$ if and only if $\widetilde{M}^+(\mu_1) \leq c\mu_2$. In particular, a weight μ belonging to the class $A_1^+(\mathbb{R}_+^2)$ is equivalent to $\widetilde{M}^+(\mu) \leq c\mu$.

Remark 6.1.2 It is easy to see, for $z, z' \in \mathbb{R}_+^2$, that

$$\widetilde{M}^+(f)(z') \geq \widetilde{M}^+(f)(z)$$

whenever $\Re(z') = \Re(z)$ and $\Im(z') \leq \Im(z)$. Note that the "absolute value" of the Bergman projection on the upper half plane

$$\widetilde{B}(f)(z) = \int_{\mathbb{R}_+^2} \frac{1}{|z - \bar{w}|^2} f(w) dV(w),$$

also has the same property above as \widetilde{M}^+ for $f \geq 0$.

Associated to the special maximal function operator \widetilde{M}^+ and the corresponding special disks, we also introduce a collection of special squares.

Definition 6.1.3 Let \mathcal{S} be the collection of all the special squares of form

$$\widetilde{S}_{j,k} = \{x + iy \in \mathbb{R}_+^2 : j \cdot 2^k \leq x \leq (j+1) \cdot 2^k \text{ and } 0 \leq y \leq 2^k\},$$

where $j, k \in \mathbb{Z}$. Given a special square $\widetilde{S}_{j,k}$, we define

$$\widetilde{S}_{j,k}^* = \{x + iy \in \mathbb{R}_+^2 : (j-2) \cdot 2^k \leq x \leq (j+3) \cdot 2^k \text{ and } 0 \leq y \leq 5 \cdot 2^k\}.$$

6.2 Properties of the Special Maximal Function Operator

Before investigating properties of the special maximal function operator, we first give a simple observation for a special product of two A_1^+ weights.

Proposition 6.2.1 *Suppose that μ_1 and μ_2 are two weights and $\mu_j \in A_1^+(\mathbb{R}_+^2)$, $j = 1, 2$. Then, for $1 \leq p < \infty$, we have $\mu_1\mu_2^{1-p} \in A_p^+(\mathbb{R}_+^2)$.*

Proof By definition, for $j = 1, 2$, we have

$$\frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu_j(z) dV(z) \leq c \inf_{z \in D \cap \mathbb{R}_+^2} \mu_j(z),$$

for all special disks D . Then we see that

$$\frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu_1(z) \mu_2(z)^{1-p} dV(z) \leq c \inf_{z \in D \cap \mathbb{R}_+^2} \mu_1(z) \left(\inf_{z \in D \cap \mathbb{R}_+^2} \mu_2(z) \right)^{1-p}$$

and

$$\left(\frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} (\mu_1(z) \mu_2(z)^{1-p})^{-\frac{p'}{p}} dV(z) \right)^{\frac{p}{p'}} \leq c \left(\inf_{z \in D \cap \mathbb{R}_+^2} \mu_1(z) \right)^{-1} \left(\inf_{z \in D \cap \mathbb{R}_+^2} \mu_2(z) \right)^{p-1}.$$

Combining these two inequalities above, we obtain $\mu_1\mu_2^{1-p} \in A_p^+(\mathbb{R}_+^2)$. \blacksquare

Remark 6.2.2 *From Proposition 6.2.1, we see as long as we have two $A_1^+(\mathbb{R}_+^2)$ weights, we can construct an $A_p^+(\mathbb{R}_+^2)$ weight by taking a special product of the two $A_1^+(\mathbb{R}_+^2)$ weights.*

To construct an $A_1^+(\mathbb{R}_+^2)$ weight and consider some mapping properties of \widetilde{M}^+ , we need two lemmas. The first lemma we introduce is an analogue of [Muc72, Lemma 7].

Lemma 6.2.3 *Let $f \geq 0$ be an integrable function on \mathbb{R}_+^2 , and suppose $\alpha > 0$. Then there is a sequence of measurable sets $\{W_l\}$, and a sequence of special squares $\{\widetilde{S}_l\}$ such that*

- (a) *The intersection of different W_l 's has measure 0.*

(b) $\tilde{S}_l \subset W_l \subset \tilde{S}_l^*$.

(c) $\frac{\alpha}{16} |\tilde{S}_l| \leq \int_{W_l} f(\zeta) dV(\zeta)$.

(d) If $\tilde{M}^+(f)(z) > \alpha$, then $z \in \cup W_l$.

Proof Following the idea in [Muc72], we argue as in the classical Calderón-Zygmund lemma. Since $\int_{\mathbb{R}_+^2} f(\zeta) dV(\zeta) < \infty$, there is a $k_0 \in \mathbb{Z}^+$ so that, for all $k \geq k_0$ and all $j \in \mathbb{Z}$, we have

$$\frac{1}{|\tilde{S}_{j,k}|} \int_{\tilde{S}_{j,k}} f(\zeta) dV(\zeta) \leq \frac{\alpha}{16}. \quad (6.1)$$

For the $k = k_0 - 1$ level, to each $j \in \mathbb{Z}$, we have either (6.1) still true or

$$\frac{\alpha}{16} < \frac{1}{|\tilde{S}_{j,k}|} \int_{\tilde{S}_{j,k}} f(\zeta) dV(\zeta) \leq \frac{\alpha}{4}. \quad (6.2)$$

The right hand side of (6.2) follows from (6.1) in the $k + 1$ level. If (6.2) holds for this j , we collect this special square $\tilde{S}_{j,k}$ into the sequence $\{\tilde{S}_l\}$, otherwise we continue this process to the $k - 1$ level in this $\tilde{S}_{j,k}$. Therefore, we obtain a sequence of almost disjoint special squares $\{\tilde{S}_l\}$ satisfying (6.2).

Define $W_1 = \tilde{S}_1^* \setminus \left(\cup_{m \neq 1} \tilde{S}_m \right)$, and successively let

$$W_l = \tilde{S}_l^* \setminus \left(\cup_{m \neq l} \tilde{S}_m \cup \cup_{l' < l} W_{l'} \right),$$

for $l > 1$. Properties (a) and (b) are easy to check from this definition. Property (c) follows from $\tilde{S}_l \subset W_l$ and \tilde{S}_l satisfies (6.2). If $\tilde{M}^+(f)(z) > \alpha$, then there is a special disk $D_z = D_r(x_0)$ centered at $x_0 \in \mathbb{R}$ with radius $r > 0$ so that $z \in D_z$ and

$$\frac{1}{|D_z \cap \mathbb{R}_+^2|} \int_{D_z \cap \mathbb{R}_+^2} f(\zeta) dV(\zeta) > \alpha.$$

If $2^{k_1-1} \leq r < 2^{k_1}$ for some $k_1 \in \mathbb{Z}$, then D_z intersects at most three special squares \tilde{S}_{j,k_1} 's and it is contained in the union of these squares. Moreover, we have

$$|D_z \cap \mathbb{R}_+^2| \leq \frac{\pi}{2} |\tilde{S}_{j,k_1}| < 4 |D_z \cap \mathbb{R}_+^2|.$$

Therefore at least one such special square, say \tilde{S}_{j_1, k_1} , satisfies

$$\int_{D_z \cap \mathbb{R}_+^2 \cap \tilde{S}_{j_1, k_1}} f(\zeta) dV(\zeta) > \frac{1}{3} \alpha |D_z \cap \mathbb{R}_+^2|.$$

So we obtain

$$\int_{\tilde{S}_{j_1, k_1}} f(\zeta) dV(\zeta) > \frac{\pi}{24} \alpha |\tilde{S}_{j_1, k_1}| > \frac{\alpha}{16} |\tilde{S}_{j_1, k_1}|.$$

From our construction of the sequence $\{\tilde{S}_l\}$, \tilde{S}_{j_1, k_1} cannot be any of those satisfying (6.1), so \tilde{S}_{j_1, k_1} is contained in one of the special squares $\{\tilde{S}_l\}$. Since \tilde{S}_{j_1, k_1} intersects D_z , we must have $z \in D_z \subset \tilde{S}_{j_1, k_1}^* \subset \tilde{S}_{l_1}^*$ for some l_1 . By the definition of $\{W_l\}$, if z is not in W_1, \dots, W_{l_1} , then z must be in \tilde{S}_{m_1} for some m_1 , hence $z \in W_{m_1}$, which implies (d). ■

Lemma 6.2.4 *Let f be a measurable function on \mathbb{R}_+^2 . Then either $\tilde{M}^+(f)(z) = \infty$ for all $z \in \mathbb{R}_+^2$, or $\tilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}_+^2$.*

Proof Assuming that $\tilde{M}^+(f)(z) = \infty$ for some $z \in \mathbb{R}_+^2$, we show that $\tilde{M}^+(f)(z') = \infty$ for any $z' \in \mathbb{R}_+^2$. By definition, there is a sequence of special disks $\{D_n\}$ with $z \in D_n$ and

$$\frac{1}{|D_n \cap \mathbb{R}_+^2|} \int_{D_n \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta) > n, \quad (6.3)$$

for all $n \in \mathbb{Z}^+$. Let r_n be the radius of D_n , and let x_n be the center, then $D_n = D_{r_n}(x_n)$.

Since $z \in D_n$, we see $r_n > \mathfrak{S}(z) > 0$.

If $\{r_n\}$ is not bounded above then, by selecting a subsequence, we may assume that $\lim r_n = \infty$. Then, given any $z' \in \mathbb{R}_+^2$, we have $r_n \geq |z' - z|$ for n sufficiently large. In this case, it is easy to see $z' \in D_{2r_n}(x_n)$, the special disk centered at x_n with radius $2r_n$.

From (6.3), we see that

$$\frac{1}{|D_{2r_n}(x_n) \cap \mathbb{R}_+^2|} \int_{D_{2r_n}(x_n) \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta) > \frac{1}{4} n,$$

for n sufficiently large. This implies $\tilde{M}^+(f)(z') = \infty$.

If $\{r_n\}$ is bounded above then, by selecting a subsequence, we may assume that $\lim r_n = r$, for some r with $\Im(z) \leq r < \infty$. Note that, since $z \in D_n$, we have $\Re(z) \in D_n$, so $D_n \subset D_{3r}(\Re(z))$ for n sufficiently large, where $D_{3r}(\Re(z))$ is a special disk centered at $\Re(z)$ with radius $3r$. Therefore, from (6.3), we see that

$$\int_{D_{3r}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta) > \frac{1}{2} n \pi r_n^2 \geq cn,$$

where $c = \frac{1}{2} \pi \Im(z) > 0$, for n sufficiently large. So we obtain

$$\int_{D_{3r}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta) = \infty.$$

Now, for any $z' \in \mathbb{R}_+^2$, it is easy to see that $z' \in D_{3r+|z'-z|}(\Re(z))$, the special disk centered at $\Re(z)$ with radius $3r + |z' - z|$. From the equality above, we have

$$\frac{1}{|D_{3r+|z'-z|}(\Re(z)) \cap \mathbb{R}_+^2|} \int_{D_{3r+|z'-z|}(\Re(z)) \cap \mathbb{R}_+^2} |f(\zeta)| dV(\zeta) = \infty,$$

which implies $\widetilde{M}^+(f)(z') = \infty$. This completes the proof. ■

Now we are ready to apply \widetilde{M}^+ to construct $A_1^+(\mathbb{R}_+^2)$ weights.

Theorem 10 *Let f be a measurable function on \mathbb{R}_+^2 . Then, for any $0 < q < 1$, the function $(\widetilde{M}^+(f))^q$ is in $A_1^+(\mathbb{R}_+^2)$.*

Proof It suffices to show the conclusion for $f \geq 0$. By Lemma 6.2.4, we can assume $\widetilde{M}^+(f)(z) < \infty$ for all $z \in \mathbb{R}_+^2$; otherwise, the conclusion is trivial. If $\widetilde{M}^+(f)(z) = 0$ for some $z \in \mathbb{R}_+^2$, then it is easy to see that $f = 0$ on \mathbb{R}_+^2 . In this case, the conclusion is trivial again. So we may assume that $0 < \widetilde{M}^+(f) < \infty$ on \mathbb{R}_+^2 .

We use an analogue of the argument in [Ste93, Chapter 5.2]. Let $\mu(z) = (\widetilde{M}^+(f)(z))^q$; we show that $\widetilde{M}^+(\mu)(z) \leq c\mu(z)$. That implies $\mu \in A_1^+(\mathbb{R}_+^2)$.

Fixing $z \in \mathbb{R}_+^2$, we normalize f by dividing by $\widetilde{M}^+(f)(z)$, so we may assume that $\widetilde{M}^+(f)(z) = 1$ and $\mu(z) = 1$. Hence it suffices to show that there is a $c > 0$ such that

$$\frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu(\zeta) dV(\zeta) \leq c, \quad (6.4)$$

for any special D containing z .

Given a special disk $D = D_R(x_0)$ that contains z , let $f_1 = \chi_{D_{2R}(x_0) \cap \mathbb{R}_+^2} f$ and $f_2 = f - f_1$. We first deal with f_1 . Let $V_\alpha = \{\zeta \in D \cap \mathbb{R}_+^2 : \widetilde{M}^+(f_1)(\zeta) > \alpha\}$, we have

$$\begin{aligned} \int_{D \cap \mathbb{R}_+^2} \left(\widetilde{M}^+(f_1)(\zeta) \right)^q dV(\zeta) &= \int_0^\infty q\alpha^{q-1} |V_\alpha| d\alpha \\ &= \int_0^1 + \int_1^\infty q\alpha^{q-1} |V_\alpha| d\alpha. \end{aligned} \quad (6.5)$$

Since $|V_\alpha| \leq |D|$, we see the first integral of (6.5) is bounded by cR^2 . For the second integral, since $\widetilde{M}^+(f)(z) = 1$ and $z \in D_{2R}(x_0) \cap \mathbb{R}_+^2$, we see f_1 is integrable on \mathbb{R}_+^2 . By Lemma 6.2.3, we have

$$\begin{aligned} |V_\alpha| &\leq \sum_l \left| \widetilde{S}_l^* \right| = \sum_l 25 \left| \widetilde{S}_l \right| \\ &\leq c \sum_l \frac{1}{\alpha} \int_{W_l} f_1(\eta) dV(\eta) \\ &\leq \frac{c}{\alpha} \int_{\mathbb{R}_+^2} f_1(\eta) dV(\eta) \\ &= \frac{c}{\alpha} \int_{D_{2R}(x_0) \cap \mathbb{R}_+^2} f(\eta) dV(\eta) \\ &\leq \frac{cR^2}{\alpha} \widetilde{M}^+(f)(z) \\ &= \frac{cR^2}{\alpha}. \end{aligned}$$

So the second integral of (6.5) is bounded by cR^2 . Hence, so is (6.5).

Next we deal with f_2 . For any $\zeta \in D \cap \mathbb{R}_+^2$, we consider an arbitrary special disk D'_r that contains ζ and whose radius is r . It is easy to see that $D'_r \subset D_{2r+R}(x_0)$. When

$2r < R$, we have $D'_r \subset D_{2r+R}(x_0) \subset D_{2R}(x_0)$. Since f_2 vanishes on $D_{2R}(x_0) \cap \mathbb{R}_+^2$, we see that

$$\frac{1}{|D'_r \cap \mathbb{R}_+^2|} \int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) dV(\eta) = 0 < c,$$

for any $c > 0$. When $2r \geq R$, then $(2r + R)^2 \leq 16r^2$, so we have

$$\begin{aligned} \int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) dV(\eta) &\leq \int_{D_{2r+R}(x_0) \cap \mathbb{R}_+^2} f(\eta) dV(\eta) \\ &\leq |D_{2r+R}(x_0) \cap \mathbb{R}_+^2| \widetilde{M}^+(f)(z) \\ &= c(2r + R)^2 \\ &\leq cr^2, \end{aligned}$$

since $z \in D \subset D_{2r+R}(x_0)$. In either case, we obtain

$$\frac{1}{|D'_r \cap \mathbb{R}_+^2|} \int_{D'_r \cap \mathbb{R}_+^2} f_2(\eta) dV(\eta) < c.$$

Since D'_r is arbitrary, we see $\widetilde{M}^+(f_2)(\zeta) \leq c$, for any $\zeta \in D \cap \mathbb{R}_+^2$. Therefore we obtain

$$\int_{D \cap \mathbb{R}_+^2} \left(\widetilde{M}^+(f_2)(\zeta) \right)^q dV(\zeta) \leq cR^2. \quad (6.6)$$

Combining (6.6) with the fact that (6.5) is bounded by cR^2 , we see that

$$\int_{D \cap \mathbb{R}_+^2} \left(\widetilde{M}^+(f)(\zeta) \right)^q dV(\zeta) \leq cR^2,$$

which implies (6.4). This completes the proof. ■

Following the idea in [Muc72], we now investigate the weak-type (p, p) mapping property of the special maximal function operator \widetilde{M}^+ .

Theorem 11 *Assume that $p \geq 1$. Suppose μ_1 and μ_2 are two weights on \mathbb{R}_+^2 . Then we have a weak-type (p, p) inequality: namely, there is a constant $c > 0$ so that*

$$\mu_1 \left(\{z \in \mathbb{R}_+^2 : \widetilde{M}^+(f)(z) > \alpha\} \right) \leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} |f(z)|^p \mu_2(z) dV(z) \quad (1.3)$$

for all $\alpha > 0$, if and only if $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

Proof We use an analogue of the argument in [Muc72, Theorem 8]. For the sufficient part, we only need to prove (1.3) for integrable $f \geq 0$. To see this, note that any measurable function $f \geq 0$ on \mathbb{R}_+^2 can be approximated by the increasing sequence $\{f\chi_{D_R}\}_{R>0}$, where $D_R = D_R(x_0)$ is a sequence of special disks centered at $x_0 \in \mathbb{R}$ with radius R . Note that the set $\{z \in \mathbb{R}_+^2 : \widetilde{M}^+(f)(z) > \alpha\}$ is the increasing union of the same sets formed with the $f\chi_{D_R}$'s. If we prove (1.3) for $f\chi_{D_R}$, then the monotonic convergent theorem will imply (1.3) for f . Note that any $f\chi_{D_R}$ can be approximated by an increasing sequence of simple functions. Since the support of $f\chi_{D_R}$ is bounded, these simple functions are integrable. By the same limiting argument, (1.3) for integrable functions will imply (1.3) for $f\chi_{D_R}$.

Let $V_\alpha = \{z \in \mathbb{R}_+^2 : \widetilde{M}^+(f)(z) > \alpha\}$. By Lemma 6.2.3, for $p > 1$, we have

$$\begin{aligned}
\mu_1(V_\alpha) &\leq \sum_l \mu_1(W_l) \\
&\leq \sum_l \mu_1(W_l) \left(\frac{16}{\alpha |\widetilde{S}_l|} \int_{W_l} f(z) dV(z) \right)^p \\
&\leq \sum_l \frac{c}{\alpha^p} \frac{\mu_1(W_l)}{|\widetilde{S}_l|} \int_{W_l} f(z)^p \mu_2(z) dV(z) \left(\frac{1}{|\widetilde{S}_l|} \int_{W_l} \mu_2(z)^{-\frac{p'}{p}} dV(z) \right)^{p/p'} \\
&\leq \sum_l \frac{c}{\alpha^p} \int_{W_l} f(z)^p \mu_2(z) dV(z) \frac{\mu_1(\widetilde{S}_l^*)}{|\widetilde{S}_l^*|} \left(\frac{1}{|\widetilde{S}_l^*|} \int_{\widetilde{S}_l^*} \mu_2(z)^{-\frac{p'}{p}} dV(z) \right)^{p/p'} \\
&\leq \sum_l \frac{c}{\alpha^p} \int_{W_l} f(z)^p \mu_2(z) dV(z) \\
&\leq \frac{c}{\alpha^p} \int_{\mathbb{R}_+^2} f(z)^p \mu_2(z) dV(z).
\end{aligned}$$

For $p = 1$, similarly, we have

$$\begin{aligned}
\mu_1(V_\alpha) &\leq \sum_l \frac{16\mu_1(W_l)}{\alpha|\tilde{S}_l|} \int_{W_l} f(z) dV(z) \\
&\leq \sum_l \frac{c}{\alpha} \frac{\mu_1(W_l)}{|\tilde{S}_l|} \int_{W_l} f(z)\mu_2(z) dV(z) \left(\inf_{z \in W_l} \mu_2(z) \right)^{-1} \\
&\leq \sum_l \frac{c}{\alpha} \int_{W_l} f(z)\mu_2(z) dV(z) \frac{\mu_1(\tilde{S}_l^*)}{|\tilde{S}_l^*|} \left(\inf_{z \in \tilde{S}_l^*} \mu_2(z) \right)^{-1} \\
&\leq \frac{c}{\alpha} \int_{\mathbb{R}_+^2} f(z)\mu_2(z) dV(z).
\end{aligned}$$

For the necessary part, we first consider $p > 1$. Given any special disk D , we assume that $\int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) = \infty$. Then, by duality of the space $L^p(D \cap \mathbb{R}_+^2)$, there is a $g \in L^p(D \cap \mathbb{R}_+^2)$ so that $\int_{D \cap \mathbb{R}_+^2} g(z)\mu_2(z)^{-\frac{1}{p}} dV(z) = \infty$. Let $f = g\mu_2^{-\frac{1}{p}} \chi_{D \cap \mathbb{R}_+^2}$ on \mathbb{R}_+^2 . Then $\widetilde{M}^+(f)(z) = \infty$ for all $z \in D \cap \mathbb{R}_+^2$. So (1.3) gives $\mu_1(D \cap \mathbb{R}_+^2) = 0$, which contradicts the assumption $\mu_1 > 0$ almost everywhere. We also exclude the trivial case $\mu_2 = \infty$ on $D \cap \mathbb{R}_+^2$ to see that indeed we have $0 < \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z) < \infty$.

Take $f = \mu_2^{-\frac{p'}{p}} \chi_{D \cap \mathbb{R}_+^2}$ and $\alpha = \frac{1}{|D \cap \mathbb{R}_+^2|} \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-\frac{p'}{p}} dV(z)$ in (1.3). We see that

$$\begin{aligned}
\mu_1(D \cap \mathbb{R}_+^2) &\leq \frac{c}{\alpha^p} \int_{D \cap \mathbb{R}_+^2} \mu_2(z)^{-p'} \mu_2(z) dV(z) \\
&= \frac{c |D \cap \mathbb{R}_+^2|}{\alpha^{p-1}},
\end{aligned}$$

which is equivalent to $(\mu_1, \mu_2) \in A_p^+(\mathbb{R}_+^2)$.

When $p = 1$, given any special disk D , we exclude the trivial case $\inf \mu_2 = \infty$, where the infimum is taken over all $z \in D \cap \mathbb{R}_+^2$. Then, for any $\epsilon > 0$, there must be a measurable set $U \subset D \cap \mathbb{R}_+^2$ with $|U| > 0$, so that $\mu_2(z) < \epsilon + \inf \mu_2$ on U .

Taking $f = \chi_U$ and $\alpha = \frac{|U|}{|D \cap \mathbb{R}_+^2|}$ in (1.3), we see that

$$\begin{aligned}
\mu_1(D \cap \mathbb{R}_+^2) &\leq \frac{c |D \cap \mathbb{R}_+^2|}{|U|} \int_U \mu_2(z) dV(z) \\
&\leq c |D \cap \mathbb{R}_+^2| \left(\epsilon + \inf_{z \in D \cap \mathbb{R}_+^2} \mu_2(z) \right).
\end{aligned}$$

Letting $\epsilon \rightarrow 0^+$, we see that the inequality above is equivalent to $(\mu_1, \mu_2) \in A_1^+(\mathbb{R}_+^2)$.

This completes the proof. ■

Remark 6.2.5 *For any measurable $f \geq 0$, we do not have the basic inequality $f \leq \widetilde{M}^+(f)$. So we cannot follow the classical approach to show the reverse Hölder inequality for a weight in $A_p^+(\mathbb{R}_+^2)$. Hence we cannot obtain the strong-type (p, p) inequality, nor the factorization of a A_p^+ weight.²*

²Compare to the classical results for A_p weights, see [Ste93, Chapter 5] for details.

7. Twisted-Weighted Projections

7.1 Preliminaries and Basic Definitions

To apply the previous analysis to the twisted-weighted Bergman theory, let us temporarily consider the general setting for a twisted-weighted space. Assume that Ω is a domain in \mathbb{C}^n .

Definition 7.1.1 *Suppose $\tau \in C^1(\Omega)$ and $\tau > 0$. We define the set of τ -twisted holomorphic functions on Ω by*

$$\mathcal{O}_\tau(\Omega) = \{f \in C^1(\Omega) : \bar{\partial}_\tau(f) = 0\},$$

where the τ -twisted $\bar{\partial}_\tau$ operator is defined by $\bar{\partial}_\tau(f) = \bar{\partial}(\tau f)$. Note that $\mathcal{O}_\tau(\Omega)$ is just a coset of the set of holomorphic functions on Ω , that is, $\mathcal{O}_\tau(\Omega) = \frac{1}{\tau}\mathcal{O}(\Omega)$.

Definition 7.1.2 *Let μ be a weight on Ω . Define $A_{\tau,\mu}^2(\Omega) = \mathcal{O}_\tau(\Omega) \cap L^2(\Omega, \mu)$. We say that μ is admissible with respect to τ , if for any compact subset $K \subset \Omega$, there is a constant $C_K > 0$, so that*

$$\sup_K |f(z)| \leq C_K \|f\|_{L^2(\Omega, \mu)}$$

for all $f \in A_{\tau,\mu}^2(\Omega)$. For instance, if μ is continuous and non-vanishing, then it is admissible.¹

¹For a more general class of admissible weights, see [McN12, Proposition 2.4] for details.

Remark 7.1.1 *With an admissible weight μ , $A_{\tau,\mu}^2(\Omega)$ is a closed subspace of $L^2(\Omega, \mu)$. To see this, let $\{f_n\} \subset A_{\tau,\mu}^2(\Omega)$ be a sequence that converges to f in $L^2(\Omega, \mu)$. By admissibility, for each $K \subset \Omega$, f_n converges to f uniformly on K . Since $\tau \in C^1(\Omega)$, we have $\sup_K |\tau| \leq C'_K$. Hence τf_n converges to τf uniformly on K . Noting that each τf_n is holomorphic, we finally obtain $f \in \mathcal{O}_\tau(\Omega)$.*

Throughout this chapter, we always assume that the weight μ on the underlying domain Ω is admissible with respect to the twist factor τ . With this convention, we can now talk about the orthogonal projection from $L^2(\Omega, \mu)$ to $A_{\tau,\mu}^2(\Omega)$.

Definition 7.1.3 *Define the twisted-weighted Bergman projection associated to the twist-weight pair (τ, μ) to be the orthogonal projection $\mathcal{B}_{\tau,\mu} : L^2(\Omega, \mu) \rightarrow A_{\tau,\mu}^2(\Omega)$, represented by the integral*

$$\mathcal{B}_{\tau,\mu}(f)(z) = \int_{\Omega} B_{\tau,\mu}(z, \zeta) f(\zeta) \mu(\zeta) dV(\zeta),$$

where $B_{\tau,\mu}(z, \zeta)$ on $\Omega \times \Omega$ is the twisted-weighted Bergman kernel.

Remark 7.1.2 *The twisted-weighted Bergman projection satisfies all the properties of abstract Bergman theory in a Hilbert space. For example, $\mathcal{B}_{\tau,\mu}$ is self-adjoint and it reproduces $A_{\tau,\mu}^2(\Omega)$ functions. Note that the twisted-weighted Bergman kernel $B_{\tau,\mu}(z, \zeta)$ is $\bar{\partial}_\tau$ -closed in the variable z and anti- $\bar{\partial}_\tau$ -closed in the variable ζ . Moreover, the twisted-weighted Bergman kernel is uniquely determined by these properties.²*

By applying Lemma 2.2.1, we have the following transformation formula, which generalizes Corollary 2.2.2.

Corollary 7.1.3 *Let $\Phi : \Omega_1 \rightarrow \Omega_2$ be a biholomorphism between two domains in \mathbb{C}^n .*

Suppose that Ω_j is equipped with the weight μ_j and the twist factor τ_j , $j = 1, 2$. Assume

²See [McN12, Proposition 2.8] for details.

that $\mu_2 = \mu_1 \circ \Phi^{-1}$ and $\tau_2 = \tau_1 \circ \Phi^{-1}$. Then we have the transformation formula for the twisted-weighted Bergman kernels

$$B_{\tau_1, \mu_1}^{\Omega_1}(z, \zeta) = \det J_{\mathbb{C}}\Phi(z) B_{\tau_2, \mu_2}^{\Omega_2}(\Phi(z), \Phi(\zeta)) \det \overline{J_{\mathbb{C}}\Phi(\zeta)},$$

where $(z, \zeta) \in \Omega_1 \times \Omega_1$ and $B_{\tau_j, \mu_j}^{\Omega_j}$ is the twisted-weighted Bergman kernel on $\Omega_j \times \Omega_j$, $j = 1, 2$.

Proof Repeat the argument in Corollary 2.2.2. ■

7.2 The L^p Regularity

Now let us take a closer look at the twist-weight pair (τ, μ) . Define $\tilde{\mu} = \mu/\tau^2$, and consider the space $A_{\tilde{\mu}}^2(\Omega) = L^2(\Omega, \tilde{\mu}) \cap \mathcal{O}(\Omega)$. It is easy to see that the closedness of the subspace $A_{\tilde{\mu}}^2(\Omega)$ in $L^2(\Omega, \tilde{\mu})$ follows from the closedness of $A_{\tau, \mu}^2(\Omega)$ in $L^2(\Omega, \mu)$. Then, we can consider the weighted Bergman projection $\mathcal{B}_{\tilde{\mu}}$, which is the orthogonal projection from $L^2(\Omega, \tilde{\mu})$ to $A_{\tilde{\mu}}^2(\Omega)$. Indeed, we have the following observation.

Proposition 7.2.1 *Let $\sigma = \tau^{2-p}\tilde{\mu}$. The twisted-weighted Bergman projection $\mathcal{B}_{\tau, \mu}$ is bounded on $L^p(\Omega, \mu)$ if and only if the weighted Bergman projection $\mathcal{B}_{\tilde{\mu}}$ is bounded on $L^p(\Omega, \sigma)$.*

In particular, if $\tilde{\mu} = |g|^2$ for some non-vanishing holomorphic function g on Ω , then $\mathcal{B}_{\tau, \mu}$ is bounded on $L^p(\Omega, \mu)$ if and only if \mathcal{B} is bounded on $L^p(\Omega, \tilde{\sigma})$, where \mathcal{B} is the ordinary Bergman projection on Ω and $\tilde{\sigma} = \mu^{1-\frac{p}{2}}$.

Proof Consider the isometry $F : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \tilde{\mu})$ via $F(f) = \tau f$. By Lemma 2.2.1, we have the twisted-weighted Bergman kernel

$$B_{\tau, \mu}(z, \zeta) = \frac{1}{\tau(z)} \frac{1}{\tau(\zeta)} B_{\tilde{\mu}}(z, \zeta),$$

where $(z, \zeta) \in \Omega \times \Omega$ and $B_{\tilde{\mu}}$ is the weighted Bergman kernel on $(\Omega, \tilde{\mu})$. By this relation, we see that

$$\int_{\Omega} \left| \int_{\Omega} B_{\tau, \mu}(z, \zeta) f(\zeta) \mu(\zeta) dV(\zeta) \right|^p \mu(z) dV(z) \leq C \int_{\Omega} |f(z)|^p \mu(z) dV(z)$$

is equivalent to

$$\int_{\Omega} \left| \int_{\Omega} B_{\tilde{\mu}}(z, \zeta) \tilde{f}(\zeta) \tilde{\mu}(\zeta) dV(\zeta) \right|^p \sigma(z) dV(z) \leq C \int_{\Omega} |\tilde{f}(z)|^p \sigma(z) dV(z), \quad (7.1)$$

where $\tilde{f} = \tau f$. This proves the first part of the proposition.

Now we assume that $\tilde{\mu} = |g|^2$ for some non-vanishing holomorphic function g on Ω . Applying Lemma 2.2.1 once again to the isometry $\tilde{F} : L^2(\Omega, |g|^2) \rightarrow L^2(\Omega)$ via $\tilde{F}(f) = fg$, we see that

$$B_{\tilde{\mu}}(z, \zeta) = \frac{1}{g(z)\overline{g(\zeta)}} B(z, \zeta),$$

where $(z, \zeta) \in \Omega \times \Omega$ and B is the ordinary Bergman kernel on Ω . Hence, (7.1) is equivalent to

$$\int_{\Omega} \left| \int_{\Omega} B(z, \zeta) h(\zeta) dV(\zeta) \right|^p \tilde{\sigma}(z) dV(z) \leq C \int_{\Omega} |h(z)|^p \tilde{\sigma}(z) dV(z), \quad (7.2)$$

where $h = \tilde{f}g$. This completes the proof. ■

Remark 7.2.2 *When $\mu/\tau^2 = |g|^2$, μ is always admissible with respect to τ . An important observation is that the $L^p(\Omega, \tilde{\sigma})$ boundedness of \mathcal{B} is independent of the choice of τ and g . Note that, in general, for different τ the coset \mathcal{O}_{τ} is different.*

As an application, we now focus on simply connected proper planar domains in \mathbb{C} , and apply Theorem 6 to investigate the L^p regularity of the twisted-weighted Bergman projection.

Theorem 12 *Let Ω be a proper simply connected domain in \mathbb{C} , and let $\phi : \mathbb{R}_+^2 \rightarrow \Omega$ be a biholomorphism. For a weight $\mu \in C^1(\Omega)$ and for any non-vanishing $g \in \mathcal{O}(\Omega)$, define $\tau = \mu^{\frac{1}{2}}/|g|$. Then the twisted-weighted Bergman projection $\mathcal{B}_{\tau,\mu}$ is $L^p(\Omega, \mu)$ bounded if and only if $\mu(\phi(z))^{1-\frac{p}{2}} |\phi'(z)|^{2-p} \in A_p^+(\mathbb{R}_+^2)$.*

Proof By Proposition 7.2.1, we see that the twisted-weighted Bergman projection $\mathcal{B}_{\tau,\mu}$ is bounded on $L^p(\Omega, \mu)$ if and only if the ordinary Bergman projection \mathcal{B} on Ω is bounded on $L^p(\Omega, \mu^{1-\frac{p}{2}})$, i.e., (7.2) holds.

Consider the biholomorphism $\phi : \mathbb{R}_+^2 \rightarrow \Omega$. By the transformation formula for ordinary Bergman kernels (a special case of Corollary 2.2.2 or Corollary 7.1.3), we see that (7.2) is equivalent to

$$\int_{\mathbb{R}_+^2} \left| \int_{\mathbb{R}_+^2} -\frac{f(w)}{(z-\bar{w})^2} dV(w) \right|^p \sigma(z) dV(z) \leq C \int_{\mathbb{R}_+^2} |f(z)|^p \sigma(z) dV(z), \quad (7.3)$$

where $\sigma(z) = |\tau(\phi(z))g(\phi(z))\phi'(z)|^{2-p} = \mu(\phi(z))^{1-\frac{p}{2}} |\phi'(z)|^{2-p}$. Note that (7.3) is nothing but (1.2) with both μ_1 and μ_2 replaced by σ . Therefore, by Theorem 6, we see that (7.3) holds if and only if $\sigma \in A_p^+(\mathbb{R}_+^2)$. This completes the proof. ■

8. The L^p Sobolev Regularity

In this chapter, as the last application, we study the L^p Sobolev mapping property of the Bergman projection on the Hartogs triangle, and generalize it to higher dimension.

8.1 Preliminaries and Basic Definitions

Throughout this chapter, \mathcal{B} will denote the Bergman projection on the Hartogs triangle and B will be its associated Bergman kernel, that is,

$$\mathcal{B}(f)(z) = \int_{\mathbb{H}} B(z, \zeta) f(\zeta) dV(\zeta)$$

for $z \in \mathbb{H}$. We then make a precise definition of the weighted Sobolev space on \mathbb{H} .

Definition 8.1.1 *On the Hartogs triangle \mathbb{H} , for each $k \in \mathbb{Z}^+ \cup \{0\}$, $s \in \mathbb{R}$, and $p \in (1, \infty)$, we define the weighted Sobolev space by*

$$L_k^p(\mathbb{H}, |z_2|^s) = \{f \in L_{loc}^1(\mathbb{H}) : \|f\|_{p,k,s} < \infty\},$$

where the norm is defined as

$$\|f\|_{p,k,s} = \left(\int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D_{z,\bar{z}}^\alpha(f)(z)|^p |z_2|^s dV(z) \right)^{\frac{1}{p}}.$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multi-index running over all $|\alpha| \leq k$, and

$$D_{z,\bar{z}}^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \partial \bar{z}_1^{\alpha_3} \partial \bar{z}_2^{\alpha_4}}$$

denotes the differential operator in the weak sense. We also denote the ordinary (unweighted) Sobolev space by $L_k^p(\mathbb{H})$, with its usual norm

$$\|f\|_{p,k} = \left(\int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D_{z,\bar{z}}^\alpha(f)(z)|^p dV(z) \right)^{\frac{1}{p}}.$$

Remark 8.1.1 By Definition 8.1.1 above, Theorem 13 can be rephrased in the following way. For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $p \in (\frac{4}{3}, 4)$, there exists a constant $C_{p,k} > 0$, so that

$$\|\mathcal{B}(f)\|_{p,k,pk} \leq C_{p,k} \|f\|_{p,k},$$

for any $f \in L_k^p(\mathbb{H})$.

We also adopt the following notation.

Definition 8.1.2 Let $\beta = (\beta_1, \beta_2)$ be a multi-index. We use the notation below to denote the differential operators

$$D_z^\beta = \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}}$$

and

$$D_{z_j, \bar{z}_j}^\beta = \frac{\partial^{|\beta|}}{\partial z_j^{\beta_1} \partial \bar{z}_j^{\beta_2}}$$

for $j = 1, 2$.

By Theorem 1, we see that $\mathcal{B}(f) \in A^p(\mathbb{H}) = L^p(\mathbb{H}) \cap \mathcal{O}(\mathbb{H})$ whenever $p \in (\frac{4}{3}, 4)$ and $f \in L^p(\mathbb{H})$. So we can rewrite the weighted L^p Sobolev norm of $\mathcal{B}(f)$ as

$$\|\mathcal{B}(f)\|_{p,k,pk}^p = \sum_{|\beta| \leq k} \int_{\mathbb{H}} |D_z^\beta(\mathcal{B}(f))(z)|^p |z_2|^{pk} dV(z), \quad (8.1)$$

where β and D_z^β are as in Definition 8.1.2.

8.2 Transfer to the Product Model

In order to transfer \mathbb{H} to the product model, we first recall the transformation formula for the Bergman kernels.

Proposition 8.2.1 *Let Ω_j be domains in \mathbb{C}^n and B_j be their Bergman kernels on $\Omega_j \times \Omega_j$, $j = 1, 2$. Suppose that $\Psi : \Omega_1 \rightarrow \Omega_2$ is a biholomorphism. Then, for $(w, \eta) \in \Omega_1 \times \Omega_1$, we have*

$$\det J_{\mathbb{C}}\Psi(w)B_2(\Psi(w), \Psi(\eta)) \det \overline{J_{\mathbb{C}}\Psi(\eta)} = B_1(w, \eta).$$

Proof This is a special case of Corollary 2.2.2 and Corollary 7.1.3.¹ ■

Now let us consider the biholomorphism

$$\Phi : \mathbb{H} \rightarrow \mathbb{D} \times \mathbb{D}^*$$

with its inverse

$$\Psi : \mathbb{D} \times \mathbb{D}^* \rightarrow \mathbb{H},$$

where

$$\Phi(z_1, z_2) = \left(\frac{z_1}{z_2}, z_2 \right) \quad \text{and} \quad \Psi(w_1, w_2) = (w_1 w_2, w_2).$$

A simple computation shows that $\det J_{\mathbb{C}}\Psi(w) = w_2$, for $w = (w_1, w_2) \in \mathbb{D} \times \mathbb{D}^*$. Therefore, by Proposition 8.2.1, we have

$$B(\Psi(w), \Psi(\eta)) = \frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1 - w_1 \bar{\eta}_1)^2} \cdot \frac{1}{(1 - w_2 \bar{\eta}_2)^2}, \quad (8.2)$$

where B is the Bergman kernel on $\mathbb{H} \times \mathbb{H}$ and $(w, \eta) \in \mathbb{D} \times \mathbb{D}^* \times \mathbb{D} \times \mathbb{D}^*$.

We next need to transfer the differential operators D_z^β to the ones in the new variable w , so we need a lemma.

¹See also [Kra01, Proposition 1.4.12].

Lemma 8.2.2 *Under the above biholomorphism $\Phi(z) = w$, for each β let $m = |\beta|$, we have*

$$D_z^\beta = \sum_{a+b \leq m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \quad (8.3)$$

where $p_{a,b,\beta}(w_1)$ is a polynomial of degree at most m in variable w_1 . In addition, if $|\beta| \leq k$ for some $k \in \mathbb{Z}^+ \cup \{0\}$, then $|p_{a,b,\beta}(w_1)| \leq C_k$ on \mathbb{D} uniformly in β , a , and b , for some constant $C_k > 0$ depending only on k .

Proof We prove (8.3) by induction on $m = |\beta|$. The case $m = 0$ is trivial. When $m = 1$, a direct computation shows that

$$\frac{\partial}{\partial z_1} = \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1}$$

and

$$\frac{\partial}{\partial z_2} = -\frac{w_1}{w_2} \cdot \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2}.$$

It is obvious that both of $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ are of the form (8.3).

Suppose, for all β with $|\beta| = m$, the D_z^β 's are of the form (8.3). We now check the case $|\beta'| = m + 1$. Note that $D_z^{\beta'} = \frac{\partial}{\partial z_1} \circ D_z^\beta$ or $D_z^{\beta'} = \frac{\partial}{\partial z_2} \circ D_z^\beta$, for some β . By the inductive assumption, we have

$$\begin{aligned} \frac{\partial}{\partial z_1} D_z^\beta &= \frac{1}{w_2} \cdot \frac{\partial}{\partial w_1} \sum_{a+b \leq m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} \\ &= \sum_{a+b \leq m} \frac{p'_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^{a+1} \partial w_2^b} \\ &= \sum_{a+b \leq m+1} \frac{p_{a,b,\beta'}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}, \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z_2} D_z^\beta &= \left(-\frac{w_1}{w_2} \cdot \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} \right) \sum_{a+b \leq m} \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} \\
&= \sum_{a+b \leq m} \frac{-w_1 p'_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{-w_1 p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^{a+1} \partial w_2^b} + \\
&\quad \frac{(b-m) p_{a,b,\beta}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b} + \frac{p_{a,b,\beta}(w_1)}{w_2^{m-b}} \cdot \frac{\partial^{a+b+1}}{\partial w_1^a \partial w_2^{b+1}} \\
&= \sum_{a+b \leq m+1} \frac{p_{a,b,\beta'}(w_1)}{w_2^{m+1-b}} \cdot \frac{\partial^{a+b}}{\partial w_1^a \partial w_2^b}.
\end{aligned}$$

We see that $p_{a,b,\beta'}(w_1)$ is obviously a polynomial of degree at most $m+1$, and $D_z^{\beta'}$ has the form in (8.3).

When $|\beta| \leq k$, all the possible combinations of derivatives in D_z^β are finite, so there are finitely many different coefficients in all of the $p_{a,b,\beta}(w_1)$'s. Note that $|w_1| \leq 1$ on \mathbb{D} and $a, b \leq m \leq k$. So we see that $|p_{a,b,\beta}(w_1)| \leq C_k$ on \mathbb{D} as desired. \blacksquare

Now we transfer \mathbb{H} to the product model $\mathbb{D} \times \mathbb{D}^*$ by the biholomorphism Φ , combine (8.2) and (8.3), and we see the right hand side of (8.1) becomes

$$\sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| \sum_{a+b \leq |\beta|} \int_{\mathbb{D} \times \mathbb{D}^*} K_{a,b,\beta}(w, \eta) f(\Psi(\eta)) |\eta_2|^2 dV(\eta) \right|^p |w_2|^{pk+2} dV(w), \quad (8.4)$$

where

$$K_{a,b,\beta}(w, \eta) = \frac{p_{a,b,\beta}(w_1)}{w_2^{|\beta|-b}} \cdot \frac{\partial^a}{\partial w_1^a} \left(\frac{1}{(1-w_1 \bar{\eta}_1)^2} \right) \cdot \frac{\partial^b}{\partial w_2^b} \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} \right).$$

8.3 Convert the Differential Operators on \mathbb{D}^*

Since \mathbb{D}^* is a Reinhardt domain, we can apply a result from [Str86].

Lemma 8.3.1 *As in (8.4), for the last factor in $K_{a,b,\beta}(w, \eta)$ we have*

$$\frac{\partial^b}{\partial w_2^b} \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} \right) = \frac{\bar{\eta}_2^b}{w_2^b} \cdot \frac{\partial^b}{\partial \bar{\eta}_2^b} \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} \right). \quad (8.5)$$

Proof By (3.1) in Chapter 3, we see that the kernel in (8.5) is the weighted Bergman kernel on $(\mathbb{D}^*, |z|^2)$. So we can argue step by step as in [Str86, Lemma 2.1] and its following remark in [Str86, Remark 2.3] to complete the proof. \blacksquare

Now we focus on the integration over \mathbb{D}^* in (8.4). We first define a "tangential" operator.

Definition 8.3.1 Let $S_w = w \frac{\partial}{\partial w}$ be the complex normal differential operator on a neighborhood of $\partial\mathbb{D}$. We define the tangential operator by

$$T_w = \Im(S_w) = \frac{1}{2i} \left(w \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial \bar{w}} \right).$$

Remark 8.3.2 Indeed, T_w is well-defined on a neighborhood of $\bar{\mathbb{D}}$. Moreover, for any disk $\mathbb{D}_r = \{|w| < r\}$ of radius $r < 1$ with defining function $\rho_r(w) = |w|^2 - r^2$, we have

$$T_w(\rho_r) = 0 \tag{8.6}$$

on $\partial\mathbb{D}_r$. That is, T_w is tangential on $\partial\mathbb{D}_r$ for all $r < 1$.

In order to make use of integration by parts, we need the following lemma.

Lemma 8.3.3 Let T_w be as above. For $b \in \mathbb{Z}^+ \cup \{0\}$, we have

$$T_w^b \equiv \sum_{j=0}^b c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} \pmod{\frac{\partial}{\partial w}}, \tag{8.7}$$

where the c_j s are constants, $c_b \neq 0$, and T_w^b denotes the composition of b copies of the T_w s.

Proof We prove (8.7) by induction on b . The case $b = 0$ is trivial. When $b = 1$, it is easy to see that

$$T_w \equiv -\frac{1}{2i} \bar{w} \frac{\partial}{\partial \bar{w}} \pmod{\frac{\partial}{\partial w}}.$$

Suppose that (8.7) holds for some b . Then we see that

$$T_w^b = \sum_{j=0}^b c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A \circ \frac{\partial}{\partial w},$$

for some operator A . So, for the case $b+1$, we have

$$\begin{aligned} T_w \circ T_w^b &= \frac{1}{2i} \left(w \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial \bar{w}} \right) \circ \left(\sum_{j=0}^b c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A \circ \frac{\partial}{\partial w} \right) \\ &= \frac{1}{2i} \left(\sum_{j=0}^b c_j w \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} \frac{\partial}{\partial w} - j c_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} - c_j \bar{w}^{j+1} \frac{\partial^{j+1}}{\partial \bar{w}^{j+1}} \right) + T_w \circ A \circ \frac{\partial}{\partial w} \\ &= \sum_{j=0}^{b+1} c'_j \bar{w}^j \frac{\partial^j}{\partial \bar{w}^j} + A' \circ \frac{\partial}{\partial w}, \end{aligned}$$

for some constants c'_j 's with $c'_{b+1} = -\frac{1}{2i}c_b \neq 0$ and some operator A' . Therefore, (8.7) holds for T_w^{b+1} . ■

Combining (8.5) and (8.7), we note that the kernel in (8.5) is anti-holomorphic in η_2 .

The inside integration over \mathbb{D}^* with respect to variable η_2 in (8.4) denoted by I becomes

$$\begin{aligned} I &= \int_{\mathbb{D}^*} \frac{\partial^b}{\partial w_2^b} \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1 - w_2 \bar{\eta}_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 dV(\eta_2) \\ &= \int_{\mathbb{D}^*} \frac{\bar{\eta}_2^b}{w_2^b} \cdot \frac{\partial^b}{\partial \bar{\eta}_2^b} \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1 - w_2 \bar{\eta}_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 dV(\eta_2) \\ &= \frac{1}{w_2^b} \int_{\mathbb{D}^*} \sum_{j=0}^b c_j T_{\eta_2}^j \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1 - w_2 \bar{\eta}_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 dV(\eta_2) \\ &= \frac{1}{w_2^b} \sum_{j=0}^b c_j \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{D} - \mathbb{D}_\epsilon} T_{\eta_2}^j \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1 - w_2 \bar{\eta}_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 dV(\eta_2). \end{aligned}$$

Let us assume in addition for a moment that $f(\Psi(\eta))$ belongs to $C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ in the variable η_2 . Then, by (8.6), we see that

$$\begin{aligned}
I &= \frac{1}{w_2^b} \sum_{j=0}^b c_j \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{D}-\mathbb{D}_\epsilon} T_{\eta_2}^j \left(\frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} \right) f(\Psi(\eta)) |\eta_2|^2 dV(\eta_2) \\
&= \frac{1}{w_2^b} \sum_{j=0}^b c_j (-1)^j \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{D}-\mathbb{D}_\epsilon} \frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} T_{\eta_2}^j \left(f(\Psi(\eta)) |\eta_2|^2 \right) dV(\eta_2) \quad (8.8) \\
&= \frac{1}{w_2^b} \sum_{j=0}^b (-1)^j c_j \int_{\mathbb{D}^*} \frac{1}{w_2 \bar{\eta}_2} \cdot \frac{1}{(1-w_2 \bar{\eta}_2)^2} T_{\eta_2}^j \left(f(\Psi(\eta)) \right) |\eta_2|^2 dV(\eta_2),
\end{aligned}$$

where the last line follows from the fact $T_{\eta_2}(|\eta_2|^2) = 0$.

Definition 8.3.2 *We use the following notation:*

$$F_j(\eta) = T_{\eta_2}^j \left(f(\Psi(\eta)) \right) \cdot \eta_2,$$

$$\mathcal{B}_{0,a}(g)(w_1) = \int_{\mathbb{D}} \frac{\partial^a}{\partial w_1^a} \left(\frac{1}{(1-w_1 \bar{\eta}_1)^2} \right) g(\eta_1) dV(\eta_1),$$

for any g whenever the integral is well-defined, and

$$\mathcal{B}_0(h)(w_2) = \int_{\mathbb{D}^*} \frac{h(\eta_2)}{(1-w_2 \bar{\eta}_2)^2} dV(\eta_2),$$

for any h whenever the integral is well-defined.

By (8.8) and the notation above, we see that (8.4) becomes

$$\sum_{|\beta| \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| \sum_{a+b \leq |\beta|} \frac{p_{a,b,\beta}(w_1)}{w_2^{|\beta|+1}} \sum_{j=0}^b (-1)^j c_j \mathcal{B}_{0,a}(\mathcal{B}_0(F_j))(w) \right|^p |w_2|^{pk+2} dV(w). \quad (8.9)$$

8.4 Proof of Theorem 13

To prove Theorem 13, we first need two propositions.

Proposition 8.4.1 *The Bergman projection on \mathbb{D} is bounded from $L_k^p(\mathbb{D})$ to itself for $p \in (1, \infty)$ and all $k \in \mathbb{Z}^+ \cup \{0\}$.*

Proof This is a special case of the classical result, the regularity of Bergman projection on bounded smooth domains with strongly pseudoconvex boundary. See [PS77] for details, or [KR14] for treatment of domains with more general boundary. ■

Proposition 8.4.2 *The integral operator \mathcal{B}_0 , defined as in Definition 8.3.2, is bounded from $L^p(\mathbb{D}^*, |w|^{2-p})$ to itself for $p \in (\frac{4}{3}, 4)$, where $w \in \mathbb{D}^*$.*

Proof This statement is equivalent to the assertion that the weighted Bergman projection on the weighed space $(\mathbb{D}^*, |w|^2)$ is bounded from $L^p(\mathbb{D}^*, |w|^2)$ to itself for $p \in (\frac{4}{3}, 4)$. See Theorem 2. ■

Now we are ready to prove Theorem 13 under the additional assumption $f(\Psi(\eta)) \in C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ in the variable η_2 .

Theorem 13 *The Bergman projection $\mathcal{B}_{\mathbb{H}}$ on the Hartogs triangle \mathbb{H} maps continuously from $L_k^p(\mathbb{H})$ to $L_k^p(\mathbb{H}, |z_2|^{pk})$ for $p \in (\frac{4}{3}, 4)$.*

Proof [Under the additional assumption $f(\Psi(\eta)) \in C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ in the variable η_2 .]

By (8.1), (8.4), (8.9) and Lemma 8.2.2, we obtain

$$\begin{aligned} \|\mathcal{B}(f)\|_{p,k,pk}^p &\leq \sum_{|\beta| \leq k} \sum_{a+b \leq |\beta|} \sum_{j=0}^b C_{p,k} \int_{\mathbb{D} \times \mathbb{D}^*} |\mathcal{B}_{0,a}(\mathcal{B}_0(F_j))(w)|^p |w_2|^{pk+2-p(|\beta|+1)} dV(w) \\ &\leq C_{p,k} \sum_{a+b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} |\mathcal{B}_{0,a}(\mathcal{B}_0(F_b))(w)|^p |w_2|^{2-p} dV(w). \end{aligned}$$

By Proposition 8.4.1 and Definition 8.3.2, we see that $\mathcal{B}_{0,a}$ is bounded from $L_a^p(\mathbb{D})$ to $L^p(\mathbb{D})$. Therefore, for $p \in (1, \infty)$, we have

$$\begin{aligned}
\|\mathcal{B}(f)\|_{p,k,pk}^p &\leq C_{p,k} \sum_{a+b \leq k} \int_{\mathbb{D}^*} \left(\int_{\mathbb{D}} \sum_{|\beta| \leq a} \left| D_{w_1, \bar{w}_1}^\beta (\mathcal{B}_0(F_b))(w) \right|^p dV(w_1) \right) |w_2|^{2-p} dV(w_2) \\
&\leq C_{p,k} \sum_{|\beta|+b \leq k} \int_{\mathbb{D}} \left(\int_{\mathbb{D}^*} \left| \mathcal{B}_0(D_{w_1, \bar{w}_1}^\beta (F_b))(w) \right|^p |w_2|^{2-p} dV(w_2) \right) dV(w_1).
\end{aligned}$$

Similarly, by Proposition 8.4.2 and Definition 8.3.2, for $p \in (\frac{4}{3}, 4)$ we have

$$\begin{aligned}
\|\mathcal{B}(f)\|_{p,k,pk}^p &\leq C_{p,k} \sum_{|\beta|+b \leq k} \int_{\mathbb{D}} \left(\int_{\mathbb{D}^*} \left| D_{w_1, \bar{w}_1}^\beta (F_b)(w) \right|^p |w_2|^{2-p} dV(w_2) \right) dV(w_1) \\
&= C_{p,k} \sum_{|\beta|+b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| D_{w_1, \bar{w}_1}^\beta T_{w_2}^b (f(\Psi(w))) \cdot w_2 \right|^p |w_2|^{2-p} dV(w) \\
&= C_{p,k} \sum_{|\beta|+b \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| D_{w_1, \bar{w}_1}^\beta T_{w_2}^b (f(\Psi(w))) \right|^p |w_2|^2 dV(w) \\
&\leq C_{p,k} \sum_{|\beta|+|\beta'| \leq k} \int_{\mathbb{D} \times \mathbb{D}^*} \left| D_{w_1, \bar{w}_1}^\beta D_{w_2, \bar{w}_2}^{\beta'} (f(\Psi(w))) \right|^p |w_2|^2 dV(w),
\end{aligned} \tag{8.10}$$

where the last line follows from $T_{w_2} = \frac{1}{2i} \left(w_2 \frac{\partial}{\partial w_2} - \bar{w}_2 \frac{\partial}{\partial \bar{w}_2} \right)$, $|w_2| < 1$ for $w_2 \in \mathbb{D}^*$, and an equation similar to (8.7).

Note that, under the biholomorphism $\Psi(w) = z$ defined in §8.2, we have

$$\frac{\partial}{\partial w_1} = w_2 \frac{\partial}{\partial z_1} \quad \text{and} \quad \frac{\partial}{\partial \bar{w}_1} = \bar{w}_2 \frac{\partial}{\partial \bar{z}_1},$$

and also

$$\frac{\partial}{\partial w_2} = w_1 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \quad \text{and} \quad \frac{\partial}{\partial \bar{w}_2} = \bar{w}_1 \frac{\partial}{\partial \bar{z}_1} + \frac{\partial}{\partial \bar{z}_2}.$$

Again, since $(w_1, w_2) \in \mathbb{D} \times \mathbb{D}^*$, we see that $|w_1|, |w_2| < 1$. Therefore, by (8.10) and the transfer of $\mathbb{D} \times \mathbb{D}^*$ back to \mathbb{H} , we finally arrive at

$$\|\mathcal{B}(f)\|_{p,k,pk}^p \leq C_{p,k} \sum_{|\alpha| \leq k} \int_{\mathbb{H}} |D_{z, \bar{z}}^\alpha (f)(z)|^p dV(z)$$

as desired. ■

To remove the additional assumption $f(\Psi(\eta)) \in C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ in variable η_2 , we need the following lemma.

Lemma 8.4.3 *The subspace $C^\infty(\overline{\mathbb{D}} \setminus \{0\}) \cap L_k^p(\mathbb{D}^*, |w|^2)$ is dense in $L_k^p(\mathbb{D}^*, |w|^2)$ with respect to the weighted norm in $L_k^p(\mathbb{D}^*, |w|^2)$.*

Proof The argument is based on the ideas from [Eva98, §5.3 Theorem 2 and Theorem 3].

Given any $g \in L_k^p(\mathbb{D}^*, |w|^2)$, we fix a $\delta > 0$. On $V_0 = \mathbb{D} \setminus \overline{\mathbb{D}}_{\frac{1}{2}}$, the weighted norm $L_k^p(V_0, |w|^2)$ is equivalent to the unweighted norm $L_k^p(V_0)$. Arguing as in the proof of [Eva98, §5.3 Theorem 3], we see that there is a $g_0 \in C^\infty(\overline{V_0})$, so that

$$\|g_0 - g\|_{L_k^p(V_0, |w|^2)} < \delta.$$

Define $U_j = \mathbb{D}_{\rho - \frac{1}{j}} \setminus \overline{\mathbb{D}}_{\frac{1}{j}}$ for some $1 > \rho > \frac{1}{2}$ and for $j \in \mathbb{Z}^+$ ($U_1 = \emptyset$). Let $V_j = U_{j+3} \setminus \overline{U}_{j+1}$. Then we see that $\bigcup_{j=1}^\infty V_j = \mathbb{D}_\rho \setminus \{0\}$. Arguing as in the proof of [Eva98, §5.3 Theorem 2], we can find a smooth partition of unity $\{\psi_j\}_{j=1}^\infty$ subordinate to $\{V_j\}_{j=1}^\infty$, so that $\sum_{j=1}^\infty \psi_j = 1$ on $\mathbb{D}_\rho \setminus \{0\}$. Moreover, for each j , the support of $\psi_j g$ lies in V_j (so $|w| > \frac{1}{j+3}$), and hence $\psi_j g \in L_k^p(\mathbb{D}_\rho \setminus \{0\})$. Therefore we can find smooth function g_j , with support in $U_{j+4} \setminus \overline{U}_j$, so that

$$\|g_j - \psi_j g\|_{L_k^p(\mathbb{D}_\rho \setminus \{0\})} \leq \frac{\delta}{2^j},$$

see [Eva98, §5.3 Theorem 2] for details. Write $\tilde{g}_0 = \sum_{j=1}^\infty g_j$. It is easy to see that $\tilde{g}_0 \in C^\infty(\mathbb{D}_\rho \setminus \{0\})$ and

$$\|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_\rho \setminus \{0\}, |w|^2)} \leq \|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_\rho \setminus \{0\})} \leq \delta,$$

since $|w| < 1$ on $\mathbb{D}_\rho \setminus \{0\}$.

Let V'_0 be an open set so that $\partial\mathbb{D} \subset V'_0$ and $V'_0 \cap \mathbb{D} = V_0$. Then $V'_0 \cup \mathbb{D}_\rho$ cover $\overline{\mathbb{D}}$. Take a smooth partition of unity $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ on $\overline{\mathbb{D}}$ subordinate to $\{V'_0, \mathbb{D}_\rho\}$. Then $h = \tilde{\psi}_1 g_0 + \tilde{\psi}_2 \tilde{g}_0$ belongs to $C^\infty(\overline{\mathbb{D}} \setminus \{0\})$, and

$$\begin{aligned} \|h - g\|_{L_k^p(\mathbb{D}^*, |w|^2)} &\leq C \left(\|g_0 - g\|_{L_k^p(V_0, |w|^2)} + \|\tilde{g}_0 - g\|_{L_k^p(\mathbb{D}_\rho \setminus \{0\}, |w|^2)} \right) \\ &< 2C\delta \end{aligned}$$

as desired. ■

Now we are ready to remove the extra assumption and prove Theorem 13.

Proof of Theorem 13:

For any $f \in L_k^p(\mathbb{H})$, we have $f(\Psi(w)) \in L_k^p(\mathbb{D}^*, |w_2|^2)$ in the variable w_2 . Then, by Lemma 8.4.3, we can find a sequence $\{h_j(w)\} \subset C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ tending to $f(\Psi(w))$ in the variable w_2 with respect to the norm in $L_k^p(\mathbb{D}^*, |w_2|^2)$. We have already seen that (8.10) holds for each $h_j(w)$ replacing $f(\Psi(w))$. Indeed, if we focus on the integration over \mathbb{D}^* , by comparing with (8.4), we see that (8.10) is just the following. For each $b = 0, 1, \dots, k$,

$$\int_{\mathbb{D}^*} \left| w_2^b \frac{\partial^b}{\partial w_2^b} (\mathcal{B}_2(h_j)) \right|^p |w_2|^2 dV(w_2) \leq C_{p,k} \|h_j\|_{L_k^p(\mathbb{D}^*, |w_2|^2)}^p, \quad (8.11)$$

where \mathcal{B}_2 is the weighted Bergman projection on the weighted space $(\mathbb{D}^*, |w_2|^2)$.

Now let $j \rightarrow \infty$. In view of the boundedness of \mathcal{B}_2 (by Theorem 2 (a)), we see that $w_2^b \frac{\partial^b}{\partial w_2^b} (\mathcal{B}_2(h_j))$ indeed tends to $w_2^b \frac{\partial^b}{\partial w_2^b} (\mathcal{B}_2(f(\Psi)))$ in $L^p(\mathbb{D}^*, |w_2|^2)$ for each $b = 0, 1, \dots, k$. Therefore (8.11) is valid for general $f(\Psi(w)) \in L_k^p(\mathbb{D}^*, |w_2|^2)$, which completes the proof for general $f \in L_k^p(\mathbb{H})$. ■

Remark 8.4.4 *If we consider a general weighted space $L_k^p(\mathbb{H}, |z_2|^s)$ rather than $L_k^p(\mathbb{H}, |z_2|^{pk})$, we can use the same idea and apply the two-weight inequality in Chapter 4 to derive the boundedness of \mathcal{B} for $p \in (\frac{4}{3}, \frac{s+4}{k+1})$ and $s \leq pk$. When $s = 0$ and $k = 1$, we see that we*

cannot derive the boundedness even for $p = 2$. This coincides with the fact that \mathcal{B} is not bounded from $W^1(\mathbb{H})$ to itself for the unweighted case, see [CS13].²

Remark 8.4.5 We may also consider the weighted Sobolev space with weights inhomogeneous with respect to derivatives of different orders.

To be precise, for each $k \in \mathbb{Z}^+ \cup \{0\}$, $p \in (1, \infty)$, and $t = \{t_m\}_{m=0}^\infty \in \mathbb{R}^\omega$, we define the inhomogeneous weighted Sobolev space by

$$L_k^p(|z_2|^t) = \{f \in L_{loc}^1(\mathbb{H}) : \|f\|_{p,k,t} < \infty\},$$

where the norm is defined as

$$\|f\|_{p,k,t} = \left(\int_{\mathbb{H}} \sum_{|\alpha| \leq k} |D_{z,\bar{z}}^\alpha(f)(z)|^p |z_2|^{t|\alpha|} dz \right)^{\frac{1}{p}}.$$

Then a variant of Theorem 13 can be stated as follows. (The proof is almost the same.)

For each $k \in \mathbb{Z}^+ \cup \{0\}$ and $p \in (\frac{4}{3}, 4)$, suppose that $t = \{t_m\}_{m=0}^\infty \in \mathbb{R}^\omega$ and $t_m = mp$.

Then there exists a constant $C_{p,k} > 0$ so that

$$\|\mathcal{B}(f)\|_{p,k,t} \leq C_{p,k} \|f\|_{p,k}$$

for any $f \in L_k^p(\mathbb{H})$.

8.5 Generalization to n -dimensional Hartogs Triangle

The method also applies to the n -dimensional Hartogs triangle. Using the notation in (1.1), we have the following result.

Theorem 14 The Bergman projection on the n -dimensional Hartogs triangle $\mathbb{H}_{\phi_j}^n$ maps continuously from $L_k^p(\mathbb{H}_{\phi_j}^n)$ to $L_k^p(\mathbb{H}_{\phi_j}^n, |z_1''|^{pk})$ for $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$.

²See also [Zey13] for variants of Hartogs triangle.

The idea of the proof remains the same, but requires more complicated computations.

We outline it very briefly here.

Proof Using the same argument as in the proof of Theorem 5 in Chapter 3, it suffices to prove the theorem for the domain

$$\mathbb{H}^n = \{(z', z'') \in \mathbb{C}^{m_1 + \dots + m_l + d} : \max_{1 \leq j \leq l} |\tilde{z}_j| < |z'_1| < |z''_2| < \dots < |z''_d| < 1\}.$$

As in §8.2, we transfer \mathbb{H}^n to a product model by $\Phi : \mathbb{H}^n \rightarrow \mathbb{B}^{m_1} \times \dots \times \mathbb{B}^{m_l} \times (\mathbb{D}^*)^{\times d}$, with

$$\Phi(z', z'') = \left(\frac{z'}{z''_1}, \frac{z''_1}{z''_2}, \dots, \frac{z''_{d-1}}{z''_d}, z''_d \right) = (w', w''_1, \dots, w''_d).$$

By induction, (8.3) becomes

$$D_z^\beta = \sum_{|a|+|b| \leq |\beta|} \frac{p_{a,b,\beta}(w)}{w'^{\tilde{b}}} \cdot \frac{\partial^{|a|+|b|}}{\partial w'^a \partial w''^b}, \quad (8.12)$$

where β is an n -dimensional multi-index, a is m -dimensional, and both b and \tilde{b} are d -dimensional with $\tilde{b}_j + b_j \leq |\beta|$ for $j = 1, \dots, d-1$ and $\tilde{b}_d + b_d = |\beta|$.

While converting the differential operators on \mathbb{D}^* , (8.5) applies to the weighted Bergman kernel on the weighted space $(\mathbb{D}^*, |w|^{2(j-1)})$, for $j = n-d+1, n-d+2, \dots, n$. By using (8.5), (8.6), (8.7), and integration by parts, we arrive at a similar expression as (8.9).

Classical result tells us that, for $j = 1, \dots, l$, the L_k^p boundedness of the Bergman projection on \mathbb{B}^{m_j} holds for $p \in (1, \infty)$. Thus we only need to take care of the L^p boundedness of the weighted Bergman projection on $(\mathbb{D}^*, |w|^{2(j-1)})$, for $j = n-d+1, n-d+2, \dots, n$. By Theorem 2, all these projections will be bounded when $p \in (\frac{2n}{n+1}, \frac{2n}{n-1})$.

The proof will be complete once we pass from $C^\infty(\overline{\mathbb{D}} \setminus \{0\})$ to $L_k^p(\mathbb{D}^*, |w|^{2(j-1)})$ for $j = n-d+1, n-d+2, \dots, n$, by applying a general version of Lemma 8.4.3. ■

Remark 8.5.1 *The reason we consider the weight $|z_1''|^{pk}$ rather than $|z_d''|^{pk}$ is that, if we look at (8.12) and apply (8.5), then we will obtain a factor $\frac{1}{|w_j''|^{\tilde{b}_j+b_j}}$ with $\tilde{b}_j+b_j \leq |\beta| \leq k$ for $j = 1, \dots, d$. For each j , the equality will hold in some cases. Therefore, for each j , it is required to consider a weight $|w_j''|^{pk}$ to cancel out with the factor. By transferring the product model back to \mathbb{H}^n , we obtain a weight $|z_1''|^{pk}$.*

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