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Essays on Margin Requirements, Endogenous Illiquidity, and Portfolio Choice

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ESSAYS ON MARGIN REQUIREMENTS, ENDOGENOUS ILLIQUIDITY, AND PORTFOLIO CHOICE

by

Yajun Wang

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ABSTRACT OF THE THESIS

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This dissertation includes three essays. The first essay studies the effects of margin requirements. The second essay studies how asymmetric information and imperfect competition affect equilibrium illiquidity. The third essay derives new comparative statics results for the distribution of portfolio payoffs.

Margin requirements have long been implemented in almost all financial markets and are often used as an important regulatory tool for improving market conditions. However, their economic impact beyond affecting default risk is still largely unknown. The first essay proposes a tractable and flexible equilibrium model with and without information asymmetry to examine how margin requirements on both long and short stock positions affect asset prices, market volatility, market illiquidity and the welfare of market participants. Most of my main results are obtained in closed-form. Contrary to one of the main regulatory goals, I find that margin requirements can significantly *increase* market volatility. In addition, margin requirements always increase market illiquidity (as measured by price impact) and can lead to a greater return reversal

exactly when they amplify market volatility. I also find that information asymmetry may reverse or dampen the impact of margin requirements. Moreover, margin requirements always make unconstrained investors worse off and can make constrained investors better off. The model provides new testable implications.

The second essay proposes a novel and tractable equilibrium model to study how information asymmetry, competition among market makers, and investors' risk aversion affect asset pricing, market illiquidity and welfare. The main innovation is that market makers compete through choosing simultaneously quantities to buy at the bid and to sell at the ask and accordingly market clears separately at the bid and at the ask. Equilibrium bid and ask prices, bid and ask depths, trading volume and market makers' inventory levels are all derived in closed-form. Our model can help explain some of the puzzling empirical findings, such as bid-ask spreads can be lower with asymmetric information and can be positively correlated with trading volume. In addition, we find that information asymmetry may make informed investors worse off, may reduce the welfare loss due to market power and may increase the competition among market makers in equilibrium.

Hart(1975) proved the difficulty of deriving general comparative statics in portfolio weights. Instead, in the third essay, we derive new comparative statics for the distribution of payoffs: A is less risk averse than B iff A's payoff is always distributed as B's payoff plus a non-negative random variable plus conditional-mean-zero noise. If either agent has nonincreasing absolute risk aversion, the non-negative part can be chosen to be constant. The main result also holds in some incomplete markets with two assets or two-fund separation, and in multiple periods for a mixture of payoff distributions over time (but not at every point in time).

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Chapter 1

So What Else Will Margin Requirements Do?

"I guarantee you that if you want to get rid of the bubble, whatever it is, that [raising margin requirements] will do it. My concern is that I am not sure what else it will do."

Greenspan, Sept. 24, 1996, Fed Policy Meeting

1.1 Introduction

In the wake of the 1929 Crash, the Securities Exchange Act of 1934 gave the Federal Reserve System the authority to regulate margin requirements.[1] Since then, all investors must maintain centrally mandated minimum collateral for any short or

[1] Before 1934, each broker/firm followed its own custom in setting initial and minimum margin requirements.

leveraged long positions.[2] Proponents argued that margin requirements would reduce market volatility and make market participants better off.[3] In 2000, margin loan reached a historically high level of $278 billion (2.9% of market capitalization[4]) and stock market experienced dramatic increase in volatility. This rekindled the debate about using margin requirements as an instrument for reducing market volatility. More recently, in response to significant financial market volatility surrounding the collapse of Lehman Brothers in September 2008, many countries put short-sale restrictions (a special form of margin requirements) on some listed securities. Proponents of short-sales restrictions in 2008 cited lowering volatility as a justification for such restrictions. Clearly, more stringent margin requirements would reduce margin credit, stock trading, and default risk.[5] However, whether more stringent margin requirements would indeed reduce market volatility and under what conditions this might happen are still unknown.[6] Even less is known about what else margin requirements can do to the market. For example, how do margin requirements affect

[2]Specifically, Regulation T of the Federal Reserve Banks determines the initial margin requirement for stock positions undertaken through brokers-dealers. Currently, the initial margin requirement is 50% for a long equity position and 150% for a short equity position. For a long position, this means that an investor can only borrow up to 50% of the market value of the stock. For a short position, 102% of the short sale proceeds must typically be held in cash as noted by Geczy et al. (2002) and Duffie et al. (2002). The remaining 48% needed to cover the margin requirement can be held in other securities such as U.S. Treasury Bills.

[3]Moore (1966) summarizes the discussion that transpired in the congressional hearings on margin authority.

[4]Hardouvelis and Peristiani (1990) find that, in Japan, where data on margin trading are collected regularly, margin trading represents approximately 20 percent of trading volume despite the fact that margin accounts are, as in the United States, less than 2 percent of the capitalized value of the country's stock market. Therefore, even though the size of margin debt represents a small fraction of the market capitalization, volatility can be very sensitive to the presence of margin accounts.

[5]As noted by Hardouvelis and Theodossiou (2002), some may argue that there are many innovations in the market (e.g., futures and options), which may help circumvent the regulatory restrictions. These financial innovations are usually costly for many investors who are constrained by margin requirements since, for the purpose of buying stocks, margin loan is easier and cheaper transaction than any other type of loan.

[6]See, for example, Ferris and Chance (1988), Schwert (1989), Hsieh and Miller (1990), Hardouvelis (1990), Seguin (1990), and Kupiec (1998).

market illiquidity? What is their impact on asset prices and the welfare of the market participants?

In this paper, we propose a tractable and flexible equilibrium model with and without asymmetric information to examine the impact of margin requirements (on both long and short stock positions) on asset prices, market volatility, market illiquidity and the welfare of market participants. We show that contrary to one of the objectives of the regulators, margin requirements can significantly increase market volatility. In addition, margin requirements always increase market illiquidity (as measured by price impact) and can lead to a greater return reversal exactly when they amplify market volatility. Furthermore, margin requirements can make all market participants worse off even when they reduce market volatility. Interestingly, margin requirements always make unconstrained investors worse off and can make constrained investors better off.

More specifically, there are two types of investors, "liquidity demanders" and "liquidity suppliers," who can trade a risk-free asset and a risky asset ("stock") on dates 0 and 1 and are both subject to margin requirements. Different from liquidity suppliers, liquidity demanders are endowed with a non-traded asset (such as labor income) whose payoff is correlated with the stock payoff.[7] Therefore, liquidity demanders have extra hedging demand for trading the stock due to the non-traded asset risk. Under the assumption that stock price is equal to its conditional expected payoff, Diamond and Verrecchia (1987) show that binding short-sale constraints do not affect

[7]This is for expositional simplicity, in general, we only need that these two types of investors have heterogenous endowment shocks. Endowment shocks have been modeled as a risk-sharing motive to trade in various forms in market microstructure literature. See, for example, Grossman and Stiglitz (1980), Bhattacharya and Spiegel (1991), Wang (1994), O'Hara (2003), Bai, Chang and Wang (2006), and Vayanos and Wang (2010).

asset price conditional on the same public information because rational uninformed agents take the constraints into account. In contrast, we show that consistent with Miller (1977), Harrison and Kreps (1978), and Scheinkman and Xiong (2003), binding short margin requirements always increase stock price with and without asymmetric information.[8] The main difference from Diamond and Verrecchia (1987) is that in our model all investors are risk averse and subject to short-sale constraints, while in Diamond and Verrecchia (1987) risk neutral market makers are not subject to short-sale constraints. Our findings of the impact of short margin requirements on asset prices are strongly supported by extensive empirical evidence (e.g., Asquith, Pathak and Ritter (2005), Boehme, Danielsen and Sorescu (2005), Chen, Hong and Stein (2002), Jones and Lamont (2002) and Nagel (2005)). In addition, we show that more stringent long margin requirements decrease the price. This is confirmed by the empirical studies on the impact of long margin requirements on stock prices (e.g., Largay (1973), Eckardt and Rogoff (1976), Seguin (1990), Hardouvelis (1990), and Hardouvelis and Peristiani (1992)).

We find that when margin requirements constrain liquidity demanders, they reduce market volatility and lead to a smaller return reversal. However, when margin requirements constrain liquidity suppliers, they can significantly increase market volatility and lead to a greater return reversal. Intuitively, binding long margin requirements reduce purchases and thus drive price lower and binding short margin requirements reduce sales and thus drive price higher. Therefore, if long margin requirements bind when price is low and short margin requirements bind when price is high, then the price fluctuation is amplified and market volatility is increased. Liquidity demanders

[8]For convenience, we refer to the constraints on borrowing (short selling) implied by the margin requirements as the "long (short) margin requirement".

buy (sell) the stock when hedging demand is positive (negative) and drive the price up (down). Thus, they buy when the stock price is high and sell when the stock price is low. As the counterparty, liquidity suppliers sell when the stock price is high and buy when the stock price is low. Therefore, if liquidity suppliers are constrained by margin requirements, then market volatility is increased. Since in this case margin requirements exacerbate the price fluctuation, the liquidity demanders' trades on date 0 drive the price further away from intrinsic value, while the two coincide on date 1. Therefore, margin requirements lead to a greater return reversal exactly when they increase market volatility.

In addition, we show that with and without asymmetric information, even when liquidity demanders are constrained by margin requirements, margin requirements always increase market illiquidity as measured by the average price impact of an exogenous additional trade. Intuitively, when margin requirements bind for some investors, the unconstrained investors have to absorb the entire additional trade and thus require greater price change (in the right direction) to induce them to accommodate the extra trade.

Even when margin requirements do lower market volatility, we show that they can make all market participants worse off, which suggests that volatility may not be a good measure for welfare. More specifically, we show that binding margin requirements have an adverse price effect on unconstrained investors and have an adverse quantity effect and a favorable price effect on constrained investors. The quantity effect hurts constrained investors because they are restricted from trading the optimal amount. The price effect hurts unconstrained investors and benefits constrained investors. For example, short margin requirements reduce sales and thus increase the equilibrium price. Therefore, short margin requirements hurt unconstrained investors who are buying and benefit constrained investors who are selling. Clearly, margin requirements always make unconstrained investors worse off. Whether constrained investors are better or worse off depends on which effect dominates. If margin requirements are stringent, then the quantity effect dominates, and therefore margin requirements make constrained investors also worse off. If margin requirements are not stringent, then the price effect dominates, and margin requirements make constrained investors better off. In some sense, margin requirements are like a cartel: they protect constrained investors from competition with each other and allow constrained investors to enjoy favorable trading prices. In addition, we show that the welfare gain of the constrained never exceeds the welfare loss of the unconstrained and thus margin requirements always reduce the total welfare of the constrained and the unconstrained.[9]

We further analyze how asymmetric information and default risk affect the impact of margin requirements. All our main results still hold. Moreover, the presence of asymmetric information can reverse, magnify or reduce the impact of margin requirements on market volatility.

Our model generates some policy implications. If regulators' goal is to reduce market volatility, then they may loosen long margin requirements and tighten short margin requirements in declining markets. This would soften the downward pressure on prices because less stringent long margin requirements encourage the technical investors or more knowledgeable investors to enter the market and purchase stocks and more stringent short margin requirements discourage the short sellers from selling

[9]The change in welfare depends on the choice of welfare function. A useful canonical choice of welfare function is total surplus measured by taking the sum of certainty equivalent across agents.

more shares of the stock when the prices are low, thus smoothing the decline in prices and reducing volatility.[10] On the other hand, volatility may be reduced if we tighten long margin requirements and loosen short margin requirements in advancing markets. In addition, our model implies that market volatility can also be reduced if we set less stringent margin requirements for liquidity suppliers (e.g., market makers and hedge funds) or encourage more participation of liquidity suppliers because it would be more likely for liquidity demanders who are destabilizing investors to be constrained by margin requirements. Since tightening margin requirements always reduce market liquidity, policy makers need to balance any benefit from a lower volatility and the cost of worse liquidity. If a higher volatility is not a big concern, then relaxing margin requirements can increase market liquidity. Moreover, our paper also sheds lights on how to determine the optimal margin requirements in practice. Margin requirements should be determined by balancing the cost of restricting mutually beneficial trading and the benefit of avoiding the default cost from potential systemic risk.

Our model also generates some unique testable empirical implications. (1) Market volatility is reduced by long margin requirements that bind when the price is high and by short margin requirements that bind when the price is low; (2) Market volatility is increased by long margin requirements that bind when the price is low[11] and by short margin requirements that bind when the price is high; (3) If a stock is mainly owned by liquidity suppliers (e.g., market makers and hedge funds), then more stringent

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[10] Seguin and Jarrell (1993) examine the relative return and volume behavior of marginable and nonmarginable stocks during the October 1987 stock market crash, they find that the price declines recorded by marginable securities were less severe (returns were 0.8% greater) than those recorded by nonmarginable securities. Hardouvelis and Peristiani (1992) find that, in the Japanese stock market, following a price decline, the authorities reduce long margin requirements and subsequently prices rebound immediately.

[11] Moore (1966) finds that when the stock market has risen, margin loans are lower than if the stock market had declined. This suggests that margin requirements may actually inhibit the stabilizing influence of investors.

margin requirements tend to reduce market volatility; (4) If a stock is mainly owned by liquidity demanders (e.g., portfolio insurers and individuals), then more stringent margin requirements tend to increase market volatility. These unique implications may explain why empirical analysis of the effect of margin requirements on market volatility has been generally inconclusive. For example, a number of studies (e.g., Hardouvelis (1988, 1990), Hardouvelis and Peristinani (1989,1992)) find that margin requirements indeed reduce stock price volatility. On the other hand, other studies (e.g., Ferris and Chance (1988), Kupiec (1989), Schwert (1989), Hsieh and Miller (1990)) find either no relationship or a positive relationship between margin requirements and market volatility. If we have data on stock ownership or data on which stocks are binding at long or short margin requirements, then we can use a cross sectional regression to study the relationship between margin requirements and market volatility. Consistent with our prediction, Hardouvelis and Theodossiou (2002) find that, following large declines in stock prices, more stringent long margin requirements increase market volatility; and, following large increases in prices, more stringent long margin requirements reduce market volatility.

On the theory side, Cuoco and Liu (2000) examine the impact of margin requirements on consumption choices and the cost of hedging contingent claims in a partial equilibrium setting. Gârleanu and Pedersen (2010) derive a margin-adjusted asset pricing model where securities' required returns are characterized both by their betas and their margins. Kupiec and Sharpe (1991) examine the impact of margin requirements on stock price volatility. They numerically illustrate that imposing binding margin requirements can increase market volatility in one model and can decrease it in a different model. In contrast, in this paper we show analytically that margin requirements can increase or decrease market volatility without resorting to different models. In addition, we also provide explicit sufficient conditions under which margin requirements increase or decrease market volatility. Our paper is also related to Brunnermeier and Pedersen (2008), Rytchkokv (2009), and Huang and Wang (2010). Brunnermeier and Pedersen (2008) show that market liquidity and funding liquidity are mutually reinforcing. They emphasize the importance of availability of funding to risk neutral speculators and they show that margins can increase in price volatility when financiers who set the margin cannot distinguish between fundamental shocks and liquidity shocks. In contrast to our paper, they do not study the effect of margin requirements on market depth and they assume an exogenous autoregressive conditional heteroscedasticity of fundamental volatility. Rytchkokv (2009) studies theoretical implications of time variation in long margin requirements. In contrast to our paper, he finds that binding long margin constraints always decrease return volatility and may improve market liquidity. Huang and Wang (2010) study the impact of participation costs of liquidity suppliers on market liquidity. They show that lowering the cost of supplying liquidity (e.g., relaxation of ex post margin constraints) can lower market liquidity. Our paper is also related to the literature on borrowing and short-sale constraints. For example, Yuan (2005) studies crises and contagion in an economy with information asymmetry and borrowing constraints. Bai, Chang and Wang (2006) study the impact of short-sale constraints on asset prices and market volatility assuming only liquidity demanders are subject to short-sale constraints. In their model, short-sale constraints can bind only when prices are low and therefore the stock price volatility is always reduced with short-sale constraints in the absence of asymmetric information. In addition, neither Yuan (2005) nor Bai, Chang and

Wang (2006) has examined the effect of margin requirements on the welfare of market participants.

The remainder of the paper is organized as follows. Section 1.2 describes the basic model. Section 1.3 solves the equilibrium in the absence of asymmetric information, analyzes the effects of margin requirement on market illiquidity and the level and volatility of stock return, and conducts the welfare analysis for both types of investors. Section 1.4 solves the equilibrium in the presence of asymmetric information and analyzes the effects of margin requirements. Section 1.5 verifies our main results in the presence of default risk. Section 1.6 concludes. All proofs are in the Appendix.

1.2 The Model

In a one period setting, a continuum of investors with a total population mass of 1 can trade a risk-free asset and a risky asset ("stock") on date 0 to maximize their expected constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. There is a zero net supply for the risk-free asset and the risk-free interest rate is normalized to 0. The total supply of the stock is $\bar{\theta}$ shares and the date 1 payoff of each share is $\tilde{V} = \bar{V} + \tilde{F} + \tilde{u}$, where $\bar{V}$ is a constant representing the publicly known expected payoff, $\tilde{F}$ is a zero-mean random variable that is realized on date 0 and may be observed by some or all of the investors on date 0 and $\tilde{u}$ is an independent zero-mean random variable that no one can observe before date 1.

Every investor is endowed with $\bar{\theta}$ shares of the stock. There are two types of investors: liquidity demanders (LD) with a population mass of $\omega \in [0, 1]$, and liquidity suppliers (LS) with a population mass of $1 - \omega$. In addition to the stock, on date 0 a liquidity demander is also endowed with $\tilde{X}_{LD}$ units of a non-traded risky asset with per-unit payoff of $\tilde{M}$ on date 1. We allow $\tilde{u}$ and $\tilde{M}$ to be correlated with a covariance of σ_{uM}. A liquidity supplier does not have any endowment of the non-traded asset, i.e., $\tilde{X}_{LS} = 0$.[12] Both liquidity demanders and liquidity suppliers are subject to margin requirements when trading the stock.[13] Specifically, let θ_i $(i = LD, LS)$ be the number of shares an investor holds in the stock. Then it must satisfy[14]

$$-c_s\bar{\theta} \leq \theta_i - \bar{\theta} \leq c_b\bar{\theta}, \quad i = LD, LS \tag{1.1}$$

where $c_s \geq 1$ and $c_b \geq 0$. In other words, investors who are subject to margin requirements cannot borrow to buy more than c_b times their collateral $(\bar{\theta})$ and cannot sell more than c_s times their collateral.[15] We refer to the constraints on short selling (borrowing) implied by the margin requirements as the short (long) margin requirements and use c_b (c_s) measuring the stringency of long (short) margin requirements.

For tractability, we assume that $\tilde{F}$, $\tilde{u}$, $\tilde{X}_{LD}$ and $\tilde{M}$ are all zero-mean normally distributed random variables with variances σ_F^2, σ_u^2, σ_{LD}^2 and σ_M^2, respectively. On date 1, random variables $\tilde{u}$ and $\tilde{M}$ are realized and become publicly known.

[12] Assuming all investors have the same stock endowment is without loss of generality because what matters is the total endowment of each type. Assuming some investors do not have non-traded assets is only for expositional simplicity, because $\tilde{X}_{LD}$ can be more or less than $\tilde{X}_{LS}$ (=0).

[13] As we assume a one-period setting, we do not incorporate maintenance margin requirements in our model. Fortune (2003) has shown that maintenance margin requirements on equities rarely come into play, only in the event of extreme price declines, and are therefore of minor relevance.

[14] As shown by Cuoco and Liu (2000), the standard margin requirement in the case of one risky asset reduces to the form in (1.1). (1) is equivalent to say that margin requirements for long position is $\frac{\bar{\theta}P}{(c_b+1)\bar{\theta}P} = \frac{1}{c_b+1}$ and margin requirements for short position is $1 + \frac{\bar{\theta}P}{(c_s-1)\bar{\theta}P} = \frac{c_s}{c_s-1}$, where P is the equilibrium stock price at time 0, *i.e.*, to buy $(c_b+1)\bar{\theta}$ shares of stock at price P on margin, we need to put $\bar{\theta}P$ as collateral, to short sell $(c_s-1)\bar{\theta}$ shares of stock at price P on margin, in addition to the short-sale proceeds $(c_s-1)\bar{\theta}P$, we also need to deposit $\bar{\theta}P$ as collateral. The current margin requirement corresponds to $c_b = 1$ and $c_s = 3$ in our model.

[15] In contrast to Bai, Chang and Wang (2006), we assume all investors are subject to margin requirements. As we show later, this modeling difference reverses some of the important findings in Bai, Chang and Wang (2006).

Let P be the equilibrium price on date 0 of the stock. In the sequel, we first consider the symmetric information case where $\tilde{F}$ is observed by all investors on date 0 and then examine the asymmetric information case where only the liquidity demanders observe $\tilde{F}$ on date 0.[16] The information set of liquidity demanders on date 0 is given by $I_{LD} = \{\tilde{X}_{LD}, \tilde{F}, P\}$, and that of liquidity suppliers is given by $I_{LS} = \{\tilde{F}, P\}$ in the symmetric information case and $I_{LS} = \{P\}$ in the asymmetric information case.

For $i \in \{LD, LS\}$, investor i's problem is

$$\max_{\theta_i} E[-e^{-\delta \tilde{W}_i} | I_i], \tag{1.2}$$

subject to the budget constraint

$$\tilde{W}_i = \bar{\theta} P + \theta_i(\tilde{V} - P) + \tilde{X}_i \tilde{M}, \tag{1.3}$$

and the margin requirement (1.1), where $\delta > 0$ is the absolute risk-aversion parameter.

With the substitution of (3.2) into (3.1), the investor's problem becomes equivalent to

$$\max_{\theta_i} -e^{\delta\theta_i(P-\bar{V})+\frac{1}{2}\delta^2(\theta_i^2\sigma_u^2+\tilde{X}_i^2\sigma_M^2+2\theta_i\tilde{X}_i\sigma_{uM})} \times E[e^{-\delta\theta_i\tilde{F}} | I_i], \tag{1.4}$$

subject to the margin requirement (1.1).

In this incomplete economy, we consider the following competitive equilibrium (i.e., all investors are price takers)

Definition 1.1 *A competitive equilibrium* $(\theta_{LD}, \theta_{LS}, P)$ *is such that*

[16] The case where only the liquidity suppliers observe $\tilde{F}$ reduces to symmetric information case because their trading will fully reveal the private information.

1. θ_i ($i \in \{LD, LS\}$) solves investor i's problem (3.1)*; and*

2. both the risk-free asset market and the stock market clear.

1.3 The Equilibrium under Symmetric Information

In this section we examine the effect of margin requirements in the absence of information asymmetry. To this end, we divide this case into two subcases: with and without margin requirements.

1.3.1 Symmetric Information without Margin Requirements

Let P_{s0}^* denote the equilibrium price under symmetric information and without margin requirements. In this case, an investor's information set is $I_{LD} = I_{LS} = \{\tilde{F}, \tilde{X}_{LD}, P_{s0}^*\}$.[17] Therefore, for $i \in \{LD, LS\}$, investor i's objective function is equivalent to

$$\max_{\theta_i} -\delta\theta_i(P_{s0}^* - \bar{V} - \tilde{F}) - \frac{1}{2}\delta^2(\theta_i^2\sigma_u^2 + \tilde{X}_i^2\sigma_M^2 + 2\theta_i\tilde{X}_i\sigma_{uM}). \tag{1.5}$$

The following proposition provides the equilibrium price and equilibrium stock holdings.[18]

[17] Even though liquidity suppliers do not know the liquidity demanders' endowment of the non-traded asset, they can infer it from the equilibrium price and $\tilde{F}$.

[18] The proofs of all the analytical results in the text are relegated to the Appendix.

Proposition 1.1 *In the absence of asymmetric information and margin requirements, the date 0 equilibrium price of the stock is*

$$P_{s0}^* = \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D}, \tag{1.6}$$

and the equilibrium stock holdings are

$$\theta_{LD_s}^* = \bar{\theta} + (1-\omega)\tilde{D}, \quad \theta_{LS_s}^* = \bar{\theta} - \omega\tilde{D},$$

where $\tilde{D} = -\frac{\sigma_{uM}\tilde{X}_{LD}}{\sigma_u^2}$ *represents liquidity demanders' hedging demand, i.e.,* $\tilde{D}$ *is the optimal number of shares a liquidity demander wants to buy on margin or sell to hedge their risk from the non-traded asset and* $\frac{\sigma_{uM}}{\sigma_u^2}$ *is the conditional (on* $\tilde{F}$ *and* $\tilde{X}_{LD}$*) beta of the non-traded asset payoff (*$\tilde{M}$*) with respect to the stock payoff (*$\tilde{V}$*).*

Proposition 1.1 implies that the equilibrium price increases with the expected payoff ($\bar{V} + \tilde{F}$), decreases with the volatility of the payoff and the supply of the stock. In addition, liquidity demanders' hedging demand also impacts the equilibrium price. In particular, suppose liquidity demanders have a positive endowment of the non-traded asset, i.e., $\tilde{X}_{LD} > 0$. If the stock is negatively correlated with the non-traded asset payoff, i.e., $\sigma_{uM} < 0$, then liquidity demanders have positive hedging demand ($\tilde{D} > 0$) and they are willing to buy at a higher price to induce liquidity suppliers to sell more so that liquidity demanders can hedge their risk from the non-traded asset and thus the equilibrium price gets higher than the case without non-traded asset ($\tilde{X}_{LD} = 0$). As it is well known in the literature, equilibrium price decreases with risk aversion, because the demand for the stock decreases as risk aversion increases. Interestingly, the equilibrium price can increase with the risk aversion in our model. This is because for liquidity demanders the risk from the non-traded asset may dominate the risk from the stock and thus they may be willing to buy more shares of the stock to hedge the non-traded asset risk as they become more risk averse.

We study market illiquidity using the price impact of some exogenous additional trade (Kyle's lambda). Suppose there is some extra exogenous trade ε, the market clearing condition becomes

$$\omega\theta^*_{LD_s} + (1-\omega)\theta^*_{LS_s} + \varepsilon = \bar{\theta}. \tag{1.7}$$

The equilibrium price of the stock becomes

$$P^*_{s0} = \tilde{F} + \bar{V} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D} + \delta\sigma_u^2\varepsilon, \tag{1.8}$$

and the price impact without margin requirements is

$$\lambda_s = \frac{\partial P^*_{s0}}{\partial \varepsilon} = \delta\sigma_u^2. \tag{1.9}$$

This implies that illiquidity increases in agents' risk aversion and stock payoff volatility.

1.3.2 Symmetric Information with Margin Requirements

Since both groups are subject to margin requirements, if the population weight of liquidity demanders is very small, then the maximum number of shares that liquidity demanders as a group can buy (sell) is always less than the maximum number of shares that the liquidity suppliers as a group can sell (buy). For example, suppose that there are 50 liquidity demanders and 200 liquidity suppliers, and each of them is endowed with 1 share of the stock. The current margin requirements imply that $c_b = 1$ and $c_s = 3$. Liquidity demanders as a group can buy on margin 50 shares and can sell 150 shares, while liquidity suppliers as a group can sell 600 shares and can buy on margin 200 shares. Therefore, when the population weight of liquidity demanders is very small, it is only possible for margin requirements to bind for liquidity demanders. In general, depending on the population weight of liquidity demanders, there are four different cases in equilibrium: margin requirements bind only for liquidity demanders, margin requirements bind only for liquidity suppliers, only long margin requirements can bind, and only short margin requirements can bind. Let P_s^* denote the equilibrium price under symmetric information and with margin requirements. Solving for the equilibrium subject to the margin requirement (1.1), we have[19]

Proposition 1.2 *1. if $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements can never bind for liquidity suppliers but can bind for liquidity demanders, and the equilibrium stock price is*

$$P_s^* = \begin{cases} \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta} & \tilde{D} \geq \frac{c_b}{1-\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D} & -\frac{c_s}{1-\omega}\bar{\theta} < \tilde{D} < \frac{c_b}{1-\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} - \delta\omega\frac{c_s}{1-\omega}\sigma_u^2\bar{\theta} & \tilde{D} \leq -\frac{c_s}{1-\omega}\bar{\theta}; \end{cases}$$

2. if $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements never bind for liquidity demanders but can bind for liquidity suppliers, and the equilibrium stock price is

$$P_s^* = \begin{cases} \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\sigma_u^2\tilde{D} - \delta(1-\omega)\frac{c_s}{\omega}\sigma_u^2\bar{\theta} & \tilde{D} \geq \frac{c_s}{\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D} & -\frac{c_b}{\omega}\bar{\theta} < \tilde{D} < \frac{c_s}{\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\sigma_u^2\tilde{D} + \delta(1-\omega)\frac{c_b}{\omega}\sigma_u^2\bar{\theta} & \tilde{D} \leq -\frac{c_b}{\omega}\bar{\theta}; \end{cases}$$

[19] If $\omega = \frac{c_b}{c_b+c_s}$ or $\omega = \frac{c_s}{c_b+c_s}$ then one type of agents is binding in long margin requirements while the other type of agents is binding in short margin requirements, the equilibrium price is indeterminate. We assume that these two equalities are not true.

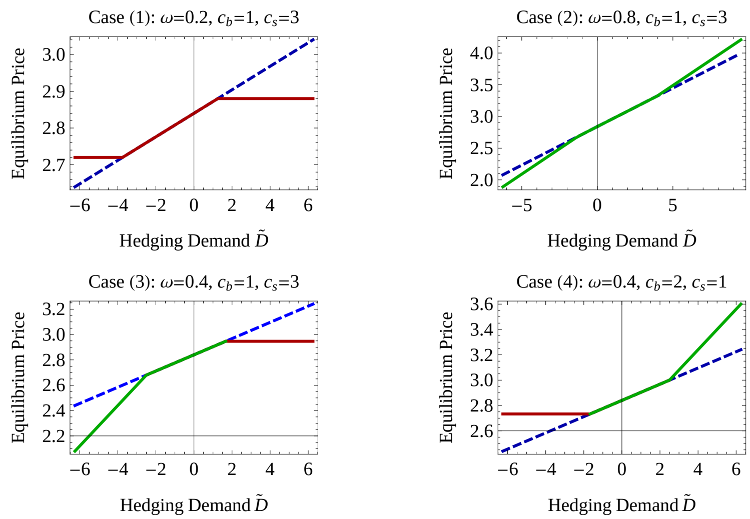


Figure 1.1: Equilibrium Prices with and without Margin Requirements

The straight (resp. dashed) line denotes the equilibrium stock price with (resp. without) margin requirements. The parameter values are $\bar{V} = 3, \bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.4, \delta = 1, \tilde{F} = 0$.

3. if $\frac{c_b}{c_b+c_s} < \omega < \frac{c_s}{c_b+c_s}$, then only long margin requirements can bind, and the equilibrium stock price is

$$P_s^* = \begin{cases} \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta} & \tilde{D} \geq \frac{c_b}{1-\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D} & -\frac{c_b}{\omega}\bar{\theta} < \tilde{D} < \frac{c_b}{1-\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\sigma_u^2\tilde{D} + \delta(1-\omega)\frac{c_b}{\omega}\sigma_u^2\bar{\theta} & \tilde{D} \leq -\frac{c_b}{\omega}\bar{\theta}; \end{cases}$$

4. if $\frac{c_s}{c_b+c_s} < \omega < \frac{c_b}{c_b+c_s}$, then only short margin requirements can bind, and the equilibrium stock price is

$$P_s^* = \begin{cases} \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\sigma_u^2\tilde{D} - \delta(1-\omega)\frac{c_s}{\omega}\sigma_u^2\bar{\theta} & \tilde{D} \geq \frac{c_s}{\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} + \delta\omega\sigma_u^2\tilde{D} & -\frac{c_s}{1-\omega}\bar{\theta} < \tilde{D} < \frac{c_s}{\omega}\bar{\theta}, \\ \bar{V} + \tilde{F} - \delta\sigma_u^2\bar{\theta} - \omega\frac{c_s}{1-\omega}\delta\sigma_u^2\bar{\theta} & \tilde{D} \leq -\frac{c_s}{1-\omega}\bar{\theta}. \end{cases}$$

Corollary 1.1 *1. If long margin requirements are binding, then $P_{s0}^* \geq P_s^*$;*

2. If short margin requirements are binding, then $P^*_{s0} \leq P^*_s$;

3. If neither constraints are binding, then $P^*_{s0} = P^*_s$.

To help understand the results in Proposition 1.2, we define $\eta^i_{LS} = (1-\omega)c_i\bar{\theta}$ and $\eta^i_{LD} = \omega c_i \bar{\theta}$, for $i \in \{b, s\}$. Then by (1.1), η^b_{LS} represents the maximum number of shares that liquidity suppliers as a group can buy-on-margin, whereas η^s_{LD} is the maximum number of shares that the liquidity demanders as a group can sell. In equilibrium the total number of shares bought on margin must be equal to the total number of shares sold. So if $\eta^b_{LS} > \eta^s_{LD}$, then the long margin requirements never bind for liquidity suppliers. Similarly if $\eta^s_{LS} > \eta^b_{LD}$, then the short margin requirements never bind for liquidity suppliers. In this way, the comparisons of η^i_{LS} and η^j_{LD}, where $i, j \in \{b, s\}$ and $i \neq j$, yield the four cases in this proposition.

The equilibrium prices reflect that when margin requirements are not binding then the prices stay the same as the case without margin requirements. If long margin requirements bind for an investor, then the equilibrium price decreases because the demand is reduced. If short margin requirements bind for an investor, then the equilibrium price increases because the sales are reduced. This finding implies that in contrast to Diamond and Verrecchia (1987), imposing only short-sale constraints would increase the expected equilibrium stock price. Diamond and Verrecchia (1987) assume that all agents are risk neutral and the price is set to the conditional expectation of the payoff. Therefore, binding short-sale constraints do not affect asset price conditional on the same public information because rational uninformed agents take the constraints into account. In contrast, we show that consistent with Miller (1977), Harrison and Kreps (1978), and Scheinkman and Xiong (2003), binding short margin requirements always increase stock price. The main difference from Diamond and Verrecchia (1987) is that in our model all investors are risk averse and subject to short-sale constraints, while in Diamond and Verrecchia (1987) risk neutral market makers are not subject to short-sale constraints.

Figure 1.1 illustrates this comparison of the equilibrium stock prices with and without margin requirements for Cases *1-4* in Proposition 1.2. In Case *1*, the population weight of liquidity demanders is small and thus margin requirements can bind only for liquidity demanders. More specifically, when hedging demand $\tilde{D} \leq -3.75$, the short margin requirements bind for liquidity demanders and when hedging demand $\tilde{D} \geq 1.25$, the long margin requirements bind for liquidity demanders. In Case *2*, the population weight of liquidity suppliers is small and thus margin requirements can bind only for liquidity suppliers. More specifically, when hedging demand $\tilde{D} \leq -1.25$, the long margin requirements bind for liquidity suppliers and when hedging demand $\tilde{D} \geq 3.75$, the short margin requirements bind for liquidity suppliers. In Case *3* (Case *4*), long (short) margin requirements are more stringent than short (long) margin requirements and therefore only long (short) margin requirements can bind for either liquidity demanders or liquidity suppliers.

As we can see from Figure 1.1, any binding long margin requirements always reduce the equilibrium stock price and any binding short margin requirements always increase the equilibrium stock price. More importantly, margin requirements bind for liquidity demanders and liquidity suppliers at different prices. Specifically, short margin requirements bind for liquidity demanders when the equilibrium price is low but they bind for liquidity suppliers when the equilibrium price is high. Conversely, long margin requirements bind for liquidity demanders when the equilibrium price is high but they bind for liquidity suppliers when the equilibrium price is low. Therefore, in our model margin requirements can bind both when stock price is high and when it is low, depending on the hedging demand and the relative wealth of investors. This finding is consistent with empirical evidence and is the critical driving force of the result that margin requirements can increase or decrease return volatility, as we will show in the next section. In contrast, short-sale constraints as modeled in Bai et. al. (2006) can only bind when stock price is low.

1.3.3 The Impact of Margin Requirements on the Expected Price, Return Volatility and Market Illiquidity under Symmetric Information

Since conditional on $\tilde{F}$ and $\tilde{X}_{LD}$, binding long margin requirements always decrease stock price and binding short margin requirements always increase stock price, it is interesting to investigate ex-ante (i.e., before the realizations of $\tilde{F}$ and $\tilde{X}_{LD}$) whether margin requirements increase or decrease the expected equilibrium price. To this extent, we have the following result:

Proposition 1.3 *The expected date 0 stock price with and without margin requirements have the following relation:*

$$E[P_s^*] \leq E[P_{s0}^*] \text{ if and only if } c_b \leq c_s.$$

Corollary 1.2 $E[P_s^*]$ *increases in* c_b *and decreases in* c_s*.*

Because long and short margin requirements have opposite effects on the equilibrium stock price, whether margin requirements increase or decrease the stock price depends on which constraint has a greater effect. By (1.1), if $c_b = c_s$, then it is equally likely for long and short margin requirements to be binding. Therefore, the effects of long and short margin requirements cancel out in expectation and the expected stock price remains the same with and without margin requirements. When $c_b < (>) c_s$, the long (short) margin requirements are more likely to be binding and therefore margin requirements tend to decrease (increase) the equilibrium stock price.

The result of Corollary 1.2 is also driven by the opposite effects of long and short margin requirements. The result that $E[P_s^*]$ increases in c_b is consistent with earlier empirical studies concentrated on the effect of long margin requirements on the level of the market prices. For example, Largay (1973), Eckardt and Rogoff (1976), Hardouvelis (1990), and Hardouvelis and Peristiani (1992) find that less stringent long margin requirements (larger c_b) tend to increase stock prices. The result that $E[P_s^*]$ decreases in c_s is consistent with most empirical studies which focus on the impact of short-sale constraints on stock prices. For example, Asquith, Pathak and Ritter (2005), Boehme, Danielsen and Sorescu (2005), Chen, Hong and Stein (2002), Jones and Lamont (2002) and Nagel (2005) find that more stringent short-sale constraints (smaller c_s) tend to increase stock prices.

Next we examine the effects of margin requirements on the volatility of stock returns. In our set-up, we measure stock returns by the price differences. Then we have:

Proposition 1.4 *1. If $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements decrease the volatility of stock returns on both date 0 and date 1, i.e.,*

$$Var[P_s^* - P_{s,-1}^*] < Var[P_{s0}^* - P_{s0,-1}^*] \quad and \quad Var[\tilde{V} - P_s^*] < Var[\tilde{V} - P_{s0}^*], \quad (1.10)$$

which implies that margin requirements lead to a smaller return reversal (less negative stock return auto-covariance), i.e.,

$$Cov(P_{s0}^* - P_{s0,-1}^*, \tilde{V} - P_{s0}^*) < Cov(P_s^* - P_{s,-1}^*, \tilde{V} - P_s^*) < 0, \qquad (1.11)$$

where $P_{s,-1}^*$ and $P_{s0,-1}^*$ denote the equilibrium stock prices with and without margin requirements before date 0, which are constants.[20]

2. *If $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements increase the volatility of stock returns on both date 0 and date 1, which implies that margin requirements lead to a greater return reversal (more negative stock return auto-covariance).*[21]

3. *If $\min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\} < \omega < \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements can increase or decrease market volatility.*

As illustrated in Figure 1.1, in Case *1*, margin requirements can only bind for liquidity demanders. From this Figure, we can see that, from left to right, $P_s^* - E[P_s^* - P_{s0}^*]$ crosses P_{s0}^* only once[22] and from above. This implies that $P_s^* - E[P_s^* - P_{s0}^*]$ second-order stochastically dominates P_{s0}^* and therefore $Var[P_s^* - P_{s,-1}^*] < Var[P_{s0}^* - P_{s0,-1}^*]$. More specifically, when liquidity demanders have positive hedging demand, to hedge their non-traded asset risk, liquidity demanders want to buy more shares of the stock. The positive hedging demand drives up the stock price. When the positive hedging

[20] More specifically, assuming agents are identical on date -1, and a fraction ω of agents receive $\tilde{X}_{LD}$ units of non-traded risky asset on date 0, we solved the equilibrium prices on date -1 with and without margin requirements in closed-form. We find that the equilibrium price on date -1 decreases in the volatility of the amount of non-traded asset σ_{LD}^2, risk aversion δ, the absolute value of the covariance between the payoff of stock and non-traded risky asset $|\sigma_{uM}|$, the volatility of stock payoff σ_u, the volatility of the non-traded risky asset payoff σ_M, and the proportion of liquidity demanders ω.

[21] The auto-covariance is negative because the liquidity demanders' trades on date 0 drive the price away from the stock's intrinsic value (the equilibrium price when $\tilde{X}_{LD} = 0$), while the two coincide on date 1.

[22] If $E[P_s^*] = E[P_{s0}^*]$, then P_s^* cross P_{s0}^* over a line from above, we can pick any point on the line.

demand is very large, long margin requirements bind for liquidity demanders when stock price is high. On the other hand, if liquidity demanders have negative hedging demand, and they want to sell more shares of the stock to hedge their risk. The negative hedging demand drives down the stock price. When the negative hedging demand is very large, short margin requirements bind for liquidity demanders when stock price is low. Therefore, in this Scenario, the long margin requirements bind and thus reduce price when the stock price is high and the short margin requirements bind and thus increase price when the stock price is low. Margin requirements reduce the overall fluctuation of the stock price and thus decrease market volatilities.

Conversely, in Case *2*, margin requirements can only bind for liquidity suppliers. From this Figure, we can see that, from left to right, $P_s^* - E[P_s^* - P_{s0}^*]$ crosses P_{s0}^* only once and from below. This implies that $P_s^* - E[P_s^* - P_{s0}^*]$ is second-order stochastically dominated by P_{s0}^* and therefore $Var[P_s^* - P_{s,-1}^*] > Var[P_{s0}^* - P_{s0,-1}^*]$. More specifically, liquidity demanders' positive hedging demand drives up the stock price, and when liquidity demanders buy a lot to hedge their risk, the short margin requirements can bind for liquidity suppliers (who are selling) when the stock price is high. On the other hand, the negative hedging demand drives down the stock price, and when liquidity demanders sell a lot to hedge their risk, the long margin requirements can bind for liquidity suppliers (who are buying) when the price is low. Therefore, in this Scenario, the long margin requirements bind and thus reduce price when the stock price is low and the short margin requirements bind and thus increase price when the stock price is high. Contrary to one of the stated regulatory goals, margin requirements can *exacerbate* the overall fluctuation of the stock price and thus increase market volatilities.

In Case *3*, either the long margin requirements or the short margin requirements bind both when stock price is low and when stock price is high (for different investors). Therefore margin requirements have opposite effects on price volatility depending on whether they bind in the high price region or in the low price region. Thus the net effect of margin requirements in this case depends on the distribution of the equilibrium price which is in turn determined by the distribution of the hedging demand.

In our model, liquidity demanders are destabilizing traders since they are buying when price is high and they are selling when price is low. Therefore, when margin requirements bind for liquidity demanders, market volatility is reduced. Liquidity suppliers in our model are like those technical investors or knowledgeable investors who take positions that stabilize the market. They may buy on margin when they think the prices are too low and they may short-sell when they think the prices are too high. When they are constrained by margin requirements, market volatility is increased.

In contrast, Bai, Chang and Wang (2006) show that short-sale constraints can bind only when prices are low and the stock price volatility is always reduced with short-sale constraints in the absence of asymmetric information. The main reason for this difference is that we assume both liquidity demanders and liquidity suppliers are subject to short margin requirements, while they assume only liquidity demanders are subject to short-sale constraints. As we have shown that short margin requirements have opposite impact on market volatility when they are binding for different type of investors. Kupiec and Sharpe (1991) numerically illustrate that imposing binding margin requirements can increase market volatility in one model and can decrease it in a different model. In contrast, in this paper we show analytically that margin requirements can increase or decrease market volatility without resorting to different models. In addition, we derive sufficient conditions under which margin requirements increase or decrease market volatility. Specifically, if $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements decrease market volatility. If $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements increase market volatility.

Our finding may explain why empirical studies which focus on the impact of margin requirements on market volatility have been generally inconclusive (e.g., Ferris and Chance (1988), Kupiec (1989,1998), Schwert (1989), Hsieh and Miller (1990), Hardouvelis (1988, 1990)). In addition, our model generates some unique empirically testable implications. For example, if we have data on which stocks are binding on short margin requirements and we divide stocks into two groups: one with short margin requirements more likely binding at high prices and the other more likely binding at low prices, then the volatilities of the stock returns in the first group are increased while those in the second are decreased. Consistent with our prediction, Hardouvelis and Theodossiou (2002) find that, following large declines in stock prices, more stringent long margin requirements increase market volatility; and, following large increases in prices, more stringent long margin requirements reduce market volatility. Alternatively, our model predicts that, if a stock is mainly owned by liquidity suppliers (e.g., market makers and hedge funds), then more stringent margin requirements tend to reduce market volatility; and if a stock is mainly owned by liquidity demanders (e.g., portfolio insurers and individuals), then more stringent margin requirements tend to increase market volatility.

If regulators' goal is to reduce market volatility, then they may loosen long margin requirements and tighten short margin requirements in declining markets. This would soften the downward pressure on prices because less stringent long margin requirements encourage the technical investors or more knowledgable investors to enter the market and purchase stocks and more stringent short margin requirements discourage the short sellers to sell more shares of the stock when the prices are low, thus smoothing the decline in prices and reducing volatility. On the other hand, volatility may be reduced if we tighten long margin requirements and loosen short margin requirements in advancing markets. This would encourage short sellers to sell more shares of the stock when the prices are high and discourage investors to buy more shares of the stock, thus dampening the upward pressure on prices and reducing volatility. In addition, our model implies that market volatility can also be reduced if we set less stringent margin requirements for liquidity suppliers (e.g., market makers and hedge funds) or encourage more participants of liquidity suppliers because it would be more likely for liquidity demanders who are destabilizing investors to be constrained by margin requirements.

We now examine how margin requirements affect market liquidity using the price impact of some exogenous additional trade on date 0. We define Kyle's lambda with margin requirements, λ_s^m as the average price impact of per unit of the additional trade, ε. We have

Proposition 1.5 *1. Price impact with margin requirements* $\lambda_s^m =$

$$\begin{cases} \frac{\delta\sigma_u^2}{1-\omega}\Big(1+\omega N(-\frac{c_s}{1-\omega}\bar{\theta}+\frac{\varepsilon}{1-\omega})-\omega N(\frac{c_b}{1-\omega}\bar{\theta}+\frac{\varepsilon}{1-\omega})\Big) & \omega<\min\{\frac{c_b}{c_b+c_s},\frac{c_s}{c_b+c_s}\}, \\ \frac{\delta\sigma_u^2}{\omega}\big(1+(1-\omega)N(-\frac{c_b}{\omega}\bar{\theta}-\frac{\varepsilon}{\omega})-(1-\omega)N(\frac{c_s}{\omega}\bar{\theta}-\frac{\varepsilon}{\omega})\big) & \omega>\max\{\frac{c_b}{c_b+c_s},\frac{c_s}{c_b+c_s}\}, \\ \frac{\delta\sigma_u^2}{\omega(1-\omega)}\Big(\omega-\omega^2 N(\frac{c_b}{1-\omega}\bar{\theta}+\frac{\varepsilon}{1-\omega})+(1-\omega)^2 N(-\frac{c_b}{\omega}\bar{\theta}-\frac{\varepsilon}{\omega})\Big) & \frac{c_b}{c_b+c_s}<\omega<\frac{c_s}{c_b+c_s}, \\ \frac{\delta\sigma_u^2}{\omega(1-\omega)}\Big(1-\omega+\omega^2 N(-\frac{c_s}{1-\omega}\bar{\theta}+\frac{\varepsilon}{1-\omega})-(1-\omega)^2 N(\frac{c_s}{\omega}\bar{\theta}-\frac{\varepsilon}{\omega})\Big) & \frac{c_s}{c_b+c_s}<\omega<\frac{c_b}{c_b+c_s}; \end{cases}$$

2. $\lambda_s^m > \lambda_s$;

3. λ_s^m *decreases in* c_s *and* c_b.

Proposition 1.5 implies that margin requirements always increase market illiquidity measured by price impact of per unit of some extra exogenous trade. Intuitively, when some investors are constrained by margin requirements, the unconstrained investors need to absorb all the extra net demand. Therefore, unconstrained investors require greater price change (in the right direction) to induce them to accommodate the extra trade. In addition, Proposition 1.5 also implies that market liquidity decreases when margin requirements become more stringent (smaller c_b and c_s). Since tightening margin requirements always reduce market liquidity, policy makers need to balance any benefit from a possible lower volatility (when margin requirements constrain liquidity demanders) and cost of worse liquidity.

1.3.4 Welfare Analysis under Symmetric Information

In this subsection, we analyze how margin requirements affect market participants' welfare by comparing the expected utilities investors achieve in equilibrium with and without margin requirements. We first define the "stringency" of margin requirements as below:

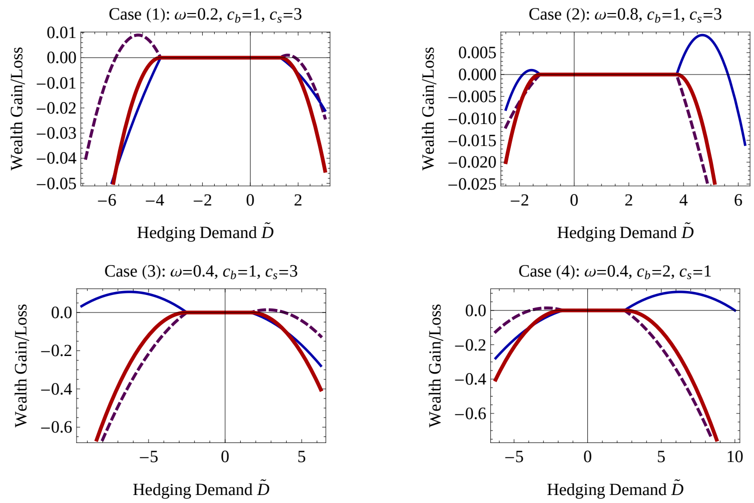


Figure 1.2: Certainty Equivalent Wealth Gain/Loss with Margin Requirements

The dashed (resp. thin solid) curve denotes the certainty equivalent wealth gain/loss with margin requirements of liquidity demanders (resp. liquidity suppliers) and the thick solid curve denotes the change in total surplus with margin requirements. The parameter values are $\bar{V} = 3, \bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.4, \delta = 1, \tilde{F} = 0$.

Definition 1.2 *For any given hedging demand $\tilde{D}$, margin requirements are stringent for liquidity demanders if*

$$c_b < \frac{(1-w)^2\tilde{D}}{(1+w)\bar{\theta}} \text{ or } c_s < -\frac{(1-w)^2\tilde{D}}{(1+w)\bar{\theta}};$$

margin requirements are stringent for liquidity suppliers if

$$c_b < -\frac{\omega^2\tilde{D}}{(2-\omega)\bar{\theta}} \text{ or } c_s < \frac{\omega^2\tilde{D}}{(2-w)\bar{\theta}}.$$

Proposition 1.6 *1. Binding margin requirements always hurt the unconstrained investors;*

2. Stringent margin requirements hurt the constrained investors;

3. Binding but not stringent margin requirements benefit constrained investors.

Binding margin requirements have a price effect on the unconstrained investors and have two opposite effects on the constrained investors, a quantity effect and a price effect. The quantity effect hurts the constrained investors because they are restricted from trading the optimal amount. The price effect hurts unconstrained investors and benefits constrained investors. This is because margin requirements always move the price in favor of the constrained investors. For example, short margin requirements reduce sales and thus increase the equilibrium price. Therefore they hurt unconstrained investors who are buying and benefit constrained investors who are selling. Thus, somewhat surprisingly, binding margin requirements always make *unconstrained* investors worse off and may make the *constrained* investors better off. Whether constrained investors are better or worse off depends on which effect dominates. If margin requirements are stringent, then the quantity effect dominates, and therefore constrained investors are also worse off. If margin requirements are not stringent, then the price effect may dominate, and thus constrained investors may be better off. In some sense, margin requirements are like a cartel: constrained investors are protected from competition with each other and thus enjoy better trading prices.

Figure 1.2 illustrates the certainty equivalent wealth gain/loss with margin requirements as a function of hedging demand $\tilde{D}$ for liquidity demanders (dashed curve) and liquidity suppliers (thin solid curve) for Cases *1-4* in Proposition 1.2.[23] As we can see from the graph, in Case *1*, liquidity suppliers are never better off. When hedging demand is large (for $\tilde{D} > 1.9$, or $\tilde{D} < -5.6$), liquidity demanders are also worse off. This

[23]The certainty equivalent wealth is defined to be the extra initial wealth required for an investor to be indifferent between facing margin requirements or not.

figure also shows that liquidity demanders can be better off with margin requirements if and only if the hedging demand is neither too small nor too large. Intuitively, if the hedging demand is too small, then margin requirements do not bind. As the hedging demand increases to a certain level, margin requirements start to bind. At this point the price effect dominates and therefore liquidity demanders are better off. As we discussed before, when margin requirements reduce too much hedging, the quantity effect dominates and therefore liquidity demanders become worse off. In contrast, in Case *2*, liquidity demanders are never better off. When hedging demand is large (for $\tilde{D} > 5.6$, or $\tilde{D} < -1.9$), liquidity suppliers are also worse off. Liquidity suppliers are better off if $\tilde{D}$ is neither too small nor too large.

Theorem 1 *Under symmetric information, imposition of margin requirements is Pareto-dominated by some lump-sum transfer from unconstrained investors to constrained investors.*

Corollary 1.3 *Under symmetric information, any binding margin requirements reduce market participants' total welfare measured using certainty equivalents.*

Intuitively, when margin requirements are binding, the marginal rates of substitution differ across investors in equilibrium, which implies that the equilibrium is Pareto suboptimal. Therefore there exists a redistribution of wealth that Pareto-dominates the imposition of margin requirements. In the case when constrained investors are better off with margin requirements, this implies that the welfare gain (measured by certainty equivalent wealth increase) of the constrained investors is insufficient to offset the welfare loss of the unconstrained investors. In this sense, the imposition of margin requirements is always socially suboptimal. The thick solid curve in Figure 1.2 illustrates the change of total welfare with margin requirements measured using certainty equivalents. As we can see from the graph, any binding margin requirements always reduce the total welfare.

1.4 The Equilibrium under Asymmetric Information

We now consider the impact of information asymmetry on the above analysis. Specifically, we assume that $\tilde{F}$ and $\tilde{X}_{LD}$ are private information to the liquidity demanders who trade for both risk-sharing and private information.[24] We first look at the benchmark case, the equilibrium without margin requirements.

1.4.1 Asymmetric Information without Margin Requirements

To understand how equilibrium price may depend on the state variables, it is helpful to first derive the optimal demand of the liquidity demanders which is the same as in the symmetric information case,

$$\theta^*_{LD_a} = \frac{\bar{V} + \tilde{F} - \delta\sigma_{uM}\tilde{X}_{LD} - P^*_{a0}}{\delta\sigma^2_u}. \tag{1.12}$$

Other market participants can only observe $\theta^*_{LD_a}$ and accordingly the equilibrium price can only depend on $\tilde{S} \equiv \frac{1}{2}\left(\tilde{F} - \delta\sigma_{uM}\tilde{X}_{LD}\right)$. Thus we conjecture that there exists an equilibrium where the equilibrium stock price depends on $\tilde{F}$ and $\tilde{X}_{LD}$ only through $\tilde{S}$.

[24]The case where the liquidity suppliers are informed is reducible to the case with symmetric information because the liquidity demanders can fully back out the private information from the market price.

Let P^*_{a0} denote the equilibrium price with asymmetric information but without margin requirements. The information sets for liquidity demanders and liquidity suppliers are respectively $I_{LD} = \{\tilde{F}, \tilde{X}_{LD}, P^*_{a0}\}$ and $I_{LS} = \{P^*_{a0}\}$. We have

Proposition 1.7 *With asymmetric information but without margin requirements, there exists an equilibrium where the equilibrium price is*

$$P^*_{a0} = \bar{V} + A_2\tilde{S} - B_2\bar{\theta}, \tag{1.13}$$

The liquidity demanders and liquidity suppliers' optimal stock demands are

$$\theta^*_{LD_a} = \bar{\theta} + (1-\omega)a_1(\tilde{S} + \frac{1}{2}\delta\sigma_F^2\bar{\theta}), \ \ \theta^*_{LS_a} = \bar{\theta} - \omega a_1(\tilde{S} + \frac{1}{2}\delta\sigma_F^2\bar{\theta}), \tag{1.14}$$

where $A_2 > 0$, $B_2 > 0$, $a_1 > 0$ and $b_1 > 1$ are constants defined in (1.36)-(1.37) in the Appendix.

Since $A_2 > 0$ and $B_2 > 0$, the equilibrium price increases with the combined demand of liquidity demanders and decreases with the stock supply. In addition, while liquidity demanders' optimal stock holding increases with $\tilde{S}$, the liquidity suppliers' demand decreases with $\tilde{S}$ because of the increase in the equilibrium price.

As in the symmetric information case, we study market illiquidity using the price impact of some exogenous additional trade (Kyle's lambda). Suppose there is some extra exogenous trade ε, the market clearing condition becomes

$$\omega\theta^*_{LD_a} + (1-\omega)\theta^*_{LS_a} + \varepsilon = \bar{\theta}. \tag{1.15}$$

The market lambda without margin requirements in the presence of asymmetric information,

$$\lambda_a = \frac{P^*_{a0}}{\partial \varepsilon} = \delta\sigma_u^2 \left(1 + \frac{1-\omega}{\omega + \left(\frac{1}{\delta^2\sigma_{uM}^2\sigma_{LD}^2} + \frac{1}{\sigma_F^2}\right)\sigma_u^2}\right). \tag{1.16}$$

This implies that λ_a increases in σ_u^2, σ_F^2, σ_{uM}, σ_{LD}, and δ, and decreases in ω.

Next we examine the effect of information asymmetry on the expected equilibrium price. We have:

Proposition 1.8 *Asymmetric information reduces the expected equilibrium stock price, i.e.,* $E[P^*_{a0}] \leq E[P^*_{s0}]$.

The presence of asymmetric information decreases the expected stock price because investors require a higher risk premium for trading the stock.

Letting $\xi = \frac{\tilde{D}}{\tilde{F}+\delta\sigma_F^2\bar{\theta}}$, *i.e.*, ξ is the ratio between liquidity demanders' trading due to hedging demand to the trading due to private information. We have:

Proposition 1.9 *In the absence of margin requirements, liquidity demanders are worse off in the presence of asymmetric information if*

$$\xi > C_1 \text{ or } \xi < -C_2, \tag{1.17}$$

where $C_1 > 0$ and $C_2 > 0$ are constants and defined in (1.39) in Appendix.[25]

[25] Clearly, liquidity suppliers can also be worse off in the presence of asymmetric information in certain states.

To understand the intuition behind Proposition 1.9, we consider those states when $\tilde{F} > -\delta\sigma_F^2\bar{\theta}$. From Propositions 1.1 and 1.7, for any $\tilde{F}$, if $\xi > C_1$, *i.e.*, the hedging demand is very large, then the equilibrium stock price is higher with asymmetric information. This is because, observing a large combined demand, liquidity suppliers rationally attribute some of the trade due to hedging demand to the trade due to private information and therefore they are willing to sell the stock only at a higher price, *i.e.*, $P_{a0}^* > P_{s0}^*$. Consequently, liquidity demanders will optimally buy less shares of the stock due to the higher stock price and they are worse off with asymmetric information. Conversely, if $\xi < -C_2$, then liquidity demanders are worse off with asymmetric information because they sell less shares than in the case with symmetric information due to the lower stock price.

Proposition 1.10 *Asymmetric information increases the variance of the stock return on date 1 and the total variance of the stock returns on date 0 and 1, i.e.,*

$$Var[\tilde{V} - P_{a0}^*] > Var[\tilde{V} - P_{s0}^*],$$

$$Var[P_{a0}^* - P_{a0,-1}^*] + Var[\tilde{V} - P_{a0}^*] > Var[P_{s0}^* - P_{s0,-1}^*] + Var[\tilde{V} - P_{s0}^*],$$

which implies that asymmetric information leads to a greater return reversal (more negative stock return auto-covariance), i.e., $Cov(P_{a0}^* - P_{a0,-1}^*, \tilde{V} - P_{a0}^*) < Cov(P_{s0}^* - P_{s0,-1}^*, \tilde{V} - P_{s0}^*) < 0$, *where* $P_{a0,-1}^*$ *and* $P_{s0,-1}^*$ *denote the equilibrium stock prices with and without asymmetric information before date 0, which are constants.*

Because liquidity suppliers can only observe $\tilde{S}$ in the presence of asymmetric information, they rationally attribute some hedging motivated trades to information motivated ones and vise versa. Therefore, comparing to the case with symmetric information, for a given shock in $\tilde{F}$, the equilibrium return on date 0, $(P^*_{a0} - P^*_{a0,-1})$ changes less, and for a given shock in $\tilde{X}_{LD}$, the equilibrium return on date 0 changes more. This implies that the date 0 return is less sensitive to $\tilde{F}$ and more sensitive to $\tilde{X}_{LD}$ and therefore the date 0 return volatility may increase or decrease with asymmetric information, depending on which effect dominates. Since the date 1 stock price is equal to $\tilde{V} = \bar{V} + \tilde{F} + \tilde{u}$, the date 1 return $\tilde{V} - P^*_{a0}$ is more sensitive to both $\tilde{F}$ and $\tilde{X}_{LD}$ and therefore the date 1 return volatility always increases with asymmetric information. The total variance also increases with asymmetric information because the volatility reduction due to the decreased sensitivity of date 0 return to $\tilde{F}$ is dominated by the sum of the volatility increase due to the increased sensitivity of date 0 and date 1 returns to $\tilde{X}_{LD}$ and the increased sensitivity of date 1 return to $\tilde{F}$. This is similar to the main result of Wang (1993), which uses a dynamic asset-pricing model under asymmetric information and also finds that information asymmetry among investors can increase price volatility and negative auto-covariance in returns.

1.4.2 Asymmetric Information with Margin Requirements

In this subsection, we examine the effect of margin requirements in the presence of asymmetric information. In the presence of margin requirements, it is easy to show that a liquidity demander's optimal stock demand is:

$$\theta^*_{LD_a} = \min\left\{\max\left[\frac{\bar{V} + 2\tilde{S} - P^*_a}{\delta\sigma_u^2}, -(c_s - 1)\bar{\theta}\right], (c_b + 1)\bar{\theta}\right\}, \tag{1.18}$$

and a liquidity supplier's optimal stock demand is:

$$\theta^*_{LS_a} = \min\left\{\max\left[\frac{\bar{V} + E[\tilde{F}|\tilde{S}] - P^*_a}{\delta\left(\sigma_u^2 + Var[\tilde{F}|\tilde{S}]\right)}, -(c_s - 1)\bar{\theta}\right], (c_b + 1)\bar{\theta}\right\}, \tag{1.19}$$

where P_a^* denotes the equilibrium stock price with asymmetric information and margin requirements. When liquidity demanders are constrained, there may be multiple equilibria, in this section, we focus on one type of equilibrium,[26] under which the equilibrium stock price is an affine function of $\tilde{S}$, as in the case without margin requirements.[27]

We summarize this equilibrium in the following proposition:

Proposition 1.11 *In the presence of asymmetric information, there are four cases:*

(1) if $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements can never bind for liquidity suppliers but can bind for liquidity demanders, and the equilibrium stock price is

$$P_a^* = \begin{cases} \bar{V} + A_1\tilde{S} - B_1\bar{\theta} + \frac{\omega c_b}{1-\omega}B_1\bar{\theta} & \tilde{S} \geq S_{LD1}^*, \\ \bar{V} + A_2\tilde{S} - B_2\bar{\theta} & S_{LD2}^* < \tilde{S} < S_{LD1}^*, \\ \bar{V} + A_1\tilde{S} - B_1\bar{\theta} - \frac{\omega c_s}{1-\omega}B_1\bar{\theta} & \tilde{S} \leq S_{LD2}^*; \end{cases}$$

(2) if $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements can never bind for liquidity demanders but can bind for liquidity suppliers, and the equilibrium stock price is

$$P_a^* = \begin{cases} \bar{V} + 2\tilde{S} - \delta\sigma_u^2\bar{\theta} - \frac{(1-\omega)c_s}{\omega}\delta\sigma_u^2\bar{\theta} & \tilde{S} \geq S_{LS1}^*, \\ \bar{V} + A_2\tilde{S} - B_2\bar{\theta} & S_{LS2}^* < \tilde{S} < S_{LS1}^*, \\ \bar{V} + 2\tilde{S} - \delta\sigma_u^2\bar{\theta} + \frac{(1-\omega)c_b}{\omega}\delta\sigma_u^2\bar{\theta} & \tilde{S} \leq S_{LS2}^*; \end{cases}$$

[26] There may exist another type of equilibrium, under which the equilibrium stock price is an affine function of $\tilde{S}$ in certain region and independent on $\tilde{S}$ beyond that region as in Bai, Chang and Wang (2006). There seems to be no closed-form solution in this case, but all the numerical results are consistent with the main results of the equilibrium we focus on in this section. Detailed analysis of this case is available from the author.

[27] Implicitly we assume that there is an auctioneer to whom all investors submit their optimal orders disregarding the existence of margin requirements and then the their optimal demand is censored by the margin requirements.

(3) if $\frac{c_b}{c_b+c_s} < \omega < \frac{c_s}{c_b+c_s}$, then only long margin requirements can bind, and the equilibrium stock price is

$$P_a^* = \begin{cases} \bar{V} + A_1\tilde{S} - B_1\bar{\theta} + \frac{\omega c_b}{1-\omega} B_1\bar{\theta} & \tilde{S} \geq S_{LD1}^*, \\ \bar{V} + A_2\tilde{S} - B_2\bar{\theta} & S_{LS2}^* < \tilde{S} < S_{LD1}^*, \\ \bar{V} + 2\tilde{S} - \delta\sigma_u^2\bar{\theta} + \frac{(1-\omega)c_b}{\omega}\delta\sigma_u^2\bar{\theta} & \tilde{S} \leq S_{LS2}^*; \end{cases}$$

(4) if $\frac{c_s}{c_b+c_s} < \omega < \frac{c_b}{c_b+c_s}$, then only short margin requirements can bind, and the equilibrium stock price is

$$P_a^* = \begin{cases} \bar{V} + A_1\tilde{S} - B_1\bar{\theta} - \frac{\omega c_s}{1-\omega} B_1\bar{\theta} & \tilde{S} \leq S_{LD2}^*, \\ \bar{V} + A_2\tilde{S} - B_2\bar{\theta} & S_{LD2}^* < \tilde{S} < S_{LS1}^*, \\ \bar{V} + 2\tilde{S} - \delta\sigma_u^2\bar{\theta} - \frac{(1-\omega)c_s}{\omega}\delta\sigma_u^2\bar{\theta} & \tilde{S} \geq S_{LS1}^*. \end{cases}$$

where S_{LD1}^*, S_{LD2}^*, S_{LS1}^*, S_{LS2}^*, A_1, B_1, A_2, and B_2 are constants defined in (1.48)-(1.52) and (1.36) in Appendix.

S_{LD1}^* (resp. S_{LD2}^*) is the critical point at which the long (resp. short) margin requirements start to bind for liquidity demanders. Similarly, S_{LS1}^* (resp. S_{LS2}^*) is the critical point at which the short (resp. long) margin requirements start to bind for liquidity suppliers. As in Proposition 1.2, the comparisons of η_{LS}^i and η_{LD}^j, where $i, j \in \{b, s\}$ and $i \neq j$, yield the four cases in this proposition. Since $A_1 < A_2 < 1$, Proposition 1.11 suggests that even though the equilibrium price is always informative about the state variable $\tilde{S}$, it becomes less sensitive when margin requirements bind for liquidity demanders and more sensitive when margin requirements bind for liquidity suppliers. This proposition also implies that, as in the symmetric information case, long margin requirements tend to decrease the equilibrium price and short margin requirements tend to increase it. Figure 1.3 illustrates the comparison of the

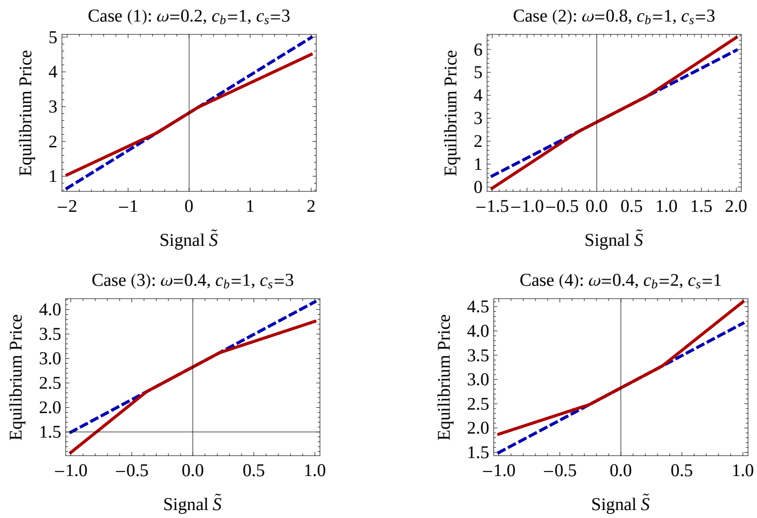


Figure 1.3: Equilibrium Prices with and without Margin Requirements in the presence of Asymmetric Information

The solid (resp. dashed) line denotes the equilibrium stock price with (resp. without) margin requirements. The parameter values are $\bar{V} = 3, \bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.4, \delta = 1, \sigma_{LD} = 0.6, \sigma_F = 0.2, \sigma_{uM} = 0.4$.

equilibrium stock prices with and without margin requirements for Cases (1)-(4) in Proposition 1.11. For example, in Case (1), when $\tilde{S} \leq -0.5$, the short margin requirements bind for liquidity demanders and thus the equilibrium price is higher with margin requirements. On the other hand, if $\tilde{S} \geq 0.15$, then long margin requirements bind for liquidity demanders and thus the equilibrium price is lower. Figure 1.3 shows that, as in the symmetric information case, short margin requirements bind for liquidity demanders when the equilibrium price is low and bind for liquidity suppliers when the equilibrium price is high. Similarly, long margin requirements bind for liquidity demanders when the equilibrium price is high and bind for liquidity suppliers when the equilibrium price is low.

1.4.3 The Impact of Margin Requirements on the Expected Stock Price, Return Volatility and Market Illiquidity

We now examine the impact of margin requirements on the initial expected stock price under asymmetric information. We have the following results:

Proposition 1.12 *Under asymmetric information, the expected date 0 stock price with and without margin requirements have the following relation:*

$$E[P_a^*] < E[P_{a0}^*] \text{ if and only if } c_b < c_s + d,$$

where $d > 0$ *when* $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, $d < 0$ *when* $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, *and* $d = 0$ *when* $\frac{c_b}{c_b+c_s} < \omega < \frac{c_s}{c_b+c_s}$, *and* d *is defined in (1.54) in the Appendix.*

Corollary 1.4 $E[P_a^*]$ *increases in* c_b *and decreases in* c_s*.*

The basic intuition of Proposition 1.12 is the same as that of Proposition 1.3 in symmetric information case. The two propositions differ in the thresholds for $c_b - c_s$ in Part (1). In symmetric information case $d = 0$ and we combine the three cases. From margin requirements (1.1), when $c_b = c_s + d$, it is equally likely for the long and short margin requirements to be binding, and therefore the equilibrium prices with and without margin requirements are the same, where $d > 0$ for the case when margin requirements can only bind for liquidity demanders and $d < 0$ for the case when margin requirements can only bind for liquidity suppliers. The intuition is as follows. From Proposition 1.7, the expected net trade for liquidity demanders, $E[\theta^*_{LD_a}] - \bar{\theta} > 0$, and the expected net trade for liquidity suppliers, $E[\theta^*_{LS_a}] - \bar{\theta} < 0$.

Liquidity suppliers require a higher risk premium to hold the stock. Therefore, ex-ante, in the presence of asymmetric information, liquidity demanders are buying and liquidity suppliers are selling.

Next we examine whether margin requirements reduce or increase the volatility of stock return in the presence of asymmetric information. We have

Proposition 1.13 *1. If margin requirements can only bind for liquidity demanders, then margin requirements reduce the volatility of the initial equilibrium stock return, i.e.,* $Var[P_a^* - P_{a,-1}^*] < Var[P_{a0}^* - P_{a0,-1}^*]$, *where* $P_{a,-1}^*$ *and* $P_{a0,-1}^*$ *denote the equilibrium stock prices with and without margin requirements before date 0, which are constants. In addition, if* $A_1 > 1$,[28] *then margin requirements also reduce the total variance of stock returns on dates 0 and 1, i.e.,*

$$Var[P_a^* - P_{a,-1}^*] + Var[\tilde{V} - P_a^*] < Var[P_{a0}^* - P_{a0,-1}^*] + Var[\tilde{V} - P_{a0}^*],$$

which implies that margin requirements lead to a smaller return reversal (less negative stock return auto-correlation),i.e., $Cov(P_{a0}^* - P_{a0,-1}^*, \tilde{V} - P_{a0}^*) < Cov(P_a^* - P_{a,-1}^*, \tilde{V} - P_a^*) < 0$;

2. *If margin requirements can only bind for liquidity suppliers, then margin requirements increase the volatility of the date 0 stock return, in addition, if* $A_2 > 1$, *then margin requirements also increase the total variance of stock returns on dates 0 and 1;*

3. *If margin requirements can bind for both liquidity demanders and liquidity suppliers, then margin requirements can increase or decrease market volatility.*

[28] Although we are not able to prove that the total variance is also reduced even when $A_1 \leq 1$, all our numerical results suggest that this is likely to be true.

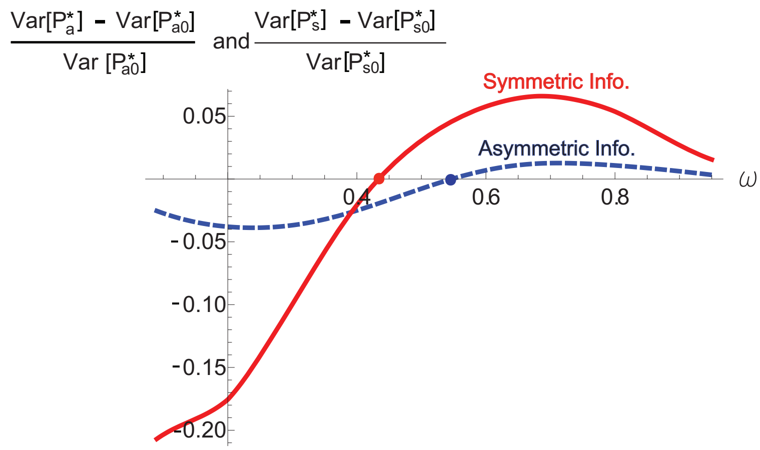


Figure 1.4: The Percentage Volatility Changes with Margin Requirements with and without Asymmetric Information

The dashed (resp. solid) curve denotes the percentage volatility changes with margin requirements with (resp. without) asymmetric information. The parameter values are $\bar{V} = 3, \bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.4, \delta = 1, \sigma_{LD} = 0.6, \sigma_F = 0.2, \sigma_{uM} = 0.4, c_s = 3, c_b = 1$.

Proposition 1.13 shows that the presence of asymmetric information does not change the conclusion that margin requirements may increase or decrease market volatility. More specifically, market volatility increases when short margin requirements bind at high prices or when long margin requirements bind at low prices. Conversely, market volatility decreases when short margin requirements bind at low prices or when long margin requirements bind at high prices. However, margin requirements may have a smaller impact on market return volatility in the presence of asymmetric information. Figure 1.4 illustrates the difference of the percentage volatility changes due to margin requirements with and without asymmetric information. The dashed curve denotes the percentage volatility change $((Var[P_a^*] - Var[P_{a0}^*])/Var[P_{a0}^*])$ due to margin requirements with asymmetric information, and the solid curve denotes the percentage volatility change $((Var[P_s^*] - Var[P_{s0}^*])/Var[P_{s0}^*])$ due to margin requirements with symmetric information. As we can see from the graph, in the example, in the case when margin requirements increase market volatility with and without asymmetric information (for $0.55 < \omega < 1$), market volatility increase less in the presence of asymmetric information. Similarly, in the case when margin requirements decrease market volatility with and without asymmetric information (for $0 < \omega < 0.39$), market volatility decreases less in the presence of asymmetric information. Intuitively, as the degree of information asymmetry measured by σ_F increases, the adverse selection problem becomes more severe and liquidity demanders may buy less or sell less due to the price pressure. For example, suppose liquidity demanders optimally buy shares on margin, if σ_F is large, then liquidity suppliers optimally attribute more of the combined demand to the private information about stock payoff and thus liquidity suppliers are only willing to sell their shares at a higher price. Liquidity demanders optimally buy less on margin because of this adverse price impact. Therefore, margin requirements may become less likely to bind and have a smaller impact on market volatility with asymmetric information.

Interestingly, margin requirements may have opposite effect on market volatility with and without asymmetric information. In addition, margin requirements may also have a larger impact on volatility in the presence of asymmetric information. In our example with $c_b = 1$ and $c_s = 3$, when $0.43 < \omega < 0.55$, margin requirements decrease market volatility with asymmetric information while they increase market volatility with symmetric information. For $0.39 < \omega < 0.43$, margin requirements decrease more of the market volatility with asymmetric information. The intuition is as following. With $c_b < c_s$, only long margin requirements can bind for either liquidity suppliers or liquidity demanders. Ex-ante, liquidity demanders are buyers since liquidity suppliers require a higher risk premium for holding the stock. Therefore, in the presence of asymmetric information, the same long margin requirements bind for liquidity demanders more often than they bind for liquidity suppliers comparing to the case with symmetric information case. Since long margin requirements reduce market volatility when they bind for liquidity demanders and increase market volatility when they bind for liquidity suppliers, long margin requirements may decrease market volatility with asymmetric information while they increase market volatility with symmetric information, or margin requirements decrease more of the market volatility with asymmetric information.

We now examine how margin requirements affect market illiquidity in the presence of asymmetric information using the average price impact (λ_a^m) of some additional exogenous trade on date 0. We have

Proposition 1.14 *1. Price impact with margin requirements $\lambda_a^m > \lambda_a$;*

2. λ_a^m decreases in c_s and c_b.

Proposition 1.14 implies that the presence of asymmetric information does not change our main results: margin requirements always increase market illiquidity measured by price impact of some extra exogenous trade and market illiquidity increases when margin requirements become more stringent (smaller c_b and c_s). The intuition is the same as the symmetric information case.

1.4.4 Welfare Analysis under Asymmetric Information

In this subsection, we examine the impact of margin requirements on market participants' welfare in the presence of asymmetric information. We also compare the welfare impact of margin requirements with and without asymmetric information. We first define the "stringency" of margin requirements with asymmetric information as below:

Definition 1.3 *Under asymmetric information, given $\tilde{S}$, margin requirements are stringent for investors i ($i = LD, LS$) when long margin requirements $c_b < b_i(\tilde{S}) - 1$, or short margin requirements $c_s < 1 - b_i(\tilde{S})$, where $b_i(\tilde{S})$, $i = LD, LS$, is defined (1.63) in Appendix.*

Proposition 1.15 *In the presence of asymmetric information,*

(1) stringent margin requirements make all market participants worse off;

(2) binding but not stringent margin requirements make constrained investors better off but unconstrained investors worse off.

As in the symmetric information case, the following theorem shows that even when constrained investors are better off with margin requirement, margin requirements are always socially suboptimal.

Theorem 2 *In the presence of asymmetric information, any binding margin requirements are Pareto-dominated by some lump-sum transfer from unconstrained investors to constrained investors.*

Corollary 1.5 *In the presence of asymmetric information, any binding margin requirements reduce total welfare measured using certainty equivalents.*

The basic intuition of Proposition 1.15 and Theorem 2 is the same as that in the case with symmetric information. However, the presence of asymmetric information has a significant impact on this welfare analysis. Due to asymmetric information, informed agents also trade to explore their private information. Therefore the presence of asymmetric information can magnify the negative welfare impact of margin requirements on the agents when the trade due to private information $\tilde{F}$ is more significant than the trade due to the hedging demand. The darker shade area in Figure 1.5 illustrates those states when the same margin requirements reduce more of the total surplus in the presence of asymmetric information for the case when margin requirements can only bind for liquidity demanders,[29] *i.e.*, $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$. The lighter shaded area in Figure 1.5 illustrates those states when margin requirements are not binding in either symmetric or asymmetric information case. Clearly, we can see that, in the presence of asymmetric information, the same margin requirements reduce more of the total welfare surplus in certain states when $\tilde{F}$ is far away from its ex ante mean for a given realization of $\tilde{X}_{LD}$. For example, if $\tilde{X}_{LD} = 1$, then when $\tilde{F} > 0.7$ or $\tilde{F} < -0.7$, margin requirements reduce more of the total welfare surplus in the presence of asymmetric information.

1.5 Margin Requirements with Limited Liability

In this section, we check our main results in the presence of limited liability, *i.e.*, investors can default if the wealth on the margin account is negative.[30] Our main

[29] The graphes comparing the welfare impact of margin requirements with and without asymmetric information for Cases (2)-(4) in Proposition 1.11 are similar to Case (1), we skip them to save space and they are available from the author.

[30] For simplicity, we assume that investors' non-traded assets are protected if they default.

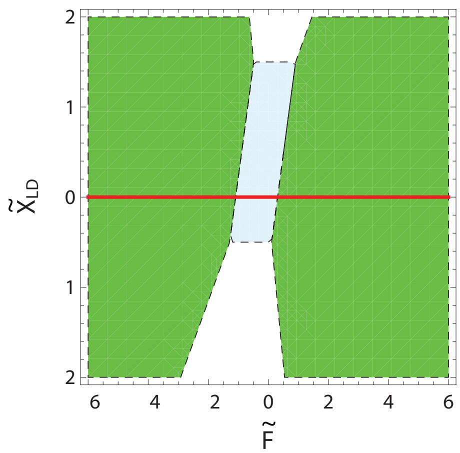


Figure 1.5: The Comparison of the Total Certainty Equivalent Wealth Gain/Loss with Margin Requirements with and without Asymmetric Information

The darker shaded area illustrates those states when the same margin requirements reduce more of the total surplus in the presence of asymmetric information. The parameter values are $\bar{V} = 3, \bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.4, \delta = 1, \sigma_{LD} = 0.6, \sigma_F = 0.2, \sigma_{uM} = 0.4, c_s = 3, c_b = 1, \omega = 0.2$.

results on the impact of margin requirements on market volatility and equilibrium stock prices still hold. Furthermore, the systemic cost of default could provide a justification for having margin regulation.

Specifically, when liquidity demanders buy and liquidity suppliers sell some of their endowed shares of the stock, liquidity demanders' problem is:

$$\max_{\theta_{LD} \leq c_b \bar{\theta}} E[-e^{-\delta \tilde{W}_{LD}}]$$

$$s.t. \quad \tilde{W}_{LD} = \Big((\bar{\theta} - \theta_{LD})P + \theta_{LD}\tilde{V}\Big)^+ + \tilde{X}_{LD}\tilde{M}, \tag{1.20}$$

and liquidity suppliers' problem is:

$$\max_{\theta_{LS}} E[-e^{-\delta \tilde{W}_{LS}}]$$

$$s.t.\quad \tilde{W}_{LS} = (\bar{\theta} - \theta_{LS})P + \theta_{LS}\tilde{V} - \frac{\omega}{1-\omega}\left((\bar{\theta} - \theta_{LD})P + \theta_{LD}\tilde{V}\right)^{-}. \tag{1.21}$$

To incorporate limited liability in our model, we assume that the payoff of the risky asset $\tilde{V} = e^{\tilde{v}}$ and the payoff of the non-traded asset $\tilde{M} = e^{\tilde{m}} - e^{\mu_m + \frac{1}{2}\sigma_m^2}$,[31] where $\tilde{v}$ and $\tilde{m}$ follow a bivariate normal distribution, *i.e.*, the joint *p.d.f.* of $\tilde{v}$ and $\tilde{m}$ is:

$$f(\tilde{v}, \tilde{m}) = \frac{1}{2\pi\sigma_v\sigma_m\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(\tilde{v}-\mu_v)^2}{\sigma_v^2} + \frac{(\tilde{m}-\mu_m)^2}{\sigma_m^2} - \frac{2\rho(\tilde{v}-\mu_v)(\tilde{m}-\mu_m)}{\sigma_v\sigma_m}\right)}, \tag{1.22}$$

where $\rho = \frac{\sigma_{vm}}{\sigma_v\sigma_m}$ is the correlation coefficient between $\tilde{v}$ and $\tilde{m}$.

In equilibrium, both stock market and bond market clear,

$$\omega\theta_{LD}^{*} + (1-\omega)\theta_{LS}^{*} = \bar{\theta}. \tag{1.23}$$

From (1.20), (1.21) and (1.23), we can numerically solve the equilibrium stock prices with different level of long margin requirements c_b. We find that the stricter margin requirement may significantly reduce investors' default probability. As we can see from Figure 1.6, with one share of stock as collateral in the margin account, if borrowers are allowed to borrow to buy 2 shares of the stock, then the borrower's default probability is 14%. If buyers are only allowed to borrow to buy 1 share of the stock, then the default probability is reduced to 3%.

Consistent with our previous results, the equilibrium stock price is lower with the stricter long margin requirement.[32] In Figure 1.7, long margin requirements bind for

[31] We demean the payoff of the non-traded asset, so that the expected payoff of the non-traded asset is zero and the average endowment shock is zero ex-ante.

[32] We use the equilibrium with less strict long margin requirements as the benchmark in this section while we use the equilibrium without margin requirements as the benchmark in the previous sections.

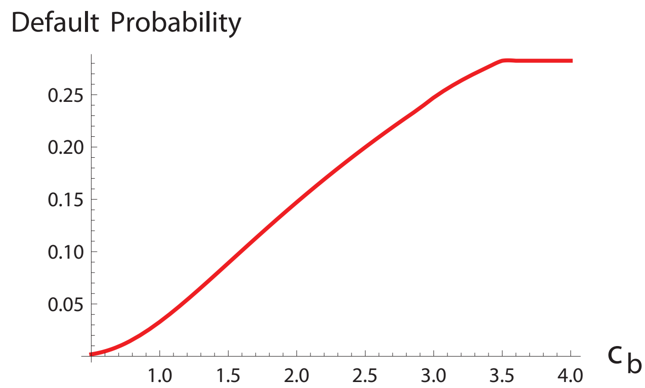


Figure 1.6: The Borrower's Default Probability Against the Level of Long Margin Requirements

The parameter values are $\bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.6, \sigma_{uM} = 0.2, \delta = 1, \omega = 0.2, \mu_M = 0, \mu_V = 0, \tilde{X}_{LD} = -1$.

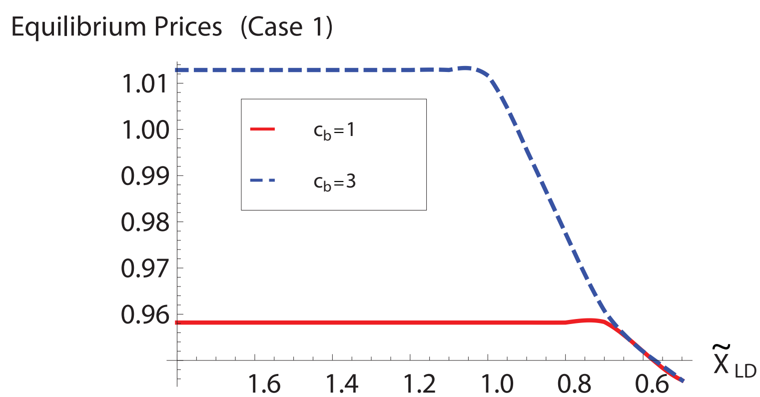


Figure 1.7: Equilibrium Prices with Long Margin Requirements $c_b = 1$ and $c_b = 3$ (Case 1)

The dashed (resp. solid) curve denotes the equilibrium price with long margin requirements $c_b = 3$ (resp. $c_b = 1$). The parameter values are $\bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.6, \sigma_{uM} = 0.2, \delta = 1, \omega = 0.2, \mu_M = 0, \mu_V = 0$.

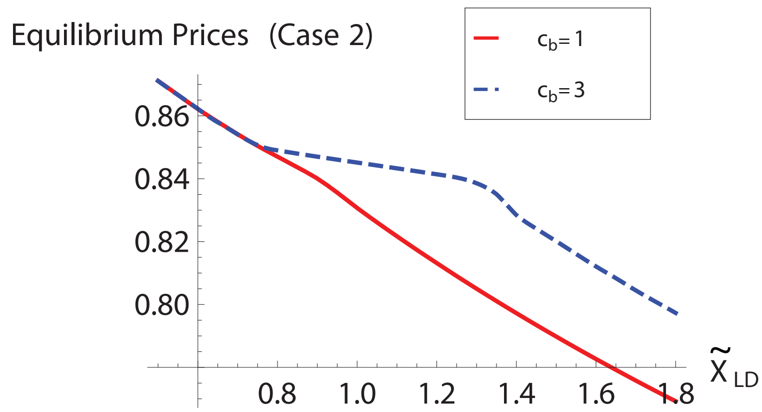


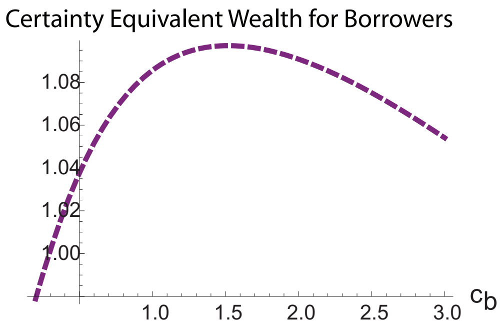
Figure 1.8: Equilibrium Prices with Long Margin Requirements $c_b = 1$ and $c_b = 3$ (Case 2)

The dashed (resp. solid) curve denotes the equilibrium price with long margin requirements $c_b = 3$ (resp. $c_b = 1$). The parameter values are $\bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.6, \sigma_{uM} = 0.2, \delta = 1, \omega = 0.8, \mu_M = 0, \mu_V = 0$.

liquidity demanders and the long margin requirements bind when equilibrium stock price is relatively high. Therefore the long margin requirements reduce volatility. In Figure 1.8,[33] long margin requirements bind for liquidity suppliers and the long margin requirements bind when equilibrium stock price is relatively low. Therefore the long margin requirements increase volatility.

Interestingly, in the presence of limited liability, as we can see from Figure 1.9, there is an optimal long margin requirement for margin buyers and sellers respectively. In our example, $c_b = 1.52$ is optimal for margin buyers and $c_b = 1.04$ is optimal for sellers. As we discussed in the welfare analysis in Section 3.4, long margin requirements have two effects, a quantity effect and a price effect on margin buyers. If long margin requirements are not very stringent, then margin buyers can be better off because the price effect dominates the quantity effect. Therefore, in the presence of limited

[33] This is the case when liquidity demanders sell some of their endowed shares of the stock and liquidity suppliers buy.



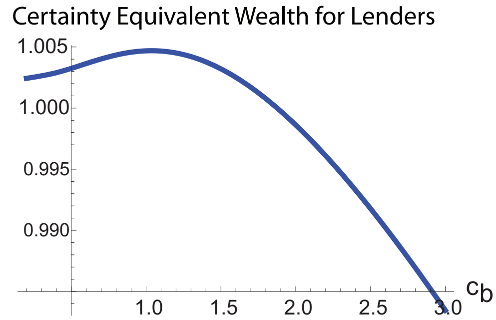


Figure 1.9: The Certainty Equivalent Wealth of Borrowers and Lenders Against Long Margin Requirements

The parameter values are $\bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.6, \sigma_{uM} = 0.2, \delta = 1, \omega = 0.2, \mu_M = 0, \mu_V = 0, \tilde{X}_{LD} = -1$.

liability, the optimal margin requirements for margin buyers are determined by balancing the price effect, the quantity effect and default benefit. On the other hand, long margin requirements have an adverse price effect on the sellers because sellers have to sell at a lower price. However, in the presence of limited liability, long margin requirements protect sellers who are also lenders from borrowers' default. Therefore, the optimal margin requirements for sellers are determined by balancing the price effect and buyers' default risk.

In addition, regulators solve an optimal margin requirement which maximizes social welfare by incorporating some systemic cost of default.[34] We use a simplest reduced-form model to capture the systemic cost of default. More specifically, we assume that

[34]Schwarcz (2008): systemic risk results from a type of tragedy of the commons in which market participants lack sufficient incentive, absent regulation, to limit risk taking in order to reduce the systemic danger to others. For example, default in paying debts might well cause the institution's failure, as well as trigger a potential chain of defaults as other institutions are not paid amounts owned them (and in turn, if highly leveraged, such other institutions might then be unable to pay amounts owed to other institutions.) These costs would likely be high because they include not only direct economic costs but also indirect social costs (poverty, unemployment and crime as potential social costs).

regulator's problem is

$$\max_{c_b} \quad \omega E[-e^{-\delta \tilde{W}_{LD}}]+(1-\omega)E[-e^{-\delta \tilde{W}_{LS}}]+E\left[c\omega\left(\theta_{LD}\tilde{V}+(\bar{\theta}-\theta_{LD})P-z(\theta_{LD}-\bar{\theta})P\right)^{-}\right]. \tag{1.24}$$

The sum of the first two terms in (3.39) is the weighted average utility of buyers and sellers. The third term in (3.39) captures the externality of financial crisis and c measures the severity of the externality imposed the the economy when borrowers as a group are in distress.[35] We define that borrowers are in financial distress when all borrowers' total wealth is below a fraction of their total debt,[36] *i.e.*, $\theta_{LD}\tilde{V}+(\bar{\theta}-\theta_{LD})P < z(\theta_{LD}-\bar{\theta})P$, and each borrower's systemic expected shortfall as the amount that the wealth drops below a fraction z of the debt in case of a systemic crisis, *i.e.*, $E\left[\left(\theta_{LD}\tilde{V}+(\bar{\theta}-\theta_{LD})P-z(\theta_{LD}-\bar{\theta})P\right)^{-}\right]$.

The optimal margin requirement for the regulator is more stringent than the sellers would prefer, as illustrated in Figure 1.10, the optimal margin requirement for the regulator in our example is $c_b = 0.95$. Therefore, this systemic cost of default could provide a justification for having margin regulation.

The results with default shed some lights on how to determine the optimal margin requirements in practice. Margin requirements should be determined by balancing two facts: the cost of restricting mutually beneficial trading and the benefit of avoiding the default cost from potential systemic risk. If it is unlikely that defaulting on margin can lead to systemic risk, then the damage may be greater than the potential

[35] This is similar to the reduced-form model of measuring systemic risk used in Acharya, Pedersen, Philippon and Richardson (2010).

[36] Though in our model all borrowers are identical and there is one risky asset, and thus all borrowers default at the same time, our simplified model captures the economics without bearing complicated numerical computations since borrowers who buy on margin on multiple risky assets may default at the same time when they face a systematic shock.

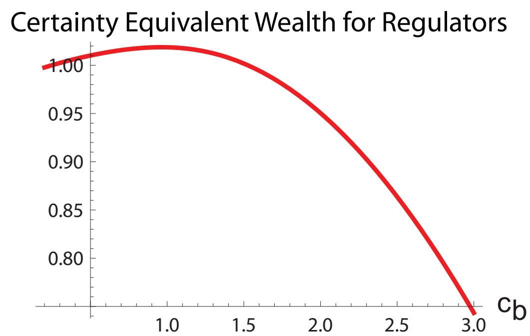


Figure 1.10: The Certainty Equivalent Wealth of Regulators Against Long Margin Requirements

The parameter values are $\bar{\theta} = 1, \sigma_u = 0.4, \sigma_M = 0.6, \sigma_{uM} = 0.2, \delta = 1, \omega = 0.2, \mu_M = 0, \mu_V = 0, \tilde{X}_{LD} = -1$.

benefit from margin regulation. It might be optimal to leave the private sectors to determine their own margin requirements. On the other hand, as we experienced in financial crisis in 1929, private sectors' margin requirements are extremely loose and that encourage margin buying and might have caused the following market crash. It might be social optimal for the regulators to set margin requirements.

1.6 Concluding Remarks

We propose a tractable and flexible equilibrium model with and without asymmetric information to examine the impact of margin requirements (on both long and short stock positions) on asset prices, market volatility, market illiquidity and the welfare of market participants. We show that margin requirements reduce market volatility and lead to a smaller return reversal when they constrain liquidity demanders (e.g., portfolio insurers and individuals) who are buying when prices are high or selling when prices are low. However, margin requirements can significantly increase market volatility and lead to a greater return reversal when they constrain liquidity suppliers (e.g., market markers and hedge funds) who are selling when prices are high or buying when prices are low.

In addition, margin requirements always increase market illiquidity measured by price impact even when they constrain liquidity demanders. Moreover, stringent margin requirements make both participants worse off and less stringent margin requirements benefit constrained investors and hurt unconstrained investors. Our main results remain the same in the presence of asymmetric information and default risk while the presence of asymmetric information may reverse or reduce the impact of margin requirements on market volatility.

1.7 Appendix

We provide all the proofs for the case when margin requirements never bind for liquidity suppliers and can only bind for liquidity demanders $i.e., \omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, the proofs for the case when $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$ are very similar and we skip them here to save space and they are available from the author.

Proof of Proposition 1.1:

From investor i's objective function (1.5), it is straightforward to get i's optimal demand for the risky asset:

$$\theta_i^* = \frac{(\tilde{F} + \bar{V} - P_{s0}^*) - \delta\sigma_{uM}\tilde{X}_i}{\delta\sigma_u^2}, \quad i = LD, LS. \tag{1.25}$$

Substituting (1.25) into the market clearing condition, $\omega\theta^*_{LD_s} + (1-\omega)\theta^*_{LS_s} = \bar{\theta}$, we get that the equilibrium price is

$$P^*_{s0} = \bar{V} + \tilde{F} - \delta\omega\sigma_{uM}\tilde{X}_{LD} - \delta\sigma_u^2\bar{\theta}. \tag{1.26}$$

Plugging (1.26) into (1.25), we get $\theta^*_{LD_s} = \bar{\theta} - (1-\omega)\frac{\sigma_{uM}}{\sigma_u^2}\tilde{X}_{LD}$, and $\theta^*_{LS_s} = \bar{\theta} + \omega\frac{\sigma_{uM}}{\sigma_u^2}\tilde{X}_{LD}$.

Q.E.D.

Proof of Proposition 1.2 and Corollary 1.1:

Liquidity demanders' optimal demand is given by (1.25), if

$$-(c_s-1)\bar{\theta}\delta\sigma_u^2 < \bar{V} + \tilde{F} - P^*_s - \delta\sigma_{uM}\tilde{X}_{LD} < (c_b+1)\bar{\theta}\delta\sigma_u^2, \tag{1.27}$$

then neither long nor short margin requirements are binding, the equilibrium price is the same as that without margin requirement, *i.e.*, $P^*_s = P^*_{s0}$. Plugging (1.26) into (1.27), we get that when $-\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta} < \sigma_{uM}\tilde{X}_{LD} < \frac{c_s}{1-\omega}\sigma_u^2\bar{\theta}$, neither long or short margin requirements are binding for liquidity demanders; if $\sigma_{uM}\tilde{X}_{LD} > \frac{c_s}{1-\omega}\sigma_u^2\bar{\theta}$, then short margin requirements are binding for liquidity demanders; if $\sigma_{uM}\tilde{X}_{LD} < -\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta}$, then long margin requirements are binding for liquidity demanders.

Similarly, we have: (i) if $-\frac{c_s}{\omega}\sigma_u^2\bar{\theta} < \sigma_{uM}\tilde{X}_{LD} < \frac{c_b}{\omega}\sigma_u^2\bar{\theta}$, then margin requirements are not binding for liquidity suppliers; (ii) if $\sigma_{uM}\tilde{X}_{LD} > \frac{c_b}{\omega}\sigma_u^2\bar{\theta}$, then long margin requirements are binding for liquidity suppliers; (iii) if $\sigma_{uM}\tilde{X}_{LD} < -\frac{c_s}{\omega}\sigma_u^2\bar{\theta}$, then short margin requirements are binding for liquidity suppliers. Therefore, we have the following cases: (1) if $\frac{\omega}{1-\omega} < \min\{\frac{c_b}{c_s}, \frac{c_s}{c_b}\}$, then margin requirements can bind only for liquidity demanders; (2) if $\frac{\omega}{1-\omega} > \max\{\frac{c_b}{c_s}, \frac{c_s}{c_b}\}$, then margin requirements can bind only for liquidity suppliers; (3) if $\frac{c_b}{c_s} < \frac{\omega}{1-\omega} < \frac{c_s}{c_b}$, then long margin requirements are more stringent than short margin requirements. Therefore, only long margin requirements can bind. (4) if $\frac{c_s}{c_b} < \frac{\omega}{1-\omega} < \frac{c_b}{c_s}$, then short margin requirements are more stringent than long margin requirements. Therefore, only short margin requirements can bind.

We now prove the Proposition for case (1). For case (1), if $\sigma_{uM}\tilde{X}_{LD} \leq -\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta}$, long margin requirements are binding. In equilibrium, $\theta^*_{LD_s} = (c_b+1)\bar{\theta}$ and $\theta^*_{LS_s} = \frac{1-\omega-\omega c_b}{1-\omega}\bar{\theta}$. From (1.25), liquidity suppliers' optimal demand is $\theta^*_{LS_s} = \frac{\bar{V}+\tilde{F}-P^*_s}{\delta\sigma_u^2}$. It follows that the equilibrium stock price is $P^*_s = \bar{V} + \tilde{F} - \frac{1-\omega-\omega c_b}{1-\omega}\delta\sigma_u^2\bar{\theta}$. It can be shown that $P^*_{s0} - P^*_s = -\delta\omega\sigma_{uM}\tilde{X}_{LD} - \omega\frac{c_b}{1-\omega}\delta\sigma_u^2\bar{\theta} \geq 0$, *i.e.*, $P^*_{s0} \geq P^*_s$, long margin requirements tend to decrease stock price. If $\sigma_{uM}\tilde{X}_{LD} \geq \frac{c_s}{1-\omega}\sigma_u^2\bar{\theta}$, short margin requirements are binding. In equilibrium, $\theta^*_{LD_s} = -(c_s-1)\bar{\theta}$ and $\theta^*_{LS_s} = \frac{1-\omega+\omega c_s}{1-\omega}\bar{\theta}$, from (1.25), liquidity suppliers' optimal demand is $\theta^*_{LS_s} = \frac{\bar{V}+\tilde{F}-P^*_s}{\delta\sigma_u^2}$. It follows that the equilibrium stock price is $P^*_s = \bar{V} + \tilde{F} - \frac{1-\omega+\omega c_s}{1-\omega}\delta\sigma_u^2\bar{\theta}$. It can be shown that $P^*_{s0} - P^*_s = -\delta\omega\sigma_{uM}\tilde{X}_{LD} + \omega\frac{c_s}{1-\omega}\delta\sigma_u^2\bar{\theta} \leq 0$, *i.e.*, $P^*_{s0} \leq P^*_s$, short margin requirements tend to increase stock price. *Q.E.D.*

Proof of Proposition 1.3 and Corollary 1.2:

$$E[P^*_s] - E[P^*_{s0}] = \delta\omega \times \frac{|\sigma_{uM}|\sigma_{LD}}{\sqrt{2\pi}} \times \left[-e^{-\frac{(-c_b\sigma_u^2\bar{\theta})^2}{2(1-\omega)^2\sigma_{uM}^2\sigma_{LD}^2}} + e^{-\frac{(c_s\sigma_u^2\bar{\theta})^2}{2(1-\omega)^2\sigma_{uM}^2\sigma_{LD}^2}} \right]$$

$$+\frac{\delta\omega c_b\sigma_u^2\bar{\theta}}{1-\omega} N\left(-\frac{c_b\sigma_u^2\bar{\theta}}{(1-\omega)|\sigma_{uM}|\sigma_{LD}}\right) - \frac{\delta\omega c_s\sigma_u^2\bar{\theta}}{1-\omega} N\left(-\frac{c_s\sigma_u^2\bar{\theta}}{(1-\omega)|\sigma_{uM}|\sigma_{LD}}\right).$$

Note that $E[P_s^*] = E[P_{s0}^*]$, when $c_b = c_s$. We now prove that $E[P_s^*]$ increases in c_b and decreases in c_s, then, it follows that $E[P_s^*] \geq E[P_{s0}^*]$, iff $c_b \geq c_s$. We have

$$\frac{\partial E[P_s^*]}{\partial c_b} = \frac{\omega\delta\sigma_u^2\bar{\theta}}{1-\omega} N\left(-\frac{c_b\sigma_u^2\bar{\theta}}{(1-\omega)|\sigma_{uM}|\sigma_{LD}}\right) > 0,$$

$$\frac{\partial E[P_s^*]}{\partial c_s} = -\frac{\omega\delta\sigma_u^2\bar{\theta}}{1-\omega} N\left(-\frac{c_s\sigma_u^2\bar{\theta}}{(1-\omega)|\sigma_{uM}|\sigma_{LD}}\right) < 0. \tag{1.28}$$

Therefore, $E[P_s^*] - E[P_{s0}^*]$ increases in c_b and decreases in c_s. *Q.E.D.*

Lemma 1.1 *S is continuously distributed in $[-\infty, +\infty]$ with probability density function $f_S(s)$, we define*

$$Y = \begin{cases} kS + (1-k)n & S \geq n \\ S & m < S < n \\ kS + (1-k)m & S \leq m \end{cases}$$

where m, n and k are constants with $n > m$ and $0 \leq k < 1$. Then, $Var(Y) < Var(S)$

Proof of Lemma 1.1: We define $Y_1 = Y + E[S] - E[Y]$, then $E[Y_1] = E[S]$. From Figure 1.11, there exists s^*, (not necessarily unique, pick one) such that $Y_1 > S$, for $S < s^*$ and $Y_1 < S$ for $S > s^*$. Let $F_{Y_1}(s)$ be the *c.d.f.* for Y_1 and $G_S(s)$ be the *c.d.f.* for S. $\forall s < s^*$, $Y_1 < s \Rightarrow S < s$. This implies that $G_S(s) - F_{Y_1}(s) \geq 0$ for $s < s^*$. And $\forall s \geq s^*$, $Y_1 \geq s \Rightarrow S \geq s$. This implies $Prob(S > s) \geq Prob(Y_1 > s)$, *i.e.*, $G_S(s) - F_{Y_1}(s) \leq 0$ for $s \geq s^*$. Now, consider the stochastic dominance integral $I(s) = \int_{\tau=-\infty}^{s} (G_S(\tau) - F_{Y_1}(\tau))\, d\tau$. Since $E[S] = E[Y_1]$, we have $I(+\infty) = 0$. The sign pattern of $G_S(s) - F_{Y_1}(s)$ implies that $\forall s$, we have $I(s) \geq 0$. *i.e.*, $\int_{-\infty}^{s} G_S(\tau)d\tau \geq \int_{-\infty}^{s} F_{Y_1}(\tau)d\tau, \forall s$. This means that Y_1 second-order stochastically dominates S, we immediately get $Var(Y_1) < Var(S)$, note that $Var(Y_1) = Var(Y)$,

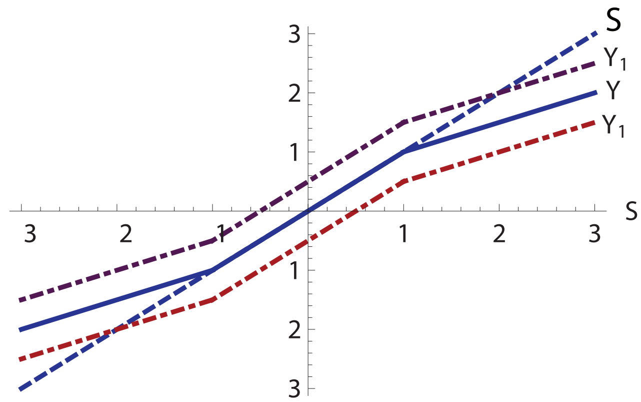


Figure 1.11: Random Variables Y_1, Y and S

The dot-dashed line denotes Y_1. The solid line denotes Y. The dashed line denotes S.

therefore, $Var(Y) < Var(S)$. $Q.E.D.$

Proof of Proposition 1.4:

$$P_s^* = \begin{cases} \bar{V} + \tilde{F} - \frac{1-\omega-\omega c_b}{1-\omega}\delta\sigma_u^2\bar{\theta} & P_{s0}^* \geq \bar{V} + \tilde{F} - \frac{1-\omega-\omega c_b}{1-\omega}\delta\sigma_u^2\bar{\theta} \\ P_{s0}^* & \bar{V} + \tilde{F} - \frac{1-\omega+\omega c_s}{1-\omega}\delta\sigma_u^2\bar{\theta} < P_{s0}^* < \bar{V} + \tilde{F} - \frac{1-\omega-\omega c_b}{1-\omega}\delta\sigma_u^2\bar{\theta} \\ \bar{V} + \tilde{F} - \frac{1-\omega+\omega c_s}{1-\omega}\delta\sigma_u^2\bar{\theta} & P_{s0}^* \leq \bar{V} + \tilde{F} - \frac{1-\omega+\omega c_s}{1-\omega}\delta\sigma_u^2\bar{\theta} \end{cases}$$

And we have $Var[\tilde{V} - P_s^*] = Var[\tilde{F} - P_s^*]$ and $Var[\tilde{V} - P_{s0}^*] = Var[\tilde{F} - P_{s0}^*]$. For any fixed $\tilde{F}$, using Lemma 3.1 with $k = 0$, we get $Var[P_s^*] < Var[P_{s0}^*]$, and $Var[\tilde{F} - P_s^*] < Var[\tilde{F} - P_{s0}^*]$. Since the sum of the variances of stock returns of dates 0 and 1 with and without margin requirements are respectively $Var(P_s^* - P_{s,-1}^*) + Var(\tilde{V} - P_s^*) = Var(\tilde{V}) - 2Cov(P_s^* - P_{s,-1}^*, \tilde{V} - P_s^*)$, and $Var(P_{s0}^* - P_{s0,-1}^*) + Var(\tilde{V} - P_{s0}^*) = Var(\tilde{V}) - 2Cov(P_{s0}^* - P_{s0,-1}^*, \tilde{V} - P_{s0}^*)$. Under symmetric information, since $\tilde{F}$ is public information, we have $Cov(P_s^* - P_{s,-1}^*, \tilde{V} - P_s^*) = -Var(P_s^*) < 0$, $Cov(P_{s0}^* - P_{s0,-1}^*, \tilde{V} -$

$P_{s0}^*) = -Var(P_{s0}^*) < 0$. Therefore margin requirements lead to a less negative auto-correlation of stock returns. *Q.E.D.*

Proof of Proposition 1.5: We prove the first case, other cases are very similar. If $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$, then margin requirements can only bind for liquidity demanders. When long margin requirements bind for liquidity demanders, *i.e.*, when $\tilde{D} > \frac{c_b}{1-\omega}\bar{\theta} + \frac{\varepsilon}{1-\omega}$, we have $P_s^* = \tilde{F} + \bar{V} - \delta\sigma_u^2\bar{\theta} + \delta\omega\frac{c_b}{1-\omega}\sigma_u^2\bar{\theta} + \frac{\delta\sigma_u^2}{1-\omega}\varepsilon$. Therefore, $\frac{\partial P_s^*}{\partial \varepsilon} = \frac{\delta\sigma_u^2}{1-\omega}$. Similarly, we can show that $\lambda_s^m = \frac{\partial P_s^*}{\partial \varepsilon} = \frac{\delta\sigma_u^2}{1-\omega}$ when short margin requirements bind for liquidity demanders, *i.e.*, when $\tilde{D} < -\frac{c_s\bar{\theta}}{1-\omega} + \frac{\varepsilon}{1-\omega}$. Therefore, the average price impact $\lambda_s^m = \frac{\delta\sigma_u^2}{1-\omega}\left(1 + \omega N(-\frac{c_s}{1-\omega}\bar{\theta} + \frac{\varepsilon}{1-\omega}) - \omega N(\frac{c_b}{1-\omega}\bar{\theta} + \frac{\varepsilon}{1-\omega})\right) > \delta\sigma_u^2 = \lambda_s$. In addition, we have $\frac{\partial \lambda_s^m}{\partial c_b} < 0$ and $\frac{\partial \lambda_s^m}{\partial c_s} < 0$. *Q.E.D.*

Proof of Proposition 1.6: We first show that stringent margin requirements make liquidity demanders worse off while binding but less stringent margin requirements make liquidity demanders better off. More specifically, let W_{LD}^* and W_{LD0}^* denote the equilibrium terminal wealth of liquidity demanders with and without margin requirements respectively. We want to show the following: (1) if $\tilde{D} < -\frac{1+\omega}{(1-\omega)^2}c_s\bar{\theta}$ or $\tilde{D} > \frac{1+\omega}{(1-\omega)^2}c_b\bar{\theta}$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] > E[-e^{-\delta\tilde{W}_{LD}^*}]$; (2) if $-\frac{1+\omega}{(1-\omega)^2}c_s\bar{\theta} < \tilde{D} \le -\frac{\bar{\theta}}{1-\omega}c_s$ or $\frac{\bar{\theta}}{1-\omega}c_b \le \tilde{D} < \frac{1+\omega}{(1-\omega)^2}c_b\bar{\theta}$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] \le E[-e^{-\delta\tilde{W}_{LD}^*}]$; (3) if $-\frac{\bar{\theta}}{1-\omega}c_s \le \tilde{D} \le \frac{\bar{\theta}}{1-\omega}c_b$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] = E[-e^{-\delta\tilde{W}_{LD}^*}]$.

From Proposition 1.1 and Proposition 1.2, we can compute $E[-e^{-\delta\tilde{W}_{LD0}^*}] = -e^{d_1}$, and when short margin requirements bind for liquidity demanders, $E[-e^{-\delta\tilde{W}_{LD}^*}] =$

$-e^{d_2}$. We have:

$$d_1 - d_2 = -\frac{\delta^2(1-\omega)^2\sigma_u^2}{2}\left(-\tilde{D} - c_s\frac{\bar{\theta}}{1-\omega}\right)\left(-\tilde{D} - \frac{1+\omega}{(1-\omega)^2}c_s\bar{\theta}\right). \tag{1.29}$$

Therefore, if $\tilde{D} < -\frac{1+\omega}{(1-\omega)^2}c_s\bar{\theta}$ then $d_1 < d_2$; and if $-\frac{1+\omega}{(1-\omega)^2}c_s\bar{\theta} \le \tilde{D} \le -c_s\frac{\bar{\theta}}{1-\omega}$, then $d_1 \ge d_2$. When long margin requirements bind for liquidity demanders, we can compute $E[-e^{-\delta\tilde{W}^*_{LD}}] = -e^{d_3}$. And we have

$$d_1 - d_3 = -\frac{\delta^2(1-\omega)^2\sigma_u^2}{2}\left(-\tilde{D} + \frac{\bar{\theta}}{1-\omega}c_b\right)\left(\tilde{D} + \frac{1+\omega}{(1-\omega)^2}c_b\bar{\theta}\right). \tag{1.30}$$

Therefore, liquidity demanders are worse off if margin requirements are stringent and they are better off if margin requirements are binding but not stringent.

We now show that liquidity suppliers are always worse off with binding margin requirements. Let W^*_{LS} and W^*_{LS0} denote the equilibrium terminal wealth of liquidity suppliers with and without margin requirements respectively. We can compute $E[-e^{-\delta\tilde{W}^*_{LS0}}] = -e^{d_4}$, and if short margin requirements bind for liquidity demanders, $E[-e^{-\delta\tilde{W}^*_{LS}}] = -e^{d_5}$, we have

$$d_4 - d_5 = -\frac{1}{2}\delta^2\omega^2\sigma_u^2\left(\tilde{D}^2 - c_s^2\frac{\bar{\theta}^2}{(1-\omega)^2}\right). \tag{1.31}$$

Obviously, if $\tilde{D} \le -\frac{\bar{\theta}}{1-\omega}c_s$, then $d_4 \le d_5$. Therefore, liquidity suppliers are always worse off when short margin requirements bind for liquidity demanders. Similarly, we can prove that liquidity suppliers are always worse off when long margin requirements bind for liquidity demanders.

Q.E.D.

Proof of Theorem 1 and Corollary 1.3: We assume that liquidity suppliers are indifferent between having margin requirements and lifting margin requirements after transferring ΔW_{LS} to liquidity demanders, *i.e.*, $-e^{d_5} = -e^{d_4+\delta\Delta W_{LS}}$. We get

$$\Delta W_{LS} = \frac{d_5 - d_4}{\delta} = \frac{1}{2}\delta w^2 \sigma_u^2 \left(\tilde{D}^2 - c_s^2 \frac{\bar{\theta}^2}{(1-\omega)^2}\right). \tag{1.32}$$

To make liquidity demanders better off, we need $-e^{d_1-\delta\Delta W_{LD}} \geq -e^{d_2}$, *i.e.*, $d_1 - d_2 \leq \delta\Delta W_{LD} = \delta\frac{1-\omega}{\omega}\Delta W_{LS}$, which is equivalent to $\left(\tilde{D} - c_s\frac{\bar{\theta}}{1-\omega}\tilde{D}\right)^2 \geq 0$. Therefore, even though the liquidity demanders are better off with binding but not stringent short margin requirements, the total surplus measured by certainty equivalent is reduced. This implies that short margin requirements are dominated by a lump-sum transfer scheme without constraints. Similarly, we can prove that long margin requirements are also dominated by some lump-sum wealth transfer.

Q.E.D.

Proof of Proposition 1.7: Liquidity demanders' optimal stock demand is

$$\theta^*_{LD_a} = \frac{\bar{V} + \tilde{F} - \delta\sigma_{uM}\tilde{X}_{LD} - P^*_{a0}}{\delta\sigma_u^2} = \frac{\bar{V} + 2\tilde{S} - P^*_{a0}}{\delta\sigma_u^2}. \tag{1.33}$$

Liquidity suppliers' information set is: $I_{LS} = \{P^*_{a0}\} = \{\tilde{S}\}$, therefore, liquidity suppliers' problem is

$$\min_{\theta_{LS}} e^{\delta\theta_{LS}(P^*_{a0}-\bar{V})+\frac{1}{2}\delta^2(\theta^2_{LS}\sigma^2_u+\tilde{X}^2_{LS}\sigma^2_M+2\theta_{LS}\tilde{X}_{LS}\sigma_{uM})} \times E[e^{-\delta\theta_{LS}\tilde{F}}|\tilde{S}]. \tag{1.34}$$

Letting $a = \delta\sigma_{uM}$, and $k = \sigma_{LD}/\sigma_F$, then $\tilde{S} \sim N(0, \frac{1}{4}(\sigma_F^2 + a^2\sigma_{LD}^2))$. We have

$$E[\tilde{F}|\tilde{S}] = \frac{2\tilde{S}}{1+a^2k^2},\ Var[\tilde{F}|\tilde{S}] = \frac{a^2k^2}{1+a^2k^2}\sigma_F^2,\ E[e^{-\delta\theta_{LS}\tilde{F}}|\tilde{S}] = e^{-\delta\theta_{LS}E[\tilde{F}|\tilde{S}]+\frac{1}{2}\delta^2\theta_{LS}^2 Var[\tilde{F}|\tilde{S}]}. \tag{1.35}$$

Liquidity suppliers' optimal stock demand given P_{a0}^* is $\theta_{LS_a}^* = \frac{\bar{V}+\frac{2\tilde{S}}{1+a^2k^2}-P_{a0}^*}{\delta\left(\sigma_u^2+\frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right)}$. Define

$$A_2 = \frac{2\left[(1+\omega a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2\right]}{(1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2}, B_2 = \frac{\delta\sigma_u^2\left[(1+a^2k^2)\sigma_u^2 + a^2k^2\sigma_F^2\right]}{(1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2}. \tag{1.36}$$

$$a_1 = \frac{2a^2k^2}{\delta\left((1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2\right)},\ b_1 = \frac{(1+a^2k^2)\sigma_u^2 + a^2k^2\sigma_F^2}{(1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2}. \tag{1.37}$$

We can solve for the equilibrium price $P_{a0}^* = \bar{V} + A_2\tilde{S} - B_2\bar{\theta}$ using the market clearing condition, $\omega\theta_{LD_a}^* + (1-\omega)\theta_{LS_a}^* = \bar{\theta}$. *Q.E.D.*

Proof of Proposition 1.8: We have $E[P_{a0}^*] = \bar{V} - B_2\bar{\theta}$, $E[P_{s0}^*] = \bar{V} - \delta\sigma_u^2\bar{\theta}$. Therefore, $E[P_{a0^*}] - E[P_{s0^*}] = (\delta\sigma_u^2 - B_2)\bar{\theta} < 0$. *Q.E.D.*

Proof of Proposition 1.9: We can compute the expected utilities of liquidity demanders with and without asymmetric information are $-e^{c_1}$ and $-e^{d_1}$. It is not difficult to get

$$c_1 - d_1 = C_3\sigma_u^2\left(\tilde{D} - C_1(\tilde{F} + \delta\sigma_F^2\bar{\theta})\right)\left(\tilde{D} + C_2(\tilde{F} + \delta\sigma_F^2\bar{\theta})\right), \tag{1.38}$$

where

$$C_3 = \frac{\delta^2(1-\omega)^2(\sigma_u^2 + \omega a^2k^2\sigma_F^2)((1+2a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2)}{2\left((1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2\right)^2\sigma_u^2} > 0,$$

$$C_1 = \frac{a^2k^2}{\delta\left(\sigma_u^2 + \omega a^2k^2\sigma_F^2\right)} > 0,\quad C_2 = \frac{a^2k^2}{\delta\left((1+2a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2\right)} > 0. \tag{1.39}$$

The results in Proposition 1.9 follow directly. *Q.E.D.*

Proof of Proposition 1.10: From Proposition 1.1 and 1.7, we have

$$Var(P_{s0}^*) = \sigma_F^2 + \omega^2 a^2 \sigma_{LD}^2,\ Var(P_{a0}^*) = \frac{1}{4}A_2^2(\sigma_F^2 + a^2\sigma_{LD}^2),\ Var(\tilde{V} - P_{s0}^*) = \omega^2 a^2 \sigma_{LD}^2 \tag{1.40}$$

$$\begin{aligned} Var(\tilde{V} - P_{a0}^*) &= \sigma_F^2 + \frac{1}{4}A_2^2(\sigma_F^2 + a^2\sigma_{LD}^2) - 2Cov(\tilde{F}, A_2\tilde{S}) \\ &= (1 - \frac{1}{2}A_2)^2\sigma_F^2 + \frac{1}{4}A_2^2 a^2 \sigma_{LD}^2 > \omega^2 a^2 \sigma_{LD}^2. \end{aligned} \tag{1.41}$$

It is not difficult to get that

$$(Var(P_{s0}^* - P_{-s1}) + Var(\tilde{V} - P_{s0}^*)) - (Var(P_{a0}^* - P_{-a1}) + Var(\tilde{V} - P_{a0}^*)), \tag{1.42}$$

is equivalent to

$$\frac{2\omega(\omega-1)a^2\sigma_{LD}^2}{\left((1+a^2k^2)\sigma_u^2 + \omega a^2k^2\sigma_F^2\right)^2}\left((1+a^2k^2)\sigma_u^4 + \omega(\omega+1)a^4k^4\sigma_F^4 + a^2k^2\left(2\omega(1+a^2k^2)+1\right)\sigma_u^2\sigma_F^2\right) < 0 \tag{1.43}$$

The increase of the total variance implies that $Cov(P_{s0}^* - P_{-s1}, \tilde{V} - P_{s0}^*) > Cov(P_{a0}^* - P_{-a1}, \tilde{V} - P_{a0}^*)$. We also have

$$Cov(P_{s0}^* - P_{-s1}, \tilde{V} - P_{s0}^*) = -Var(P_{s0}^*) < 0,$$

$$Cov(P_{a0}^* - P_{-a1}, \tilde{V} - P_{a0}^*) = A_2\left(\frac{1}{2} - \frac{1}{4}A_2(1 + a^2k^2)\right)\sigma_F^2 < 0. \tag{1.44}$$

Therefore, the presence of asymmetric information tends to increase market volatility and leads to a more negative market return auto-correlation. *Q.E.D.*

Proof of Proposition 1.11: Similarly to the proof of Proposition 1.2, we have four cases. For case (1), when margin requirements can bind only for liquidity demanders, liquidity demanders' optimal stock demand is:

$$\theta^*_{LD_a} = \min\{\max[\frac{\bar{V} + 2\tilde{S} - P^*_a}{\delta\sigma_u^2}, -(c_s - 1)\bar{\theta}], (c_b + 1)\bar{\theta}\}. \tag{1.45}$$

From the proof of Proposition 1.7, we know that $\theta^*_{LS_a} = \frac{\bar{V} + \frac{2\tilde{S}}{1+a^2k^2} - P^*_a}{\delta\left(\sigma_u^2 + \frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right)}$. If neither long or short margin requirements are binding, then the equilibrium price is the same as that without margin requirements, *i.e.*, $P^*_a = \bar{V} + A_2\tilde{S} - B_2\bar{\theta}$. If long margin requirements are binding for liquidity demanders, then $\theta^*_{LD_a} = (c_b+1)\bar{\theta}$, and therefore in equilibrium, using market clearing condition, we get that the equilibrium price is

$$P^*_a = \bar{V} + \frac{2\tilde{S}}{1+a^2k^2} - \delta\left(\sigma_u^2 + \frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right)(1-\omega-\omega c_b)\frac{\bar{\theta}}{1-\omega}. \tag{1.46}$$

If short margin requirements are binding for liquidity demanders, then $\theta^*_{LD_a} = -(c_s - 1)\bar{\theta}$, and therefore in equilibrium, using market clearing condition, we get that the equilibrium price is

$$P^*_a = \bar{V} + \frac{2\tilde{S}}{1+a^2k^2} - \delta\left(\sigma_u^2 + \frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right)(1-\omega+\omega c_s)\frac{\bar{\theta}}{1-\omega}. \tag{1.47}$$

Define

$$A_1 = \frac{2}{1+a^2k^2}, B_1 = \delta\left(\sigma_u^2 + \frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right), \tag{1.48}$$

we can write the equilibrium stock price as in the Proposition. S^*_{LD1} and S^*_{LD2} can be solved directly by the continuity of the equilibrium stock price. Specifically, we have

$$S^*_{LD1} = \left[\frac{1+a^2k^2}{2a^2k^2}c_b\sigma_u^2 - \frac{1}{2}(1-\omega-\omega c_b)\sigma_F^2\right]\frac{\delta\bar{\theta}}{1-\omega}, \tag{1.49}$$

$$S^*_{LD2} = \left[-\frac{1+a^2k^2}{2a^2k^2}c_s\sigma_u^2 - \frac{1}{2}(1-\omega+\omega c_s)\sigma_F^2\right]\frac{\delta\bar{\theta}}{1-\omega}, \tag{1.50}$$

$$S^*_{LS1} = \left[\frac{1+a^2k^2}{2a^2k^2}c_s\sigma_u^2 + \frac{1}{2}\omega(c_s-1)\sigma_F^2\right]\frac{\delta\bar{\theta}}{\omega}, \tag{1.51}$$

$$S^*_{LS2} = \left[-\frac{1+a^2k^2}{2a^2k^2}c_b\sigma_u^2 - \frac{1}{2}\omega(c_b+1)\sigma_F^2\right]\frac{\delta\bar{\theta}}{\omega}. \tag{1.52}$$

Proof of Proposition 1.12 and Corollary 1.4:

$$E[P_a^*] = E[P_{a0}^*] + (A_1 - A_2) \times \frac{\sqrt{1+a^2k^2}\sigma_F}{2\sqrt{2\pi}} \times \left[e^{-\frac{2S^{*\,2}_{LD1}}{(1+a^2k^2)\sigma_F^2}} - e^{-\frac{2S^{*\,2}_{LD2}}{(1+a^2k^2)\sigma_F^2}}\right]$$

$$+\left(B_2\bar{\theta} - B_1\frac{(1-\omega-\omega c_b)\bar{\theta}}{1-\omega}\right)N\left(-\frac{2S^*_{LD1}}{\sqrt{1+a^2k^2}\sigma_F}\right)$$
$$+\left(B_2\bar{\theta} - B_1\frac{(1-\omega+\omega c_s)\bar{\theta}}{1-\omega}\right)N\left(\frac{2S^*_{LD2}}{\sqrt{1+a^2k^2}\sigma_F}\right), \tag{1.53}$$

where A_1, A_2, B_1, B_2 are defined in (1.48) and (1.36). We notice that $E[P_a^*] = E[P_{a0}^*]$, when $c_b = c_s + d$, where

$$d = \frac{d_1 a^2\sigma_{LD}^2\sigma_F^2}{(\sigma_F^2 + a^2\sigma_{LD}^2)\sigma_u^2 + \omega a^2\sigma_{LD}^2\sigma_F^2}, \tag{1.54}$$

where $d_1 = 2(1-\omega)$ for $\omega < \min\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$ and $d_1 = -2\omega$ for $\omega > \max\{\frac{c_b}{c_b+c_s}, \frac{c_s}{c_b+c_s}\}$. Now, we want to prove that $E[P_a^*]$ increases in c_b and decreases in c_s. Then, it follows that $E[P_a^*] > E[P_{a0}^*]$ iff $c_b > c_s + d$. It is not difficult to see that

$$\frac{\partial E[P_a^*]}{\partial c_b} = (A_1 - A_2) \times \left[\frac{1+a^2k^2}{2a^2k^2}\sigma_u^2 + \frac{\omega}{2}\sigma_F^2\right](-S^*_{LD1})f(S^*_{LD1})$$

$$+(B_2\bar{\theta} - B_1(1-\omega-\omega c_b)\frac{\bar{\theta}}{1-\omega}) \times \left[\frac{1+a^2k^2}{2a^2k^2}\sigma_u^2 + \frac{\omega}{2}\sigma_F^2\right](-f(S^*_{LD1}))$$

$$+B_1\omega\frac{\bar{\theta}}{1-\omega}N\left(-\frac{2S^*_{LD1}}{\sqrt{1+a^2k^2}\sigma_F}\right) = B_1\omega\frac{\bar{\theta}}{1-\omega} \times N\left(-\frac{2S^*_{LD1}}{\sqrt{1+a^2k^2}\sigma_F}\right) > 0. \tag{1.55}$$

The last equality follows from $S^*_{LD1}(A_1-A_2)=-B_2\bar{\theta}+B_1(1-\omega-\omega c_b)\frac{\bar{\theta}}{1-\omega}$. Similarly, we can prove that $\frac{\partial E[P_a^*]}{\partial c_s}<0$. Therefore, $E[P_a^*]$ increases in c_b and decreases in c_s.

Q.E.D.

Proof of Proposition 1.13:

$$P_a^*=\begin{cases}\frac{A_1}{A_2}P_{a0}^*+\frac{A_1}{A_2}B_2\bar{\theta}-B_1\frac{(1-\omega-\omega c_b)\bar{\theta}}{1-\omega}+\left(1-\frac{A_1}{A_2}\right)\bar{V} & P_{a0}^*\geq\bar{V}+A_2S^*_{LD1}-B_2\bar{\theta}\\ P_{a0}^* & \bar{V}+A_2S^*_{LD2}-B_2\bar{\theta}<P_{a0}^*<\bar{V}+A_2S^*_{LD1}-B_2\bar{\theta}\\ \frac{A_1}{A_2}P_{a0}^*+\frac{A_1}{A_2}B_2\bar{\theta}-B_1\frac{(1-\omega+\omega c_s)\bar{\theta}}{1-\omega}+\left(1-\frac{A_1}{A_2}\right)\bar{V} & P_{a0}^*\leq\bar{V}+A_2S^*_{LD2}-B_2\bar{\theta}\end{cases}$$

It is not hard to show that $\frac{A_1}{A_2}<1$, using Lemma 3.1, it is obvious that $Var[P_a^*-P_{-a1}^*]<Var[P_{a0}^*-P_{-a1}]$. In order to prove the reduction of total variance, we need the following Lemma

Lemma 1.2 *For two functions $f(x),g(x)$ with $f'(x)\geq 0$ and $g'(x)\leq 0$, where x is randomly distributed in set Ω with probability density function $p(x)$. We have $Cov(f(x),g(x))\leq 0$.*

Proof :

$$\begin{aligned}&Cov(f(x),g(x))=E(f(x)g(x))-E(f(x))E(g(x))\\&=\int_\Omega f(x)g(x)p(x)dx-\int_\Omega f(x)p(x)dx\int_\Omega g(x)p(x)dx\\&=\int_\Omega p(y)dy\int_\Omega f(x)g(x)p(x)dx-\int_\Omega f(y)p(y)dy\int_\Omega g(x)p(x)dx\\&=\int_\Omega\int\left(f(x)g(x)-f(y)g(x)\right)p(x)p(y)dxdy\\&=\frac{1}{2}\int_\Omega\int\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)p(x)p(y)dxdy.\end{aligned}$$

From $f'(x) \geq 0$ and $g'(x) \leq 0$, we know that $(f(x) - f(y))(g(x) - g(y)) \leq 0$ is always true. $Q.E.D.$

The difference of total variance with and without margin requirements is:

$$Var(\tilde{V} - P_a^*) + Var(P_a^* - P_{-a1}^*) - Var(\tilde{V} - P_{a0}^*) - Var(P_{a0}^* - P_{-a1})$$

$$= 2Cov(P_a^* - P_{a0}^*, P_a^* + P_{a0}^* - \tilde{F}). \tag{1.56}$$

It is not hard to see,

$$f(\tilde{F}, \tilde{X}_{LD}) \equiv P_a^* - P_{a0}^*$$

$$= \begin{cases} \frac{1}{2}(A_1 - A_2)\tilde{F} - \frac{1}{2}(A_1 - A_2)a\tilde{X}_{LD} - B_1(1 - \omega - \omega c_b)\frac{\bar{\theta}}{1-\omega} + B_2\bar{\theta} & \tilde{F} - a\tilde{X}_{LD} \geq 2S_{LD1}^* \\ 0 & 2S_{LD2}^* < \tilde{F} - a\tilde{X}_{LD} < 2S_{LD1}^* \\ \frac{1}{2}(A_1 - A_2)\tilde{F} - \frac{1}{2}(A_1 - A_2)a\tilde{X}_{LD} - B_1(1 - \omega + \omega c_s)\frac{\bar{\theta}}{1-\omega} + B_2\bar{\theta} & \tilde{F} - a\tilde{X}_{LD} \leq 2S_{LD2}^* \end{cases}$$

and

$$g(\tilde{F}, \tilde{X}_{LD}) \equiv P_a^* + P_{a0}^* - \tilde{F}$$

$$= \begin{cases} \left(\frac{1}{2}(A_1 + A_2) - 1\right)\tilde{F} - \frac{1}{2}(A_1 + A_2)a\tilde{X}_{LD} - B_1(1 - \omega - \omega c_b)\frac{\bar{\theta}}{1-\omega} - B_2\bar{\theta} & \tilde{F} - a\tilde{X}_{LD} \geq 2S_{LD1}^* \\ (A_2 - 1)\tilde{F} - A_2 a\tilde{X}_{LD} - 2B_2\bar{\theta} & 2S_{LD2}^* < \tilde{F} - a\tilde{X}_{LD} < 2S_{LD1}^* \\ \left(\frac{1}{2}(A_1 + A_2) - 1\right)\tilde{F} - \frac{1}{2}(A_1 + A_2)a\tilde{X}_{LD} - B_1(1 - \omega + \omega c_s)\frac{\bar{\theta}}{1-\omega} - B_2\bar{\theta} & \tilde{F} - a\tilde{X}_{LD} \leq 2S_{LD2}^* \end{cases}$$

We know that $A_2 > A_1 > 0$, if $A_1 \geq 1$, it is easy to see that

$$\frac{\partial f}{\partial \tilde{F}} \leq 0, \quad \frac{\partial f}{\partial \tilde{X}_{LD}} \geq 0, \quad \frac{\partial g}{\partial \tilde{F}} \geq 0, \quad \frac{\partial g}{\partial \tilde{X}_{LD}} \leq 0 \tag{1.57}$$

The strict inequalities in (1.57) hold for non-zero measure sets.

$$Cov\left(f(\tilde{F}, \tilde{X}_{LD}), g(\tilde{F}, \tilde{X}_{LD})\right) = E_{\tilde{F}\tilde{X}_{LD}}\left(f(\tilde{F}, \tilde{X}_{LD})g(\tilde{F}, \tilde{X}_{LD})\right) - E_{\tilde{F}\tilde{X}_{LD}}f(\tilde{F}, \tilde{X}_{LD}) \times E_{\tilde{F}\tilde{X}_{LD}}g(\tilde{F}, \tilde{X}_{LD})$$

$$= E_{\tilde{X}_{LD}}\left(Cov_{\tilde{F}}\left(f(\tilde{F}, \tilde{X}_{LD}), g(\tilde{F}, \tilde{X}_{LD})\right)\right) + Cov_{\tilde{X}_{LD}}\left(E_{\tilde{F}}f(\tilde{F}, \tilde{X}_{LD}), E_{\tilde{F}}g(\tilde{F}, \tilde{X}_{LD})\right). \tag{1.58}$$

From Lemma 1.2 and (1.57), we know that for any $\tilde{X}_{LD}$, we have

$$Cov_{\tilde{F}}\left(f(\tilde{F},\tilde{X}_{LD}),g(\tilde{F},\tilde{X}_{LD})\right)<0,$$

Therefore, the first term in (1.58) is negative. Also, we have

$$\frac{\partial E_{\tilde{F}}f(\tilde{F},\tilde{X}_{LD})}{\partial\tilde{X}_{LD}}=E_{\tilde{F}}\frac{\partial f(\tilde{F},\tilde{X}_{LD})}{\partial\tilde{X}_{LD}}\geq 0,\ \frac{\partial E_{\tilde{F}}g(\tilde{F},\tilde{X}_{LD})}{\partial\tilde{X}_{LD}}=E_{\tilde{F}}\frac{\partial g(\tilde{F},\tilde{X}_{LD})}{\partial\tilde{X}_{LD}}\leq 0. \tag{1.59}$$

The strict inequalities in (1.59) hold for non-zero measure sets. Using the above Lemma, we get that the second term in (1.58) is negative too. Therefore, we have showed that, for $A_1>1$, $Cov(P_a^*-P_{a0}^*,P_a^*+P_{a0}^*-\tilde{F})<0$, and from (1.56), the total variance of stock returns on date 0 and 1 is reduced with margin requirements. The auto-correlation between stock returns on date 0 and 1 without margin requirements is:

$$Cov(P_{a0}^*-P_{-a1},\tilde{V}-P_{a0}^*)=Cov(A_2\tilde{S},\tilde{F}-A_2\tilde{S})=A_2\left(\frac{1}{2}-\frac{1}{4}A_2(1+a^2k^2)\right)\sigma_f^2<0$$

Using similar argument, when $A_1>1$, it is not hard to show that the auto-correlation between stock returns on date 0 and 1 with margin requirements $Cov(P_a^*-P_{-a1}^*,\tilde{V}-P_a^*)<0$. The reduction of total variance implies that margin requirements lead to a less negative auto-correlation between stock returns under asymmetric information. *Q.E.D.*

Proof of Proposition 1.14: If margin requirements bind for liquidity demanders, then $\lambda_a^m = \frac{P_a^*}{\partial \varepsilon} = \frac{\delta\left(\sigma_u^2 + \frac{a^2k^2}{1+a^2k^2}\sigma_F^2\right)}{1-\omega} > \lambda_a$. If margin requirements bind for liquidity suppliers, then $\lambda_a^m = \frac{\delta\sigma_u^2}{\omega} > \lambda_a$. Therefore, the average $\lambda_a^m > \lambda_a$. *Q.E.D.*

Proof of Proposition 1.15: We first show that stringent margin requirements make liquidity demanders worse off while binding but less stringent margin requirements make liquidity demanders better off. More specifically, let W_{LD}^* and W_{LD0}^* denote the equilibrium terminal wealth of liquidity demanders with and without margin requirements respectively. We want to show the following: (1) if $\tilde{S} > S_{LD1}^{**}$ or $\tilde{S} < S_{LD2}^{**}$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] > E[-e^{-\delta\tilde{W}_{LD}^*}]$; (2) if $S_{LD1}^* < \tilde{S} \le S_{LD1}^{**}$ or $S_{LD2}^{**} \le \tilde{S} < S_{LD2}^*$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] \le E[-e^{-\delta\tilde{W}_{LD}^*}]$; (3) if $S_{LD2}^* \le \tilde{S} \le S_{LD1}^*$, then $E[-e^{-\delta\tilde{W}_{LD0}^*}] = E[-e^{-\delta\tilde{W}_{LD}^*}]$. S_{LD1}^* and S_{LD2}^* are defined in (1.49) and (1.50), $S_{LD2}^{**} < S_{LD2}^*$ and $S_{LD1}^{**} > S_{LD1}^*$ are constants depending on parameters in this model, we will define them below. From Proposition 1.7, we can compute$E[-e^{-\delta\tilde{W}_{LD0}^*}] = -e^{c_1}$, and when short margin requirements bind for liquidity demanders, we can compute $E[-e^{-\delta\tilde{W}_{LD}}] = -e^{c_2}$. It is not difficult to compute

$$c_1 - c_2 \equiv f_1(\tilde{S}) = -\frac{2(1-\omega)^2}{\omega^2\sigma_F^4}\left(1 - \frac{B_2}{B_1}\right)^2 \sigma_u^2(\tilde{S} - S_{LD2}^*)(\tilde{S} - S_{LD2}^{**}),$$

where

$$S_{LD2}^{**} = -\frac{2B_1}{2-A_1}\frac{\bar{\theta}}{1-\omega}\left(\frac{1}{2} + \frac{\omega}{1-\omega}\frac{B_1}{B_2}\right) + \frac{\omega\sigma_F^2}{1-\frac{B_2}{B_1}}\left(\frac{1}{2} - \frac{B_1}{(1-\omega)B_2}\right)\delta(c_s - 1)\frac{\bar{\theta}}{1-\omega}. \tag{1.60}$$

$$S_{LD2}^{**} - S_{LD2}^* = -\frac{\omega}{1-\omega}\frac{2B_1}{2-A_1}\frac{B_1}{B_2}\frac{\bar{\theta}}{1-\omega} + \frac{\omega\sigma_F^2}{1-\frac{B_2}{B_1}}\left(1 - \frac{B_1}{(1-\omega)B_2}\right)\delta(c_s - 1)\frac{\bar{\theta}}{1-\omega} < 0. \tag{1.61}$$

Therefore, if $\tilde{S} < S^{**}_{LD2}$, then $c_1 < c_2$; and if $S^{**}_{LD2} \leq \tilde{S} \leq S^{*}_{LD2}$, then $c_1 \geq c_2$. Similarly, we can prove the arguments for the long margin requirements and

$$S^{**}_{LD1} = -\frac{2B_1}{2-A_1}\frac{\bar{\theta}}{1-\omega}\left(\frac{1}{2}+\frac{\omega}{1-\omega}\frac{B_1}{B_2}\right) - \frac{\omega\sigma_F^2}{1-\frac{B_2}{B_1}}\left(\frac{1}{2}-\frac{B_1}{(1-\omega)B_2}\right)\delta(c_b+1)\frac{\bar{\theta}}{1-\omega}. \tag{1.62}$$

Therefore, liquidity demanders are worse off if margin requirements are stringent and they are better off if margin requirements are binding but not stringent. Letting

$$b_{LD}(\tilde{S}) = \frac{\tilde{S}+\frac{2B_1}{2-A_1}\frac{\bar{\theta}}{1-\omega}\left(\frac{1}{2}+\frac{\omega}{1-\omega}\frac{B_1}{B_2}\right)}{\frac{\omega\sigma_F^2}{1-\frac{B_2}{B_1}}\left(\frac{B_1}{(1-\omega)B_2}-\frac{1}{2}\right)\delta\frac{\bar{\theta}}{1-\omega}},\ \ b_{LS}(\tilde{S}) = -\frac{\tilde{S}+\delta\bar{\theta}\sigma_F^2-\frac{\delta^2\sigma_F^2\sigma_u^2\bar{\theta}}{\omega B_2\left(1-\frac{B_2}{B_1}\right)}+\frac{\delta\sigma_u^2-\omega B_2}{(2-A_2)\omega}\bar{\theta}}{\frac{\delta\sigma_F^2\bar{\theta}}{\left(1-\frac{B_2}{B_1}\right)}\left(\frac{\delta\sigma_u^2}{\omega B_2}-\frac{1}{2}\right)}, \tag{1.63}$$

where A_1, B_1 and B_2 are defined as in (1.48) and (1.36). Margin requirements are stringent for investors i when $c_b < b_i(\tilde{S}) - 1$ or $c_s < 1 - b_i(\tilde{S}),\ i = LD, LS$.

We now show that liquidity suppliers are always worse off with binding margin requirements. Let W^*_{LS} and W^*_{LS0} denote the equilibrium terminal wealth of liquidity suppliers with and without margin requirements respectively. We can compute $E[-e^{-\delta\tilde{W}^*_{LS0}}] = -e^{c_4}$, and if short margin requirements bind for liquidity demanders, $E[-e^{-\delta\tilde{W}^*_{LS}}] = -e^{c_5}$, we have

$$c_4 - c_5 \equiv g_1(\tilde{S}) = -(2-A_1)\left(1-\frac{B_2}{B_1}\right)\frac{\omega B_2}{\delta\sigma_F^2\sigma_u^2}(\tilde{S}-S^*_{LD2})(\tilde{S}-S'_{LD2}).$$

It is not difficult to show that

$$S'_{LD2} - S^*_{LD2} = \left(\frac{1+a^2k^2}{a^2k^2}c_s\sigma_u^2 + (1-\omega+\omega c_s)\sigma_F^2 - (1-\omega)\sigma_F^2\right)\delta\frac{\bar{\theta}}{1-\omega} > 0.$$

Obviously, if $\tilde{S} \leq S^*_{LD2}$, then $c_4 \leq c_5$, *i.e.*, liquidity suppliers are always worse off with binding short margin requirements under asymmetric information. Similarly, we can prove that liquidity suppliers are worse off with binding long margin requirements.

Q.E.D.

Proof of Theorem 2 and Corollary 1.5: Given the hedging demand, when short margin requirements are not stringent, *i.e.*, $S^*_{LD1} < \tilde{S} \leq S^{**}_{LD1}$, liquidity demanders are better off and liquidity suppliers are worse off, *i.e.*, $-e^{c_1} \leq -e^{c_2}, \quad -e^{c_4} \geq -e^{c_5}$. First, we assume liquidity suppliers are indifferent between having margin requirements and lifting margin requirements after transferring ΔW_{LS} to liquidity demanders, *i.e.*, $-e^{c_5} = -e^{c_4+\delta\Delta W_{LS}}$. Therefore, $\Delta W_{LS} = \frac{c_5-c_4}{\delta}$, to make liquidity demanders better off, we need $-e^{c_1-\delta\Delta W_{LD}} \geq -e^{c_2}$, *i.e.*,

$$c_1 - c_2 \leq \delta\Delta W_{LD} = \delta\frac{1-\omega}{\omega}\Delta W_{LS} = \frac{1-\omega}{\omega}(c_5 - c_4).$$

This is equivalent to

$$\left(\tilde{S} + \frac{1}{2}\left(\frac{\omega\sigma_F^2}{1-\frac{B_2}{B_1}}(c_s - 1) + \frac{2B_1}{(2-A_1)\delta}\right)\delta\frac{\bar{\theta}}{1-\omega}\right)^2 \geq 0. \tag{1.64}$$

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Chapter 2

Asymmetric Information, Endogenous Illiquidity, and Asset Pricing With Imperfect Competition[1]

2.1 Introduction

How do information asymmetry, competition among market makers and risk aversion affect asset pricing, market illiquidity and welfare? How are bid and ask prices, bid and ask depths, and market makers' inventory levels jointly determined in equilibrium? What is the impact of information asymmetry on the equilibrium degree of competition among market makers? Is the value of private information to informed investors always positive? In this paper, we develop a novel and tractable equilibrium model that can help answer questions like these.

Specifically, we consider an economy with three types of risk averse investors: informed investors, uninformed investors, and potential market makers who are also

[1] This is a joint work with Hong Liu.

uninformed. All investors optimally choose how to trade a risk-free asset and a stock to maximize their expected utility and all are endowed with some shares of a stock but no risk-free asset. Informed investors can privately observe the expected payoff of the stock before the terminal date and thus they have trading demand motivated by the private information. They are also subject to a liquidity shock modeled as a random endowment of a nontraded asset (e.g., labor income, highly illiquid asset) that is correlated with the stock. Accordingly, informed investors also have trading demand motivated by the liquidity needs for hedging.[2] Neither the informed nor the uninformed trade strategically. Any trades informed and uninformed investors choose to make must be with market makers at the bid or ask prices. A potential market maker can choose to be an uninformed investor or to pay a fixed utility cost to become a (uninformed) market maker.[3] As in Kyle (1985), informed and uninformed investors submit buy or sell orders simultaneously to market makers who then choose how to trade. In contrast to the standard literature which assumes Bertrand (or perfect) competition among market makers, we model the competition among market makers as a Cournot competition: they choose simultaneously how much to buy at the bid and

[2]As in Glosten (1989) and Vayanos and Wang (2009), the assumption that the informed have both information and liquidity motivated trades is a simple way to keep the private information not fully revealed in equilibrium. All we need is that some uninformed investors with liquidity needs trade in the same direction as the informed so that market makers only see the pooled order flow. Indeed, we analyzed an alternative model where we have four types of investors: (1) informed who observe the expected stock payoff, but do not have any liquidity shock; (2) uninformed without any liquidity shock; (3) uninformed with a privately observed liquidity shock; and (4) uninformed market makers without any liquidity shock. In the non-fully revealing equilibrium, the equilibrium price is a linear combination of the private signal and the liquidity shock. The uninformed with a privately observed liquidity shock can infer the private information from the market price and thus become informed. We show that our qualitative results in this alternative model stay the same as in our current model. However, the alternative model involves much more notations and makes the key intuitions less transparent.

[3]Allowing informed and uninformed investors to be strategic does not change our main results. For example, the bid-ask spread could still be lower with asymmetric information. If the market making cost were a pecuniary payment, then one would need to model where this payment goes in the economy. We assume it is a utility cost so that we can focus on our main points of interest.

how much to sell at the ask, taking into account the price impact of their trades. The equilibrium bid and ask prices are then determined by the market clearing conditions at the bid and at the ask, i.e., the total amount market makers buy at the bid is equal to the total amount other investors sell, and the total amount market makers sell at the ask is equal to the total amount other investors buy. In equilibrium, both the stock market and the risk-free asset market clear. We solve the equilibrium bid and ask prices, bid and ask depths, trading volume, and inventory levels in closed forms.

In addition to the methodological contribution, our model can also help explain some puzzling empirical findings, such as the bid-ask spread can be lower with asymmetric information (e.g., Kini and Mian (1995), Brooks (1996), Huang and Stoll (1997), Acker, Stalker and Tonks (2002)) and the bid-ask spread can be positively correlated with trading volume (e.g., Brock and Kleidon (1992), Lin, Sanger and Booth (1995), Chordia, Roll, and Subrahmanyam (2001)).[4] To help explain the main intuitions behind these findings, consider the case where the informed buy the stock while the uninformed sell it in equilibrium. Unlike "noise traders" as modeled in most of the microstructure models, uninformed investors in our model optimally react to market prices in determining their trades. Define the reservation price as the critical price such that an investor buys (sells) the stock if and only if the ask (bid) is lower (higher) than this critical price. Since the informed buy and the uninformed sell, we must have the reservation price of the informed > ask > bid > the reservation price of the uninformed. Similar to the standard result in the classical Cournot competition models, the equilibrium spread is equal to the absolute value of the difference

[4]For example, Brooks (1996) find a negative relationship between bid-ask spreads and information asymmetry around earnings and dividends announcements. Similarly, Acker, Stalker and Tonks (2002) find that bid-ask spreads start to narrow about two weeks before earnings announcements. Chordia, Roll, and Subrahmanyam (2001) find that the effective bid and ask spread is positively correlated with trading volume unconditionally (Table III).

between the informed's and the uninformed's reservation prices, divided by one plus the number of competitors, i.e., market makers. Because the uninformed do not have liquidity shock and must estimate the expected stock payoff, the difference is the sum of three differences across the informed and the uninformed: (1) the difference due to hedging demand for liquidity shock ("hedging demand effect"); (2) the difference in the estimation of the expected payoff ("estimation error effect"); and (3) the difference in the risk premium required for estimation risk ("estimation risk effect"). Since the uninformed are risk averse, they require a positive estimation risk premium, thus the estimation risk effect lowers the reservation price. In contrast, because the uninformed can overestimate or underestimate the expected payoff, the estimation error effect can increase or decrease their reservation price. When the uninformed overestimate, the estimation error effect drives up the reservation price of the uninformed. Thus, if the estimation error effect dominates the estimation risk effect, then the absolute value of the reservation price difference in the asymmetric information case can be lower than in the symmetric information case where only the first effect is present. Therefore, the bid-ask spread with asymmetric information can be lower than with symmetric information. On the other hand, if the uninformed underestimate, then the estimation error effect drives down the reservation price and thus makes it further away from that of the informed. then the absolute value of the reservation price difference is greater and so is the spread in the asymmetric information case. Because the difference between the bid (ask) and the seller's (buyer's) reservation price is also proportional to the absolute value of the reservation price difference, the trading volume from both the informed and the uninformed also increases in this case. Therefore, the trading volume can be higher with asymmetric information and can be positively correlated with the bid-ask spread.

In addition, we show that as competition among market makers (measured by the number of market makers) increases, the equilibrium bid-ask spread decreases and trading volume increases. Therefore, our model can also allow for negative correlation between the bid-ask spread and trading volume. As the number of market makers increases, the net benefit from being a market maker decreases, therefore the maximum number of market markers that can exist in equilibrium is finite if market making cost is positive. We find that the maximum number of market makers in equilibrium increases in the trading demand and decreases in the market making cost. When the bid-ask spread increases in information asymmetry, so does the maximum number of market makers in equilibrium.

Unlike the standard models with noise traders, we find that the value of private information to the informed can be negative. This is because the uninformed can over-attribute the informed's trading to the private information and thus the market prices can be worse for the informed investors with asymmetric information. We also find that even though market makers gain from their market power, both the informed and the uninformed investors lose. More importantly, the market makers' welfare gain is smaller than the welfare losses of other investors and thus social welfare is reduced by the presence of market power. This finding suggests the importance of increasing competition among market makers through some systematic mechanism (e.g., improving electronic markets). It also suggests that some limits on bid-ask spreads and the bid-ask depths with appropriate compensation for market making may increase social welfare. In addition, consistent with the finding that asymmetric information may reduce the bid-ask spread, we find that greater information asymmetry can reduce the social welfare loss due to market power.

While our analysis focuses on a pure dealership market, our main results also apply to designated market makers in hybrid markets (e.g., NYSE). As found by Venkataraman and Waisburd (2007) and Saar (2010), in many limit order markets, designated market makers for less active securities can improve market quality and are indeed commonly hired to facilitate trading in these securities.

In contrast to our model, most of the existing models in market microstructure literature assume market makers engage in Bertrand competition, have unlimited capital, and are risk neutral (e.g., Copeland and Galai (1983), Kyle (1985), Glosten and Milgrom (1985)). As is well-known, it takes only two Bertrand competitors to reach the perfect competition equilibrium prices. However, market prices can be far from the perfect competition ones (e.g., Christie and Schultz (1994), Chen and Ritter (2000), and Biais, Bisière and Spatt (2003)). In addition, the capital of market makers is likely finite and market makers can be risk-averse (e.g., Garman (1976), Lyons (1995)).[5] The existing literature also assumes that market makers acquire a net inventory position after each trading. As shown by the existing literature (e.g., Sofianos (1993)), market makers on average lose money from inventory positions and they tend to offset trades at the bid and the ask to avoid significant net inventory positions. In addition, in contrast to standard theories (e.g., Amihud and Mendelson (1980)) which predict that dealers will use their price quotes to control their inventories, Madhavan and Sofianos (1998) find that market makers mainly adjust quote depths to manage inventories. Our model provides a simple framework for analyzing how market makers vary the bid and ask depths (and then market determines the prices) to offset trades to control inventories.

[5]The popularity of various hedging trades (e.g., delta hedging) by market makers also suggests they are typically risk averse.

Our model is also related to Kyle (1989), Subrahmanyam (1991), Diamond and Verrecchia (1991), Naik, Neuberger, and Viswanathan (1999), Back and Baruch (2004), Vayanos and Wang (2009), and Rasu (2010). Kyle (1989) considers the imperfect competition among risk averse informed investors. He shows that informed investors reveal less information when competition is imperfect. Subrahmanyam (1991) finds that increasing the precision of private information intensifies competition between risk averse informed investors and thus can increase market liquidity. Diamond and Verrecchia (1991) show that reducing information asymmetry can increase liquidity and security prices may be nonmonotonic in information asymmetry because of the potential exit of market makers. In all these three papers, market makers post a single price, the trading needs of some of the uninformed investors (i.e., "noise investors") are exogenous and thus do not respond to price changes. Therefore if market makers were allowed to post bid and ask prices, then in contrast to our predictions, the bid-ask spread would always be increasing in information asymmetry. Naik, Neuberger, and Viswanathan (1999) examine whether full and prompt disclosure of public-trade details improves the welfare of a risk-averse investor in a two-stage dealership market. Similar to the other three papers, market makers post a single price which is the conditional expected payoff of the stock. Back and Baruch (2004) solve a version of the Glosten-Milgrom model with a single informed investor, in which the informed investor chooses his trading times optimally. As in the original Glosten-Milgrom model, they find that the bid-ask spread is greater with asymmetric information. Vayanos and Wang (2009) examine how liquidity and asset prices are affected by market imperfections and find asymmetric information always increases market illiquidity measured by price impact. In contrast to Vayanos and Wang (2009), in our model, market makers are strategic and transaction costs are endogenous. Rasu (2010) finds that bid-ask spread can decrease with the fraction of informed investors because the competition among them intensifies (similar to the effect of signal precision on competition in Subrahmanyam (1991)) and they can trade with impatient investors who are assumed to always submit market orders. In contrast to our setting and the markets where bid-ask spreads were empirically found to be smaller with asymmetric information (e.g., NYSE, Nasdaq), Rasu (2010) considers a pure limit order book market where there is no designated market maker who must post reasonable quotes.

There also exists a large literature on the effect of illiquidity on portfolio choice and asset pricing (e.g., Constantinides (1986), Vayanos (1998), Liu and Loewenstein (2002), Lo, Mamaysky and Wang (2004), Liu (2004), Acharya and Pedersen (2005)). In this literature, illiquidity is generally modeled as exogenous transaction costs and therefore the fundamental question of what affects illiquidity (which in turn affects asset pricing) is largely unanswered.[6]

The remainder of the paper proceeds as follows. In Section 2.2 we present the model. In Section 2.3 we solve the case with symmetric information, and in Section 2.4 we derive the equilibrium under asymmetric information. In Section 2.5 we provide some comparative statics on asset prices, illiquidity, and welfare. We conclude in Section 2.6. All proofs are in the Appendix.

[6]In addition, in most of this literature, it is not clear where transaction costs paid by the investors go and the impact of the agents who receive these transaction costs is thus not examined.

2.2 The model

In a one period setting, there are N investors who maximize their expected constant absolute risk aversion (CARA) utility from the terminal wealth on date 1. They can trade one risk-free asset and one risky asset ("stock") on date 0. There is a zero net supply for the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the stock is $N\bar{\theta}$ shares and the date 1 payoff of each share is $\tilde{V} = \bar{V} + \hat{F} + \tilde{u}$, where $\bar{V}$ is a constant representing the publicly known expected payoff, $\hat{F} \sim \mathbf{N}(\bar{F}, \sigma_F^2)$ is realized on date 0 and may be observed only by informed investors on date 0, and $\tilde{u} \sim \mathbf{N}(0, \sigma_u^2)$ cannot be observed by anyone until it becomes public on date 1, where $\bar{F}$ is a constant, $\sigma_F > 0$, and $\mathbf{N}(\cdot)$ denotes the normal distribution.[7]

There are three types of investors: N_I informed investors (I), N_U uninformed investors (U), and $N_M \equiv N - (N_I + N_U)$ potential market makers (M) who are also uninformed. Every investor is endowed with $\bar{\theta}$ shares of the stock but no risk-free asset. To become a market maker, an investor must be a potential market maker and must pay a fixed market-making utility cost c on date 0 before making the market. We assume that both N_U and N_I are large such that all I and U investors are price takers and there are no strategic interactions among them or with market makers. In addition to the stock, a type I investor is also subject to a liquidity shock that is modeled as a random endowment of $\hat{X}_I \sim \mathbf{N}(0, \sigma_I^2)$ units of a non-traded risky asset on date 0, with $\hat{X}_I$ realized and only directly known to the investor on date 0. The

[7] $\bar{F}$ can be set to zero and will not affect any of our results. Allowing $\bar{F}$ to be nonzero makes it clear that the asymmetric information case nests the symmetric information case. Throughout this paper, "bar" variables are constant, "tilde" variables are realized on date 1 and "hat" variables are realized on date 0.

non-traded asset has a per-unit payoff of $\tilde{N} \sim \mathbf{N}(0, \sigma_N^2)$ that has a covariance of σ_{uN} with $\tilde{u}$ and is realized and becomes public on date 1.[8] The correlation between the nontraded asset and the stock results in a liquidity demand for hedging the nontraded asset payoff. Assuming that the one who is subject to a liquidity shock is also informed is for simplicity: even if he does not observe the private signal $\hat{F}$, because he observes liquidity shock, he can infer it perfectly from the equilibrium price that reflects the sum of private information and liquidity shock. Asymmetric information can therefore exist only if some investors who do not have any liquidity shock are uninformed. We assume that these investors are all uninformed for simplicity.[9]

All trades must go through market makers. Specifically, given market bid price B and ask price A, I and U investors sell to market makers at the bid or buy from them at the ask or do not trade at all.

For each $i \in \{I, U, M\}$, investors of type i are ex ante identical. Accordingly, we restrict our analysis to symmetric equilibria where all type i investors adopt the same trading strategy. Let $\mathcal{I}_i$ represent a type i investor's information set on date 0 for $i \in \{I, U, M\}$. Given B and A, for $i \in \{I, U\}$, a type i investor's problem is

$$\max_{\theta_i} E[-e^{-\delta \tilde{W}_i} | \mathcal{I}_i], \tag{2.1}$$

subject to the budget constraint

$$\tilde{W}_i = (\bar{\theta} - \theta_i)^+ B - (\theta_i - \bar{\theta})^+ A + \theta_i \tilde{V} + \hat{X}_i \tilde{N}, \tag{2.2}$$

[8] The random endowment can represent any shock in the demand for the stock, such as a liquidity shock or a change in the labor income or a change in a highly illiquid asset.

[9] Alternatively, we can view an informed investor as a broker who combines the information motivated trades and liquidity shock motivated trades.

where $\hat{X}_U = 0$, $\delta > 0$ is the absolute risk-aversion parameter, θ_i is the number of shares held until date 1 by the investor, and $x^+ \equiv \max(0, x)$.[10]

As in Kyle (1985), informed and uninformed investors submit buy or sell orders simultaneously to market makers who then choose how to trade. Since other investors buy from market makers at ask and sell to them at bid, we can view these trades occur in two separate markets: the "ask" market and the "bid" market. In the "ask" market, other investors are demanders and market makers are suppliers and the opposite is true in the "bid" market. As market makers supply (sell) more in the "ask" market, the ask price goes down and as market makers demand (buy) more in the "bid" market, the bid price goes up. Accordingly, in contrast to the standard microstructure literature where market makers directly choose market prices, we assume market makers directly choose how much to buy at bid given the inverse supply function (a function of the market makers' purchasing quantity) of all other participants and how much to sell at ask given the inverse demand function (a function of the market makers' selling quantity) of all other participants.[11] Since all trades must go through market makers, market makers can have market powers especially when the number of market makers is small. To model the oligopolistic competition among the market makers, we use the notion of the Cournot competition that is well studied and understood in economics. Specifically, we assume that market makers simultaneously choose the optimal number of shares to sell at ask and to buy at bid, taking into

[10] For the more general case where all investors have liquidity shocks or different risk aversions, there are eight different subcases. We also obtain closed-form solutions and our main results still hold in this more general case. We focus on the current case where all investors have the same risk aversion and only an I investor has liquidity shock to make the main intuitions as clear as possible and to save space.

[11] We view the posted bid and ask prices as the required prices to achieve the optimal amount market makers choose to trade.

account the price impact of their trades. This is the key innovation of this model and it drives our main results.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{N_M})^\top$ and $\beta = (\beta_1, \beta_2, ..., \beta_{N_M})^\top$ be the vector of the number of shares market makers buy at bid (i.e., bid depth) and sell at ask (i.e., ask depth) respectively. The bid price $B(\alpha)$ (i.e., the inverse supply function) and the ask price $A(\beta)$ (i.e., the inverse demand function) can be determined by the following stock market clearing conditions at the bid and ask prices.[12]

$$\sum_{j=1}^{N_M} \alpha_j = \sum_{i=I,\ U} N_i(\bar{\theta} - \theta_i^*(A, B))^+, \ \sum_{j=1}^{N_M} \beta_j = \sum_{i=I,\ U} N_i(\theta_i^*(A, B) - \bar{\theta})^+, \tag{2.3}$$

where the left-hand sides represent the total purchases and sales by market makers respectively and the right-hand sides represent the total sales and purchases by other investors respectively.

Then for $j = 1, 2, ..., N_M$, the potential market maker M_j's problem is

$$\max_{\alpha_j \geq 0, \beta_j \geq 0, R_j \in \{0,1\}} E\left[\left(-e^{-\delta \tilde{W}_{M_j}} - c\right) R_j + \left(-e^{-\delta \tilde{W}_{M_j}}\right)(1 - R_j) | \mathcal{I}_M\right], \tag{2.4}$$

subject to the budget constraint

$$\begin{aligned} \tilde{W}_{M_j} &= \left(\beta_j A(\beta) - \alpha_j B(\alpha) + (\bar{\theta} + \alpha_j - \beta_j)\tilde{V}\right) R_j \\ &\quad + \left(\beta_j B - \alpha_j A + (\bar{\theta} + \alpha_j - \beta_j)\tilde{V}\right)(1 - R_j), \end{aligned} \tag{2.5}$$

where $R_j \in \{1, 0\}$ indicates the choice of being a market maker or not. Note that if potential market maker M_j chooses to be a market maker (i.e., $R_j = 1$), then she takes into account the price impact of her own trades, i.e., recognizing both A and B

[12]The risk-free asset market will be automatically cleared by the Walras' law. A buyer's (seller's) trade only depends on ask A (bid B). So A only depends on β and B only depends on α.

will be affected by her trades. On the other hand, if M_j chooses not to be a market maker (i.e., $R_j = 0$), then she takes prices B and A as given and has exactly the same problem as an uninformed investor.[13]

This leads to our definition of the Nash equilibrium of the Cournot competition where all potential market makers choose to be market makers.[14]

Definition 2.1 *An equilibrium* $(\theta_I^*, \theta_U^*, \alpha^*, \beta^*, A^*(\beta^*), B^*(\alpha^*))$ *is such that*

1. *given* $A^*(\beta^*)$ *and* $B^*(\alpha^*)$, θ_i^* *solves a type* i *investor's Problem* (2.1) *for* $i \in \{I, U\}$*;*

2. *given* θ_I^* *and* θ_U^*, α_j^*, β_j^* *and* $R_j^* = 1$ *solve potential market maker* M_j*'s Problem (2.4), for* $j = 1, 2, ..., N_M$*; and*

3. $A^*(\beta^*)$ *and* $B^*(\alpha^*)$ *clear both the stock and the risk-free asset markets.*

2.3 The equilibrium with symmetric information

As a benchmark, in this section we study the case with symmetric information where $\hat{F}$ is publicly known at date 0 and therefore other investors can also infer a type I

[13]Note that other investors' problems can also be written in terms of the amount to sell at bid (β_i) and the amount to buy at ask (α_i), for $i \in \{I, U\}$. However, different from market makers, either β_i or α_i must be zero for any of other investors. For simplicity, we use the after-trade position θ_i to describe the problems for other investors.

[14]This is without loss of generality, because the case where some potential market makers choose not to be market makers is equivalent to the case with less potential market makers. Deviations by undercutting prices can be prevented by matching prices by other market makers in subsequent periods in a repeated games setting. As in a standard Cournot competition, varying prices is not in the strategy space.

investor's liquidity shock from the equilibrium stock price. In this case, the equilibrium illiquidity arises from the market making cost and the market power of market makers.

2.3.1 Perfect competition with symmetric information

We first examine the simplest subcase where all investors are price takers and there is no market-making cost.[15] With perfect competition and zero market-making cost, equilibrium bid and ask prices must be the same and thus all investors trade at the same price. Let P_s^* denote the equilibrium stock price. In this subcase a market maker has exactly the same problem as an uninformed, with $\theta_M \equiv \bar{\theta} + \alpha_j - \beta_j$ (recall that all market makers are identical and use the same trading strategy). In addition, the market clearing condition becomes

$$N_I(\theta_I^* - \bar{\theta}) + N_U(\theta_U^* - \bar{\theta}) + N_M(\theta_M^* - \bar{\theta}) = 0. \tag{2.6}$$

With symmetric information, investors' information sets are such that $\mathcal{I}_I = \mathcal{I}_U = \mathcal{I}_M = \{\hat{F}, \hat{X}_I, P_s^*\}$. Therefore, a type i ($i = I, U, M$) investor's problem is equivalent to

$$\max_{\theta_i} \quad -e^{-\delta(\bar{\theta}-\theta_i)P_s^* - \delta\theta_i\bar{V} - \delta\theta_i\hat{F}} E[e^{-\delta\theta_i\tilde{u} - \delta\hat{X}_i\tilde{N}} | \mathcal{I}_i], \tag{2.7}$$

[15] With positive market making cost, no competitive equilibrium exists. This is because on one hand market makers need compensation in terms of a positive bid-ask spread for the market making cost, on the other hand, a positive bid-ask spread implies infinite demand and supply by market makers since as price takers they no longer internalize their trades' price impact.

which can be simplified into

$$\min_{\theta_i} \quad \delta\theta_i(P_s^* - \bar{V} - \hat{F}) + \frac{1}{2}\delta^2(\theta_i^2\sigma_u^2 + \hat{X}_i^2\sigma_N^2 + 2\theta_i\sigma_{uN}\hat{X}_i), \tag{2.8}$$

where $\hat{X}_M = \hat{X}_U = 0$. Let

$$\hat{h}_i = -\frac{\sigma_{uN}}{\sigma_u^2}\hat{X}_i \tag{2.9}$$

be a type i investor's hedging demand and

$$\hat{H}_i = \delta\sigma_u^2\hat{h}_i \tag{2.10}$$

be the premium that a type I investor is willing to pay for hedging. From the first order condition, we get:

$$P_s^* - \bar{V} - \hat{F} - \hat{H}_i + \delta\sigma_u^2\theta_i^* = 0, \tag{2.11}$$

which leads to the optimal position

$$\theta_i^* = \frac{\bar{V} + \hat{F} + \hat{H}_i - P_s^*}{\delta\sigma_u^2}, \quad i = I, U, M. \tag{2.12}$$

The following concept is helpful for understanding many main results of this paper.

Definition 2.2 *The reservation price of an investor for a stock is the critical price such that the investor buys (sells, respectively) the stock if and only if the ask price is lower (the bid price is greater, respectively) than this critical price.*

Equation (3.41) then implies that the reservation price of a type-i investor is

$$P_i^R \equiv \bar{V} + \hat{F} + \hat{H}_i - \delta\sigma_u^2\bar{\theta}, \quad i = I, U, M. \tag{2.13}$$

Equation (2.13) implies that the reservation price of a type-i investor increases with expected stock payoff and the premium for hedging and decreases with stock payoff volatility. Then (3.41) can be rewritten as

$$\theta_i^* = \bar{\theta} + \frac{P_i^R - P_s^*}{\delta\sigma_u^2}, \quad i = I, U, M. \tag{2.14}$$

Let ΔRP_s denote the difference in the reservation prices of the I and U investors, i.e.,

$$\Delta RP_s \equiv P_I^R - P_U^R = \hat{H}_I. \tag{2.15}$$

The following theorem provides the equilibrium price and equilibrium stock holdings.

Theorem 2.1 *With symmetric information, zero market-making cost, and perfect competition,*

1. *the equilibrium price of the stock is*

$$P_s^* = \frac{N_I}{N}P_I^R + \frac{N_U}{N}P_U^R + \frac{N_M}{N}P_M^R = \bar{V} + \hat{F} - \delta\sigma_u^2\bar{\theta} + \frac{N_I}{N}\Delta RP_s; \mathit{and} \tag{2.16}$$

2. *the equilibrium stock holdings are*

$$\theta_I^* = \bar{\theta} + \left(1 - \frac{N_I}{N}\right)\frac{\Delta RP_s}{\delta\sigma_u^2}, \quad \theta_U^* = \theta_M^* = \bar{\theta} - \frac{N_I}{N}\frac{\Delta RP_s}{\delta\sigma_u^2}. \tag{2.17}$$

Theorem 2.1 shows that the equilibrium price is the population weighted average of the reservation prices of all the investors, which follows directly from (2.14) and the market clearing condition (2.6). The equilibrium price can also be rewritten as the reservation price of the uninformed investor plus the difference in the reservation prices ΔRP_s. Theorem 2.1 implies that the equilibrium price increases with the expected payoff $(\bar{V} + \hat{F})$, but decreases with the payoff volatility and stock supply. Since $\Delta RP_s = \delta\sigma_u^2\hat{h}_I$, which increases with risk aversion, the equilibrium price can increase with the risk aversion in our model. This is because the risk from the non-traded asset may dominate the risk from the stock and thus investors may be willing to buy more shares of the stock to hedge the non-traded asset risk as they become more risk averse, and thus drive up the stock price. Theorem 2.1 implies that I investors buy and U investors sell if and only if I investors have a higher reservation price than U investors. Later we show that this result carries through the cases with imperfect competition and with asymmetric information.

2.3.2 Imperfect competition with symmetric information

When the market-making cost is positive or the competition among market makers is imperfect, the equilibrium bid-ask spread will no longer be zero. As the number of market makers increases, the competition among market makers increases and the benefit from market making decreases. When the number of market makers is so high that the benefit of market making is lower than the cost of market making, some potential market makers will choose not to make the market. The following proposition shows that if the market making cost is below the utility gain from being the monopolistic market maker and the number of potential market makers is small enough, then there always exists a unique equilibrium (where all potential market makers choose to be market makers).

Proposition 2.1 *For any given $c \in [0, \bar{c}]$, where $\bar{c}$ is a monopolistic market maker's utility gain from making the market in equilibrium, there exists a unique positive integer N_M^* such that: there is a unique equilibrium if and only if $N_M \leq N_M^*$, where N_M^* represents the maximum number of market makers that can exist in equilibrium for the given market-making cost c.*[16]

As in the perfect competition case, we conjecture that I investors buy and U investors sell if and only if I investors have a higher reservation price than U investors. The following theorem shows that this conjecture is indeed correct.[17]

Theorem 2.2 *Suppose $N_M \leq N_M^*$. In the presence of market-making cost and market power,*

1. *the equilibrium ask and bid prices are*

$$A_s^* = \bar{V} + \hat{F} - \delta\sigma_u^2\bar{\theta} + \frac{N_M N_I}{(N+1)(N_M+1)}\Delta RP_s + \frac{1}{N_M+1}(\Delta RP_s)^+, \quad (2.18)$$

$$B_s^* = \bar{V} + \hat{F} - \delta\sigma_u^2\bar{\theta} + \frac{N_M N_I}{(N+1)(N_M+1)}\Delta RP_s - \frac{1}{N_M+1}(\Delta RP_s)^-, \quad (2.19)$$

which implies that $A_s^* > P_s^* > B_s^*$, where P_s^* is the perfect competition equilibrium price as defined in (2.16), $x^- \equiv \max(0, -x)$, and the bid-ask spread is

$$A_s^* - B_s^* = \frac{|\Delta RP_s|}{N_M+1} = \frac{|\hat{H}_I|}{N_M+1};$$

[16] A mixed-strategy equilibrium can exist only when a potential market maker is indifferent between being a market maker and being an uninformed investor and $N_M = N_M^*$. In these rare cases, we pick the pure strategy equilibrium.

[17] Since market makers are identical, we use notations without the subscript j to save notation. We also use subscript s to indicate the symmetric information case and subscript a to indicate the asymmetric information case in the next section.

2. the equilibrium stock holdings are

$$\theta_I^* = \bar{\theta} + \frac{N_M(N_U+N_M+1)}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right), \quad \theta_U^* = \bar{\theta} - \frac{N_I N_M}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right), \tag{2.20}$$

$$\theta_M^* = \bar{\theta} - \frac{N_I}{N+1}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right); and \tag{2.21}$$

the equilibrium quote depths are

$$\alpha_s^* = \frac{N_I(N_M+N_U+1)}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right)^- + \frac{N_I N_U}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right)^+,$$

$$\beta_s^* = \frac{N_I(N_M+N_U+1)}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right)^+ + \frac{N_I N_U}{(N+1)(N_M+1)}\left(\frac{\Delta RP_s}{\delta\sigma_u^2}\right)^-,$$

which implies that the equilibrium trading volume is

$$N_M(\alpha_s^* + \beta_s^*) = \frac{N_I N_M(N_M+2N_U+1)}{(N_M+1)(N+1)}\left(\frac{|\Delta RP_s|}{\delta\sigma_u^2}\right). \tag{2.22}$$

Theorem 2.2 implies that both the bid and the ask prices increase in the reservation price difference ΔRP_s. In addition, similar to the results of classical Cournot competition models of multiple firms who compete through choosing the amount of output of a homogeneous product, the bid and ask spread is equal to the absolute value of the reservation price difference ΔRP_s, divided by the number of market makers plus one. This implies that market makers equally split the market making benefit, which increases in $|\Delta RP_s|$ and decreases in competition among market makers.

To help understand this result, suppose I investors buy and U investors sell. The market clearing condition (2.3) implies that the inverse demand and supply functions faced by the market makers are respectively

$$A = P_I^R - k_1\beta_s, \quad B = P_U^R + k_2\alpha_s,$$

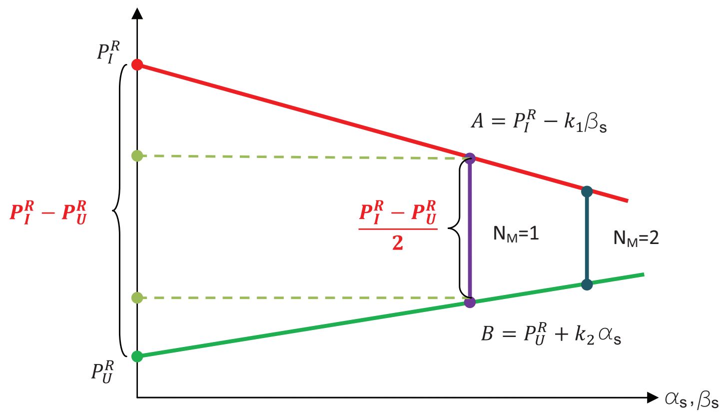

Figure 2.1: Inverse Demand and Supply Functions and Bid and Ask Spread.

where

$$k_1 = \frac{N_M \delta \sigma_u^2}{N_I}, \quad k_2 = \frac{N_M \delta \sigma_u^2}{N_U}.$$

We plot the inverse demand and supply functions and equilibrium spreads in Figure 2.1 for the case $P_I^R > P_U^R$. Figure 2.1 shows that as market makers buy (sell) more at the bid (ask), the bid (ask) price goes up (down). In the monopolistic case, the equilibrium spread is equal to half of the reservation price difference. In addition, Figure 2.1 also shows that the difference between P_I^R (P_U^R) and the ask (bid) price is also proportional to the reservation price difference $\Delta RP_s = P_I^R - P_U^R$. Therefore the trading amount of both I and U investors and thus the trading volume are also proportional to ΔRP_s.

As in the perfect competition case, I investors buy and U investors sell if and only if I investors have a higher reservation price than U investors. Because market makers have the same reservation price as the U investors, in the net they trade in the same direction as U investors. Because market makers trade at more favorable prices due to market power, they trade more in the net than U investors. More specifically, we have

$$\theta_M^* - \bar{\theta} = \frac{N_M + 1}{N_M}(\theta_U^* - \bar{\theta}),$$

then by the market clearing condition, the net trade of an I investor satisfies

$$\theta_I^* - \bar{\theta} = -\frac{N_U + N_M + 1}{N_I}(\theta_U^* - \bar{\theta}).$$

Therefore, informed and uninformed investors always trade in the opposite directions. (2.20) and (2.17) imply that investors buy less and sell less for the same hedging demand $\Delta RP_s/(\delta\sigma_u^2)$ than the perfect competition case, due to the market illiquidity resulted from the market power. As we show in the next section, all these properties hold in the presence of asymmetric information.

2.4 The equilibrium with asymmetric information

We now assume that both $\hat{F}$ and $\hat{X}_I$ are only observable to the informed investors. Therefore, informed investors' trades can be motivated by both liquidity shock and private information. As before, we first consider the perfect competition case without market making cost.

2.4.1 Perfect competition with asymmetric information

Let P_a^* denote the competitive equilibrium price with asymmetric information. The optimal demand of an informed investor is then

$$\theta_I^* = \frac{\bar{V} + \hat{S} - P_a^*}{\delta\sigma_u^2}, \tag{2.23}$$

where $\hat{S} \equiv \hat{F} + \hat{H}_I$ and $\hat{S}/(\delta\sigma_u^2)$ measures the combined demand from private information about the expected payoff and hedging needs. (2.23) implies that the reservation price for I investors is

$$P_{Ia}^R = \bar{V} + \hat{S} - \delta\sigma_u^2\bar{\theta}, \tag{2.24}$$

which is, as expected, the same as the reservation price P_I^R in the symmetric information case.

Since the informed investor's demand is a monotonically increasing function of $\hat{S}$, his order reveals the value of $\hat{S}$ to market makers. Thus we conjecture that the equilibrium price depends on $\hat{S}$. Since the uninformed investors can then infer the value of $\hat{S}$ from the market price, the information sets for the informed, the uninformed investors and market makers are $\mathcal{I}_I = \{\hat{F}, \hat{X}_I, P_a^*\}$ and $\mathcal{I}_U = \mathcal{I}_M = \{P_a^*\} = \{\hat{S}\}$ respectively. Therefore, the uninformed investor's problem is

$$\max_{\theta_U} -e^{-\delta\bar{\theta}P_a^* + \delta\theta_U(P_a^* - \bar{V}) + \frac{1}{2}\delta^2\theta_U^2\sigma_u^2} \times E[e^{-\delta\theta_U\hat{F}}|\mathcal{I}_U]. \tag{2.25}$$

Let $\sigma_H^2 \equiv \delta^2\sigma_{uN}^2\sigma_I^2$, assumed to be strictly positive, be the variance of the premium for hedging $\hat{H}_I$. Then the conditional expectation of $\hat{F}$ is

$$E[\hat{F}|\hat{S}] = \bar{F} + \frac{\sigma_F^2(\hat{S} - \bar{F})}{\sigma_F^2 + \sigma_H^2}, \tag{2.26}$$

and the conditional variance of $\hat{F}$ is

$$\text{Var}[\hat{F}|\hat{S}] = \frac{\sigma_F^2\sigma_H^2}{\sigma_F^2 + \sigma_H^2}. \tag{2.27}$$

Let

$$\bar{\sigma}_u^2 = \text{Var}[\tilde{V}|\hat{S}] = \sigma_u^2 + \frac{\sigma_H^2\sigma_F^2}{\sigma_F^2 + \sigma_H^2}$$

be the conditional variance of the stock payoff of the uninformed (and the market makers) and

$$\nu = \frac{\bar{\sigma}_u^2}{\sigma_u^2} > 1$$

be the ratio of the conditional variance of the stock payoff of the uninformed to that of the informed. Then, the optimal position of a U-investor is:

$$\theta_U^* = \frac{\bar{V} + \bar{F} + \frac{\sigma_F^2(\hat{S} - \bar{F})}{\sigma_F^2 + \sigma_H^2} - P_a^*}{\delta \bar{\sigma}_u^2}. \tag{2.28}$$

As in the symmetric information case with perfect competition, market makers solve exactly the same problem as the uninformed and have the same reservation price as the uninformed. Equation (2.28) then implies that the reservation price for a U investor and an M investor is now

$$P_{Ua}^R = P_{Ma}^R = \bar{V} + \bar{F} + \frac{\sigma_F^2(\hat{S} - \bar{F})}{\sigma_F^2 + \sigma_H^2} - \delta \bar{\sigma}_u^2 \bar{\theta}. \tag{2.29}$$

Thus the difference in the reservation prices is:

$$\Delta RP_a = P_{Ia}^R - P_{Ua}^R = \hat{H}_I + (\hat{F} - E[\hat{F}|\hat{S}]) + \delta \bar{\theta} Var[\hat{F}|\hat{S}] = \frac{\sigma_H^2}{\sigma_F^2 + \sigma_H^2}(\hat{S} - \bar{F} + \delta \sigma_F^2 \bar{\theta}). \tag{2.30}$$

Remark 1. As $\sigma_F^2 \to 0$, since $\hat{F} \sim \mathbf{N}(\bar{F}, \sigma_F^2)$, we must have $\hat{F} \to \bar{F}$. By (2.27), we have $\bar{\sigma}_u^2 \to \sigma_u^2$ and $\nu \to 1$. Then $P_{ia}^R \to P_i^R$ for $i = I, U$, $\Delta RP_a \to \Delta RP_s$, and therefore the equilibrium with asymmetric information converges to the equilibrium with symmetric information. This convergence holds with or without perfect competition and with or without market making costs. In this sense, the symmetric information case is a special case of the asymmetric information case. Accordingly, we only provide proofs for the asymmetric information case.

As in the symmetric information case, we conjecture that I investors buy and U and M investors sell if and only if $\Delta RP_a > 0$. The following theorem provides the equilibrium price and equilibrium stock holdings, confirming our conjecture.

Theorem 2.3 *In the presence of asymmetric information, there exists a unique competitive equilibrium with stock price being linear in $\hat{S}$, where the equilibrium price is*

$$P_a^* = \frac{\nu N_I}{N_a} P_{Ia}^R + \frac{N_U}{N_a} P_{Ua}^R + \frac{N_M}{N_a} P_{Ma}^R = \bar{V} + \bar{F} - \delta(\sigma_F^2 + \sigma_u^2)\bar{\theta} + \left(\frac{\sigma_F^2}{\sigma_H^2} + \frac{\nu N_I}{N_a} \right) \Delta RP_a, \tag{2.31}$$

and the investors' optimal stock positions are given by

$$\theta_I^* = \bar{\theta} + \left(1 - \frac{\nu N_I}{N_a}\right) \frac{\Delta RP_a}{\delta \sigma_u^2}, \tag{2.32}$$

$$\theta_U^* = \theta_M^* = \bar{\theta} - \frac{\nu N_I}{N_a} \frac{\Delta RP_a}{\delta \bar{\sigma}_u^2}, \tag{2.33}$$

where

$$N_a \equiv \nu N_I + N_M + N_U > N \tag{2.34}$$

is the information weighted total population.

As noted above, when $\sigma_F^2 \to 0$, the equilibrium quantities in Theorem 2.3 converge to those in the symmetric information case. Since the difference in reservation prices ΔRP_a is linear in $\hat{S}$, so is the equilibrium price. This implies that in equilibrium all investors can indeed infer the unique value of $\hat{S}$ from observing the market price.[18] As shown by (2.31), similar to the symmetric information case, the equilibrium price

[18]In our model, market makers observe order flow and can infer how much informed investors are trading. However, they do not know how much of the informed investor's order is due to information on the stock's payoff or how much is due to the hedging demand. This is similar to the set-up of Glosten (1989) and Vayanos and Wang (2009). See Footnote 2.

is again a weighted average of the reservation prices of the investors in the economy. Compared to the symmetric information case, however, since $\nu > 1$, the weight of the reservation price of the informed investors ($\nu N_I/N_a$) is greater, because they have more information about the stock payoff. Accordingly, the information weighted total population N_a puts more weight on the informed investors, which justifies the interpretation of N_a as the information weighted total population. Since *ex ante* $E[\Delta RP_a] = \frac{\sigma_H^2}{\sigma_F^2+\sigma_H^2}\delta\sigma_F^2\bar{\theta} > 0$, on average informed investors buy in equilibrium because the uninformed investors require a risk premium for estimation risk and thus on average value the stock lower than the informed. Since $N_a > N$, Theorems 2.3 and 2.1 imply that the informed trade less than in the symmetric information case given the same difference in reservation prices.

2.4.2 Imperfect competition with asymmetric information

As in the symmetric information case, we first show that when the market-making cost c is not too large, there exists a maximum number of market makers $N_{M_a}^*$ below which a unique equilibrium exists.

Proposition 2.2 *For any given $c \in [0, \bar{c}_a]$, where $\bar{c}_a$ is a monopolistic market maker's equilibrium utility gain from making the market in the presence of asymmetric information, there exists a unique positive integer $N_{M_a}^*$ such that: there is a unique equilibrium if and only if $N_M \leq N_{M_a}^*$, where $N_{M_a}^*$ represents the maximum number of market makers that can exist in equilibrium for the given market-making cost c.*

From now on, we assume $N_M \leq N^*_{M_a}$. Let B^*_a and A^*_a be the equilibrium bid price and ask price respectively. Define

$$C_I \equiv \frac{N_M(N_U + N_M + 1)}{(N_M + 1)\,(N_a + 1)}, \quad C_U \equiv \frac{\nu N_M N_I}{(N_M + 1)\,(N_a + 1)}, \tag{2.35}$$

$$C_M \equiv \frac{\nu N_I}{N_a + 1}, \quad C_D \equiv \frac{N_I(N_U + N_M + 1)}{(N_M + 1)\,(N_a + 1)}, \quad \text{and} \quad \kappa \equiv \frac{\sigma_F^2}{\sigma_H^2} + C_U. \tag{2.36}$$

The following theorem provides the equilibrium bid and ask prices and equilibrium stock holdings in the presence of asymmetric information and market power.

Theorem 2.4 *Suppose $N_M \leq N^*_{M_a}$. In the presence of asymmetric information and market power, we have that*

1. *the equilibrium bid and ask prices are*

$$A^*_a = \bar{V} + \bar{F} - \delta(\sigma_F^2 + \sigma_u^2)\bar{\theta} + \kappa \Delta RP_a + \frac{\Delta RP_a^+}{N_M + 1},$$

$$B^*_a = \bar{V} + \bar{F} - \delta(\sigma_F^2 + \sigma_u^2)\bar{\theta} + \kappa \Delta RP_a - \frac{\Delta RP_a^-}{N_M + 1}.$$

The bid and ask spread is

$$A^*_a - B^*_a = \frac{|\Delta RP_a|}{N_M + 1} = \frac{\sigma_H^2 |\hat{S} - \bar{F} + \delta \sigma_F^2 \bar{\theta}|}{(N_M + 1)(\sigma_F^2 + \sigma_H^2)}, \tag{2.37}$$

and we have

$$A^*_a > P^*_a > B^*_a; \tag{2.38}$$

2. *the equilibrium stock holdings are*

$$\theta^*_I = \bar{\theta} + C_I \frac{\Delta RP_a}{\delta \sigma_u^2}, \quad \theta^*_U = \bar{\theta} - C_U \frac{\Delta RP_a}{\delta \bar{\sigma}_u^2}, \quad \theta^*_M = \bar{\theta} - C_M \frac{\Delta RP_a}{\delta \bar{\sigma}_u^2}; \tag{2.39}$$

the equilibrium quote depths are

$$\alpha_a^* = C_D \left(\left(\frac{\Delta RP_a}{\delta \sigma_u^2} \right)^- + \frac{\nu N_U}{N_U + N_M + 1} \left(\frac{\Delta RP_a}{\delta \bar{\sigma}_u^2} \right)^+ \right), \tag{2.40}$$

and

$$\beta_a^* = C_D \left(\frac{\nu N_U}{N_U + N_M + 1} \left(\frac{\Delta RP_a}{\delta \bar{\sigma}_u^2} \right)^- + \left(\frac{\Delta RP_a}{\delta \sigma_u^2} \right)^+ \right), \tag{2.41}$$

which implies that the equilibrium trading volume is

$$N_M(\alpha_a^* + \beta_a^*) = \frac{N_M N_I (N_M + 2N_U + 1)}{(N_M + 1)(N_a + 1)} \left(\frac{|\Delta RP_a|}{\delta \sigma_u^2} \right). \tag{2.42}$$

Theorem 2.4 implies that both the bid and ask prices increase in ΔRP_a and thus also in $\hat{S}$.[19] As in the symmetric information case, the bid and ask spread is equal to the absolute value of the reservation price difference, divided by $N_M + 1$. Thus the bid-ask spread decreases in competition among market makers and increases in $|\Delta RP_a|$, as illustrated in Figure 2.2.

Both Theorem 2.2 and Theorem 2.4 imply that the bid price converges from below and ask price converges from above to the competitive market equilibrium price as N_M increases when the market making cost is zero, as illustrated in Figure 2.3. Moreover, the equilibrium bid price always increases in N_M while the equilibrium ask price always decreases in N_M due to the more intensive competition among market makers.[20] Because the equilibrium bid (ask) price is lower (higher) than the competitive equilibrium price, the equilibrium trading volume is lower than that in

[19]Theorem 2.4 also implies that in equilibrium, all investors always trade unless the reservation prices of the I investors and U investors are exactly the same. In the more general case where different types of investors have different risk aversions or different liquidity shocks, then some types of non-market-makers might not trade in equilibrium.

[20]If we measure the stock return of a non-market-maker by $\frac{\tilde{V}}{A^*}$, these results suggest that market maker competition increases expected return and return volatility, but does not affect the Sharpe-ratio.

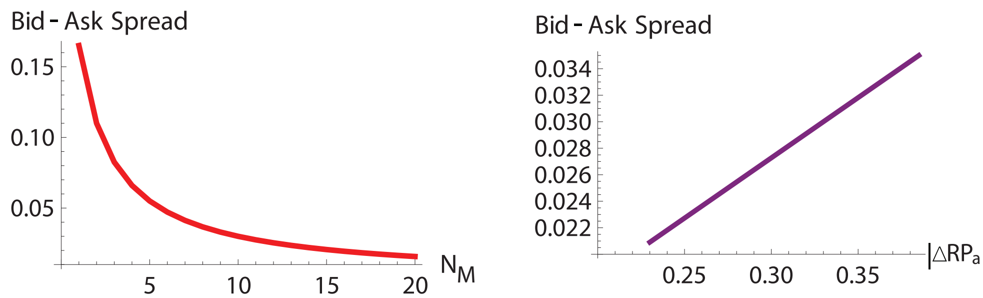

Figure 2.2: Bid and Ask Spread Against N_M and $|\Delta RP_a|$

The default parameter values are: $\bar{\theta} = 1, \delta = 1, N_M = 10, \sigma_F = 0.4, \sigma_H = 0.4, \sigma_u = 0.4, \hat{S} = 0.5$, and $\bar{F} = 0$.

the perfect competition case. Therefore, market power and market making cost increase the spread and decrease the equilibrium trading volume. Thus market power and market making cost tend to make the bid-ask spread negatively correlated with trading volume, as expected. On the other hand, because both spread and trading volume increase in $|\Delta RP_a|$, the bid-ask spread can also be positively correlated with trading volume. Lin, Sanger and Booth (1995) find that trading volume and effective spreads are positively correlated at the beginning and the end of the day. Chordia, Roll, and Subrahmanyam (2001) find that the effective bid-ask spread is positively correlated with trading volume. Our model suggests that these positive correlation may be caused by a change in the valuation difference of investors.

2.5 Comparative statics

In this section, we provide some comparative statics on asset prices, market illiquidity, and welfare.

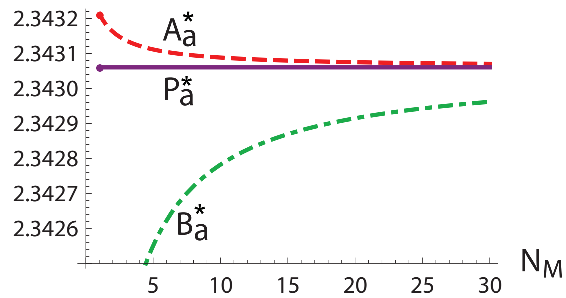

Figure 2.3: The Bid (lower curve), Ask (upper curve) and Competitive Market Equilibrium Price (middle line) When N_M Increases (with fixed $N_U + N_M$)

The default parameter values are: $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_H = 0.4, \sigma_u = 0.4$, $\hat{S} = -0.5$, and $\bar{F} = 0$.

2.5.1 Bid-ask spread, market depths, and trading volume

First we compare bid-ask spread with and without asymmetric information.

Proposition 2.3 $A_a^* - B_a^* < A_s^* - B_s^*$ *iff* $|\Delta RP_a| < |\Delta RP_s|$.

Proposition 2.3 implies that the bid-ask spread with asymmetric information can be smaller than with symmetric information. This occurs if and only if the reservation price difference with asymmetric information is smaller than with symmetric information. Figure 4 shows that this occurs when $\hat{F}$ is relatively small for a given premium for hedging $\hat{H}_I$. To help understand this result, we can rewrite (2.30) as

$$\Delta RP_a = \hat{H}_I + \left(\hat{F} - \bar{F} - \frac{\sigma_F^2(\hat{S} - \bar{F})}{\sigma_F^2 + \sigma_H^2}\right) + \left(\delta\bar{\theta}\frac{\sigma_H^2\sigma_F^2}{\sigma_F^2 + \sigma_H^2}\right), \tag{2.43}$$

where the first term is from the difference in the hedging demand ("hedging demand effect"), the second term is the difference in the estimation of the expected stock payoff ("estimation error effect"), and the third term is the difference in the risk premium required for the estimation risk ("estimation risk effect"). Since only the uninformed

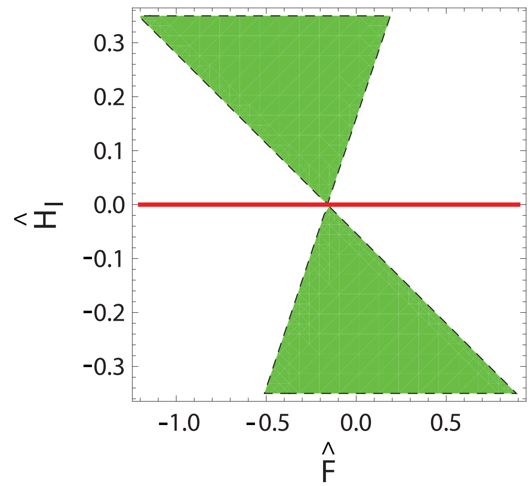

Figure 2.4: The Bid-Ask Spread with and without Asymmetric Information

The colored area denotes those states where the bid-ask spread is narrower with asymmetric information. The default parameter values are: $\bar{\theta} = 1$, $\delta = 1$, $\bar{V} = 3$, $N_M = 10$, $\sigma_H = 0.4$, $\sigma_F = 0.4$, $\sigma_u = 0.4$, and $\bar{F} = 0$.

are subject to the estimation risk and they are risk averse, they require a higher risk premium, which drives their reservation price down and thus the estimation risk effect always drives up the reservation price difference ΔRP_a. In contrast, since the uninformed can overestimate or underestimate the expected stock payoff, the estimation error effect can drive ΔRP_a down or up. When the uninformed overestimate and thus the estimation error effect is negative, which occurs when the realized $\hat{F}$ is relatively small, the net of the estimation error effect and the estimation risk effect can cancel out some of the hedging demand effect. In these cases, the reservation price difference with asymmetric information can be lower than with symmetric information, and accordingly the bid-ask spread with asymmetric information can be lower than with symmetric information. For example, if $\hat{F} = F^* \equiv \bar{F} - \hat{H}_I - \delta\sigma_F^2\bar{\theta}$, then the reservation prices are the same for I and U, because the net of the estimation error effect and the estimation risk effect exactly cancels out the hedging demand effect.

Therefore, the equilibrium bid-ask spread must be zero (and no trade). When $\hat{F}$ is near F^*, then investors' reservation prices are close, and thus the bid-ask spread can be smaller than with symmetric information. On the other hand, when $\hat{F}$ is far from F^*, then investors' reservation prices are significantly different from each other, thus market makers can take advantage of this difference by increasing the bid-ask spread. Therefore, when $\hat{F}$ is far from F^*, the bid-ask spread with asymmetric information is wider than with symmetric information.

The bid-ask spread comparison in Proposition 2.3 is an ex-post result which is dependent on the realized values of $\hat{F}$ and $\hat{H}_I$. We next provide an ex-ante comparison of the expected bid-ask spreads before the realization of $\hat{F}$ and $\hat{H}_I$ with and without asymmetric information.

Proposition 2.4 *1. The expected bid-ask spreads under symmetric and asymmetric information are:*

$$E[A_s^* - B_s^*] = \frac{2}{N_M + 1}\frac{\sigma_H}{\sqrt{2\pi}}, \tag{2.44}$$

$$E[A_a^* - B_a^*] = \frac{\sigma_H^2}{(N_M+1)b^2}\left(\frac{2b}{\sqrt{2\pi}}e^{-\frac{\delta^2\sigma_F^4\bar{\theta}^2}{2b^2}} + \delta\sigma_F^2\bar{\theta}\left(2\boldsymbol{N}\left(\frac{\delta\sigma_F^2\bar{\theta}}{b}\right) - 1\right)\right), \tag{2.45}$$

where $b = \sqrt{\sigma_F^2 + \sigma_H^2}$ and $\boldsymbol{N}$ is the cdf of the standard normal distribution.

2. If $0 < \sigma_H < \frac{\sigma_F}{1+\delta\sigma_F\bar{\theta}\sqrt{\frac{\pi}{2}}}$, then $E[A_a^* - B_a^*] < E[A_s^* - B_s^*]$.

3. If

$$\sigma_F^2 > \frac{2\sigma_H^2}{\delta\sigma_H\bar{\theta}\sqrt{2\pi} - 2} \text{ and } \sigma_H > \frac{2}{\sqrt{2\pi}\delta\bar{\theta}}, \tag{2.46}$$

then $E[A_a^* - B_a^*] > E[A_s^* - B_s^*]$.

Proposition 2.4 shows that in the presence of asymmetric information, if the uncertainty about the hedging demand (σ_H) is small and the uncertainty about the private information (σ_F) is large, then the average bid-ask spread with asymmetric information is smaller than with symmetric information. This is because (1) $E[\Delta RP_a] > 0$ and so the informed buy on average; (2) when the uncertainty of the private information is much greater than that of the hedging demand, the uninformed can significantly overestimate the expected stock payoff and thus the estimation error effect offsets some hedging demand effect and the estimation risk effect, making the expected spread smaller.

Next we further examine how bid-ask spread changes with the degree of information asymmetry. The difference in the stock payoff conditional variances of the informed and the uninformed is

$$\mathrm{Var}(\tilde{V}|\mathcal{I}_U) - \mathrm{Var}(\tilde{V}|\mathcal{I}_I) = \frac{\sigma_F^2 \sigma_H^2}{\sigma_F^2 + \sigma_H^2}.$$

Since uncertainty about the hedging demand is unrelated to stock payoff, we will fix σ_H and use σ_F to measure the degree of information asymmetry. The larger σ_F is, the greater the information asymmetry between the informed and the uninformed is. For example, as noted before, if $\sigma_F \to 0$, then the asymmetric information case converges to the symmetric information case because there would be no uncertainty about stock payoff $\tilde{V}$.[21]

[21] Alternatively, as in Subrahmanyam (1991) and Vayanos and Wang (2009), one can assume that the informed observes a private signal s about the stock payoff $\tilde{V}$ in period 0, where $s = \tilde{V} + \tilde{\varepsilon}$, and $\tilde{\varepsilon}$ is independently normally distributed with mean zero and variance σ_ε^2. Then $Var[\tilde{V}|s] = \frac{(\sigma_u^2+\sigma_F^2)\sigma_\varepsilon^2}{\sigma_u^2+\sigma_F^2+\sigma_\varepsilon^2}$. In our model, $Var[\tilde{V}|\hat{F}] = \sigma_u^2$. So, the measure of the precision of private information, $\sigma_\varepsilon^2 = \sigma_u^2(1 + \frac{\sigma_u^2}{\sigma_F^2})$. Therefore, increasing the precision of the signal (decreasing σ_ε) in the alternative model is qualitatively equivalent to increasing σ_F^2 in our model.

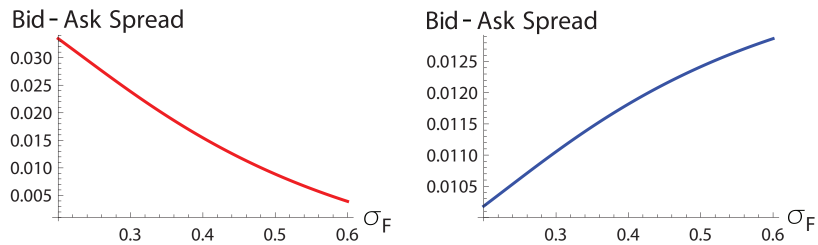

Figure 2.5: The Bid-Ask Spread in Equilibrium Against σ_F

The default parameter values are $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_M = 10, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_u = 0.4, \sigma_H = 0.4$, and $\bar{F} = 0$. $\hat{S} = -0.5$ in the left graph, and $\hat{S} = 0.1$ in the right one.

The following proposition shows that in contrast to most of the existing literature (e.g., Glosten and Milgrom (1985)), the bid-ask spread can decrease as the degree of information asymmetry increases.

Proposition 2.5 $\frac{\partial(A_a^* - B_a^*)}{\partial \sigma_F} < 0$ *iff* $\Delta RP_a < 0$ *or* $\Delta RP_a > \delta \sigma_H^2 \bar{\theta}$.

Proposition 2.5 implies that the bid-ask spread decreases with the information asymmetry if the informed sell or the informed buy a sufficient amount. To help understand the intuition, note that σ_F does not affect the informed's reservation price P_{Ia}^R and we rewrite the reservation price of the uninformed (2.29) as

$$P_{Ua}^R = \bar{V} + \left(\bar{F} + \frac{\sigma_F^2(\hat{S} - \bar{F})}{\sigma_F^2 + \sigma_H^2} \right) - \delta \left(\sigma_u^2 + \frac{\sigma_F^2 \sigma_H^2}{\sigma_F^2 + \sigma_H^2} \right) \bar{\theta}. \qquad (2.47)$$

If $\Delta RP_a < 0$, then the reservation price of the uninformed is above that of the informed and I investors sell, U investors buy. In addition, in this case $\hat{S} - \bar{F} < 0$ by (2.30), which implies that as σ_F increases, the conditional mean (the second term in (2.47)) decreases and the risk premium (the third term in (2.47)) increases. Therefore as σ_F increases, the reservation price of the uninformed decreases and gets closer to the reservation price of the informed as long as $\Delta RP_a < 0$, which reduces the reservation price difference and hence also the spread.

If $\Delta RP_a > \delta\sigma_H^2\bar{\theta}$, then the reservation price of the informed is above that of the uninformed and I investors buy, U investors sell. As σ_F increases, the risk premium still increases and thus drives down the reservation price of the uninformed and makes it further away from the reservation price of the informed. However, because in this case $\hat{S} - \bar{F} > \delta\sigma_H^2\bar{\theta} > 0$ by (2.30), the conditional mean *increases* as σ_F increases, which drives up the reservation price of the uninformed and thus makes it closer to the reservation price of the informed. When $\hat{S} - \bar{F} > \delta\sigma_H^2\bar{\theta}$, the effect of σ_F on the conditional mean dominates its effect on the risk premium and thus drives up the reservation price of the uniformed. Therefore, the reservation price difference decreases and so does the spread. On the other hand, if $0 < \Delta RP_a < \delta\sigma_H^2\bar{\theta}$, then the effect of σ_F on the risk premium dominates its effect on conditional mean and thus drives down the reservation price of the uniformed. Therefore, the reservation price difference increases and so does the spread. These cases are shown in Figure 2.5.

Next we examine how market depths, trading volume and net order size change with information asymmetry, reservation price difference, the number of market makers and the stock payoff volatility.

Proposition 2.6 *1.* $\alpha_a^* > \alpha_s^*$*,* $\beta_a^* > \beta_s^*$*, and* $N_M(\alpha_a^* + \beta_a^*) > N_M(\alpha_s^* + \beta_s^*)$ *iff* $|\Delta RP_a| > \frac{N_a+1}{N+1}|\Delta RP_s|$.

2. $\frac{\partial(N_M(\alpha_a^*+\beta_a^*))}{\partial\sigma_F} > 0$ *iff*

$$0 < \Delta RP_a < \frac{(N_a+1)\delta\sigma_u^2\bar{\theta}}{N_I + (N+1)\sigma_u^2/\sigma_H^2}(< \delta\sigma_H^2\bar{\theta}).$$

The same is true for the net order size $|\alpha_a^* - \beta_a^*|$.

3. *As* $|\Delta RP_a|$ *increases, the bid depth* α_a^**, the ask depth* β_a^**, the net order size* $|\alpha_a^* - \beta_a^*|$*, and the trading volume* $N_M(\alpha_a^* + \beta_a^*)$ *all increase.*

4. *Fixing* $N_U + N_M$*, as* N_M *increases, both the bid depth and the ask depth decrease, but trading volume increases when the number of uninformed investors is large.*

5. *As the stock payoff volatility* σ_u *increases, the bid depth, the ask depth, the net order size, and the trading volume all decrease.*

Part 1 of Proposition 2.6 shows that the equilibrium market depths and trading volume can be higher with asymmetric information when the reservation price difference is large relative to the symmetric information case. Intuitively, with a greater reservation price difference, the difference between a buyer's (seller's) reservation price and the ask (bid) price also increases, so the trading demand of the investor increases and therefore both the market depths and the market trading volume increase. In addition, Part 2 suggests that both the trading volume and the net order size can increase with information asymmetry when I investors buy a moderate amount. In contrast to Easley and O'Hara (1987, 1992), Proposition 2.5 and Part 2 of Proposition 2.6 imply that net order size can be negatively correlated with the bid-ask spread. For example, if

$$\frac{(N_a + 1)\delta\sigma_u^2\bar{\theta}}{N_I + (N+1)\sigma_u^2/\sigma_H^2} < \Delta RP_a < \delta\sigma_H^2\bar{\theta},$$

then as information asymmetry σ_F increases, the net order size decreases by Part 2 of Proposition 2.6, but bid-ask spread increases by Proposition 2.5. However, since the bid-ask spread also increases with $|\Delta RP_a|$, Part 3 implies that the bid-ask spread and net order size can also be positively correlated. A typical justification of this positive correlation (e.g., Easley and O'Hara (1987, 1992)) is that as the net order size increases, the adverse effect of information asymmetry increases and thus the bid-ask spread increases. In contrast, we view the net order size as the net trade that the market makers are willing to make, because $|\alpha_a^* - \beta_a^*| = |\theta_M^* - \bar{\theta}|$. As the reservation price difference increases, the spread increases and thus the market makers are willing to sell or buy more in the net at the better price.

Part 4 shows that when the number of the uninformed is large, then competition increases market trading volume, although it decreases the quote depths of individual market maker. Part 5 implies that as the stock payoff volatility increases, market depths, trading volume, and net order size all decrease due to the increased risk.

2.5.2 Value of private information and utility loss due to market power

In the standard microstructure models with noise investors, the value of private information to the informed is always positive, because the informed can always profit from trading with noise investors. The following result shows it can be negative in our model because the uninformed optimally react to market prices.

Proposition 2.7 *1. The informed investors are worse off in the asymmetric information case than in the symmetric information case iff* $|\Delta RP_a| < \frac{N_a+1}{N+1}|\Delta RP_s|$.

2. The expected utility of the informed decreases with information asymmetry σ_F *iff*

$$\Delta RP_a < 0 \text{ or } \Delta RP_a > \frac{(N_a+1)\delta\sigma_u^2\bar{\theta}}{N_I + (N+1)\sigma_u^2/\sigma_H^2}.$$

Proposition 2.7 implies that for any given liquidity shock, if the private information $\hat{F}$ is relatively small in magnitude, then the informed are worse off with asymmetric information, and therefore the value of the private information is negative in these cases (similar to Figure 4). Intuitively, when the private information $\hat{F}$ is relatively small, the uninformed over-attribute the informed's trading to the private information $\hat{F}$ and thus the market prices are worse for the informed investors than in the symmetric information case. In addition, the expected utility of the informed can decrease with information asymmetry when I investors sell or buy a large amount, because the uninformed over-attribute more the informed's trading to the private information as the information asymmetry increases.

Next we analyze the welfare loss due to market power. To isolate the effect of market power on welfare, in this subsection, we assume that the market-making cost $c = 0$. Let U_i and $\bar{U}_i$ denote the utility of i ($i = I, U, M$) investors with imperfect and perfect competition respectively and f_i and $\bar{f}_i$ be the corresponding certainty equivalent wealth, i.e., $U_i = -\exp(-\delta f_i)$, and $\bar{U}_i = -\exp(-\delta \bar{f}_i)$.

Definition 2.3 *The certainty equivalent wealth loss of a type i investor ($i = I, U, M$) due to market power is $\bar{f}_i - f_i$.*

The following proposition shows how market power affects the welfare of the informed, the uninformed and market makers.[22]

[22]Closed-form expressions for the equivalent wealth losses are available from the authors.

Proposition 2.8 *1. Market makers' market power makes themselves better off and non-market-makers worse off. More importantly, the sum of their welfare is reduced.*

2. Both the certainty equivalent wealth losses for I and U investors and the certainty equivalent wealth gain for M investors decrease with N_M, and increase with $|\Delta RP_a|$.

Not surprisingly, market makers benefit from their market power by earning a higher bid-ask spread. Other investors are worse off because they have to trade at a worse price. More importantly, Proposition 2.8 shows that market makers' welfare gain is less than the welfare loss of the other investors. This is because when determining their trades, market makers do not internalize other investors' losses. As N_M increases, market power decreases and thus both market makers' utility gain and other investors' utility loss decrease. This implies that there exists a Pareto improvement wealth transfer and market regulation mechanism that limits market bid-ask spreads and depths and makes all investors (including market makers) strictly better off. It also suggests the importance of promoting competition among market makers on improving market liquidity and social welfare.

If the difference between reservation prices increases, then investors trade more with market makers and therefore investors' certainty equivalent wealth loss increases as we can see in Figure 2.6.

Next we compare investors' total certainty equivalent wealth loss due to market power with and without asymmetric information and examine how the loss changes

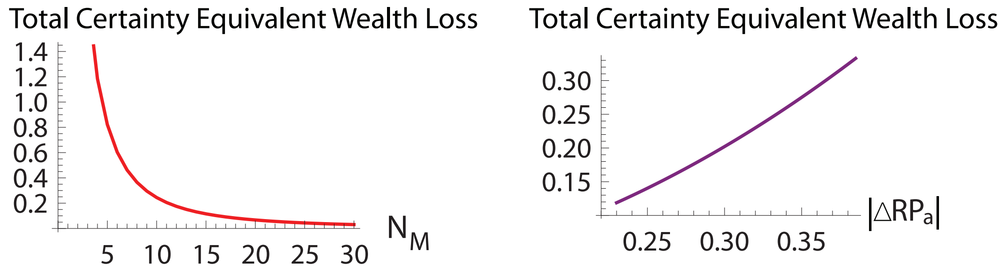

Figure 2.6: The Total Certainty Equivalent Wealth Loss due to Market Power Against N_M and $|\Delta RP_a|$.

The default parameter values are: $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_M = 10, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_H = 0.4, \sigma_u = 0.4, \hat{S} = 0.5$, and $\bar{F} = 0$.

with information asymmetry. Since the bid-ask spread can decrease with information asymmetry and investors' welfare can increase when the spread is smaller, one expects that the presence of asymmetric information may *decrease* the welfare loss from market power. The following proposition confirms this expectation.

Proposition 2.9 *Let WL_a and WL_s be the certainty equivalent wealth loss due to market power with and without asymmetric information respectively, then*

1. *$WL_a < WL_s$ if and only if $|\Delta RP_a| < C_1|\Delta RP_s|$, where $C_1 \geq 1$ is as defined in (2.68) in the Appendix.*

2. *If $\Delta RP_a < 0$, then the total certainty equivalent wealth loss due to market power decreases with information asymmetry σ_F.*

Proposition 2.9 implies that the presence of asymmetric information indeed may *decrease* the investors' total certainty equivalent wealth loss due to market power. This decrease typically occurs when $\hat{F}$ is relatively small for a given hedging premium

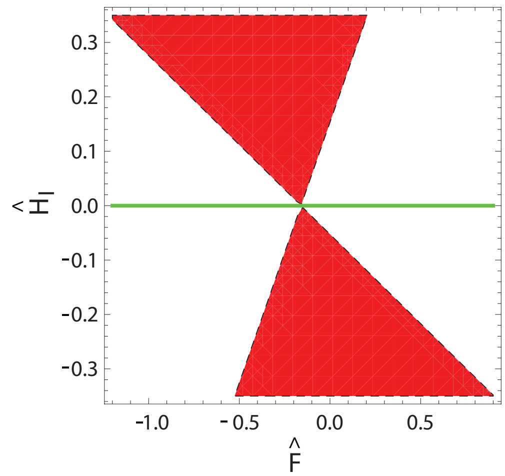

Figure 2.7: The Total Certainty Equivalent Wealth Loss with and without Asymmetric Information.

The colored area denotes those states where the total certainty equivalent wealth loss is greater with symmetric information. The default parameter values are: $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_M = 10, N_I = 100, N_U = 1000, \sigma_H = 0.4, \sigma_F = 0.4, \sigma_u = 0.4$ and $\bar{F} = 0$.

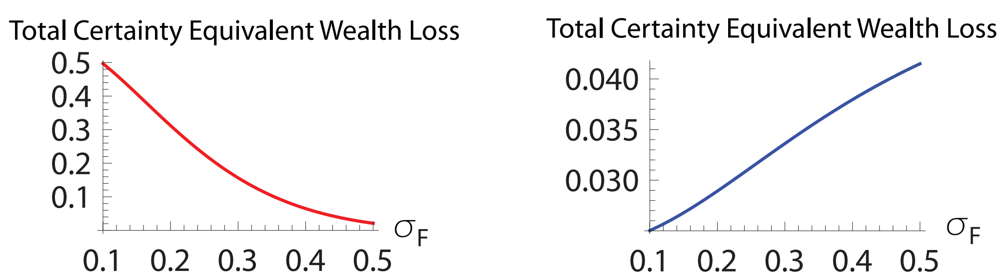

Figure 2.8: The Total Certainty Equivalent Wealth Loss due to Market Power Against σ_F.

The default parameter values are: $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_M = 10, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_H = 0.4, \sigma_u = 0.4$, and $\bar{F} = 0$. $\hat{S} = -0.5$ in the left graph, and $\hat{S} = 0.1$ in the right graph.

$\hat{H}_I$, as illustrated in Figure 2.7. In most of these cases, the bid-ask spread is smaller in the presence of asymmetric information.

In addition, Part 2 of Proposition 2.9 shows that the total certainty equivalent wealth loss due to the market power can decrease in information asymmetry σ_F, as illustrated in Figure 2.8. Intuitively, since the bid-ask spread can increase or decrease with information asymmetry, so can the total welfare loss.

2.5.3 Maximum number of market makers in equilibrium

As shown in Propositions 1 and 2, as long as the market making cost is not too large, there exists a positive maximum number of market makers in equilibrium.

Figure 2.9 shows that as the market making cost c or the competition increases, the maximum number decreases, because market making becomes less profitable. Figure 2.10 shows the same cases as Figure 2.5. When $|\hat{S}|$ is large, the maximum number of market makers decreases with information asymmetry, because the spread decreases with information asymmetry (as shown in Figure 2.5) and the profitability of market making declines.

2.6 Concluding remarks

In this paper we develop a novel framework to study how asymmetric information, competition among market makers, and risk aversion affect equilibrium illiquidity and asset pricing. All our results are obtained in closed-form. In contrast to most of the existing models, our model can help explain many puzzling empirical findings such as

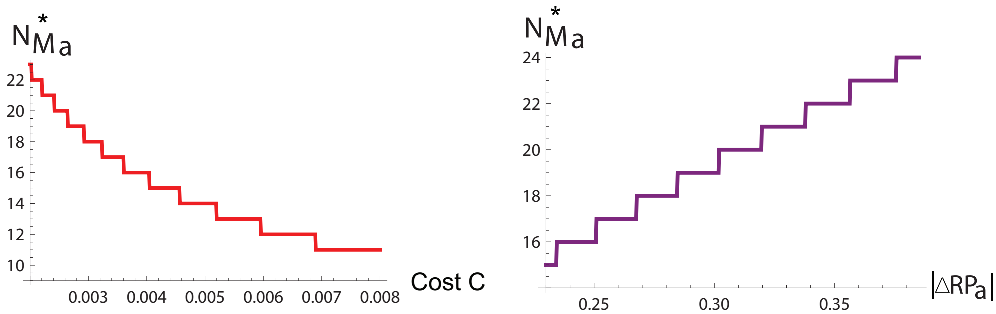

Figure 2.9: The Maximum Number of Market Makers in Equilibrium Against Market Making Utility Cost c and $|\Delta RP_a|$

The default parameter values are $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_u = 0.4, \sigma_H = 0.4, c = 0.005, \hat{S} = -0.5$, and $\bar{F} = 0$.

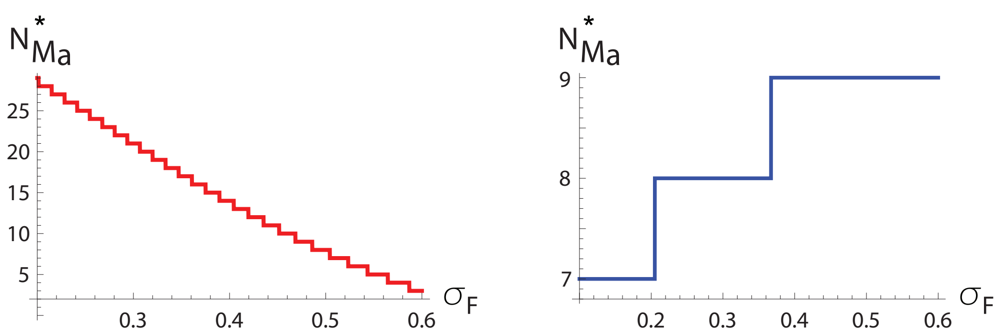

Figure 2.10: The Maximum Number of Market Makers in Equilibrium Against σ_F

The default parameter values are $\bar{\theta} = 1, \delta = 1, \bar{V} = 3, N_M = 10, N_I = 100, N_U = 1000, \sigma_F = 0.4, \sigma_u = 0.4, \sigma_H = 0.4$, and $\bar{F} = 0$. $\hat{S} = -0.5$ in the left graph, and $\hat{S} = 0.1$ in the right one.

the bid-ask spread may decrease with asymmetric information, and trading volume may be positively correlated with market illiquidity. The main departure from the existing literature where market makers directly compete through prices is that in our model market makers choose simultaneously how much to sell at the ask and how much to buy at the bid through Cournot competition and then the market clearing condition determines both the bid and the ask prices. Our new framework is flexible, tractable and can be applied to analyze many interesting questions on the effect of asymmetric information, competition, trading constraints on asset prices and market illiquidity.

2.7 Appendix

Proof of Theorem 2.1: This is a special case of the proof of Theorem 2.3 with $\sigma_F^2 = 0$ and $\bar{F} = \hat{F}$. *Q.E.D.*

Proof of Theorem 2.2: This is a special case of the proof of Theorem 2.4 with $\sigma_F^2 = 0$ and $\bar{F} = \hat{F}$. *Q.E.D.*

Proof of Proposition 2.1: This is a special case of the proof of Proposition 2.2 with $\sigma_F^2 = 0$ and $\bar{F} = \hat{F}$. *Q.E.D.*

Proof of Proposition 2.2: Let U_M denote the utility of the N_M^{th} potential market maker with N_U uninformed investors and U_U denote the utility of the $(N_U+1)^{th}$ uninformed investor with $N_M - 1$ market makers. We compare their certainty equivalent wealths f_M and f_U, where f_M and f_U are such that $U_M = -\exp(-\delta f_M) - c$, and $U_U = -\exp(-\delta f_U)$. To save space, we prove the case when U investors are buyers in equilibrium. The proof for the other case is similar. First we assume that given N_M and N_U, all potential market makers choose to be market makers, then the proof of Theorem 2.4 implies the existence of a unique (linear) equilibrium and provides explicit expressions for the equilibrium. Later we show that when the market making cost c is small enough, the assumption indeed holds. Under the assumption, we have

$$f_M = (\beta_a^* A_a^* - \alpha_a^* B_a^*) + (\bar{\theta} + \alpha_a^* - \beta_a^*)\left(\bar{V} + \frac{\sigma_F^2 \hat{S} + \sigma_H^2 \bar{F}}{\sigma_F^2 + \sigma_H^2}\right) - \frac{1}{2}\delta(\bar{\theta} + \alpha_a^* - \beta_a^*)^2 \sigma_u^2, \quad (2.48)$$

and

$$f_U = (\bar{\theta} - \theta_U^*)A_a^* + \theta_U^*\left(\bar{V} + \frac{\sigma_F^2 \hat{S} + \sigma_H^2 \bar{F}}{\sigma_F^2 + \sigma_H^2}\right) - \frac{1}{2}\delta {\theta_U^*}^2 \sigma_u^2, \quad (2.49)$$

where $\alpha_a^*, \beta_a^*, A_a^*$ and B_a^* are as given in Theorem 2.4.

$$f_M(N_M, N_U) - f_U(N_M - 1, N_U + 1) = N_I(\Delta RP_a)^2 \times$$
$$\frac{2N_M^2(N_M + N_U + 1)^2 + \nu N_I(N_M^2(2N_M + 2N_U + 3) - 1)}{2N_M^2(N_M + 1)^2(N_a + 1)^2 \delta \sigma_u^2} > 0,$$

and $f_M(N_M, N_U) \to f_U(N_M - 1, N_U + 1)$ as $N_M \to \infty$. It can be verified that $f_M(N_M, N_U) - f_U(N_M - 1, N_U + 1)$ strictly decreases in N_M. Therefore, for small enough $c > 0$, $-e^{-\delta f_M(N_M,\, N_U)} - c - \left(-e^{-\delta f_U(N_M - 1,\, N_U + 1)}\right) > 0$ when $N_M < \infty$, and $-e^{-\delta f_M(N_M,\, N_U)} - c < -e^{-\delta f_U(N_M - 1,\, N_U + 1)}$, when $N_M \to \infty$. It follows that for small enough $c > 0$, there is a unique $0 < x^* < \infty$, such that $-e^{-\delta f_M(x^*,\, N_U)} - c =$

$-e^{-\delta f_U(x^*-1,\ N_U+1)}$. Then for any $N_M \leq N^*_{Ma} \equiv [x^*]$, all potential market makers choose to be market makers and then Theorem 2.4 implies there exists a unique equilibrium. It can be shown that for a fixed $N_U + N_M$, the equivalent wealth of a market maker decreases with competition (N_M) and the equivalent wealth of an uninformed investor increases with competition, i.e., $f_M(n,\ N_U + N_M - n)$ is decreasing in n and $f_U(n-1,\ N_U + N_M - n + 1)$ is increasing in n. This implies that a monopolistic market maker's equivalent wealth gain from making the market is the greatest. Let $\bar{c} = \left(-e^{-\delta f_M(1,\ N_U+N_M-1)}\right) - \left(-e^{-\delta f_U(0,\ N_U+N_M)}\right)$ be the monopolistic market maker's equivalent wealth gain from making the market, where $f_U(0,\ N_U + N_M)$ is the certainty equivalent wealth of the $(N_U + N_M)^{th}$ uninformed investor when there is no trade. Then for any $c < \bar{c}$, there exists a unique N^*_{Ma} such that for any $N_M \leq N^*_{Ma}$, there exists an equilibrium where all potential market makers choose to be market makers in equilibrium and then Theorem 2.4 implies the existence of a unique equilibrium. *Q.E.D.*

Proof of Theorem 2.3: From our assumption that $\hat{F}$ and $\sigma_{uN}\hat{X}_I$ are *i.i.d* normally distributed, we know that $\hat{S}$ is normally distributed with mean $\bar{F}$ and variance $\sigma_F^2 + \sigma_H^2$. The covariance between $\hat{F}$ and $\hat{S}$ is $Cov(\hat{F}, \hat{S}) = Cov(\hat{F}, \hat{F} - \delta\sigma_{uN}\hat{X}_I) = Var(\hat{F}) = \sigma_F^2$, therefore, the correlation coefficient of $\hat{F}$ and $\hat{S}$ is $\rho_{\hat{F},\hat{S}} = \frac{\sigma_F}{\sqrt{\sigma_F^2+\sigma_H^2}}$. $E[\hat{F}|\hat{S}] = \frac{\sigma_F^2\hat{S}+\sigma_H^2\bar{F}}{\sigma_F^2+\sigma_H^2}$, and $Var[\hat{F}|\hat{S}] = \frac{\sigma_H^2\sigma_F^2}{\sigma_F^2+\sigma_H^2}$. The optimal stock holding of an uninformed investor is given in (2.28), and similarly, for an market maker we get:

$$\theta^*_M = \frac{\bar{V} + \frac{\sigma_F^2\hat{S}+\sigma_H^2\bar{F}}{\sigma_F^2+\sigma_H^2} - P^*_a}{\delta\bar{\sigma}_u^2}. \tag{2.50}$$

Substituting (2.23), (2.28) and (3.47) into the market clearing condition $N_I\theta_I^* + N_U\theta_U^* + N_M\theta_M^* = N\bar{\theta}$, we get the equilibrium stock price P_a^*. Substituting P_a^* into (2.23), (2.28) and (3.47), we can get I, U and M investors' optimal stock holdings. *Q.E.D.*

Proof of Theorem 2.4: We prove the case when $\Delta RP_a < 0$. In this case, we conjecture that I investors sell at the bid and U investors buy at the ask. Given bid price B and ask price A, the optimal demand of I and U are:

$$\theta_I^* = \frac{\bar{V} + \hat{S} - B}{\delta\sigma_u^2} \quad and \quad \theta_U^* = \frac{\bar{V} + \frac{\sigma_F^2\hat{S}+\sigma_H^2\bar{F}}{\sigma_F^2+\sigma_H^2} - A}{\delta\bar{\sigma}_u^2}. \tag{2.51}$$

Substituting (2.51) into the market clearing conditions (2.3), we get that the market clearing bid and ask prices are:

$$A = \bar{V} + \frac{\sigma_F^2\hat{S} + \sigma_H^2\bar{F}}{\sigma_F^2 + \sigma_H^2} - \delta\bar{\sigma}_u^2\bar{\theta} - \frac{\delta\bar{\sigma}_u^2}{N_U}\sum_{j=1}^{N_M}\beta_j, \ \ and \ \ B = \bar{V} + \hat{S} - \delta\sigma_u^2\bar{\theta} + \frac{\delta\sigma_u^2}{N_I}\sum_{j=1}^{N_M}\alpha_j, \tag{2.52}$$

where α_j and β_j are the optimal shares of stock M_j choose to buy from I investors and sell to U investors respectively. Market maker M_j's problem is:

$$\min_{\alpha_j,\beta_j} -\delta(\beta_j A - \alpha_j B) - \delta(\bar{\theta} + \alpha_j - \beta_j)\left(\bar{V} + \frac{\sigma_F^2\hat{S} + \sigma_H^2\bar{F}}{\sigma_F^2 + \sigma_H^2}\right) + \frac{1}{2}\delta^2\bar{\sigma}_u^2(\bar{\theta} + \alpha_j - \beta_j)^2, \tag{2.53}$$

where A and B are the market clearing prices given in (2.52). F.O.C with respect to α_j gives us:

$$\frac{\sigma_H^2}{\sigma_F^2 + \sigma_H^2}(\hat{S} - \bar{F}) + \delta\left(\bar{\sigma}_u^2 - \sigma_u^2\right)\bar{\theta} + \frac{\delta\sigma_u^2}{N_I}\sum_{j=1}^{N_M}\alpha_j + \left(\frac{\sigma_u^2}{N_I} + \bar{\sigma}_u^2\right)\delta\alpha_j - \delta\bar{\sigma}_u^2\beta_j = 0. \tag{2.54}$$

Sum all, we get:

$$\frac{N_M\sigma_H^2}{\sigma_F^2+\sigma_H^2}(\hat{S}-\bar{F})+N_M\delta\left(\bar{\sigma}_u^2-\sigma_u^2\right)\bar{\theta}+\left(\frac{(N_M+1)\sigma_u^2}{N_I}+\bar{\sigma}_u^2\right)\delta\sum_{j=1}^{N_M}\alpha_j-\delta\bar{\sigma}_u^2\sum_{j=1}^{N_M}\beta_j=0. \tag{2.55}$$

F.O.C with respect to β_j, we get:

$$\frac{\delta}{N_U}\sum_{j=1}^{N_M}\beta_j-\delta(\alpha_j-\beta_j)+\frac{\delta}{N_U}\beta_j=0. \tag{2.56}$$

Sum all, we get:

$$\sum_{j=1}^{N_M}\alpha_j=\frac{N_U+N_M+1}{N_U}\sum_{j=1}^{N_M}\beta_j. \tag{2.57}$$

Substituting (2.57) into (2.55), we get

$$\sum_{j=1}^{N_M}\beta_j=-\frac{N_MN_IN_U}{(N_M+1)\,(N_a+1)}\frac{\sigma_H^2\left(\hat{S}-\bar{F}+\delta\sigma_F^2\bar{\theta}\right)}{(\sigma_F^2+\sigma_H^2)\delta\sigma_u^2}=-\frac{N_MN_IN_U}{(N_M+1)\,(N_a+1)}\frac{\Delta RP_a}{\delta\sigma_u^2}. \tag{2.58}$$

Substituting (2.58) into (2.52), we can get the equilibrium ask and bid price A_a^* and B_a^*. And then substituting A_a^* and B_a^* into (2.51), we can get the optimal stock holdings of I and U investors as stated in Theorem 2.4.

It is not difficult to derive that $A_a^* < P_{aU}^*$ and $B_a^* > P_{aI}^*$ are equivalent to $\hat{S} < \bar{F}-\delta\sigma_F^2\bar{\theta}$ which is exactly the condition we conjecture for I investors to sell and U investors to buy. Similarly, we can prove the other case of this Theorem when I investors buy and U investors sell. *Q.E.D.*

Proof of Proposition 2.3: This is direct from Theorem 2.4. *Q.E.D.*

Proof of Proposition 2.4:

$$E[A_s^* - B_s^*] = \frac{1}{N_M + 1} E|\hat{H}_I| = \frac{2}{N_M + 1} \frac{\sigma_H}{\sqrt{2\pi}}. \tag{2.59}$$

We know from previous section,

$$f(\hat{S}) = \frac{1}{\sqrt{2\pi} b} e^{-\frac{(\hat{S} - \bar{F})^2}{2b^2}}, \ \hat{S} = \hat{F} + \hat{H}_I.$$

Therefore,

$$\begin{aligned} & E[A_a^* - B_a^*] \\ = & \frac{\sigma_H^2}{(N_M + 1) b^2} \int_{-\infty}^{+\infty} |\hat{S} - \bar{F} + \delta \sigma_F^2 \bar{\theta}| f(\hat{S}) d\hat{S} \\ = & \frac{\sigma_H^2}{(N_M + 1) b^2} \left(\frac{2b}{\sqrt{2\pi}} e^{-\frac{\delta^2 \sigma_F^4 \bar{\theta}^2}{2b^2}} + \delta \sigma_F^2 \bar{\theta} \left(2\mathbf{N} \left(\frac{\delta \sigma_F^2 \bar{\theta}}{b} \right) - 1 \right) \right). \end{aligned}$$

We use the fact that $\frac{x}{1+x^2} n(x) < 1 - \mathbf{N}(x) < \frac{n(x)}{x}$, for $x \geq 0$, where $n(x)$ is the *pdf* for standard normal distribution. We have:

$$E[A_a^* - B_a^*] > \frac{\delta \sigma_H^2 \sigma_F^2 \bar{\theta}}{(N_M + 1)(\sigma_F^2 + \sigma_H^2)}.$$

And we have

$$E[A_a^* - B_a^*] \leq \frac{\sigma_H^2}{(N_M + 1)(\sigma_F^2 + \sigma_H^2)} E(|\hat{F} - \bar{F}| + |\hat{H}_I| + \delta \sigma_F^2 \bar{\theta})$$

$$= \frac{\sigma_H^2}{(N_M + 1)(\sigma_F^2 + \sigma_H^2)} \left(\frac{2\sigma_F}{\sqrt{2\pi}} + \frac{2\sigma_H}{\sqrt{2\pi}} + \delta \sigma_F^2 \bar{\theta} \right)$$

Therefore, if $\frac{\delta \sigma_H^2 \sigma_F^2 \bar{\theta}}{\sigma_H^2 + \sigma_F^2} > \frac{2\sigma_H}{\sqrt{2\pi}}$, which is equivalent to (2.46), then $E[A_a^* - B_a^*] > E[A_s^* - B_s^*]$, and if $0 < \sigma_H < \frac{\sigma_F}{1 + \delta \bar{\theta} \sigma_F \sqrt{\frac{\pi}{2}}}$, then $E[A_a^* - B_a^*] < E[A_s^* - B_s^*]$. *Q.E.D.*

Proof of Proposition 2.5: If $\hat{S} < \bar{F} - \delta\sigma_F^2\bar{\theta}$, then $\frac{\partial(A_a^* - B_a^*)}{\partial\sigma_F} = \frac{2\sigma_F\sigma_H^2(\hat{S} - \bar{F} - \delta\sigma_H^2\bar{\theta})}{(\sigma_F^2 + \sigma_H^2)^2(N_M+1)} < 0$. If $\hat{S} > \bar{F} - \delta\sigma_F^2\bar{\theta}$, then $\frac{\partial(A_a^* - B_a^*)}{\partial\sigma_F} = -\frac{2\sigma_F\sigma_H^2(\hat{S} - \bar{F} - \delta\sigma_H^2\bar{\theta})}{(\sigma_F^2 + \sigma_H^2)^2(N_M+1)} < 0$, when $\hat{S} > \bar{F} + \delta\sigma_H^2\bar{\theta}$. *Q.E.D.*

Proof of Proposition 2.6: Part 1 is direct from Theorems 2.2 and 2.4.

For Part 2, if $\Delta RP_a < 0$, i.e., $\hat{S} < \bar{F} - \delta\sigma_F^2\bar{\theta}$, then

$$\frac{\partial\left(N_M(\alpha_a^* + \beta_a^*)\right)}{\partial\sigma_F} =$$

$$-\frac{2\sigma_F N_M N_I(1 + N_M + 2N_U)\sigma_H^2(-(\sigma_H^2 N_I + (N+1)\sigma_u^2)(\hat{S} - \bar{F}) + (N+1)\delta\sigma_H^2\sigma_u^2\bar{\theta})}{\delta(N_M+1)(\sigma_F^2(N+1)\sigma_u^2 + \sigma_H^2(\sigma_F^2 N_I + (N+1)\sigma_u^2))^2} < 0, \tag{2.60}$$

and if $\Delta RP_a > 0$, i.e., $\hat{S} > \bar{F} - \delta\sigma_F^2\bar{\theta}$, then

$$\frac{\partial\left(N_M(\alpha_a^* + \beta_a^*)\right)}{\partial\sigma_F} =$$

$$\frac{2\sigma_F N_M N_I(1 + N_M + 2N_U)\sigma_H^2(-(\sigma_H^2 N_I + (N+1)\sigma_u^2)(\hat{S} - \bar{F}) + (N+1)\delta\sigma_H^2\sigma_u^2\bar{\theta})}{\delta(N_M+1)(\sigma_F^2(N+1)\sigma_u^2 + \sigma_H^2(\sigma_F^2 N_I + (N+1)\sigma_u^2))^2}, \tag{2.61}$$

which is positive if $\Delta RP_a < \frac{(N_a+1)\delta\sigma_u^2\bar{\theta}}{N_I + (N+1)\sigma_u^2/\sigma_H^2}$.

Part 3 follows directly from Theorem 2.4 and noting that the net order size is

$$|\alpha_a^* - \beta_a^*| = \frac{N_I|\Delta RP_a|}{(\nu N_I + N_U + N_M + 1)\delta\sigma_u^2}.$$

For Part 4, fixing $N_U + N_M$, we have

$$\frac{\partial\alpha_a^*}{\partial N_M} = \frac{\partial\beta_a^*}{\partial N_M} = -\frac{N_I(1 + N_U + N_M)|\Delta RP_a|}{(N_M+1)^2(N_a+1)\delta\sigma_u^2} < 0,$$

$$\frac{\partial \sum_{j=1}^{N_M}(\alpha_a^* + \beta_a^*)}{\partial N_M} = \frac{N_I(2N_U + 1 - N_M^2)|\Delta RP_a|}{(N_M+1)^2(N_a+1))\delta\sigma_u^2},$$

which is positive when N_U is large.

For Part 5, if $\Delta RP_a < 0$, then

$$\frac{\partial \alpha_a^*}{\partial \sigma_u} = \frac{2N_I(N_U + N_M + 1)(N+1)\Delta RP_a}{(N_M+1)(N_a+1)^2\delta\sigma_u^3} < 0,$$

$$\frac{\partial \beta_a^*}{\partial \sigma_u} = \frac{\partial \alpha_a^*}{\partial \sigma_u}\frac{N_U}{N_M + N_U + 1} < 0.$$

The case for $\Delta RP_a > 0$ is similar. *Q.E.D.*

Proof of Proposition 2.7: The expected utility of I investors in asymmetric information case is: $U_{Ia} = -\exp(-\delta f_{Ia})$, where

$$f_{Ia} = \frac{1}{2}\Big(-\frac{\hat{H}_I^2\sigma_N^2}{\delta\sigma_{uN}^2} + 2\bar{\theta}\left(\Delta RP_a + \bar{F} + \bar{V} - \delta\bar{\theta}\sigma_F^2 + \frac{\Delta RP_a\sigma_F^2}{\sigma_H^2}\right) \tag{2.62}$$

$$+\frac{\Delta RP_a^2(1+N_M+N_U)^2N_M^2}{(1+N_a)^2(1+N_M)^2\delta\sigma_u^2} - \delta\bar{\theta}^2\sigma_u^2\Big) \tag{2.63}$$

Similarly, the expected utility of I investors in symmetric information case is: $U_{Is} = -\exp(-\delta f_{Is})$, where

$$f_{Is} = \frac{1}{2}\left(-\frac{\hat{H}_I^2\sigma_N^2}{\delta\sigma_{uN}^2} + 2\bar{\theta}\left(\Delta RP_s + \hat{F} + \bar{V}\right) + \frac{\Delta RP_s^2(1+N_M+N_U)^2N_M^2}{(1+N)^2(1+N_M)^2\delta\sigma_u^2} - \delta\bar{\theta}^2\sigma_u^2 \right).$$

Therefore, after some simplification, we have

$$f_{Ia} - f_{Is} = \frac{N_M^2(1+N_M+N_U)^2\left(-\frac{\Delta RP_s^2}{(1+N)^2} + \frac{\Delta RP_a^2}{(1+N_a)^2}\right)}{2(1+N_M)^2\delta\sigma_u^2}.$$

Therefore, the informed investors are better off if and only if $|\Delta RP_a| > \frac{N_a+1}{N+1}|\Delta RP_s|$. Part 2 follows from the proof of the second part of Proposition 2.6. *Q.E.D.*

Proof of Proposition 2.8: We will provide proof only for the case when I investors sell and U investors buy. Similar proof applies to the other case. The expected utility of I investors in the perfect competition case (i.e., without market power) is: $\bar{U}_I = -\exp(-\delta \bar{f}_I)$, where

$$\bar{f}_I = \left(\bar{\theta} - \frac{\hat{S} + \bar{V} - P_a^*}{\delta \sigma_u^2}\right) P_a^* + \frac{\hat{S} + \bar{V} - P_a^*}{\delta \sigma_u^2}(\bar{V} + \hat{F})$$

$$-\frac{1}{2}\delta \left(\left(\frac{\hat{S} + \bar{V} - P_a^*}{\delta \sigma_u^2} \right)^2 \sigma_u^2 + \hat{X}_I^2 \sigma_N^2 + 2\frac{\hat{S} + \bar{V} - P_a^*}{\delta \sigma_u^2} \sigma_{uN} \hat{X}_I \right). \tag{2.64}$$

The expected utility of I investors with market power is: $U_I = -\exp(-\delta f_I)$, where

$$f_I = \left(\bar{\theta} - \frac{\hat{S} + \bar{V} - B_a^*}{\delta \sigma_u^2}\right) B_a^* + \frac{\hat{S} + \bar{V} - B_a^*}{\delta \sigma_u^2}(\bar{V} + \hat{F})$$

$$-\frac{1}{2}\delta \left(\left(\frac{\hat{S} + \bar{V} - B_a^*}{\delta \sigma_u^2} \right)^2 \sigma_u^2 + \hat{X}_I^2 \sigma_N^2 + 2\frac{\hat{S} + \bar{V} - B_a^*}{\delta \sigma_u^2} \sigma_{uN} \hat{X}_I \right). \tag{2.65}$$

It is not difficult to see that:

$$WL_I \equiv \bar{f}_I - f_I = (P_a^* - B_a^*) \left(\left(\bar{\theta} - \frac{\hat{S} + \bar{V}}{\delta \sigma_u^2} \right) + \frac{1}{2\delta} \frac{P_a^* + B_a^*}{\sigma_u^2} \right). \tag{2.66}$$

Substituting P_a^*, B_a^* and A_a^* into (2.66) and simplifying, we have

$$WL_I = (\Delta RP_a)^2 \frac{(N_a N_U + (N_U + N_M)(N_M + 1))((N_a + 1)(2N_M + 1)(N_U + N_M) + \nu N_I N_M)}{2N_a^2 (N_a + 1)^2 (N_M + 1)^2 \delta \sigma_u^2},$$

which is greater than 0, i.e., I investors are always worse off with market power. Similarly, we can show that

$$WL_U = (\Delta RP_a)^2 \frac{N_I^2\left((N_M+1)^2 + 2N_a(N_M+1)^2 + N_a^2(2N_M+1)\right)\nu}{2N_a^2(N_a+1)^2(N_M+1)^2\delta\sigma_u^2}$$

which is greater than 0 and

$$WL_M = -(\Delta RP_a)^2 \frac{N_I\left((N_M+1)^2((2N_a+1)(N_U+N_M)-N_a) + 2N_a^2 N_U(N_a+N_M+2)\right)}{2N_a^2(N_a+1)^2(N_M+1)^2\delta\sigma_u^2},$$

which is less than 0, i.e., the uninformed are worse off and market makers are better off with market power. The total certainty equivalent wealth loss is $WL_A = N_I \times WL_I + N_U \times WL_U + N_M \times WL_M$, which can be shown to be

$$(\Delta RP_a)^2 \frac{(N_a+1)^2 N_U + N_M((N_M+1)^2 + (N_a+N_M+2)N_U)}{2N_a(N_a+1)^2(N_M+1)^2\delta\sigma_u^2},$$

which is strictly greater than 0, i.e., other investors lose more than market makers gain due to market power.

Then taking derivative of WL_i $(i = I, U, M)$ with respect to N_M yields that they all decrease with N_M. WL_i $(i = I, U, M)$ clearly increases with $|\Delta RP_a|$. *Q.E.D.*

Proof of Proposition 2.9: The total equivalent wealth loss with symmetric information is: $WL_s = (\Delta RP_s)^2 D^2$, and the total equivalent wealth loss with asymmetric information is: $WL_a = (\Delta RP_a)^2 E^2$, where D^2 and E^2 are as follows.

$$D^2 = \frac{N_I\left(N_U(N+1)^2 + N_M N_U(N+1) + N_M(N_M+1)(N_U+N_M+1)\right)}{2(N_M+1)^2 N(N+1)^2\delta\sigma_u^2}, \quad (2.67)$$

$$E^2 = \frac{N_I\left(N_U(N_a+1)^2 + N_M N_U(N_a+1) + N_M(N_M+1)(N_U+N_M+1)\right)}{2(N_M+1)^2 N_a(N_a+1)^2\delta\sigma_u^2}.$$

We then have

$$C_1 \equiv \frac{D}{E} \geq 1, \tag{2.68}$$

where the inequality holds because E is decreasing in N_a, which is increasing with information asymmetry σ_F^2. Then we have $WL_a < WL_s$ if and only if $|\Delta RP_a| < C_1|\Delta RP_s|$. For Part 2, taking derivative of WL_a with respect to σ_F (N_a is a function of σ_F) shows that WL_a decreases with σ_F when $\Delta RP_a < 0$. *Q.E.D.*

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Chapter 3

Increases in Risk Aversion and the Distribution of Portfolio Payoffs[1]

3.1 Introduction

The trade-off between risk and return arises in many portfolio problems in finance. This trade-off is more-or-less assumed in mean-variance optimization, and is also present in the comparative statics for two-asset portfolio problems explored by Arrow (1965) and Pratt (1964) (for a model with a riskless asset) and Kihlstrom, Romer, and Williams (1981) and Ross (1981) (for a model without a riskless asset). However, the trade-off is less clear in portfolio problems with many risky assets, as pointed out by Hart (1975). Assuming a complete market with many states (and therefore many assets), we show that a less risk-averse (in the sense of Arrow and Pratt) agent's portfolio payoff is distributed as the payoff for the more risk-averse agent, plus a non-negative random variable (extra return), plus conditional-mean-zero noise (risk). Therefore, the general complete-markets portfolio problem, which may not be a mean-variance problem, still trades off risk and return.

[1]This is a joint work with Philip H. Dybvig.

If either agent has non-increasing absolute risk aversion, then the non-negative random variable (extra return) can be chosen to be a constant. We also give a counter-example that shows that in general, the non-negative random variable cannot be chosen to be a constant. In this case, the less risk averse agent's payoff can also have a higher mean and a lower variance than the more risk averse agent's payoff. We further prove a converse theorem. Suppose there are two agents, such that in all complete markets, the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise. Then the first agent is less risk averse than the other agent.

Our main result applies directly in a multiple period setting with consumption only at a terminal date, and perhaps dynamic trading is the most natural motivation for the completeness we are assuming. Our main result can also be extended to a multiple period model with consumption at many dates, but this is more subtle. Consumption at each date may not be ordered when risk aversion changes, due to shifts in the timing of consumption. However, for agents with the same pure rate of time preference, we show there is a weighting of probabilities across periods that preserves the single-period result.

Our main result also extends to some special settings with incomplete markets, for example, a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. However, for a two-asset world without a risk-free asset, both parts of the proof fail in general and we have a counter-example. Therefore, our result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world using Ross's stronger measure of risk aversion. Each result from two assets can be re-interpreted as applying to parallel settings with two-fund separation identifying the two funds with the two assets.

The proofs in the paper make extensive use of results from stochastic dominance, portfolio choice, and Arrow-Pratt and Ross (1981) risk aversion. One contribution of the paper is to show how these concepts relate to each other. We use general versions of the stochastic dominance results for L^1 random variables[2] and monotone concave preferences, following Strassen (1965) and Ross (1971). To see why our results are related to stochastic dominance, note that if the first agent's payoff equals the second agent's payoff plus a non-negative random variable plus conditional-mean-zero noise, this is equivalent to saying that negative the first agent's payoff is monotone-concave dominated by negative the second agent's payoff.

Section 3.2 introduces the model setup and provides some preliminary results, Section 3.3 derives the main results. Section 3.4 extends the main results in a multiple-period model. Section 3.5 discusses the case with incomplete markets. Section 3.6 illustrates the main results using some examples and Section 3.7 concludes.

[2]We assume that the consumptions have unbounded distributions instead of compact support (e.g., Rothschild-Stiglitz (1970)). Compact support for consumption is not a happy assumption in finance because it is violated by most of our leading models. Unfortunately, as noted by Rothschild-Stiglitz (1972), the integral condition is not available in our general setting.

3.2 Model Setup and Some Standard Results

We want to work in a fairly general setting with complete markets and strictly concave increasing von Neumann-Morgenstern preferences. There are two agents A and B with von Neumann-Morgenstern utility functions $U_A(c)$ and $U_B(c)$, respectively. We assume that $U_A(c)$ and $U_B(c)$ are of class C^2, $U_A'(c) > 0$, $U_B'(c) > 0$, $U_A''(c) < 0$ and $U_B''(c) < 0$. Each agent's problem has the form:

Problem 3.1 *Choose random consumption* $\tilde{c}$ *to*

$$\max E[U_i(\tilde{c})],$$

$$s.t. \quad E[\tilde{\rho}\tilde{c}] = w_0. \tag{3.1}$$

In Problem 3.1, $i = A$ or B indexes the agent, w_0 is initial wealth (which is the same for both agents), and $\tilde{\rho} > 0$ is the state price density. We will assume that $\tilde{\rho}$ is in the class $\mathcal{P}$ for which both agents have optimal random consumptions with finite means, denoted $\tilde{c}_A$ and $\tilde{c}_B$.

The first order condition is

$$U_i'(\tilde{c}_i) = \lambda_i \tilde{\rho}, \tag{3.2}$$

i.e., the marginal utility is proportional to the state price density $\tilde{\rho}$. We have

$$\tilde{c}_i = I_i(\lambda_i \tilde{\rho}), \tag{3.3}$$

where I_i is the inverse function of $U_i'(\cdot)$. By continuity and negativity of the second order derivative $U_i''(\cdot)$, $\tilde{c}_i$ is a decreasing function of $\tilde{\rho}$.

Our main result will be that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where " $\sim$ " denotes "is distributed as," $\tilde{z} \geq 0$, and $E[\tilde{\varepsilon}|c_B + z] = 0$.[3] We firstly review and give the proofs in the Appendix of some standard results in the form needed for the proofs of our main results.

Lemma 3.1 *If B is weakly more risk averse than A, $\left(\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}\right)$, then*

1. *for any solution to (3.2) (which may not satisfy the budget constraint (3.1)), there exists some critical consumption level c^* (can be $\pm\infty$) such that $\tilde{c}_A \geq \tilde{c}_B$ when $\tilde{c}_B \geq c^*$, and such that $\tilde{c}_A \leq \tilde{c}_B$ when $\tilde{c}_B \leq c^*$;*

2. *assuming $\tilde{c}_A$ and $\tilde{c}_B$ have finite means, and A and B have equal initial wealths w_0, then $E[\tilde{c}_A] \geq E[\tilde{c}_B] \geq \frac{w_0}{E[\tilde{\rho}]}$. Note that $\frac{w_0}{E[\tilde{\rho}]}$ is the payoff to a riskless investment of w_0.*

The first result in Lemma 3.1 implies that the consumptions function of the less risk averse agent crosses that of the more risk averse agent at most once and from above. This single-crossing result is due to Pratt (1964), expressed in a slightly different way. Lemma 3.1 gives us a sense in which decreasing the agent's risk aversion takes us further from the riskless asset. In fact, we can obtain a more explicit description (our main result) of how decreasing the agent's risk aversion changes the optimal portfolio choice. The description and proof are both related to monotone concave stochastic dominance.[4] The following theorem gives a distributional characterization

[3] Throughout this paper, the letters with "tilde" denote random variables, and the corresponding letters without "tilde" denote particular values of these variables.

[4] We avoid using the term "second order stochastic dominance" in this paper because different papers use different definitions. In this paper, we follow unambiguous terminology from Ross (1971): (1) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all nondecreasing functions, then $\tilde{X}$ monotone stochastically dominates $\tilde{Y}$; (2) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave functions, then $\tilde{X}$ concave stochastically dominates $\tilde{Y}$; (3) if $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nondecreasing functions, then $\tilde{X}$ monotone-concave stochastically dominates $\tilde{Y}$.

of stochastic dominance for all monotone and concave functions of one random variable over another. The form of this result is from Ross (1971) and is a special case of a result of Strassen (1965) which generalizes a traditional result for bounded random variables to possibly unbounded random variables with finite means.

Theorem 3.1 *(**Monotone Concave Stochastic Dominance:** Strassen (1965) and Ross (1971)) Let $\tilde{X}$ and $\tilde{Y}$ be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, for all concave nondecreasing functions $V(\cdot)$, i.e., $\tilde{X}$ monotone-concave stochastically dominates $\tilde{Y}$, if and only if $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geq 0$, and $E[\tilde{\varepsilon}|X - Z] = 0$.*

Rothschild and Stiglitz (1970, 1972) popularized a similar characterization of stochastic dominance for all concave functions (which implies equal means) that is a special case of another result of Strassen's.

Theorem 3.2 *(**Concave Stochastic Dominance:** Strassen (1965), and Rothschild and Stiglitz (1970, 1972)) Let $\tilde{X}$ and $\tilde{Y}$ be two random variables defined in R^1 with finite means; then $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, for all concave functions $V(\cdot)$, i.e., $\tilde{X}$ concave stochastically dominates $\tilde{Y}$, if and only if $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$.*

Rothschild and Stiglitz (1970) also offered an integral condition for Concave Stochastic Dominance, which unfortunately does not generalize to all random variables with finite mean, as they note in Rothschild and Stiglitz (1972).[5]

[5] The integration by parts used to prove the integral condition unfortunately includes a term at the lower endpoint which needs not equal to zero in general. Therefore, the integral condition may not be sufficient or necessary condition for Concave Stochastic Dominance under unbounded distribution. As noted by Rothschild and Stiglitz (1972), the integral condition does not appear to have any natural analog in these more general cases. Ross (1971) has a sufficient condition for the integral condition to be valid, but unfortunately it is hard to interpret.

3.3 Main Results

Suppose agent A with utility function U_A and agent B with utility function U_B have identical initial wealth w_0 and solve Problem 3.1. Recall that we assume that $U_A(c)$ and $U_B(c)$ are of class C^2, $U_A'(c) > 0$, $U_B'(c) > 0$, $U_A''(c) < 0$ and $U_B''(c) < 0$. We have

Theorem 3.3 *If B is weakly more risk averse than A in the sense of Arrow and Pratt $\left(\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}\right)$, then for every $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. Furthermore, if $\tilde{c}_A \neq \tilde{c}_B$, neither $\tilde{z}$ nor $\tilde{\varepsilon}$ is identically zero.*

Proof: The first step of the proof[6] is to show that $-\tilde{c}_B$ monotone-concave stochastically dominates $-\tilde{c}_A$, *i.e.*, $E[V(-\tilde{c}_B)] \geq E[V(-\tilde{c}_A)]$ for any concave nondecreasing function $V(\cdot)$. By Lemma 3.1, $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related and there is a critical value c^* above which $\tilde{c}_A$ is weakly larger and below which $\tilde{c}_B$ is weakly larger. Let $V'(\cdot)$ be any selection from the subgradient correspondence $\nabla V(\cdot)$, then $V'(\cdot)$ is positive and nonincreasing and it is the derivative of $V(\cdot)$ whenever it exists. Recall from Rockafellar (1970), the subgradient for concave[7] $V(\cdot)$ is $\nabla V(x_1) \equiv \{s|(\forall x), V(x) \leq V(x_1) + s(x - x_1)\}$. By concavity of $V(\cdot)$, $\nabla V(x)$ is nonempty for all x_1. And if $x_2 > x_1$, then $s_2 \leq s_1$ for all $s_2 \in \nabla V(x_2)$ and $s_1 \in \nabla V(x_1)$.

The definition of subgradient for concave $V(\cdot)$ implies that

$$V(x + \Delta x) \leq V(x) + V'(x)\Delta x. \tag{3.4}$$

[6] As noted in Footnote 4, the integral condition does not hold under unbounded distributions, so that a proof using Lemma 3.1 and the intergral condition would be wrong. More specifically, because $\tilde{c}_A$ and $\tilde{c}_B$ might be unbounded, we cannot get that $-\tilde{c}_B$ monotone concave stochastically dominates $-\tilde{c}_A$ directly from $\int_{q=-\infty}^{c} [F_{-c_A}(q) - F_{-c_B}(q)]\, dq \geq 0$.

[7] For convex $V(\cdot)$, the inequality is reversed.

Letting $x = -\tilde{c}_B$ and $\Delta x = -\tilde{c}_A + \tilde{c}_B$ in (3.4), we have

$$V(-\tilde{c}_A) - V(-\tilde{c}_B) \leq V'(-\tilde{c}_B)(-\tilde{c}_A + \tilde{c}_B). \tag{3.5}$$

If $\tilde{c}_B \geq c^*$, then $\tilde{c}_A \geq \tilde{c}_B$ (by Lemma 3.1), and $V'(-\tilde{c}_B) \geq V'(-c^*)$, while if $\tilde{c}_B \leq c^*$, then $\tilde{c}_A \leq \tilde{c}_B$ and $V'(-\tilde{c}_B) \leq V'(-c^*)$. In both cases, we always have $(V'(-\tilde{c}_B) - V'(-c^*))(\tilde{c}_A - \tilde{c}_B) \geq 0$. Rewriting (3.5) and substituting in this inequality, we have

$$V(-\tilde{c}_B) - V(-\tilde{c}_A) \geq V'(-\tilde{c}_B)(\tilde{c}_A - \tilde{c}_B) \geq V'(-c^*)(\tilde{c}_A - \tilde{c}_B). \tag{3.6}$$

Since $V(\cdot)$ is nondecreasing and $E[\tilde{c}_A] \geq E[\tilde{c}_B]$ (result 2 of Lemma 3.1), we have

$$E[V(-\tilde{c}_B) - V(-\tilde{c}_A)] \geq E[V'(-c^*)(\tilde{c}_A - \tilde{c}_B)] = V'(-c^*)(E[\tilde{c}_A] - E[\tilde{c}_B]) \geq 0. \tag{3.7}$$

Therefore, we have that $-\tilde{c}_B$ is preferred to $-\tilde{c}_A$ by all concave nondecreasing $V(\cdot)$, and by Theorem 3.1, this says that $-\tilde{c}_A$ is distributed as $-\tilde{c}_B - \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}| - c_B - z] = 0$. This is exactly the same as saying that $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + (-\tilde{\varepsilon})$, where $\tilde{z} \geq 0$ and $E[-\tilde{\varepsilon}|c_B + z] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have proven the first sentence of the theorem.

To prove the second sentence of the theorem, note that because $\tilde{c}_A$ and $\tilde{c}_B$ are monotonely related, $\tilde{c}_A$ is distributed the same as $\tilde{c}_B$ only if $\tilde{c}_A = \tilde{c}_B$. Therefore, if $\tilde{c}_A \neq \tilde{c}_B$, one or the other of $\tilde{z}$ or $\tilde{\varepsilon}$ is not identically zero. Now, if $\tilde{z}$ is identically zero, then $\tilde{\varepsilon}$ must not be identically zero, and $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{\varepsilon}$, by Jensen's inequality, we have $E[U_A(\tilde{c}_A)] = E[U_A(\tilde{c}_B + \tilde{\varepsilon})] = E[E[U_A(\tilde{c}_B + \tilde{\varepsilon})|\tilde{c}_B]] < E[U_A(E[\tilde{c}_B|\tilde{c}_B] + E[\tilde{\varepsilon}|\tilde{c}_B])] = E[U_A(\tilde{c}_B)]$, which contradicts the optimality of $\tilde{c}_A$ for agent A. If $\tilde{\varepsilon}$ is identically zero, then $\tilde{z}$ must not be, and $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z}$, where $\tilde{z} \geq 0$ and is not identically zero. Therefore, $\tilde{c}_A$ strictly monotone stochastically dominates $\tilde{c}_B$, contradicting optimality of $\tilde{c}_B$ for agent B. This completes the proof that if $\tilde{c}_A$ and $\tilde{c}_B$ do not have the same distribution, then neither $\tilde{\varepsilon}$ nor $\tilde{z}$ is identically 0. *Q.E.D.*

We now prove a converse result of Theorem 3.3: if in all complete markets, one agent chooses a portfolio whose payoff is distributed as a second agent's payoff plus a nonnegative random variable plus conditional-mean-zero noise, then the first agent is less risk averse than the second. Specifically, we have

Theorem 3.4 *If for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$, then B is weakly more risk averse than A $\left(\forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)}\right)$. This implies a converse result of Theorem 3.3: if for all $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then B is weakly more risk averse than A.*

Proof: We prove this theorem by contradiction. If B is not weakly more risk averse than A, then there exists a constant $\hat{c}$, such that $-\frac{U_B''(\hat{c})}{U_B'(\hat{c})} < -\frac{U_A''(\hat{c})}{U_A'(\hat{c})}$. Since U_A and U_B are of the class of C^2, from the continuity of $-\frac{U_i''(c)}{U_i'(c)}$, where $i = A, B$, we get that there exists an interval RA containing $\hat{c}$, *s.t.*, $\forall c \in RA$, $-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now from Lemma 3.5 in the Appendix, there exists hypothetical agents A_1 and B_1, so that U_{A_1} agrees with U_A and U_{B_1} agrees with U_B on $[c_1, c_2]$, but A_1 is everywhere strictly more risk averse than B_1 (and not just on $[c_1, c_2]$).

Fix any $\lambda_B > 0$ and choose $\tilde{\rho}$ to be any random variable that takes on all the values on $[\frac{U_B'(c_2)}{\lambda_B}, \frac{U_B'(c_1)}{\lambda_B}]$. Then, the corresponding $\tilde{c}_B$ solving the first order condition

$U_B'(\tilde{c}_B) = \lambda_B \tilde{\rho}$ takes on all the values on $[c_1, c_2]$. Because $U_B'' < 0$, the $F.O.C$ solution is also sufficient(expected utility exists because $\tilde{\rho}$ and $U_B(\tilde{c}_B)$ are bounded), $\tilde{c}_B$ solves the portfolio problem for utility function U_B, state price density $\tilde{\rho}$ and initial wealth $w_0 = E[\tilde{\rho}\tilde{c}_B]$. Since $U_{B_1} = U_B$ on the support of $\tilde{c}_B$, letting $\tilde{c}_{B_1} = \tilde{c}_B$, then $\tilde{c}_{B_1}$ solves the corresponding optimization for U_{B_1} for $\lambda_{B_1} = \lambda_B$.

We now show that there exists λ_{A_1} such that $\tilde{c}_{A_1} \equiv I_{A_1}(\lambda_{A_1}\tilde{\rho})$ satisfies the budget constraint $E[\tilde{\rho}\tilde{c}_{A_1}] = w_0$. Due to the choice of U_{A_1}, $I_{A_1}(\lambda_{A_1}\tilde{\rho})$ exists and is a bounded random variable for all λ_{A_1}. Letting $\underline{\rho} = \frac{U_B'(c_2)}{\lambda_B}$ and $\bar{\rho} = \frac{U_B'(c_1)}{\lambda_B}$ (so, $\tilde{\rho} \in [\underline{\rho}, \bar{\rho}]$), we define $\lambda_1 = \frac{U_{A_1}'(c_1)}{\underline{\rho}}$ and $\lambda_2 = \frac{U_{A_1}'(c_2)}{\bar{\rho}}$, then we have

$$c_1 = I_{A_1}(\lambda_1 \underline{\rho}) > I_{A_1}(\lambda_1 \tilde{\rho}) \ , \ c_2 = I_{A_1}(\lambda_2 \bar{\rho}) < I_{A_1}(\lambda_2 \tilde{\rho}). \tag{3.8}$$

The inequalities follow from $I_{A_1}(\cdot)$ decreasing. From (3.8) and $c_1 \le \tilde{c}_B \le c_2$, we have

$$E[\tilde{\rho} I_{A_1}(\lambda_1 \tilde{\rho})] < E[\tilde{\rho} c_1] \le E[\tilde{\rho}\tilde{c}_B] = w_0, \quad E[\tilde{\rho} I_{A_1}(\lambda_2 \tilde{\rho})] > E[\tilde{\rho} c_2] \ge E[\tilde{\rho}\tilde{c}_B] = w_0. \tag{3.9}$$

Since $I_{A_1}(\lambda\tilde{\rho})$ is continuous from the assumption that $U_{A_1}(\cdot)$ is in the class of C^2 and $U_{A_1}'' < 0$. By the intermediate value theorem, there exists λ_{A_1}, such that $E[\tilde{\rho} I_{A_1}(\lambda_{A_1}\tilde{\rho})] = w_0$, *i.e.*, $\tilde{c}_{A_1}$ satisfies the budget constraint for $\tilde{\rho}$ and w_0.

From the second result of Lemma 3.6 in the Appendix, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then we have that $\tilde{c}_{B_1}$ has a wider range of support than that of $\tilde{c}_{A_1}$. Let the support of A_1's optimal consumption be $[c_3, c_4] \subseteq [c_1, c_2]$. From the construction of U_{A_1}, $U_{A_1} = U_A$ on the support of $\tilde{c}_{A_1}$. Letting $\tilde{c}_A = \tilde{c}_{A_1}$, then $\tilde{c}_A$ solves the corresponding optimization for U_A for $\lambda_A = \lambda_{A_1}$.

Now, since B_1 is strictly less risk averse than A_1, from Theorem 3.3, $\tilde{c}_{B_1} \sim \tilde{c}_{A_1} + \tilde{z}_1 + \tilde{\varepsilon}_1$, where $\tilde{z}_1 \geq 0$ and $E[\tilde{\varepsilon}_1 | c_{A_1} + z_1] = 0$. Furthermore, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero. From the first result of Lemma 3.6 in the Appendix, if A_1 is strictly more risk averse than B_1, then $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$. Thus, by Theorem 3.3, neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero. Therefore, $E[\tilde{c}_{B_1}] > E[\tilde{c}_{A_1}]$, *i.e.* $E[\tilde{c}_B] > E[\tilde{c}_A]$, this contradicts the assumption that, for all $\tilde{\rho} \in \mathcal{P}$, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$. This also contradicts a stronger condition: for all $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon} | c_B + z] = 0$. *Q.E.D.*

Theorem 3.3 shows that if B is weakly more risk averse than A, then $\tilde{c}_A$ is distributed as $\tilde{c}_B$ plus a risk premium plus random noise. The distributions of the risk premium and the noise term are typically not uniquely determined. Also, it is possible that the weakly less risk averse agent's payoff can have a higher mean and a lower variance than the weakly more risk averse agent's payoff as we will see in example 3.6.2. This can happen because although adding condition-mean-zero noise always increases variance, adding the non-negative random variable decreases variance if it is sufficiently negatively correlated with the rest (Since $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + Var(\tilde{z}) + 2Cov(\tilde{c}_B, \tilde{z})$, if $Cov(\tilde{c}_B, \tilde{z}) < -\frac{1}{2}\left(Var(\tilde{z}) + Var(\tilde{\varepsilon})\right)$, then $Var(\tilde{c}_A) < Var(\tilde{c}_B)$). This should not be too surprising, given that it is well-known that in general variance is not a good measure of risk[8] for von Neumann-Morgenstern

[8]See, for example Hanoch and levy (1970), and the survey of Machina and Rothschild (2008).

utility functions,[9] and for general distributions in a complete market, mean-variance preferences are hard to justify.

Our second main result says that when either of the two agents has non-increasing absolute risk aversion, we can choose $\tilde{z}$ to be non-stochastic, in which case $z = E[\tilde{c}_A - \tilde{c}_B]$. The basic idea is as follows. If either agent has non-increasing absolute risk aversion, then we can construct a new agent A^* whose consumption equals to A's consumption plus $E[\tilde{c}_A - \tilde{c}_B]$. We can therefore get the distributional results for agent A^* and B since A^* is weakly less risk averse than B.

Theorem 3.5 *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof: Define the utility function $U_{A^*}(\tilde{c}) = U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$. In the case when A has non-increasing absolute risk aversion, A^* is weakly less risk averse than B because A is weakly less risk averse than B and non-increasing risk aversion of A implies that A^* is weakly less risk averse than A. In the case when B has non-increasing absolute risk aversion, B^* with utility $U_{B^*} = U_B(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$ is weakly less risk averse than B and A^* is weakly less risk averse than B^*. Therefore, in both cases, we have that A^* is weakly less risk averse than B.

[9] If von Neumann-Morgenstern utility functions are mean-variance preferences, then they have to be quadratic utility functions, but quadratic preferences are not appealing because they are not increasing everywhere and they have increasing risk aversion where they are increasing. Also, Dybvig and Ingersoll (1982) show that if markets are complete, mean-variance pricing of all assets implies there is arbitrage unless the payoff to the market portfolio is bounded.

Give agent A^* initial wealth $w_{A^*} = w_0 - E[\tilde{\rho}]E[\tilde{c}_A - \tilde{c}_B]$, where w_0 is the common initial wealth of agent A and B. A^*'s problem is

$$\max_{\tilde{c}} E[U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])],$$

$$s.t. \quad E[\tilde{\rho}\tilde{c}] = w_{A^*}. \tag{3.10}$$

The first order conditions are related to the optimality of $\tilde{c}_A$ for agent A. To satisfy the budget constraints, agent A^* will optimally hold $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$.

By Lemma 3.1, $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ and $\tilde{c}_B$ are monotonely related and there is a critical value c^* above which $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is weakly larger and below which $\tilde{c}_B$ is weakly larger. This implies that

$$(V'(-\tilde{c}_B) - V'(-c^*))(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B) \geq 0, \tag{3.11}$$

where $V(\cdot)$ is an arbitrary concave function and $V'(\cdot)$ is any selection from the subgradient correspondence $\nabla V(\cdot)$. The concavity of $V(\cdot)$ implies that

$$V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]) - V(-\tilde{c}_B) \leq V'(-\tilde{c}_B)(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B] + \tilde{c}_B). \tag{3.12}$$

(3.11) and (3.12) imply that

$$V(-\tilde{c}_B) - V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]) \geq V'(-c^*)(\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] - \tilde{c}_B). \tag{3.13}$$

We have

$$E[V(-\tilde{c}_B) - V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B])] \geq E[V'(-c^*)(\tilde{c}_A - \tilde{c}_B - E[\tilde{c}_A - \tilde{c}_B])] = 0. \tag{3.14}$$

Therefore, for any concave function $V(\cdot)$, we have

$$E[V(-\tilde{c}_B)] \geq E[V(-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B])]. \tag{3.15}$$

By Theorem 3.2, this says that $-\tilde{c}_A + E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $-\tilde{c}_B + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}| - c_B] = 0$. This is exactly the same as saying that $\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B]$ is distributed as $\tilde{c}_B + (-\tilde{\varepsilon})$, where $E[-\tilde{\varepsilon}|c_B] = 0$. Relabel $-\tilde{\varepsilon}$ as $\tilde{\varepsilon}$, and we have

$$\tilde{c}_A - E[\tilde{c}_A - \tilde{c}_B] \sim \tilde{c}_B + \tilde{\varepsilon}, \ \textit{i.e.}, \ \tilde{c}_A \sim \tilde{c}_B + E[\tilde{c}_A - \tilde{c}_B] + \tilde{\varepsilon}, \tag{3.16}$$

where $E[\tilde{\varepsilon}|c_B + z] = 0$. *Q.E.D.*

The non-increasing absolute risk aversion condition is sufficient but not necessary. A quadratic utility function has increasing absolute risk aversion. But, as illustrated by example 3.6.1, the non-negative random variable can still be chosen to be a constant for quadratic utility functions (which can be viewed as an implication of two-fund separation and Theorem 3.7). If the non-negative random variable can be chosen to be a constant, then we have the following Corollary:

Corollary 3.1 *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then $Var(\tilde{c}_A) \geq Var(\tilde{c}_B)$.*

Proof: From Theorem 3.5, the non-negative random variable $\tilde{z}$ can be chosen to be the constant $E[\tilde{c}_A - \tilde{c}_B]$. Then we have $E(\tilde{\varepsilon}|\tilde{c}_B) = 0$, which implies that $Cov(\tilde{\varepsilon}, \tilde{c}_B) = 0$. Therefore, $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) + 2Cov(\tilde{c}_B, \tilde{\varepsilon}) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geq Var(\tilde{c}_B)$. *Q.E.D.*

3.4 Extension to a Multiple-Period Model

We now examine our main results in a multiple period model. We assume that each agent's problem is:

Problem 3.2

$$\max_{\tilde{c}_t} E[\sum_{t=1}^{T} D_t U_i(\tilde{c}_t)],$$

$$s.t. \quad E[\sum_{t=1}^{T} \tilde{\rho}_t \tilde{c}_t] = w_0, \tag{3.17}$$

where $i = A$ or B indexes the agent, D_t is a discount factor (*e.g.*, $D_t = e^{-\kappa t}$ if the pure rate of time discount κ is constant), and $\tilde{\rho}_t$ is the state price density in period t. Again, we will assume that both agents have optimal random consumptions, denoted $\tilde{c}_{At}$ and $\tilde{c}_{Bt}$, and both $\tilde{c}_{At}$ and $\tilde{c}_{Bt}$ have finite means. The first order condition gives us

$$U_i'(\tilde{c}_{it}) = \lambda_i \frac{\tilde{\rho}_t}{D_t}, i = A, B,$$

we have

$$\tilde{c}_{it} = I_i\left(\lambda_i \frac{\tilde{\rho}_t}{D_t}\right),$$

where $I_i(\cdot)$ is the inverse function of $U_i'(\cdot)$, by negativity of the second order derivatives, $\tilde{c}_{it}$ is a decreasing function of $\tilde{\rho}_t$. By similar arguments in the one period model, we have

Lemma 3.2 *If B is weakly more risk averse than A, then*

1. *there exists some critical consumption level c_t^* (can be $\pm\infty$) such that $\tilde{c}_{At} \geq \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \geq c_t^*$, and such that $\tilde{c}_{At} \leq \tilde{c}_{Bt}$ when $\tilde{c}_{Bt} \leq c_t^*$;*

2. if it happens that the budget shares as a function of time are the same for both agents at some time t, i.e., $E[\tilde{\rho}_t\tilde{c}_{At}] = E[\tilde{\rho}_t\tilde{c}_{Bt}]$, then $E[\tilde{c}_{At}] \geq E[\tilde{c}_{Bt}]$, and we have $\tilde{c}_{At} \sim \tilde{c}_{Bt} + \tilde{z}_t + \tilde{\varepsilon}_t$, where $\tilde{z}_t \geq 0$ and $E[\tilde{\varepsilon}_t|c_{Bt} + z_t] = 0$. And if $\tilde{c}_{At} \neq \tilde{c}_{Bt}$, then neither $\tilde{z}_t$ nor $\tilde{\varepsilon}_t$ is identically zero. In particular, if the budget shares are the same for all t, then this distributional condition holds for all t.

The proof of Lemma 3.2 is essentially the same as the proof of the corresponding parts of Lemma 3.1, and Theorem 3.3 in the one-period model. If the D_t is not the same for both agents, or the same for the two agents without any restriction on budget shares, then the distributional condition may not hold in any period. For example, if the weakly more risk averse agent B spends most of the money earlier but the weakly less risk averse agent A spends more later, then the mean payoff could be higher in an earlier period for the weakly more risk averse agent, *i.e.*, $E[\tilde{c}_{Bt}] > E[\tilde{c}_{At}]$.

Now, assume both agents have the same discount factor D_t and choose the period and consumption using a mixture model: first choose t with probability $\mu_t = \frac{D_t}{\sum_{t=1}^{T} D_t}$, and then choose $\tilde{\rho}_t$ from its distribution. Then, we will show that, under this probability measure $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$.

Definition 3.1 *Suppose the original probability space has probability measure P over states Ω with filtration $\{\mathcal{F}_t\}$. We define the discrete random variable τ on associated probability space $(\Omega^*, \mathcal{F}^*, P^*)$ so that $P^*(\tau = t) = \mu_t \equiv D_t/(\sum_{t=1}^{T} D_t)$. We then define a single-period problem on a new probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Define the state of nature in the product space $(t, \omega) \in \hat{\Omega} \equiv \Omega^* \times \Omega$ with t and ω drawn independently. Let $\hat{\mathcal{F}}$ be the optional σ-algebra, which is the completion of $\mathcal{F}^* \times \mathcal{F}_\tau$. The synthetic* *probability measure is the one consistent with independence generated from* $\hat{P}(f^*, f) = P^*(f^*) \times P(f)$ *for all subsets* $f^* \in \mathcal{F}^*$ *and subsets* $f \in \mathcal{F}_\tau$.

The synthetic probability measure assigns a probability measure that looks like a mixture model, drawing time first assigning probability μ_t to time t, and then drawing from $\tilde{\rho}_t$ using its distribution in the original problem.

Recall that under the original probability measure, each agent's problem is given in (3.17). Now we want to write down an equivalent problem, in terms of the choice of distribution of each $\tilde{c}_t$, but with the new synthetic probability measure. The consumption $\tilde{c}$ under the new probability space over which synthetic probabilities are defined is a function of $\tilde{\rho}$ and t; we identify $\tilde{c}(\tilde{\rho}, t)$ with what used to be $\tilde{c}_t(\tilde{\rho})$. To write the objective function in terms of the synthetic probabilities, we can write

$$\begin{aligned} E[\sum_{t=1}^{T} D_t U(\tilde{c}_t)] &= \sum_{t=1}^{T} D_t E[U(\tilde{c}_t)] = \sum_{t=1}^{T}(\sum_{s=1}^{T} D_s)\mu_t \hat{E}[U(\tilde{c})|t] \\ &= (\sum_{s=1}^{T} D_s)\sum_{t=1}^{T} \mu_t \hat{E}[U(\tilde{c})|t] = (\sum_{s=1}^{T} D_s)\hat{E}[U(\tilde{c})], \end{aligned} \tag{3.18}$$

where $\hat{E}$ denotes the expectation under the synthetic probability. $\sum_{s=1}^{T} D_s$ is a positive constant, so the objective function is equivalent to maximizing $\hat{E}[U(\tilde{c})]$.

Now, we can write the budget constraint in terms of the synthetic probabilities,

$$w_0 = E[\sum_{t=1}^{T} \tilde{\rho}_t \tilde{c}_t] = \sum_{t=1}^{T} \mu_t E[\frac{\tilde{\rho}_t}{\mu_t}\tilde{c}_t] = \sum_{t=1}^{T} \mu_t \hat{E}[\frac{\tilde{\rho}}{\mu}\tilde{c}|t] = \hat{E}[\frac{\tilde{\rho}}{\mu}\tilde{c}]. \tag{3.19}$$

Then we can apply our single-period results (Theorem 3.3, 3.4 and 3.5) to derive that our main results holds on a mixture model of the $\tilde{c}_A$ and $\tilde{c}_B$ over time:

Theorem 3.6 *In a multiple-period model, assume agent A and B have the same discount factor D_t and solve Problem 3.2, and let $\tilde{c}_A$ and $\tilde{c}_B$ be the optimal consumption of A and B respectively under the synthetic probability measure, we have*

1. *if B is weakly more risk averse than A, then, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ under the synthetic probabilities, where $\tilde{z} \geq 0$, $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$;*

2. *If for all $\tilde{\rho} \in \mathcal{P}$, $\hat{E}[\tilde{c}_A] \geq \hat{E}[\tilde{c}_B]$, then B is weakly more risk averse than A. This implies a converse result of statement 1: if for all $\tilde{\rho} \in \mathcal{P}$, $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$, then B is weakly more risk averse than A;*

3. *If B is weakly more risk averse than A and either of the two agents has non-increasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = \hat{E}[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $\hat{E}[\tilde{\varepsilon}|c_B + z] = 0$.*

Therefore, if the budget shares are not the same for both agents at each time period t, then the distributional result may not hold period-by-period in a multiple-period model with time-separable von Neumann-Morgenstern utility having identical weights over time. However, Theorem 3.6 implies that our main results still hold under the synthetic probabilities in a multiple-period model. This results retain the spirit of our main results while acknowledging that changing risk aversion may cause consumption to shift over time.

3.5 Possibly Incomplete Market Case

Our result still holds in a two-asset world with a risk-free asset. For a two-asset world without a risk-free asset, we have a counter-example to our result holding. Therefore, our main result does not hold in general with incomplete markets. However, our result holds in a two-risky-asset world if we make enough assumptions about asset payoffs and the risk-aversion measure. Also, each two-asset result has a natural analog for models with many assets and two-fund separation, since the portfolio payoffs will be the same as in a two-asset model in which only the two funds are traded.[10] Note that while this section is intended to ask to what extent our results can be extended to incomplete markets, the results also apply to complete markets with two-fund separation.

First, we show that our main result still holds in a two-asset world with a risk-free asset. The proof is in two parts. The first part is the standard result: decreasing the risk aversion increases the portfolio allocation to the asset with higher return. The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. To show the second part, we use the following Lemma:

Lemma 3.3 *1. If $E[\tilde{q}] = 0$ and $0 \leq m_1 \leq m_2$, then $m_2\tilde{q} \sim m_1\tilde{q} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|m_1 q] = 0$.*

2. Let $E[\tilde{\chi}]$ be finite, $E[\tilde{q}|\chi] \geq 0$, and $0 \leq m_1 \leq m_2$. Then $\tilde{\chi}+m_2\tilde{q} \sim \tilde{\chi}+m_1\tilde{q}+\tilde{z}+\tilde{\varepsilon}$, where $\tilde{z} = (m_2 - m_1)E[\tilde{q}|\chi] \geq 0$ and $E[\tilde{\varepsilon}|\chi + m_1 q + z] = 0$.

[10]See Cass and Stiglitz (1970) and Ross (1978) for characterization of two-fund separation, *i.e.*, for portfolio choice to be equivalent to choice between two mutual funds of assets.

Proof: We prove 2, and 1 follows immediately by setting $\tilde{\chi} = 0$ and $E[\tilde{q}] = 0$. Let $\tilde{z}_0 \equiv E[\tilde{q}|\chi]$ and $\tilde{z} \equiv (m_2 - m_1)\tilde{z}_0$. By Theorem 3.2, we only need to show that, for any concave function $V(\cdot)$, $E[V(\tilde{\chi} + m_2\tilde{q})] \leq E[V(\tilde{\chi} + m_1\tilde{q} + \tilde{z})]$. Fix $V(\cdot)$ and let $V'(\cdot)$ be any selection from its subgradient correspondence $\nabla V(\cdot)$ (so $V'(\cdot)$ is the derivative of $V(\cdot)$ whenever it exists). The concavity of $V(\cdot)$ and the definitions of $\tilde{z}_0$ and $\tilde{z}$ imply that

$$V(\tilde{\chi} + m_2\tilde{q}) - V(\tilde{\chi} + m_1\tilde{q} + \tilde{z}) \leq V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z})(m_2 - m_1)(\tilde{q} - \tilde{z}_0). \tag{3.20}$$

Furthermore, $V'(\cdot)$ nonincreasing, $m_2 \geq m_1 \geq 0$, and the definitions of $\tilde{z}_0$ and $\tilde{z}$ imply

$$(V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z}) - V'(\tilde{\chi} + m_2\tilde{z}_0))(m_2 - m_1)(\tilde{q} - \tilde{z}_0) \leq 0. \tag{3.21}$$

From (3.20), (3.21), and the definitions of $\tilde{z}_0$ and $\tilde{z}$, we get

$$\begin{aligned} E[V(\tilde{\chi} + m_2\tilde{q})] - E[V(\tilde{\chi} + m_1\tilde{q} + \tilde{z})] &\leq E[V'(\tilde{\chi} + m_1\tilde{q} + \tilde{z})(m_2 - m_1)(\tilde{q} - \tilde{z}_0)] \\ &\leq E[V'(\tilde{\chi} + m_2\tilde{z}_0)(m_2 - m_1)(\tilde{q} - \tilde{z}_0)] \\ &= E[E[V'(\tilde{\chi} + m_2\tilde{z}_0)(m_2 - m_1)(\tilde{q} - \tilde{z}_0)|\chi]] = 0. \end{aligned}$$

Q.E.D.

Now, we consider the following portfolio choice problem:

Problem 3.3 *(Possibly Incomplete Market with Two Assets) Agent i's ($i = A, B$) problem is*

$$\max_{\alpha_i \in R} E[U_i(w_0\tilde{x} + \alpha_i w_0\tilde{v})],$$

where w_0 is the initial wealth, α_i is the proportion invested in the second asset, and $\tilde{v}$ is the excess of the return on the second asset over the first asset, i.e., $\tilde{v} = \tilde{y} - \tilde{x}$, where $\tilde{x}$ and $\tilde{y}$ are the total returns on the two assets. We assume that $E[\tilde{v}] \geq 0$, $\tilde{v}$ is nonconstant, and $E[\tilde{v}]$ and $E[\tilde{x}]$ are finite.

We denote agent A and B's respective optimal investments in the risky asset with payoff $\tilde{y}$ by α_A^* and α_B^*. The payoff for agent A is $\tilde{c}_A = w_0\tilde{x} + \alpha_A^* w_0 \tilde{v}$ and agent B's payoff is $\tilde{c}_B = w_0\tilde{x} + \alpha_B^* w_0 \tilde{v}$. We maintain the utility assumptions made earlier: $U_i'(\cdot) > 0$ and $U_i''(\cdot) < 0$, so $\tilde{v}$ nonconstant implies that α_A^* and α_B^* are unique if they exist. We have the following well-known result.

Lemma 3.4 *Suppose $\tilde{x}$ is riskless ($\tilde{x}$ nonstochastic), if B is weakly more risk averse than A, then the agents' solutions to Problem 3.3 satisfy $\alpha_A^* \geq \alpha_B^*$.*

Proof: The first-order condition of A's problem is:

$$E[U_A'(xw_0 + \alpha_A^* w_0 \tilde{v}) w_0 \tilde{v}] = 0. \tag{3.22}$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi(\alpha) \equiv E[U_B'(xw_0 + \alpha w_0 \tilde{v}) w_0 \tilde{v}]. \tag{3.23}$$

Since $U_B(\cdot) = G(U_A(\cdot))$, where $G'(\cdot) > 0$ and $G''(\cdot) \leq 0$, we have:

$$\varphi(\alpha_A^*) = E[G'(U_A(xw_0 + \alpha_A^* w_0 \tilde{v}))U_A'(xw_0 + \alpha_A^* w_0 \tilde{v}) w_0 \tilde{v}]$$

$$= G'(U_A(xw_0))E[U_A'(xw_0 + \alpha_A^* w_0 \tilde{v}) w_0 \tilde{v}]$$

$$+E[(G'(U_A(xw_0 + \alpha_A^* w_0 \tilde{v})) - G'(U_A(xw_0)))\, U_A'(xw_0 + \alpha_A^* w_0 \tilde{v}) w_0 \tilde{v}] \leq 0, \tag{3.24}$$

where the first term in (3.24) is zero by (3.22) and the expression inside the expectation in the second term is non-positive because $G''(\cdot) \leq 0$ and $U'_A(\cdot) > 0$. Finally, the concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing, and therefore from (3.23) and (3.24), we must have $\alpha_A^* \geq \alpha_B^*$. *Q.E.D.*

Lemma 3.4 implies that decreasing the risk aversion increases the portfolio allocation to the asset with higher return. Now, we show that our main result still holds in a two-asset world with a risk-free asset. We have

Theorem 3.7 *(Two-asset World with a Riskless Asset) Consider the two-asset world with a riskless asset ($\tilde{x}$ nonstochastic) of Problem 3.3, if B is weakly more risk averse than A in the sense of Arrow and Pratt, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.*

Proof: When the first asset in Problem 3.3 is riskless, then we have $\tilde{c}_A - E[\tilde{c}_A] = \alpha_A^* w_0(\tilde{y} - E[\tilde{y}])$ and $\tilde{c}_B - E[\tilde{c}_B] = \alpha_B^* w_0(\tilde{y} - E[\tilde{y}])$. From Lemma 3.4, $\alpha_A^* \geq \alpha_B^*$. Let $\tilde{q} \equiv \tilde{y} - E[\tilde{y}]$, $m_1 \equiv \alpha_B^* w_0$ and $m_2 \equiv \alpha_A^* w_0$ in the first part of Lemma 3.3, we have $\tilde{c}_A - E[\tilde{c}_A] \sim \tilde{c}_B - E[\tilde{c}_B] + \tilde{\varepsilon}$, which implies that $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. *Q.E.D.*

Theorem 3.7 generalizes in obvious ways to settings with two-fund separation since optimal consumption is the same as it would be with ordering the two funds as assets. The main requirement is that one of the funds can be chosen to be riskless, for example, in a mean-variance world with a riskless asset and normal returns for risky assets.[11] In this example, if B is weakly more risk averse than A, Theorem 3.7 tells us that $\tilde{c}_A \sim \tilde{c}_B + z + \tilde{\varepsilon}$, where $z \geq 0$ is constant and $E[\tilde{\varepsilon}|c_B + z] = 0$. We know that A's optimal portfolio is further up the frontier than $B's$, *i.e.*, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$ and $Var[\tilde{c}_A] \geq Var[\tilde{c}_B]$. This result is verified by noting that we can choose $z = E[\tilde{c}_A - \tilde{c}_B]$, $\tilde{\varepsilon} \sim N(0, Var[\tilde{c}_A] - Var[\tilde{c}_B])$, and $\tilde{\varepsilon}$ is drawn independently of $\tilde{c}_B$.

Now, we examine the case with two risky assets in Problem 3.3. For a two-asset world without a riskless asset, we have a counter-example to our result holding. In the counter-example, $\alpha_A^* > \alpha_B^*$, but the distributional result does not hold.

Example 3.6.1 We assume that there are two risky assets and four states. The probabilities for the four states are $0.2, 0.3, 0.3$ and 0.2 respectively. The payoff of $\tilde{x}$ is $(10\ 8\ 1\ 1)^T$ and the net payoff $\tilde{v}$ is $(-1\ 1\ 1\ \ -1)^T$. Agent's utility function is $U_i(\tilde{w}_i) = -e^{-\delta_i \tilde{w}_i}$, where $i = A, B$, and $\tilde{w}_i$ is agent i's terminal wealth. We assume that agent B is weakly more risk averse than A with $\delta_A = 1$ and $\delta_B = 1.5$. The agents solve Problem 3.3 with initial wealth $w_0 = 1$.

The agents' problems are:

$$\max_{\alpha_A} 0.2e^{-(10-\alpha_A)} + 0.3e^{-(8+\alpha_A)} + 0.3e^{-(1+\alpha_A)} + 0.2e^{-(1-\alpha_A)},$$

and

$$\max_{\alpha_B} 0.2e^{-1.5(10-\alpha_B)} + 0.3e^{-1.5(8+\alpha_B)} + 0.3e^{-1.5(1+\alpha_B)} + 0.2e^{-1.5(1-\alpha_B)}.$$

First-order conditions give $\alpha_A^* = \frac{1}{2}\log\left(\frac{3+3e^{-7}}{2+2e^{-9}}\right) = 0.2$, and $\alpha_B^* = \frac{1}{3}\log\left(\frac{3+3e^{-10.5}}{2+2e^{-13.5}}\right) = 0.135$. Therefore, agent A's portfolio payoff is $(9.8\ 8.2\ 1.2\ 0.8)^T$ and agent B's portfolio

[11] This example is a special case of two-fund separation in mean-variance worlds or the separating distributions of Ross (1978).

payoff is $(9.865\ 8.135\ 1.135\ 0.865)^T$. If agent A's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geq 0|c_B + z) > 0$, therefore, we have $\max \tilde{c}_A \geq \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.8$ and $\max \tilde{c}_B = 9.865$, *i.e.*, $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in general, our result does not hold in a two-asset world without a riskless asset. *Q.E.D.*

It is a natural question to ask whether our main result holds in a two risky asset world if we make enough assumptions about asset payoffs. We can, if we use Ross's stronger measure of risk aversion (see Ross (1981)) and his payoff distributional condition. We have

Theorem 3.8 *(Two Risky Assets with Ross's Measure) Consider the two-risky-asset world of Problem 3.3 with $E[\tilde{v}|x] \geq 0$ for all x. If B is weakly more risk averse than A under Ross's stronger measure of risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, and $\tilde{z} \geq 0$.*

Proof: Our proof is in two parts. The first part is from Ross (1981): if agent A is weakly less risk averse than B under Ross's stronger measure, then $\alpha_A^* \geq \alpha_B^*$. The first order condition of A's problem is

$$E[U_A'(w_0\tilde{x} + \alpha_A^* w_0\tilde{v})w_0\tilde{v}] = 0. \tag{3.25}$$

The analogous expression for B is $\varphi(\alpha_B^*) = 0$, where

$$\varphi(\alpha_B^*) \equiv E[U_B'(w_0\tilde{x} + \alpha_B^* w_0\tilde{v})w_0\tilde{v}]. \tag{3.26}$$

From Ross (1981), if B is weakly more risk averse than A under Ross's stronger measure, then there exists $\lambda > 0$ and a concave decreasing function $G(\cdot)$, such that $U_B(\cdot) = \lambda U_A(\cdot) + G(\cdot)$. Therefore,

$$\varphi(\alpha_A^*) = E[(\lambda U_A'(w_0\tilde{x} + \alpha_A^* w_0\tilde{v}) + G'(w_0\tilde{x} + \alpha_A^* w_0\tilde{v}))w_0\tilde{v}]$$

$$= E[G'(w_0\tilde{x} + \alpha_A^* w_0\tilde{v})w_0\tilde{v}] = E[E[G'(w_0\tilde{x} + \alpha_A^* w_0\tilde{v})w_0\tilde{v}|x]] \leq 0, \tag{3.27}$$

where the last inequality is a consequence of the fact that $G'(\cdot)$ is negative and decreasing while $E[\tilde{v}|x] \geq 0$. The concavity of $U_B(\cdot)$ implies that $\varphi(\cdot)$ is decreasing. Therefore, from (3.25) and (3.27), we have $\alpha_A^* \geq \alpha_B^*$.

The second part shows that the portfolio payoff for the higher allocation is distributed as the other payoff plus a constant plus conditional-mean-zero noise. Let $\tilde{q} \equiv \tilde{v}, \tilde{\chi} \equiv w_0\tilde{x}$, $m_1 \equiv \alpha_B^* w_0$ and $m_2 \equiv \alpha_A^* w_0$ in Lemma 3.3, part 2, we have $w_0\tilde{x} + \alpha_A^* w_0\tilde{v} \sim w_0\tilde{x} + \alpha_B^* w_0\tilde{v} + \tilde{z} + \tilde{\varepsilon}$, *i.e.*, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = w_0(\alpha_A^* - \alpha_B^*)E[\tilde{v}|x] \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. *Q.E.D.*

Theorem 3.8 implies that our main result holds when we use Ross's stronger measure of risk aversion with the assumption of $E[\tilde{v}|x] \geq 0$. If the condition $E[\tilde{v}|x] \geq 0$ is not satisfied, then our main result may not hold even when we use Ross's stronger measure of risk aversion as we can see in the following example.

Example 3.6.2 We assume that there are two risky assets and four states. The probabilities for the four states are $0.3, 0.2, 0.3$ and 0.2 respectively. The payoff of $\tilde{x}$ is $(10\ 8\ 1\ 1)^T$ and the net payoff $\tilde{v}$ is $(-1\ 1\ 1\ -1)^T$. Agent A's utility function is $U_A(\tilde{w}_A) = e^6\tilde{w}_A - e^{-\tilde{w}_A}$, and agent B's utility function is $U_B(\tilde{w}_B) = \tilde{w}_B - e^{6-1.5\tilde{w}_B}$, where $\tilde{w}_i$ is the terminal wealth of agent i. The agents solve Problem 3.3 with initial wealth $w_0 = 1$. We have

$$\frac{U_B''(w)}{U_A''(w)} = \frac{2.25e^{6-1.5w}}{e^{-w}} = 2.25e^{6-0.5w}, \quad \frac{U_B'(w)}{U_A'(w)} = \frac{1+1.5e^{6-1.5w}}{e^6+e^{-w}}.$$

Therefore, $\inf_w \frac{U_B''(w)}{U_A''(w)} > \sup_w \frac{U_B'(w)}{U_A'(w)}$, for any $0 \leq w \leq 10$, which implies that agent B is strictly more risk aversion than agent A under Ross's stronger measure of risk aversion.

The Agents' problems are:

$$\max_{\alpha_A} 0.3\left(e^6(10-\alpha_A) - e^{-(10-\alpha_A)}\right) + 0.2\left(e^6(8+\alpha_A) - e^{-(8+\alpha_A)}\right)$$

$$+0.3\left(e^6(1+\alpha_A) - e^{-(1+\alpha_A)}\right) + 0.2\left(e^6(1-\alpha_A) - e^{-(1-\alpha_A)}\right),$$

and

$$\max_{\alpha_B} 0.3\left(10-\alpha_B - e^{6-1.5(10-\alpha_B)}\right) + 0.2\left(8+\alpha_B - e^{6-1.5(8+\alpha_B)}\right)$$

$$+0.3\left(1+\alpha_B - e^{6-1.5(1+\alpha_B)}\right) + 0.2\left(1-\alpha_B - e^{6-1.5(1-\alpha_B)}\right).$$

From the first order condition, $e^{2\alpha_A^*} = \frac{3+2e^{-7}}{2+3e^{-9}}$, *i.e.*, $\alpha_A^* = \frac{1}{2}\log\left(\frac{3+2e^{-7}}{2+3e^{-9}}\right) = 0.2029$, and $e^{3\alpha_B^*} = \frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}}$, *i.e.*, $\alpha_B^* = \frac{1}{3}\log\left(\frac{3e^{-1.5}+2e^{-12}}{2e^{-1.5}+3e^{-15}}\right) = 0.1352$. Therefore, agent A's portfolio payoff is $(9.7971\ 8.2029\ 1.2029\ 0.7971)^T$ and agent B's portfolio payoff is $(9.8648\ 8.1352\ 1.1352\ 0.8648)^T$. If agent A's payoff $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|c_B + z] = 0$, then we have $Pr(\tilde{\varepsilon} \geq 0|c_B + z) > 0$. Therefore, we have $\max \tilde{c}_A \geq \max \tilde{c}_B$. However, in this example, we can see that $\max \tilde{c}_A = 9.7971$ and $\max \tilde{c}_B = 9.8648$, *i.e.*, $\max \tilde{c}_A < \max \tilde{c}_B$. Contradiction! Therefore, in a two-risky asset world, our main result does not hold in general even under Ross's stronger measure of risk aversion if we don't make the assumption that $E[\tilde{v}|x] \geq 0$. *Q.E.D.*

An alternative to the approach following Ross (1981) is the approach of Kihlstrom, Romer and Williams (1981) for handling random base wealth. They show that the Arrow-Pratt measure works if we restrict attention to comparisons in which (1) at least one of the utility functions has nonincreasing absolute risk aversion and (2) base wealth is independent of the other gambles. Here is how their argument works. The independence implies that we can convert a problem with random base wealth x to a problem with nonrandom base wealth by using the indirect utility functions $\hat{U}_i(w) \equiv E[U_i(\tilde{x}+w)]$, and our results for nonrandom base wealth apply directly. For this to work, the indirect utility functions $\hat{U}_A$ and $\hat{U}_B$ must inherit the risk aversion ordering from U_A and U_B, which as they point out, does not happen in general. However, letting $F(\cdot)$ be the distribution function of $\tilde{x}$, simple calculations tell us that provided integrals exist, we can write

$$-\frac{\hat{U}_i''(w)}{\hat{U}_i'(w)} = \int \frac{U_i'(\tilde{x}+w)}{\int U_i'(\tilde{y}+w)dF(\tilde{y}+w)} \left(-\frac{U_i''(\tilde{x}+w)}{U_i'(\tilde{x}+w)}\right) dF(\tilde{x}) \tag{3.28}$$

For both agents, the risk aversion of the indirect utility function is therefore a weighted average of the risk aversion of the direct utility function, but the weights are different so the risk aversion ordering is not preserved in general (since the more risk averse agent may have relatively higher weights from wealth regions where both agents have small risk aversion). However, we do know that the more risk averse agents' weights put relatively higher weight on lower wealth levels (since i's absolute risk aversion is $-d\log(U_i'(w)/dw)$), so if either agent has nonincreasing absolute risk aversion, then the risk aversion ordering of the direct utility function is inherited by the indirect utility function. Subject to existence of some integrals (ensured by compactness in their paper), their results and our Theorem 3.7 imply that if B is weakly more risk averse than A, at least one of U_A and U_B has nonincreasing absolute risk aversion, and $\tilde{v}$ is independent of $\tilde{x}$, then our main result holds: $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

As we have shown that our main result does not hold in general in the traditional type of incomplete markets where portfolio payoffs are restricted to a subspace. However, it is an open question whether the results extend to more interesting models of incomplete markets in which there is a reason for the incompleteness. For example, a market that is complete over states distinguished by security returns and incomplete over other private states (see Dybvig (1992) or Chen and Dybvig (2009)). Another type of incompleteness comes from a nonnegative wealth constraint (which is an imperfect solution to information problems when investors have private information or choices related to default), which means agents have individual incompleteness and cannot fully hedge future non-traded wealth or else they would violate the nonnegative wealth constraint (see Dybvig and Liu (2009)).

3.6 Examples

In example 3.6.1, we illustrate our main result with specific distribution of $\tilde{c}_A$, $\tilde{c}_B$ and $\tilde{\varepsilon}$. In this example, the nonnegative random variable $\tilde{z}$ can be chosen to be a constant, and therefore from Corollary 3.1 in Section 3.3, the variance of the less risk averse agent's payoff is higher.

Example 3.6.1 B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 1$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{1}{2}(4-\tilde{c})^2, \ U_B(\tilde{c}) = -\frac{1}{2}(3-\tilde{c})^2,$$

where $\tilde{c} < 4$ for agent A, and $\tilde{c} < 3$ for agent B. We assume that the state price density $\tilde{\rho}$ is uniformly distributed in $[0,1]$. The first-order conditions give us $\tilde{c}_A = 4 - \lambda_A\tilde{\rho}$, and $\tilde{c}_B = 3 - \lambda_B\tilde{\rho}$. Because $E[\tilde{\rho}] = \frac{1}{2}$ and $E[\tilde{\rho}^2] = \frac{1}{3}$, the budget constraint $E[\tilde{\rho}\tilde{c}_i] = 1$, $i = A, B$, implies that $\lambda_A = 3$ and $\lambda_B = \frac{3}{2}$. Therefore, $\tilde{c}_A$ is uniformly distributed in $[1,4]$ and $\tilde{c}_B$ is uniformly distributed in $\left[\frac{3}{2}, 3\right]$. We have $E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{4}$. Let $\tilde{\varepsilon}$ have a Bernoulli distribution drawn independently of $\tilde{c}_B$ with two equally possible outcomes $\frac{3}{4}$ and $-\frac{3}{4}$. It is not difficult to see that $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{4}$, and $\tilde{\varepsilon}$ is independent of $\tilde{c}_B$, which implies $E[\tilde{\varepsilon}|c_B + z] = 0$.

Next, in example 3.6.2, we show that in general $\tilde{z}$ may not be chosen to be a constant. Interestingly, the variance of the weakly less risk averse agent's payoff can be smaller than the variance of the weakly more risk averse agent's payoff.

Example 3.6.2 B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 1$ and the utility functions are as follows

$$U_A(\tilde{c}) = -\frac{(8-\tilde{c})^3}{3}, \quad U_B(\tilde{c}) = -\frac{(8-\tilde{c})^5}{5},$$

where $\tilde{c} < 8$. The first-order conditions give us

$$U_A'(\tilde{c}_A) = (8-\tilde{c}_A)^2 = \lambda_A\tilde{\rho}, \ U_B'(\tilde{c}_B) = (8-\tilde{c}_B)^4 = \lambda_B\tilde{\rho}. \tag{3.29}$$

Therefore,

$$\tilde{c}_A = 8 - \sqrt{\lambda_A \tilde{\rho}},\ \tilde{c}_B = 8 - (\lambda_B \tilde{\rho})^{1/4}. \tag{3.30}$$

From (3.30), we get

$$\tilde{c}_A = 8 - \sqrt{\frac{\lambda_A}{\lambda_B}}(8 - \tilde{c}_B)^2. \tag{3.31}$$

We have: $\tilde{c}_A \geq \tilde{c}_B$ iff $\tilde{c}_B \geq 8 - \sqrt{\frac{\lambda_B}{\lambda_A}}$. From Theorem 3.3, we know that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. To find an example that the variance of the less risk averse agent's payoff can be smaller, we assume that $\tilde{\rho}$ has a discrete distribution, *i.e.*, $\rho_1 = \varepsilon$ with probability $\frac{1}{2}$, $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{4}$, and $\rho_3 = \frac{1}{2}$ with probability $\frac{1}{4}$. If ε is very tiny (close to zero), then from (3.30) and the budget constraint $E[\tilde{\rho}\tilde{c}_A] = 1$. It is not difficult to compute $\lambda_A \approx 17.5, \lambda_B \approx 125.8$, $\tilde{c}_A \approx (8\ 5.91\ 5.045)$ and $\tilde{c}_B \approx (8\ 5.632\ 5.184)$. Therefore, $E[\tilde{c}_A] \approx 6.73$, $E[\tilde{c}_B] \approx 6.70$, and $Var(\tilde{c}_A) \approx 1.684 < Var(\tilde{c}_B) \approx 1.704$, *i.e.*, the variance of the weakly more risk averse agent's payoff is higher. In this example, both agents' utility functions have increasing absolute risk aversion, which gets very high at the shared satiation point $\tilde{c} = 8$. In the high-consumption (low $\tilde{\rho}$), the optimal consumptions of agent A and B are both very close to 8. To have $E[\tilde{c}_A] > E[\tilde{c}_B]$, $\tilde{z}$ is greater in the low-consumption states. Therefore $\tilde{z}$ is large when $\tilde{c}$ is small and small when $\tilde{c}$ is large, and thus $\tilde{z}$ is very negatively correlated with $\tilde{c}_B + \tilde{\varepsilon}$. As noted in Section 3.3, we know that if the non-negative random variable $\tilde{z}$ can be chosen to be a constant, then $Var(\tilde{c}_A) = Var(\tilde{c}_B) + Var(\tilde{\varepsilon}) \geq Var(\tilde{c}_B)$. Therefore, in this example, $\tilde{z}$ cannot be chosen to be a constant.

The next example shows that if the utility functions are not strictly concave, then our main result does not hold.

Example 3.6.3 B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 1$ and the utility functions are $U_A(\tilde{c}) = U_B(\tilde{c}) = \tilde{c}$. We assume there are two states with $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{3}$, and $\rho_2 = \frac{1}{2}$ with probability $\frac{2}{3}$. It is not difficult to see that $\tilde{c}_A = (0, 3)$ and $\tilde{c}_B = (4, 1)$ is an optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. We have $E[\tilde{c}_A] = E[\tilde{c}_B] = 2$ and $Var[\tilde{c}_A] = Var[\tilde{c}_B] = 2$. If $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$, then $\tilde{z} = 0$ and $\tilde{\varepsilon} = 0$, we get $\tilde{c}_A \sim \tilde{c}_B$. Contradiction! So, we cannot have $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}|c_B + z] = 0$.

Example 3.6.3 is degenerate with constant $\tilde{\rho}$ and linear utility. It is not difficult to construct a more general example (Example 3.6.4), where $\tilde{\rho}$ is random and the utility function has two straight segments. The optimal portfolio is not unique on these two straight segments taken together and therefore our payoff distributional result may not hold.

Example 3.6.4 B is weakly more risk averse than A, A and B have the same initial wealth $w_0 = 2$ and the utility functions are as follows

$$U_A(\tilde{c}) = U_B(\tilde{c}) = \begin{cases} -(\tilde{c}-1)^4 + \tilde{c} & \tilde{c} < 1 \\ \tilde{c} & 1 \leq \tilde{c} \leq 2 \\ \frac{1}{256}(\tilde{c}^4 - 16\tilde{c}^3 + 72\tilde{c}^2 + 128\tilde{c} + 80) & 2 < \tilde{c} < 6 \\ \frac{1}{2}\tilde{c} + 2 & 6 \leq \tilde{c} \leq 14 \\ \frac{1}{2}e^{-(\tilde{c}-14)} - 2e^{-(\tilde{c}-14)/2} + 9 & \tilde{c} \geq 14. \end{cases}$$

In this example, the utility function has two straight segments and the optimal portfolio is not unique on these two straight segments taken together. We assume that $\rho_1 = \frac{1}{2}$ with probability $\frac{1}{2}$ and $\rho_2 = \frac{1}{4}$ with probability $\frac{1}{2}$. Then, it is not difficult to see that $\tilde{c}_A = (2, 12)$ and $\tilde{c}_B = (1, 14)$ is the optimal consumption for agent A and B for $\lambda_A = \lambda_B = 2$. So, while A is weakly less risk averse than B (their risk aversion is equal everywhere), $\tilde{c}_A$ is not distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ with $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon} | c_B + z] = 0$.

It is natural to think of the completeness in our model as coming from dynamic trading in a continuous-time model. This is a good setting for seeing that our distributional result holds even if it is hard to interpret what is happening with portfolio weights. In the next example, we consider a continuous-time model with one-year investment horizon. There are two assets: a locally riskless bond and a one-year risky discount bond. We show that a very risk averse agent may invest all of his wealth in the one-year risky discount bond while a less risk averse agent invests part of his wealth in the locally riskless bond. Therefore, the comparative statics results in portfolio weights do not hold in a continuous-time model with two assets. However, our comparative statics results in the distribution of portfolio payoffs still hold.

Example 3.6.5 There are two assets that trade continuously: a locally riskless bond and a one-year discount bond that is locally risky because the interest rate is random. Agents are endowed with wealth w_0 at time 0 and consume $\tilde{c}$ at time 1. Each investor has constant relative risk aversion $U(\tilde{c}) = \frac{\tilde{c}^{1-\gamma}}{1-\gamma}$ (or $U(\tilde{c}) = \log(\tilde{c})$ if $\gamma = 1$), and chooses a dynamic portfolio strategy to maximize $E[U(\tilde{c})]$, where $\tilde{c}$ equals wealth at time 1. The interest rate follows the absolute[12] Vasicek process $dr_t = \sigma dZ_t$, or equivalently $r_t = r_0 + \sigma Z_t$, where Z_t is a standard Wiener process. The state price density is

$$\tilde{\rho}_t = e^{-\int_0^t (r_s + \frac{1}{2}\kappa^2)ds - \int_0^t \kappa dZ_s} = e^{-r_0 t - \int_0^t (\kappa + \sigma(t-s))dZ_s - \frac{\kappa^2}{2}t}, \tag{3.32}$$

where $\kappa > 0$ is the local Sharpe ratio. We have $\tilde{\rho}_1 = e^{-r_0 - \int_0^1 (\kappa+\sigma(1-s))dZ_s - \frac{\kappa^2}{2}}$. Agents' problem is $\max_{\tilde{c}} E[\frac{\tilde{c}^{1-\gamma}}{1-\gamma}]$, subject to the budget constraint $E[\tilde{\rho}_1 \tilde{c}] = w_0$.

The first order condition gives us $\tilde{c}^{-\gamma} = \lambda \tilde{\rho}_1$. Substituting $\tilde{c} = (\lambda \rho_1)^{-\frac{1}{\gamma}}$ into the budget constraint, we get

$$\lambda = e^{-r_0(\gamma-1) - \frac{\kappa^2}{2}(\gamma-1) + \frac{\gamma}{2}(1-\frac{1}{\gamma})^2(\kappa^2 + \frac{\sigma^2}{3} + \kappa\sigma)}.$$

Therefore, we have

$$\tilde{c} = e^{r_0 + \frac{1}{2}\kappa^2 - \frac{1}{2}(1-\frac{1}{\gamma})^2(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma) + \frac{1}{\gamma}\int_0^1 (\kappa+\sigma(1-s))dZ_s}. \tag{3.33}$$

Suppose that there are two agents A and B with risk aversion γ_A and γ_B, with $\gamma_A < \gamma_B$. For $i = A, B$, we have

$$\log \tilde{c}_i \sim \ln N(r_0 + \frac{1}{2}\kappa^2 - \frac{1}{2}(1 - \frac{1}{\gamma_i})^2(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma), \frac{1}{\gamma_i^2}(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)). \tag{3.34}$$

It is not difficult to show that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} = \tilde{c}_B \left(e^{(\frac{1}{\gamma_A} - \frac{1}{\gamma_B})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)} - 1\right) > 0$, and

$$\tilde{\varepsilon} = \tilde{c}_B e^{(\frac{1}{\gamma_A} - \frac{1}{\gamma_B})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)} \left(e^{\eta - \frac{1}{2}(\frac{1}{\gamma_A^2} - \frac{1}{\gamma_B^2})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)} - 1\right),$$

where $\eta \sim N\left(0, (\frac{1}{\gamma_A^2} - \frac{1}{\gamma_B^2})(\kappa^2 + \frac{1}{3}\sigma^2 + \kappa\sigma)\right)$ and is drawn independently of $\tilde{c}_B$. This confirms our comparative statics result for the distribution of portfolio payoffs from

[12] A more complex Vasicek process with mean reversion gives similar results but the calculations are more complex.

Theorem 3.3. However, we next show that the comparative static result in portfolio weights does not hold, *i.e.*, the more risk averse agent may invest more in the locally risky bond.

Investor's wealth at time t,

$$W_t = E_t\left[\frac{\tilde{\rho}_1 \tilde{c}}{\tilde{\rho}_t}\right] = f(t) e^{\int_0^t \left((\kappa+\sigma(t-s)) - (1-\frac{1}{\gamma})(\kappa+\sigma(1-s))\right) dZ_s}, \tag{3.35}$$

where $f(t) = e^{r_0 t + \frac{1}{2}\kappa^2 t - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \left(\frac{1}{3}\sigma^2 t^2 - \sigma(\kappa+\sigma)t + (\kappa+\sigma)^2\right)t}$. Using $Ito's$ Lemma, we get

$$\frac{dW_t}{W_t} = \left(r_t + \kappa\left(\kappa - \left(1-\frac{1}{\gamma}\right)(\kappa+\sigma(1-t))\right)\right)dt$$

$$+ \left(\kappa - \left(1-\frac{1}{\gamma}\right)(\kappa+\sigma(1-t))\right)dZ_t. \tag{3.36}$$

The discount bond price at time t,

$$B_t = E_t\left[\frac{\tilde{\rho}_1}{\tilde{\rho}_t}\right] = g(t) e^{\int_0^t \sigma(t-1) dZ_s}, \tag{3.37}$$

where $g(t) = e^{-(r_0+\frac{1}{2}\kappa^2)(1-t) + \frac{1}{6\sigma}\left((\kappa+\sigma(1-t))^3 - \kappa^3\right)}$. Using $Ito's$ Lemma, we have

$$\frac{dB_t}{B_t} = (r_t + \kappa\sigma(t-1))\, dt + \sigma(t-1) dZ_t. \tag{3.38}$$

From (3.36) and (3.38), we get that the investor with risk aversion γ optimally invests

$$\frac{\kappa - \left(1-\frac{1}{\gamma}\right)(\kappa+\sigma(1-t))}{\sigma(t-1)} = 1 - \frac{1}{\gamma}\left(1 + \frac{\kappa}{(1-t)\sigma}\right) \tag{3.39}$$

proportion of wealth in the risky discount bond. Therefore, the proportion of wealth invested in the locally risky bond increases in investors' risk aversion. It is useful to consider the intuition in a limiting case when $\kappa \downarrow 0$ and $\gamma_B \uparrow \infty$, with $\gamma_A = 1$. In this case, agent A with log utility holds approximately the locally riskless asset, because log utility is myopic, and the agent does not invest much in the risky bond when its local risk premium is small. The very risk averse agent B puts approximately 100% in the locally risky bond with a positive risk premium. This generates a nearly riskless payoff at the end, which is what a very risk averse agent wants. This example illustrates that although it is hard to get comparative statics results in portfolio weights, our comparative statics result in the distribution of portfolio payoffs still holds.

3.7 Concluding Remarks

Under some assumptions, Hart (1975) proved the impossibility of deriving general comparative statics on how portfolio weights vary with risk aversion. We have proven comparative statics results instead in the distribution of portfolio payoffs. Specifically, in a complete market, we show that an agent who is less risk averse than another will choose a portfolio whose payoff is distributed as the other's payoff plus a nonnegative random variable plus conditional-mean-zero noise. This result holds for any increasing and strictly concave C^2 utility functions. If either agent has non-increasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. The non-increasing absolute risk aversion condition is sufficient but not necessary. We also provide a counter-example showing that, in general, this non-negative random variable cannot be chosen to be a constant.

We further prove a converse theorem. If in all complete markets the first agent chooses a payoff that is distributed as the second's payoff, plus a non-negative random variable, plus conditional-mean-zero noise, then the first agent is less risk averse than the second agent. We also extend our main results to a multiple period model. Due to shifts in the timing of consumption, agents' optimal consumption at each date may not be ordered when risk aversion changes. However, for agents with the same pure rate of time preference, there is a natural weighting of probabilities across periods that preserves the single-period result.

The optimal consumption may not be ordered for agents with different risk aversion when agents' utility functions are concave but not strictly concave as we have shown in example 3.6.3 and 3.6.4. Intuitively, the problem is that even with identical preferences, two different optimal consumptions may not be ordered. We conjecture that there exists some canonical choice of optimal consumption for each agent that extends our main results to weakly concave preferences. Our paper derives comparative statics results in complete markets for agents with von Neumann-Morgenstern preferences. Machina (1989) has shown that many previous comparative statics results generalize to the broader class of Machina preferences (Machina (1982)). Our proofs do not generalize obviously to this class, but we conjecture that our results are still true.

We also show that our main result still holds in a two-asset world with a risk-free asset or more generally in a two-fund separation world with a risk-free asset. However, our main result is not true in general with incomplete markets. We further provide sufficient conditions under which our results still hold in a two-risky-asset world or a world with two-fund separation.

3.8 Appendix

Proof of Lemma 3.1: By Pratt (1964), we have the concave transform characterization[13] that there exists $G(\cdot) \in C^2$, such that

$$U_B(c) = G(U_A(c)), \tag{3.40}$$

where $G'(\cdot) > 0$ and $G''(\cdot) \leq 0$. Using the concave transform characterization of more risk averse in (3.40), the first order condition (3.2) becomes

$$U_A'(\tilde{c}_A) = \lambda_A \tilde{\rho} = \frac{\lambda_A}{\lambda_B} \lambda_B \tilde{\rho} = \frac{\lambda_A}{\lambda_B} G'(U_A(\tilde{c}_B)) U_A'(\tilde{c}_B). \tag{3.41}$$

Because marginal utility is strictly decreasing, we have: if $G' < \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A > \tilde{c}_B$; if $G' = \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A = \tilde{c}_B$; and if $G' > \frac{\lambda_B}{\lambda_A}$, then $\tilde{c}_A < \tilde{c}_B$. Choose c^* so that $G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$ if possible, or pick $c^* = -\infty$ if $G' < \frac{\lambda_B}{\lambda_A}$ everywhere or $c^* = +\infty$ if $G' > \frac{\lambda_B}{\lambda_A}$ everywhere. If $\tilde{c}_B \geq c^*$, then $G'(U_A(\tilde{c}_B)) \leq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \leq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \geq \tilde{c}_B$. If $\tilde{c}_B \leq c^*$, then $G'(U_A(\tilde{c}_B)) \geq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, *i.e.*, $G' \geq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \leq \tilde{c}_B$. This proves statement 1.

Now suppose that A and B have equal initial wealths, then the budget constraints for the agents are that

$$E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0, \tag{3.42}$$

therefore, we have $E[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)] = 0$. Since $\lambda_B \tilde{\rho} = U_B'(\tilde{c}_B)$ and $U_B'' < 0$, $\tilde{\rho}$ and $\tilde{c}_B$ are negatively monotonely related. Let $\rho^* \equiv U_B'(c^*)/\lambda_B > 0$. Then $\tilde{\rho} \geq \rho^* \Rightarrow \tilde{c}_A \leq \tilde{c}_B$

[13] This result can be obtained by defining $G(\cdot)$ implicitly from (3.40) and using the implicit function theorem to compute the derivatives of $G(\cdot)$.

and $\tilde{\rho} \leq \rho^* \Rightarrow \tilde{c}_A \geq \tilde{c}_B$. Therefore, $(\tilde{\rho} - \rho^*)(\tilde{c}_A - \tilde{c}_B) \leq 0$ and we have

$$0 = E[\tilde{\rho}(\tilde{c}_A - \tilde{c}_B)] = E[\rho^*(\tilde{c}_A - \tilde{c}_B)] + E[(\tilde{\rho} - \rho^*)(\tilde{c}_A - \tilde{c}_B)] \leq \rho^* E[\tilde{c}_A - \tilde{c}_B]. \quad (3.43)$$

Therefore, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$. This proves statement 2. *Q.E.D.*

Proof of Theorem 3.1: (*Sufficiency*) The monotonicity and concavity of the function and Jensen's inequality yield $E[V(\tilde{Y})] = E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})] = E[E[V(\tilde{X} - \tilde{Z} + \tilde{\varepsilon})|X, Z]] \leq E[V(\tilde{X} - \tilde{Z})] \leq E[V(\tilde{X})]$.

(*Necessity*) Let μ_1 be the distribution of $-\tilde{X}$, and let μ_2 be the distribution of $-\tilde{Y}$. From Theorem 9 of Strassen (1965),[14] the following two statements are equivalent.

(i) For any concave nondecreasing function $V(s)$, $\int V(-s)d\mu_1(s) \geq \int V(-s)d\mu_2(s)$.
(ii) There exists a submartingale $\tilde{\xi}_n$ $(n = 1, 2)$, *i.e.*, $E[\tilde{\xi}_2|\xi_1] \geq \tilde{\xi}_1$, such that the distribution of $\tilde{\xi}_n$ is μ_n.

Let $\tilde{Z} \equiv E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1$ and $\tilde{\varepsilon} \equiv -\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1]$, then (ii) implies that $\tilde{Z} \geq 0$. Since $\tilde{\xi}_1 + \tilde{Z} = E[\tilde{\xi}_2|\xi_1]$, we have $E[\tilde{\varepsilon}|\xi_1 + Z] = E[(-\tilde{\xi}_2 + E[\tilde{\xi}_2|\xi_1])|E[\tilde{\xi}_2|\xi_1]] = 0$. (i) implies $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$, and since $\tilde{\xi}_2 = \tilde{\xi}_1 + (E[\tilde{\xi}_2|\xi_1] - \tilde{\xi}_1) + (\tilde{\xi}_2 - E[\tilde{\xi}_2|\xi_1])$, we have $-\tilde{Y} \sim -\tilde{X} + \tilde{Z} - \tilde{\varepsilon}$, where $\tilde{Z} \sim E[-\tilde{Y}| - X] + \tilde{X} \geq 0$ and $\tilde{\varepsilon} \sim \tilde{Y} + E[-\tilde{Y}| - X]$. It follows that $\tilde{Y} \sim \tilde{X} - \tilde{Z} + \tilde{\varepsilon}$, where $\tilde{Z} \geq 0$ and $E[\tilde{\varepsilon}|X - Z] = 0$. *Q.E.D.*

Proof of Theorem 3.2: The sufficiency follows directly from Jensen's inequality. The necessity can be proved using Theorem 8 in Strassen (1965). We prove it instead

[14] In applying Strassen's result, we ignore ξ_n for $n > 2$. Formally, we set $\xi_n = \xi_2$ and $\mu_n = \mu_2$ for all $n > 2$.

using Theorem 3.1 above. We have $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave function, and in particular, $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nondecreasing functions. Therefore, by Theorem 3.1, $\tilde{Y} \sim \tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1$, where $\tilde{Z}_1 \geq 0$ and $E[\tilde{\varepsilon}_1 | X - Z_1] = 0$. We have

$$E[\tilde{Y}] = E[E[\tilde{Y}|X - Z_1]] = E[E[\tilde{X} - \tilde{Z}_1 + \tilde{\varepsilon}_1 | X - Z_1]] = E[\tilde{X}] - E[\tilde{Z}_1] \leq E[\tilde{X}]. \quad (3.44)$$

Now $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave functions also implies $E[V(\tilde{X})] \geq E[V(\tilde{Y})]$ for all concave nonincreasing functions, *i.e.*, $E[V(-\tilde{X})] \geq E[V(-\tilde{Y})]$ for all concave nondecreasing functions. From Theorem 3.1, $-\tilde{Y} \sim -\tilde{X} - \tilde{Z}_2 + \tilde{\varepsilon}_2 \Rightarrow \tilde{Y} \sim \tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2$, where $\tilde{Z}_2 \geq 0$, and $E[\tilde{\varepsilon}_2 | X + Z_2] = 0$. We have

$$E[\tilde{Y}] = E[E[\tilde{Y}|X + Z_2]] = E[E[\tilde{X} + \tilde{Z}_2 - \tilde{\varepsilon}_2 | X + Z_2]] = E[\tilde{X}] + E[\tilde{Z}_2] \geq E[\tilde{X}]. \quad (3.45)$$

Therefore, $E[\tilde{X}] = E[\tilde{Y}]$, which implies $E[\tilde{Z}_1] = 0$. Since $\tilde{Z}_1 \geq 0$, we must have $\tilde{Z}_1 = 0$. It follows that $\tilde{Y} \sim \tilde{X} + \tilde{\varepsilon}$, where $E[\tilde{\varepsilon}|X] = 0$. *Q.E.D.*

Lemma 3.5 *Suppose B is not weakly more risk averse than A, then there exists an bounded nondegenerate interval $[c_1, c_2]$ and hypothetical agents A_1 and B_1, such that A_1 strictly more risk averse than B_1 ($\forall c, -\frac{U''_{B_1}(c)}{U'_{B_1}(c)} < -\frac{U''_{A_1}(c)}{U'_{A_1}(c)}$) and $\forall c \in [c_1, c_2]$, $U_{A_1}(c) = U_A(c)$ and $U_{B_1}(c) = U_B(c)$.*

Proof of Lemma 3.5: If B is not weakly more risk averse than A, then there exists a constant $\hat{c}$, such that $-\frac{U''_B(\hat{c})}{U'_B(\hat{c})} < -\frac{U''_A(\hat{c})}{U'_A(\hat{c})}$. Since U_A and U_B are of the class of C^2 (see our assumptions in the beginning of Section 3.2), from the continuity of $-\frac{U''_i(c)}{U'_i(c)}$, where $i = A, B$, we get that there exists an interval RA containing $\hat{c}$, *s.t.*, $\forall c \in RA$,

$-\frac{U_B''(c)}{U_B'(c)} < -\frac{U_A''(c)}{U_A'(c)}$. We pick $c_1, c_2 \in RA$ with $c_1 < c_2$. Now, let

$$U_{A_1}(c) = \begin{cases} a_1 - m_1 \exp(\frac{U_A''(c_1)}{U_A'(c_1)}c) & c < c_1 \\ U_A(c) & c_1 \leq c \leq c_2 \\ a_2 - m_2 \exp(\frac{U_A''(c_2)}{U_A'(c_2)}c) & c > c_2, \end{cases}$$

and let

$$U_{B_1}(c) = \begin{cases} b_1 - n_1 \exp(\frac{U_B''(c_1)}{U_B'(c_1)}c) & c < c_1 \\ U_B(c) & c_1 \leq c \leq c_2 \\ b_2 - n_2 \exp(\frac{U_B''(c_2)}{U_B'(c_2)}c) & c > c_2, \end{cases}$$

where a_j and m_j ($j = 1, 2$) are determined by the continuity and smoothness of $U_{A_1}(c)$, and b_j and n_j ($j = 1, 2$) are determined by the continuity and smoothness of $U_{B_1}(c)$. More specifically, for $j = 1, 2$, we have

$$m_j = -\frac{(U_A'(c_j))^2}{U_A''(c_j)} \exp\left(-\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right), \; a_j = m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right) + U_A(c_j), \quad (3.46)$$

and

$$n_j = -\frac{(U_B'(c_j))^2}{U_B''(c_j)} \exp\left(-\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right), \; b_j = n_j \exp\left(\frac{U_B''(c_j)}{U_B'(c_j)}c_j\right) + U_B(c_j). \quad (3.47)$$

Now, $U_{A_1}(c)$ is in the class of C^2 since from (3.46), we have:

$$-m_j \exp\left(\frac{U_A''(c_j)}{U_A'(c_j)}c_j\right)\left(\frac{U_A''(c_j)}{U_A'(c_j)}\right)^2 = U_A''(c_j),$$

i.e., U_{A_1} is twice differentiable. Similarly, we can show that $U_{B_1}(c)$ is also in the class of C^2. Also, we have $U_{A_1}''(c) < 0$, $U_{B_1}''(c) < 0$, and $\forall c$, $-\frac{U_{B_1}''(c)}{U_{B_1}'(c)} < -\frac{U_{A_1}''(c)}{U_{A_1}'(c)}$, *i.e.*, agent A_1 is more risk averse than B_1. *Q.E.D.*

Lemma 3.6 *Suppose B is strictly more risk averse than A ($\forall c, -\frac{U_A''(c)}{U_A'(c)} < -\frac{U_B''(c)}{U_B'(c)}$), and A and B have equal initial wealths. A has an optimal choice $\tilde{c}_A$, and B has an optimal choice $\tilde{c}_B$. We assume that the state price density $\tilde{\rho}$ is not a constant. Then, we have*

1. $\tilde{c}_A \neq \tilde{c}_B$;

2. *if $\tilde{c}_A$ has a bounded support $[c_1, c_2]$, then we have* $\sup \tilde{c}_A \geq \sup \tilde{c}_B$, *and* $\inf \tilde{c}_A \leq \inf \tilde{c}_B$.

Proof of Lemma 3.6: We first prove statement 1 by contradiction. If $\tilde{c}_A = \tilde{c}_B$, then we pick any two points, for example, c_3, c_4 ($c_3 < c_4$) in the support of both $\tilde{c}_A$ and $\tilde{c}_B$. From the first order conditions, we get: $\frac{U_A'(c_3)}{U_A'(c_4)} = \frac{U_B'(c_3)}{U_B'(c_4)}$, *i.e.*, $\frac{U_A'(c_3)}{U_B'(c_3)} = \frac{U_A'(c_4)}{U_B'(c_4)}$. However, from $-\frac{U_A''(c)}{U_A'(c)} < -\frac{U_B''(c)}{U_B'(c)}$, we have: $\frac{d}{dc}\left(\log \frac{U_B'(c)}{U_A'(c)}\right) < 0$, *i.e.*, $\frac{U_B'(c)}{U_A'(c)}$ decreases in c. We have: $\frac{U_B'(c_3)}{U_A'(c_3)} > \frac{U_B'(c_4)}{U_A'(c_4)}$. Contradiction! So, $\tilde{c}_A \neq \tilde{c}_B$.

Since B is more risk averse than A, from Lemma 3.1, we know that there exists c^*, such that $\tilde{c}_A \geq \tilde{c}_B$ when $\tilde{c}_B \geq c^*$, and $\tilde{c}_A \leq \tilde{c}_B$ when $\tilde{c}_B \leq c^*$. And we have $c^* \in [c_1, c_2]$, or else either $\tilde{c}_A \leq \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ or $\tilde{c}_A \geq \tilde{c}_B$ but $\tilde{c}_A \neq \tilde{c}_B$ and both could not satisfy the budget constraint ($E[\tilde{\rho}\tilde{c}_A] = E[\tilde{\rho}\tilde{c}_B] = w_0$). Therefore, $\tilde{c}_A$ has a wider range of support than that of $\tilde{c}_B$. *Q.E.D.*

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