January 2010

Properties of truncated Toeplitz operators

Nicholas Sedlock
Washington University in St. Louis

WASHINGTON UNIVERSITY

Department of Mathematics

Dissertation Examination Committee:
Richard Rochberg, Chair
Albert Baernstein II
Ramanath Cowsik
Kabe Moen
Joseph A. O'Sullivan
Xiang Tang

PROPERTIES OF TRUNCATED TOEPLITZ OPERATORS

by

Nicholas Alexander Sedlock

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment for the degree
of Doctor of Philosophy

May 2010

Saint Louis, Missouri

Acknowledgments

Richard Rochberg taught me almost everything I know about operator theory and functional analysis, and most of what I know about real analysis, and has patiently encouraged me in my studies these last five years. He has helped me every step of the way through the process of my research. I could never have done any of this without him.

John McCarthy taught me everything else I know about operator theory and functional analysis, and provided generous financial support over the last few summers, which were some of the most productive periods of my research life.

Bill Ross and Warren Wogen gave several notes, suggestions and corrections to these results, for which I give my thanks.

My parents, William and Barbara, sacrificed a lot of time and money to make sure I got the best education they could provide for me, and have patiently waited for me to stop being a student and finally get a real job.

Thanks to Don Knuth for writing TeX and to Leslie Lamport for writing LaTeX, thus saving me a great deal of time these last 8 months.

Finally I would like to thank my wife Holly for sticking with me through my highs and lows, and giving me much needed encouragement when I was at my most discouraged.

Contents

Abstract

Let A_Φ be a truncated Toeplitz operator – the compression of the Hardy space Toeplitz operator T_Φ to the model space $H^2 \ominus uH^2$, where u is a nonconstant inner function. We find a necessary and sufficient condition that the product $A_{\Phi_1}A_{\Phi_2}$ is itself a truncated Toeplitz operator. Specifically, we show that there are algebras of truncated Toeplitz operators $\mathcal{B}^\alpha$ (depending on $\alpha \in \mathbb{C}^*$) such that two truncated Toeplitz operators have a truncated Toeplitz operator as a product if they are both in the same $\mathcal{B}^\alpha$. Some consequences of this are also discussed.

Chapter 1

Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{C}^*$ the Riemann sphere, $\mathbb{D}$ denote the unit disc, and let $\mathbb{T}$ denote the unit circle. H^2 is the usual Hardy space, the subspace of $L^2(\mathbb{T})$ of normalized Lebesgue measure m on $\mathbb{T}$ whose harmonic extensions to $\mathbb{D}$ are holomorphic (or, whose negative indexed Fourier coefficients are all zero). H^2 will interchangably refer to both the boundary functions and the functions on $\mathbb{D}$. Let P denote the projection from $L^2(\mathbb{T})$ to H^2, which is given explicitly by the Cauchy integral:

$$(Pf)(\lambda) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \lambda\bar{\zeta}} dm(\zeta), \lambda \in \mathbb{D}$$

By this expression, it makes sense to think of P as an operator from $L^1(\mathbb{T})$ into $\mathrm{Hol}(\mathbb{D})$, the space of holomorphic function on $\mathbb{D}$, which is continuous relative to the weak topology of $L^1(\mathbb{T})$ and the topology of locally uniform convergence of $\mathrm{Hol}(\mathbb{D})$. We also see that the reproducing kernel at $\lambda \in \mathbb{D}$ for the Hardy space is the the Szego kernel $K_\lambda := (1 - \overline{\lambda}z)^{-1}$. Let S denote the shift operator $f \mapsto zf$ on

H^2. Its adjoint (the backwards shift) is the operator

$$S^* f = \frac{f - f(0)}{z}$$

A Toeplitz operator is the compression of a multiplication operator on $L^2(\mathbb{T})$ to H^2. In other words, given $\Phi \in L^2(\mathbb{T})$ (called the symbol of the operator), $T_\Phi = PM_\Phi$ is the operator that sends f to $P(\Phi f)$ for all $f \in H^2$. This operator is bounded if and only if $\Phi \in L^\infty(\mathbb{T})$, and the mapping $\Phi \to T_\Phi$ from L^∞ to the space of bounded operators on H^2 is linear and one-to-one. In the case that $\Phi \in H^\infty$, the Toeplitz operator T_Φ is just the multiplication operator M_Φ. In [BH64], Brown and Halmos describe the algebraic properties of Toeplitz operators. Among other things, they found necessary and sufficient conditions for the product of two Toeplitz operators to itself be a Toeplitz operator, namely that either the first operator's symbol is antiholomorphic or the second operator's symbol is holomorphic. In either case, the symbol of the product is the product of the symbols (ie $T_\Phi T_\Psi = T_{\Phi\Psi}$). From this they derive several results about when a Toeplitz operator is invertible, unitary, or idempotent, and when the product of two Toeplitz operators is the zero operator.

More recently, Sarason [Sar07] found equivalents to several of Brown and Halmos's results for truncated Toeplitz operators on the model spaces $H^2 \ominus uH^2$, where u is some non-constant inner function. The model spaces are the backward-shift invariant subspaces of H^2 (that they are backward shift invariant follows easily from the fact that uH^2 is clearly shift invariant). Let K_u^2 denote the space $H^2 \ominus uH^2$ from here forward. Let $P_u = P - M_u P M_{\overline{u}}$ denote the projection from L^2 to K_u^2.

Given $\Phi \in L^2(\mathbb{T})$ we then define the truncated Toeplitz operator (TTO) A_Φ to be the operator that sends f to $P_u(\Phi f)$ for all $f \in K_u^2$. Truncated Toeplitz operators have many of the same properties as ordinary Toeplitz operators (for example, $A_\Phi^* = A_{\overline{\Phi}}$) but there are also striking differences. For example, there are bounded truncated Toeplitz operators with unbounded symbols (though any truncated Toeplitz operator with a bounded symbol is itself bounded). Additionally, symbols are not unique: the same operator can be generated from more than one symbol, and we say that Ψ is a symbol for A_Φ if $A_\Phi = A_\Psi$. More background about model spaces and truncated Toeplitz operators can be found in Chapter 3.

One result from the Brown-Halmos paper which does not have an equivalent in Sarason's paper is the necessary and sufficient condition for the product of two truncated Toeplitz operators to itself be a truncated Toeplitz operator. It is easy to see that the product of two bounded TTOs with holomorphic symbols is itself a TTO, but the general problem is more delicate. Our results can be found in Chapter 4. Specificaly, in Sections 4.2 and 4.3 we find two necessary and sufficient conditions for the product of two TTOs to itself be a TTO, the first of which is a condition based on the symbol of the operators in question, the second of which identifies a $\mathbb{C}^*$-indexed family of algebras of TTOs and shows that if two TTOs have a TTO for a product, then all three operators lie in the same algebra. In Section 4.4 we show that these algebras are the commutants of certain rank-one perturbations of the compressed shift and their adjoints. In Section 4.5 we also show that the TTOs in these algebras have bounded symbols in most, but not all, cases, and in all cases find a symbol algebra for the products of TTOs. Finally, we discuss invertible TTOs in Section 4.6 and eigenvectors and eigenvalues of TTOs in Section 4.7.

There are two appendices. Appendix A contains results not directly related to the main results in Chapter 4, and Appendix B applies the results of Chapter 4 to the case $u = z^n$ in which TTOs are classical Toeplitz matrices.

In what follows, I refers to the identity operator on whatever space we're considering, $\langle f, g \rangle := \int_{\mathbb{T}} f\overline{g}\, dm$ for all $f, g \in L^2(\mathbb{T})$, and $||f|| := \sqrt{\langle f, f \rangle}$. Further, for f, g in a Hilbert space, $f \otimes g$ represents the rank one operator $f \otimes g(h) := f \langle h, g \rangle$.

Chapter 2

Toeplitz operators on H^2

In [BH64], Brown and Halmos provide many results about Toeplitz operators and how they behave algebraically. For example, an operator T on H^2 is a Toeplitz operator if and only if $T = S^*TS$. These results motivate our questions about TTOs as well as the work of others.

If $\varphi \in L^\infty(\mathbb{T})$, then T_φ is bounded. For $\varphi \in L^2(\mathbb{T})$ T_φ is densely defined (on the polynomials). If T_φ is bounded, however, it follows that φ is also bounded.

Proposition 2.0.1. *Let $\varphi \in L^2(\mathbb{T})$ such that T_φ is a bounded operator. Then $\|\varphi\|_\infty = \|T_\varphi\|$.*

Proof. Let $k_\lambda = \sqrt{1-|\lambda|^2}K_\lambda$ denote the normalized reproducing kernel at $\lambda \in \mathbb{D}$, and note $\|k_\lambda\|^2 = \langle k_\lambda, k_\lambda \rangle = (1-|\lambda|^2)\, K_\lambda(\lambda) = 1$. Thus

$$| \langle T_\varphi k_\lambda, k_\lambda \rangle | \leq \|T_\varphi\|$$

By the definition of the $L^2(\mathbb{T})$ inner product we have

$$\langle T_\varphi k_\lambda, k_\lambda\rangle = \int \varphi(\zeta)\frac{1-|\lambda|^2}{|1-\overline{\lambda}\zeta|^2}\, dm(\zeta) = \varphi(\lambda)$$

where $\varphi(\lambda)$ is the Poisson extension of φ evaluated at λ. Hence $|\varphi(\lambda)| \leq \|T_\varphi\|$ for all λ and so $\|\varphi\|_\infty < \|T_\varphi\|$.

Now let $\psi \in L^\infty(\mathbb{T})$. Then $\|M_\psi\| = \|\psi\|_\infty$ on $L^2(\mathbb{T})$ and hence $\|T_\psi\| = \|PM_\psi\| \leq \|\psi\|_\infty$. The conclusion follows. □

The set of all bounded Toeplitz operators is not an algebra:

Proposition 2.0.2. *$T_\Phi T_\Psi$ is itself a Toeplitz operator if and only if either Ψ or $\overline{\Phi}$ is holomorphic. In this case, $T_\Phi T_\Psi = T_{\Phi\Psi}$.*

From this several facts follow, proofs of which are in Brown and Halmos's paper:

Corollary 1.

1. *The product of two Toeplitz operators is zero if and only if one of the two operators is itself zero.*

2. *Suppose that T_Φ is invertible. Then $(T_\Phi)^{-1}$ is a Toeplitz operator if and only if Φ is holomorphic or antiholomorphic.*

3. *The only unitary Toeplitz operators are the operators T_c for $c \in \mathbb{T}$.*

4. *T_Φ is idempotent if and only if $\Phi = 0$ or 1.*

A related question is when two Toeplitz operators commute:

Proposition 2.0.3. *Two Toeplitz operators T_Φ and T_Ψ commute if and only if both Φ and Ψ are holomorphic or both are antiholomorphic, or if one is a linear function of the other.*

Corollary 2. *The only normal Toeplitz operators are linear functions of self-adjoint ones.*

Chapter 3

K_u^2 and truncated Toeplitz operators

3.1 The Hilbert space K_u^2

From here forward, let u be a non-trivial inner function. K_u^2 is then a reproducing kernel Hilbert space with reproducing kernels $K_\lambda^u := P_u K_\lambda = \frac{1-\overline{u(\lambda)}u}{1-\bar{\lambda}z}$ for $\lambda \in \mathbb{D}$. Note that K_λ^u is bounded for all λ and since the span of the reproducing kernels is dense in K_u^2, $K_u^\infty := L^\infty(\mathbb{T}) \cap K_u^2$ is dense in K_u^2 as well. Thus for any $\Phi \in L^2$, A_Φ is densely defined, since its domain contains K_u^∞.

The function u is said to have an angular derivative in the sense of Carathádory (ADC) at the point $\zeta \in \mathbb{T}$ if u has a nontangential limit $u(\zeta)$ of unit modulus at ζ and u' has a nontangential limit $u'(\zeta)$ at ζ. It is known that u has an ADC at ζ if and only if every function in K_u^2 has a nontangential limit at ζ [Sar94]. Thus there exists a reproducing kernel function K_ζ^u such that $\langle f, K_\zeta^u \rangle = f(\zeta)$. Specifically, K_ζ^u is the limit of K_λ^u as λ approaches ζ nontangentially in the disc and so $K_\zeta^u = \frac{1-\overline{u(\zeta)}u}{1-\bar{\zeta}z}$.

If u is a finite Blaschke product, both u and u' are holomorphic in a domain which compactly contains $\mathbb{D}$ and so these boundary reproducing kernels are defined for every unimodular ζ.

Just at $S = T_z$, define $S_u := A_z$. Then $S_u^* = A_{\overline{z}}$ is simply the ordinary backwards shift, since K_u^2 is backwards shift invariant.

$H^2 = K_u^2 \oplus uH^2$, and by using this fact we can decompose H^2 into the direct sum of countably many disjoint subspaces generated by K_u^2 using Halmos' Wandering Subspace lemma [Hal61], which states that if U is an isometry on a Hilbert space $\mathcal{H}$ and $\mathcal{K} = \mathcal{H} \ominus U\mathcal{H}$, then

$$\mathcal{H} = \left(\bigoplus_{n=0}^{\infty} U^n \mathcal{K} \right) \oplus \left(\bigcap_{n=0}^{\infty} U^n \mathcal{H} \right)$$

Proposition 3.1.1.

$$H^2 = \bigoplus_{n=0}^{\infty} u^n K_u^2$$

Proof. The operator M_u is an isometry on H^2 and so we have that

$$H^2 = \left(\bigoplus_{n=0}^{\infty} u^n K_u^2 \right) \oplus \left(\bigcap_{n=0}^{\infty} u^n H^2 \right)$$

Suppose $f \in \bigcap_{n=0}^{\infty} u^n H^2$. If u has a zero at $\lambda \in \mathbb{D}$ then f has a zero of order ∞ at λ and therefore $f \equiv 0$.

If, on the other hand, u is singular, then

$$u(z) = \zeta \exp \left(- \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)$$

where $\zeta \in \mathbb{T}$ and μ is a bounded positive singular measure. It follows that

$$u^n(z) = \zeta \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} dn\mu(t)\right)$$

where $n\mu$ is a bounded positive singular measure. Let

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\nu(t)\right)$$

be the singular inner factor of f, where ν is a bounded positive singular measure. If $f \in u^n H^2$, then it follows that u^n divides S, which implies that $\nu - n\mu$ is a bounded positive singular measure. Since ν is positive and bounded and μ is positive, it follows that there is sufficienly large n such that this is not the case, and as a result f is not divisible by u^n, a contradiction. Therefore it follows that $\bigcap_{n=0}^{\infty} u^n H^2 = \{0\}$ and the claim follows. $\square$

3.2 C symmetry and K_u^2

In [Gar06, GP06, GP07] C-symmetric operators are introduced. Given a $\mathbb{C}$-Hilbert space $\mathcal{H}$ and an antilinear isometric involution C on $\mathcal{H}$, we say that a bounded operator T is a C-symmetric operator (CSO) if $T^* = CTC$. Here by isometric we mean that $\langle Cf, Cg\rangle = \langle g, f\rangle$. C is called a conjugation operator because if we look, for example, at the space $\mathbb{C}^n$ and define C to be pointwise complex conjugation, a bounded operator M on $\mathbb{C}^n$ is C-symmetric if it is complex symmetric as a matrix.

In $L^2(\mathbb{T})$, the operator $Cf = u\overline{zf}$ is a conjugation which bijectively maps uH^2 to $\overline{zH^2}$ and K_u^2 to itself, and so C can be thought of as a conjugation on K_u^2. From here on, C always refers to this operator. We will sometimes write $\widetilde{f}$ for Cf for sake of readability. An operator that will come up frequently in what follows is the conjugate reproducing kernel $\widetilde{K_\lambda^u} = \frac{u(z)-u(\lambda)}{z-\lambda}$. By the definition of a conjugation, we can see that for $f \in K_u^2$, $\widetilde{f}(\lambda) = \left\langle \widetilde{K_\lambda^u}, f \right\rangle$.

As it turns out, truncated Toeplitz operators are C-symmetric. The following result is Garcia and Putinar's:

Lemma 3.2.1. *For $\Phi \in L^2(\mathbb{T})$ such that A_Φ is bounded, $CA_\Phi C = A_{\overline{\Phi}}$.*

Proof. Let $f, g \in K_u^2$. Then

$$\langle CA_\Phi Cf, g\rangle = \langle Cg, A_\Phi Cf\rangle = \left\langle u\overline{zg}, \Phi u\overline{zf}\right\rangle = \left\langle \overline{\Phi} f, g\right\rangle$$

□

Necessary and sufficient conditions for the product of two CSOs to be a CSO are straightforward.

Proposition 3.2.2. *Let A and B be CSOs on some Hilbert space $\mathcal{H}$. Then AB is a CSO iff $AB = BA$ iff $(AB)^* = A^*B^*$.*

Proof. We will show $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

($1 \Longrightarrow 2$) $AB = C^2ABC^2 = C(AB)^*C = CB^*A^*C = C^2BC^2AC^2 = BA$.

($2 \Longrightarrow 3$) $(AB)^* = (BA)^* = A^*B^*$.

($3 \Longrightarrow 1$) $CABC = CAC^2BC = A^*B^* = (AB)^*$. □

3.3 Some algebraic properties of truncated Toeplitz operators

We will need a number of technical lemmas which can be found in [Cla72, Sar67, Sar07] which we include here for reference. The following is Theorem 4.1 in [Sar07], which gives us a Brown-Halmos-like characterization of the truncated Toeplitz operators.

Fact 3.3.1. A is a TTO iff $A - S_u A S_u^* = \Phi \otimes K_0^u + K_0^u \otimes \Psi$ for some $\Phi, \Psi \in K_u^2$, in which case $A = A_{\Phi+\overline{\Psi}}$, and hence if $\Phi \in L^2(\mathbb{T})$ then $A_\Phi = 0$ if and only if $\Phi \in uH^2 \oplus \overline{uH^2}$.

Definition 3.3.2. For two functions Φ and Ψ in $L^2(\mathbb{T})$ say $\Phi \overset{A}{\equiv} \Psi$ if $A_\Phi = A_\Psi$.

In what follows, when dealing with a TTO A we will usually use a symbol of the form $\varphi_1 + \overline{\varphi_2}$ where $\varphi_i \in K_u^2$. This symbol is not unique. For example, $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi_1+cK_0^u+\overline{\varphi_2}-c\overline{K_0^u}}$ for all $c \in \mathbb{C}$, but $c(K_0^u - \overline{K_0^u}) \neq 0$ if $u(0) \neq 0$. The following is a necessary and sufficient condition for a TTO to be zero.

Proposition 3.3.3. *Let $\varphi_1, \varphi_2 \in K_u^2$. Then $A_{\varphi_1+\overline{\varphi_2}} = 0$ if and only if $\varphi_1 = cK_0^u$ and $\varphi_2 = -\bar{c}K_0^u$ for some $c \in \mathbb{C}$.*

Proof. Let $\varphi_1 = cK_0^u$ and $\varphi_2 = -\bar{c}K_0^u$. Then

$$A_{\varphi_1+\overline{\varphi_2}} = A_{cK_0^u - c\overline{K_0^u}} = A_{cu(z)\overline{u(0)} - c\overline{u(z)}u(0)}$$

so $A_{\varphi_1+\overline{\varphi_2}} = 0$.

Now suppose $A_{\varphi_1+\overline{\varphi_2}} = 0$. Then $A - S_u A S_u^* = 0 = \varphi_1 \otimes K_0^u + K_0^u \otimes \varphi_2$, so $\varphi_1 = cK_0^u$ for some $c \in \mathbb{C}$. Hence $cK_0^u \otimes K_0^u + K_0^u \otimes \varphi_2 = 0$ and so $\varphi_2 = -cK_0^u$ as required. □

Proposition 3.3.4. $I = A_1 = A_{K_0^u} = A_{\overline{K_0^u}}$

Proof. Let $f \in K_u^2$. Then $A_1 f = P(1 \cdot f) = f$, so the first equality holds. Now $A_1 - A_{K_0^u} = A_{1-1+\overline{u(0)}u(z)} = A_{\overline{u(0)}u(z)} = 0$, so the second equality holds. Finally, this implies that $A_{K_0^u}$ is self-adjoint, and so the third equality holds. $\square$

Corollary 3. *Let φ and ψ be in H^2 such that $A_{\varphi+\overline{\psi}}$ is bounded, and let $c \in \mathbb{C}$. Then $A_{\varphi+\overline{\psi}+cK_0^u} = A_{\varphi+\overline{\psi}+c} = A_{\varphi+\overline{\psi}+c\overline{K_0^u}}$.*

Fact 3.3.5.

1. For $\lambda \in \mathbb{D}$,

$$S_u^* K_\lambda^u = \overline{\lambda} K_\lambda^u - \overline{u(\lambda)} \widetilde{K_0^u}, \; S_u \widetilde{K_\lambda^u} = \lambda \widetilde{K_\lambda^u} - u(\lambda) K_0^u$$

2. For nonzero $\lambda \in \mathbb{D}$,

$$S_u K_\lambda^u = \frac{1}{\overline{\lambda}} \left(K_\lambda^u - K_0^u \right), \; S_u^* \widetilde{K_\lambda^u} = \frac{1}{\lambda} \left(\widetilde{K_\lambda^u} - \widetilde{K_0^u} \right)$$

3. These equalities all hold for $\lambda \in \mathbb{T}$ such that u has an ADC at λ.

Fact 3.3.6.

1. $I - S_u S_u^* = K_0^u \otimes K_0^u$

2. $I - S_u^* S_u = \widetilde{K_0^u} \otimes \widetilde{K_0^u}$

The only compact TTO in H^2 is the zero operator. In K_u^2, however, there are many finite rank TTOs.

Fact 3.3.7.

1. Let $\lambda \in \mathbb{D}$. Then $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is a TTO. If $\lambda \in \mathbb{D}$, then

$$\widetilde{K_\lambda^u} \otimes K_\lambda^u = A_{\frac{u}{z-\lambda}}$$

2. Let $\lambda \in \mathbb{T}$ such that u has an ADC at λ. Then

$$K_\lambda^u \otimes K_\lambda^u = A_{K_\lambda^u + \overline{K_\lambda^u} - 1}$$

3. Let $\lambda \in \mathbb{D}$ (or let $\lambda \in \mathbb{T}$ such that u has an ADC at λ). Then

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{K_\lambda^u}}{d\lambda^j} \otimes \frac{d^{n-j-1} K_\lambda^u}{d\overline{\lambda}^{n-j-1}} \right)$$

   is a TTO. If $\lambda \in \mathbb{D}$, then

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{K_\lambda^u}}{d\lambda^j} \otimes \frac{d^{n-j-1} K_\lambda^u}{d\overline{\lambda}^{n-j-1}} \right) = A_{\frac{(n-1)!u}{(z-\lambda)^n}}$$

3.4 The H^∞ functional calculus

Definition 3.4.1. A TTO A is of holomorphic type if there is a function $\varphi \in K_u^2$ such that $A = A_\varphi$. TTOs of anti-holomorphic type are therefore the adjoints of TTOs of holomorphic type.

Corollary 4. *Let $\varphi, \psi \in K_u^2$. Then $A_{\varphi+\overline{\psi}}$ is of holomorphic type if and only if $\psi = cK_0^u$ for some $c \in \mathbb{C}$.*

The product of two TTOs of holomorphic type is itself a TTO of holomorphic type.

Proposition 3.4.2. *Let $\varphi, \psi \in H^2$ such that A_φ, A_ψ are bounded. Then $A_\varphi A_\psi = A_{P_u[\varphi P_u \psi]}$, and so $A_{\overline{\varphi}} A_{\overline{\psi}} = A_{\overline{P_u[\varphi P_u \psi]}}$.*

Proof. We proceed using Fact 3.3.1.

$$\begin{aligned} A_\varphi A_\psi - S_u A_\varphi A_\psi S_u^* &= A_\varphi A_\psi \left(I - S_u S_u^*\right) \\ &= A_\varphi A_\psi \left(K_0^u \otimes K_0^u\right) \\ &= \left(A_\varphi A_\psi K_0^u\right) \otimes K_0^u \\ &= \left(A_\varphi P_u \psi\right) \otimes K_0^u \\ &= \left(P_u[\varphi P_u \psi]\right) \otimes K_0^u \end{aligned}$$

□

Thus the TTOs of holomorphic type form an algebra. It turns out that this algebra is precisely the commutant of the compressed shift S_u. Details are laid out in [Sar67, Saw09]. We reproduce those results here for reference.

Fact 3.4.3. Let $\varphi \in H^\infty$. Then

1. $\|A_\varphi\| \leq \|\varphi\|_\infty$.

2. The map $\varphi \to A_\varphi$ is linear and multiplicative.

3. If ψ is the greatest common inner divisor of u and the inner factor ofφ, then

$$\ker A_\varphi = \frac{u}{\psi} H^2 \ominus u H^2$$

   so in particular $A_\varphi = 0$ if and only if $\varphi \in uH^2$ and A_φ is injective if and only if the inner factor of φ and u are relatively prime.

This is Sarason's characterization of the commutant of S_u.

Fact 3.4.4. Let A be a bounded operator on K_u^2 that commutes with S_u. Then there is a bounded function $\varphi \in H^\infty$ such that $A = A_\varphi$ and $\|A\| = \|\varphi\|_\infty$.

Recall that the spectrum of an operator A on a Hilbert space $\mathcal{H}$ is the set $\{\lambda \in \mathbb{C} \ : \ (\lambda I - A)$ is not invertible on $\mathcal{H}\}$.

Fact 3.4.5. λ is in the spectrum of A_φ if and only if $\inf_{z\in\mathbb{D}}(|u(z)|+|\varphi(z)-\lambda|) = 0$.

3.5 Clark operators and Clark measures

In [Cla72], Clark unitary operators and Clark measures are introduced. For reference we include a summary of these results based on their treatment in [CMR98], slightly modified to better coincide with our main results.

Definition 3.5.1. Let $\alpha \in \mathbb{T}$. Define $U_\alpha = S_u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes \widetilde{K_0^u}$. Then U_α is a unitary operator.

The spectral theorem says that U_α is therefore unitarily equivalent to M_z on $L^2(\mathbb{T}, \mu_\alpha)$ for some Borel μ_α. We call μ_α a Clark measure. The Lebesgue decomposition of μ_α with respect to m has no continuous part, so $\mu_\alpha = (\mu_\alpha)_d + (\mu_\alpha)_s$ where $(\mu_\alpha)_d$ is discrete and $(\mu_\alpha)_s$ is continuous and singular.

Fact 3.5.2. Let E_α be the subset of $\mathbb{T}$ where $u(\zeta) = \alpha$ and let E be the subset of E_α where u does not have an angular derivative. Then

$$(\mu_\alpha)_d = \sum_{\zeta \in e_\alpha \setminus E} \frac{\delta_\zeta}{|u'(\zeta)|}$$

If u is a finite Blaschke product, then $\mu_\alpha = (\mu_\alpha)_d$. From this one can deduce that the eigenvalues of U_α are the $\zeta \in \mathbb{T}$ such that $u(\zeta) = \alpha$ and $u'(\zeta) \neq \infty$, and that their associated eigenvectors are the functions K_ζ^u.

Define $V_\alpha : L^2(\mathbb{T}, \mu_\alpha) \mapsto \mathrm{Hol}(\mathbb{D})$ by letting $V_\alpha(K_\lambda) = K_\lambda^u/(1 - \alpha\overline{u(\lambda)})$ and extending to the whole domain. Then the following is true.

Fact 3.5.3.

$$V_\alpha M_z = U_\alpha V_\alpha$$

3.6 Crofoot transforms

Let $\alpha \in \mathbb{D}$. Define $u_\alpha = \frac{u-\alpha}{1-\overline{\alpha}u}$. Then $T_\alpha = T_{(1-|\alpha|^2)^{-1/2}(1-\overline{\alpha}u)}$ is a bounded operator from H^2 to itself. More is true, however. The following is due to Crofoot [Cro94], which discusses the idea of invertible maps between model spaces more generally.

Fact 3.6.1. T_α is an unitary map from $K_{u_\alpha}^2$ onto K_u^2.

Chapter 4

Results

4.1 S_uC

In what follows, the operator S_uC will feature prominently. Note that since S_u is C-symmetric, we have that $S_uC = CS_u^*$, and that therefore, since $S_u^* = S^*$ restricted to K_u^2 we can explicitly compute S_uCf.

Proposition 4.1.1. *Let $f \in K_u^2$.*

1. $S_uC|_{\text{span}(K_0^u)^\perp}$ *is an antilinear involutive isometry.*

2. $S_uCf = u\overline{(f - f(0))}$.

3. $P_u\left(u\overline{f}\right) = S_u\widetilde{f}$.

4. *If* $u(0) = 0$, $\text{Ker } S_uC = \text{sp}\left(K_0^u\right)$. *If* $u(0) \neq 0$, $\text{Ker } S_uC = \{0\}$.

5. *If* $S_u\widetilde{f} = K_0^u$, *then* $u(0) \neq 0$ *and* $f = cK_0^u$ *for some* $c \in \mathbb{C}$.

6. *If* $S_u\widetilde{f} = cK_0^u$, *then* $f \in \text{sp}\left(K_0^u\right)$.

Proof.

1. Let $f \in K_u^2$ with $f(0) = 0$. Then $S_u\widetilde{f}(0) = \left\langle S_u\widetilde{K_0^u}, f\right\rangle = 0$ and so S_uC maps $K_u^2 \ominus \operatorname{sp}(K_0^u)$ to itself. Antilinearity is obvious. Let $f, g \in \operatorname{sp}(K_0^u)^{\perp}$. Then $S_uCS_uC = S_uS_u^* = I - K_0^u \otimes K_0^u$, so if $f \perp K_0^u$, $S_uCS_uCf = f$ and so S_uC is an involution. Finally,

$$\left\langle S_u\widetilde{f}, S_u\widetilde{g}\right\rangle = \langle CS_u^*f, CS_u^*g\rangle = \langle S_u^*g, S_u^*f\rangle = \langle S_uS_u^*g, f\rangle = \langle g, f\rangle$$

2. $S_uCf = CS_u^*f = C(\overline{z}[f - f(0)]) = u\overline{(f - f(0))}$.

3. $u\overline{f} = S_u\widetilde{f} + u\overline{f(0)}$.

4. Follows from Fact 3.4.3. If $u(0) = 0$, then $\ker S_uC = C(u\overline{z}H^2 \ominus uH^2)$. Thus elements of $\ker S_uC$ are of the form $\overline{g}$ where g is a holomorphic function. Hence g is constant, which means $\ker S_uC = \operatorname{sp}(K_0^u)$. If $u(0) \neq 0$, then $\ker S_u = \{0\}$ and the conclusion follows.

5. First note that if $u(0) = 0$ then

$$S_u\widetilde{f}(0) = \left\langle S_u\widetilde{f}, K_0^u\right\rangle = \left\langle \widetilde{f}, S_u^*K_0^u\right\rangle = \left\langle \widetilde{f}, 0\right\rangle = 0$$

for all $f \in K_u^2$. So since $K_0^u(0) = 1$, $u(0) \neq 0$. Thus it remains to show that if $S_u\widetilde{f} = K_0^u$, then $f = cK_0^u$. Since $S_uC(-K_0^u/u(0)) = K_0^u$ by Fact 3.3.5 we have $f + \frac{K_0^u}{u(0)} \in \operatorname{Ker} S_uC = \operatorname{sp}(K_0^u)$, and the conclusion follows.

6. Follows from (4) and (5).

$\square$

Proposition 4.1.2. *Let $\Phi = \varphi_1 + \overline{\varphi_2}$, $\varphi_i \in K_u^2$ such that A_Φ is bounded. Then $A_\Phi K_0^u = \varphi_1 + \overline{\varphi_2(0)}K_0^u - \overline{u(0)}S_u\widetilde{\varphi_2}$ and $A_\Phi\widetilde{K_0^u} = \widetilde{\varphi_2} + \varphi_1(0)\widetilde{K_0^u} - u(0)S_u^*\varphi_1$.*

Proof.

$$\begin{aligned}
A_\Phi K_0^u &= P_u \left[(\varphi_1 + \overline{\varphi_2}) \left(1 - \overline{u(0)} u \right) \right] \\
&= P_u \left[\varphi_1 + \overline{\varphi_2} - \overline{u(0)} \varphi_1 u - \overline{u(0) \varphi_2} u \right] \\
&= \varphi_1 + P_u \left(\overline{\varphi_2(0)} \right) - \overline{u(0)} P_u \left(u \overline{\varphi_2} \right) \\
&= \varphi_1 + \overline{\varphi_2(0)} K_0^u - \overline{u(0)} S_u \widetilde{\varphi_2}
\end{aligned}$$

The second equation follows from the first. $\square$

4.2 Two conditions for the product of two TTOs to be a TTO

We are interested in the case that $A_\Phi A_\Psi$ is a TTO. Here is a necessary and sufficient condition for this to be true.

Lemma 4.2.1. *Let $\Phi = \varphi_1 + \overline{\varphi_2}$ and $\Psi = \psi_1 + \overline{\psi_2}$ where $\varphi_i, \psi_i \in K_u^2$ such that A_Φ, A_Ψ are bounded. Then $A_\Phi A_\Psi$ is a TTO if and only if*

$$\varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

for some $\Phi_0, \Psi_0 \in K_u^2$.

Proof. In what follows, Φ_0 and Ψ_0 represent functions in K_u^2 that can be different from use to use. By Fact 3.3.1, $A_\Phi A_\Psi$ is a TTO if and only if $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. It suffices to show that $A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = \varphi_1 \otimes \psi_2 - (S_u \widetilde{\varphi_2}) \otimes (S_u \widetilde{\psi_1}) + \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$. By Fact 3.3.6, $I = S_u^* S_u + \widetilde{K_0^u} \otimes \widetilde{K_0^u}$, and by Fact 3.3.1 we have that $S_u A_\Phi S_u^* = A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2$ and so

$$\begin{aligned} A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = A_\Phi A_\Psi &- \left(S_u A_\Phi \widetilde{K_0^u}\right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u}\right) \\ &- \left(A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2\right)\left(A_\Psi - \psi_1 \otimes K_0^u - K_0^u \otimes \psi_2\right) \end{aligned}$$

For sake of readability, we simplify each term in the right hand side separately. First the second term.

$$\begin{aligned} S_u A_\Phi \widetilde{K_0^u} &= S_u \left(\widetilde{\varphi_2} + \varphi_1(0)\widetilde{K_0^u} - u(0) S_u^* \varphi_1\right) \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0) S_u S_u^* \varphi_1 \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0)\varphi_1 + u(0)\left(K_0^u \otimes K_0^u\right)\varphi_1 \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1(0) K_0^u - u(0)\varphi_1 + u(0)\varphi_1(0) K_0^u \\ &= S_u \widetilde{\varphi_2} - u(0)\varphi_1 \end{aligned}$$

The first equality is by Proposition 4.1.2.

Therefore,

$$\begin{aligned} \left(S_u A_\Phi \widetilde{K_0^u}\right) \otimes \left(S_u A_{\overline{\Psi}} \widetilde{K_0^u}\right) &= \left(S_u \widetilde{\varphi_2} - u(0)\varphi_1\right) \otimes \left(S_u \widetilde{\psi_1} - u(0)\psi_2\right) \\ &= S_u \widetilde{\varphi_2} \otimes S_u \widetilde{\psi_1} - u(0)\left[\varphi_1 \otimes S_u \widetilde{\psi_1}\right] \\ &\quad - \overline{u(0)}\left[S_u \widetilde{\varphi_2} \otimes \psi_2\right] + |u(0)|^2 \left[\varphi_1 \otimes \psi_2\right] \end{aligned}$$

Next, the third term.

$$
\begin{aligned}
&\left(A_\Phi - \varphi_1 \otimes K_0^u - K_0^u \otimes \varphi_2\right)\left(A_\Psi - \psi_1 \otimes K_0^u - K_0^u \otimes \psi_2\right) \\
= &\, A_\Phi A_\Psi - \left(A_\Phi \psi_1\right) \otimes K_0^u - \left(A_\Phi K_0^u\right) \otimes \psi_2 \\
&- \varphi_1 \otimes \left(A_{\overline{\Psi}} K_0^u\right) + \psi_1(0)\left(\varphi_1 \otimes K_0^u\right) + \left(1 - |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&- K_0^u \otimes \left(A_{\overline{\Psi}} \varphi_2\right) + \langle \psi_1, \varphi_2 \rangle \left(K_0^u \otimes K_0^u\right) + \overline{\varphi_2(0)}\left(K_0^u \otimes \psi_2\right) \\
&= A_\Phi A_\Psi - \left(A_\Phi \psi_1\right) \otimes K_0^u + \psi_1(0)\left(\varphi_1 \otimes K_0^u\right) - K_0^u \otimes \left(A_{\overline{\Psi}} \varphi_2\right) \\
&+ \langle \psi_1, \varphi_2 \rangle \left(K_0^u \otimes K_0^u\right) + \overline{\varphi_2(0)}\left(K_0^u \otimes \psi_2\right) + \left(1 - |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&- \left(\varphi_1 + \overline{\varphi_2(0)} K_0^u - \overline{u(0)} S_u \widetilde{\varphi_2}\right) \otimes \psi_2 \\
&- \varphi_1 \otimes \left(\psi_2 + \overline{\psi_1(0)} K_0^u - \overline{u(0)} S_u \widetilde{\psi_1}\right)
\end{aligned}
$$

Grouping the $F \otimes K_0^u$ and $K_0^u \otimes G$ terms together, we get

$$
\begin{aligned}
&A_\Phi A_\Psi + \left[\langle \psi_1, \varphi_2 \rangle K_0^u - A_\Phi \psi_1\right] \otimes K_0^u \\
&- K_0^u \otimes \left(A_{\overline{\Psi}} \varphi_2\right) - \left(1 + |u(0)|^2\right)\varphi_1 \otimes \psi_2 \\
&+ \overline{u(0)}\left(S_u \widetilde{\varphi_2} \otimes \psi_2\right) + u(0)\left(\varphi_1 \otimes S_u \widetilde{\psi_1}\right)
\end{aligned}
$$

By combining the expanded terms together, we get

$$
\begin{aligned}
A_\Phi A_\Psi - S_u A_\Phi A_\Psi S_u^* = &\, \varphi_1 \otimes \psi_2 - S_u \widetilde{\varphi_2} \otimes S_u \widetilde{\psi_1} \\
&+ \left[A_\Phi \psi_1 - \langle \psi_1, \varphi_2 \rangle K_0^u\right] \otimes K_0^u \\
&+ K_0^u \otimes \left(A_{\overline{\Psi}} \varphi_2\right)
\end{aligned}
$$

and the result follows. □

In fact, we have found the symbol of the product of two TTOs in the event that their product is a TTO.

Proposition 4.2.2. *If $\Phi = \varphi_1+\overline{\varphi_2}$ and $\Psi = \psi_1+\overline{\psi_2}$ where $\varphi_i, \psi_i \in K_u^2$ and A_Φ, A_Ψ are bounded, and $\varphi_1\otimes\psi_2-(S_u\widetilde{\varphi_2})\otimes(S_u\widetilde{\psi_1}) = \Phi_0\otimes K_0^u+K_0^u\otimes\Psi_0$ for some $\Phi_0, \Psi_0 \in K_u^2$, then $A_\Phi A_\Psi$ is the TTO with symbol $A_\Phi\psi_1 - \langle\psi_1, \varphi_2\rangle K_0^u + \overline{A_{\overline{\Psi}}\varphi_2} + \Phi_0 + \overline{\Psi_0}$*

Lemma 4.2.3. *Let $A_\Phi A_\Psi$ be a TTO. If one of the two operators is of holomorphic (resp. antiholomorphic) type, then either that operator is actually cI or the other operator is also of holomorphic (resp. antiholomorphic) type.*

Proof. Since $A_\Phi A_\Psi$ be a TTO, A_Φ and A_Ψ commute, and by taking adjoints we have that $A_{\overline{\Phi}}A_{\overline{\Psi}}$ is a TTO as well. Thus without loss of generality we suppose A_Φ is of holomorphic type. We will show that either $A_\Phi = cI$ or A_Ψ is of holomorphic type. Let $\varphi, \psi_1, \psi_2 \in K_u^2$ such that $\Phi \overset{A}{\equiv} \varphi$ and $\Psi \overset{A}{\equiv} \psi_1 + \overline{\psi_2}$. Then $A_\phi A_{\psi_1+\overline{\psi_2}}$ is a TTO and so by Lemma 4.2.1 $\varphi\otimes\psi_2 = \Phi_0\otimes K_0^u + K_0^u\otimes\Psi_0$ for some $\Phi_0, \Psi_0 \in K_u^2$. So either $\Phi_0 = c_1K_0^u$ or $\Psi_0 = c_2K_0^u$. If $\Phi_0 = c_1K_0^u$, then it follows that $\varphi = c_3K_0^u$ and $A_\Phi = cI$. Similarly, if $\Psi_0 = c_2K_0^u$, then it follows that $\psi_2 = c_4K_0^u$ and so A_Ψ is of holomorphic type. $\square$

Definition 4.2.4. For $\alpha \in \mathbb{C}^*$, $\mathcal{B}^\alpha := \left\{A_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}+c} | \varphi \in K_u^2, c \in \mathbb{C}\right\}$ with $\mathcal{B}^\infty$ understood to mean the vector space $\{A_{\overline{\varphi}+c} | \varphi \in K_u^2, c \in \mathbb{C}\}$. Note that this makes $\mathcal{B}^0$ the vector space of TTOs of holomorphic type and $\mathcal{B}^\infty$ the vector space of TTOs of antiholomorphic type. An operator is of type α if it is in $\mathcal{B}^\alpha$.

That $\mathcal{B}^\alpha$ is a vector space for $\alpha \in \mathbb{C}\setminus\{0\}$ is straightforward to see since the operator $f \mapsto \overline{S_u\widetilde{f}}$ is $\mathbb{C}$-linear and additive.

The following is a useful alternative symbol for the operators in $\mathcal{B}^\alpha$.

Proposition 4.2.5. *If A_Φ is of type α, then there exists $\varphi_0 \in K_u^2$ and $c \in \mathbb{C}$ such that $\varphi_0(0) = 0$ and $A_\Phi = A_{\varphi_0+\alpha\overline{S_u\widetilde{\varphi_0}}+c}$*

Proof. By definition, $\Phi \stackrel{A}{\equiv} \varphi + \alpha\overline{S_u\widetilde{\varphi}} + c_1$ for some $\varphi \in K_u^2$ and $c_1 \in \mathbb{C}$. Rewrite $\varphi = \varphi_0 + c_2 K_0^u$, where $\varphi \in K_u^2$, $\varphi_0(0) = 0$ and $c_2 \in \mathbb{C}$. Then

$$\begin{aligned} S_u\widetilde{\varphi} &= S_u C(\varphi_0 + c_2 K_0^u) \\ &= S_u\widetilde{\varphi_0} + \overline{c_2} S_u \widetilde{K_0^u} \\ &= S_u\widetilde{\varphi_0} - \overline{c_2 u(0)} K_0^u \end{aligned}$$

Then by Proposition 3.3.4 the result follows. □

Definition 4.2.6. Let $\varphi \in K_u^2$ and $c \in \mathbb{C}$. Define $B_{\varphi+c}^{\alpha} := A_{\varphi+c+\alpha\overline{CS^*(\varphi+c)}} = A_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}+c} = B_{\varphi}^{\alpha} + cI$ for $\alpha \in \mathbb{C} \setminus \{0\}$. $\varphi + c$ is the $\mathcal{B}^{\alpha}$-symbol of the operator B if $B = B_{\varphi+c}^{\alpha}$.

Proposition 4.2.7. *Let B be of type α for some $\alpha \in \mathbb{C}^*$. Then B^* is of type $1/\overline{\alpha}$ using the convention $1/0 = \infty$ and $1/\infty = 0$.*

Proof. If $\alpha = 0$ or ∞ this is obvious, so assume $\alpha \in \mathbb{C} \setminus \{0\}$. There exists $\varphi \in K_u^2$ with $\varphi(0) = 0$ and $c \in \mathbb{C}$ such that $B = B_{\varphi+c}^{\alpha} = A_{\varphi+c+\alpha\overline{S_u\widetilde{\varphi}}}$. Now $S_u\widetilde{S_u\widetilde{\varphi}} = \varphi$ since $\varphi(0) = 0$. So let $\chi = \overline{\alpha} S_u\widetilde{\varphi}$. It follows that $\varphi = \frac{1}{\alpha} S_u\widetilde{\chi}$ and so

$$\begin{aligned} B^* &= A_{\overline{\varphi}+\overline{c}+\overline{\alpha}S_u\widetilde{\varphi}} \\ &= A_{\chi+\overline{c}+\frac{1}{\overline{\alpha}}\overline{S_u\widetilde{\chi}}} \\ &= B_{\chi+\overline{c}}^{1/\overline{\alpha}} \in \mathcal{B}^{1/\overline{\alpha}} \end{aligned}$$

□

Proposition 4.2.8. $B_{\varphi}^{\alpha} = cI$ *if and only if* $\varphi \in \operatorname{sp}(K_0^u)$. *Specifically,* $B_{K_0^u}^{\alpha} = \left(1 - \alpha\overline{u(0)}\right) I$, *and so if* $1 \neq \alpha\overline{u(0)}$, *then any TTO of type α can be written in* *the form B^α_φ, $\varphi \in K^2_u$. More generally, $B^\alpha_{\varphi+c} = A_{(\varphi+c)(1+\alpha\overline{u})}$ for any $\varphi \in K^2_u$ and $c \in \mathbb{C}$.*

Proof. Say

$$B^\alpha_\varphi - cI = A_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}-cK^u_0} = 0$$

Then by Proposition 3.3.3, $\varphi - cK^u_0 = c_2K^u_0$. In the other direction, $B^\alpha_{K^u_0} = A_{K^u_0+\alpha\overline{S_u\widetilde{K^u_0}}} = A_{K^u_0-\alpha\overline{u(0)K^u_0}} = \left(1-\alpha\overline{u(0)}\right)I$ by Proposition 3.3.4. Finally, $A_{K^u_0(1+\alpha\overline{u})} = A_{\left(1-\overline{u(0)}u\right)(1+\alpha\overline{u})} = A_{1-\overline{u(0)}u+\alpha\overline{u}-\alpha\overline{u(0)}} = \left(1-\alpha\overline{u(0)}\right)I$ and the result follows. Write $\varphi = \varphi_0 + c_1K^u_0$.

$$\begin{aligned} B^\alpha_{\varphi_0+c_1K^u_0+c} &= B^\alpha_{\varphi_0} + B^\alpha_{c_1K^u_0} + B^\alpha_c \\ &= A_{\varphi_0+\alpha\overline{S_u\widetilde{\varphi_0}}} + c_1B^\alpha_{K^u_0} + cI \\ &= A_{\varphi_0+\alpha\overline{u}\varphi_0} + A_{c_1K^u_0(1+\alpha\overline{u})} + A_{c(1+\alpha\overline{u})} \end{aligned}$$

The conclusion follows. □

The following is a method for determining when a TTO is of type α.

Proposition 4.2.9. *Let $A := A_{\varphi_1+\overline{\varphi_2}}$ be bounded, where $\varphi_i \in K^2_u$.*

1. *If $\alpha \in \mathbb{C}$, then A is of type α if and only if $\overline{\alpha}S_u\widetilde{\varphi_1} - \varphi_2 \in \operatorname{sp}(K^u_0)$.*

2. *A is of type ∞ if and only if $\varphi_1 \in \operatorname{sp}(K^u_0)$.*

Proof.

1. Let $A_{\varphi_1+\overline{\varphi_2}}$ be of type α. Then there is some $\varphi \in K^2_u$ and $c \in \mathbb{C}$ such that $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi+cK^u_0+\alpha\overline{S_u\widetilde{\varphi}}}$. Thus we have $A_{\varphi_1-\varphi-cK^u_0+\overline{\varphi_2}-\alpha\overline{S_u\widetilde{\varphi}}} = 0$. So by Proposition 3.3.3 we have that $\varphi_1 - \varphi \in \operatorname{sp}(K^u_0)$ and that $\varphi_2 - \overline{\alpha}S_u\widetilde{\varphi} \in$

$\mathrm{sp}\,(K_0^u)$. So then by Fact 3.3.5, we have that $S_u\widetilde{\varphi_1} - S_u\widetilde{\varphi} \in \mathrm{sp}\,(K_0^u)$ and so $\overline{\alpha}S_u\widetilde{\varphi_1} - \overline{\alpha}S_u\widetilde{\varphi} - \varphi_2 + \overline{\alpha}S_u\widetilde{\varphi} = \overline{\alpha}S_u\widetilde{\varphi_1} - \varphi_2 \in \mathrm{sp}\,(K_0^u)$.

Now suppose that $\overline{\alpha}S_u\widetilde{\varphi_1} - \varphi_2 \in \mathrm{sp}\,(K_0^u)$. Then $\varphi_2 = \overline{\alpha}S_u\widetilde{\varphi_1} + cK_0^u$ for some $c \in \mathbb{C}$ and thus $A_{\varphi_1+\overline{\varphi_2}} = A_{\varphi_1+\alpha\overline{S_u\widetilde{\varphi_1}}+\overline{cK_0^u}}$ is of type α.

2. A is of type ∞ if and only if $\varphi_1 + \overline{\varphi_2} \overset{A}{\equiv} \overline{\psi}$ for some $\psi \in K_u^2$, which is true if and only if $\varphi_1 = P_u(\overline{\psi - \varphi_2}) \overset{A}{\equiv} \overline{\psi(0) - \varphi_2(0)}$ which is true if and only if $\varphi_1 \in \mathrm{sp}\,(K_0^u)$.

□

The following is a generalization of Lemma 4.2.3.

Lemma 4.2.10. *Let $A_\Phi A_\Psi$ be a TTO and let $\alpha \in \mathbb{C}^*$. If one of the operators in the product is of type α, then either it is a constant multiple of the identity, or the other is of type α as well.*

Proof. Since $A_\Phi A_\Psi$ is a TTO, $A_\Phi A_\Psi = A_\Psi A_\Phi$ and we assume wlog that A_Φ is of type α. If $\alpha \in \{0, \infty\}$ then the conclusion follows from Lemma 4.2.3, so assume $\alpha \in \mathbb{C}$, $\alpha \neq 0$. So $\Phi \overset{A}{\equiv} \varphi_0 + \alpha\overline{S_u\widetilde{\varphi_0}} + cK_0^u$ and $\Psi \overset{A}{\equiv} \psi_1 + \overline{\psi_2}$ for some $\varphi_0, \psi_1, \psi_2 \in K_u^2$, $\varphi_0(0) = 0$, $c \in \mathbb{C}$. By Lemma 4.2.1, there exists $\Phi_0, \Psi_0 \in K_u^2$ such that

$$\begin{aligned}\Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0 &= (\varphi_0 + cK_0^u) \otimes \psi_2 - \left(S_u(\widetilde{\overline{\alpha}S_u\widetilde{\varphi_0}})\right) \otimes \left(S_u\widetilde{\psi_1}\right) \\ &= \varphi_0 \otimes \psi_2 + cK_0^u \otimes \psi_2 - \varphi_0 \otimes \left(\overline{\alpha}S_u\widetilde{\psi_1}\right) \\ &= \varphi_0 \otimes \left(\psi_2 - \overline{\alpha}S_u\widetilde{\psi_1}\right) + cK_0^u \otimes \psi_2\end{aligned}$$

So we have that $\varphi_0 \otimes \left(\psi_2 - \overline{\alpha}S_u\widetilde{\psi_1}\right) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_1$. So either Φ_0 and K_0^u are linearly dependent or Ψ_1 and K_0^u are. If Φ_0 and K_0^u are linearly dependent, then

$\Phi_0 = c_1 K_0^u$ which means $\varphi_0 = c_2 K_0^u$, but this and $\varphi_0(0) = 0$ then imply that $c_2 = 0$, and so $\varphi_0 = 0$ and $A_\Phi = cI$. Otherwise, $\Psi_1 = c_3 K_0^u$ and so $\psi_2 - \overline{\alpha} S_u \widetilde{\psi_1} = c_4 K_0^u$, which means A_Ψ is of type α by Proposition 4.2.9. $\square$

We now present our main result.

Theorem 4.2.11. *Let $\Phi, \Psi \in \mathcal{A}(K_u^2)$. Then $A_\Phi A_\Psi$ is a TTO if and only if one of two (not mutually exclusive) cases holds:*

Trivial case: Either A_Φ or A_Ψ is equal to cI for some $c \in \mathbb{C}$.

Non-trivial case: A_Φ and A_Ψ are both of type α for some $\alpha \in \mathbb{C}^*$.

Proof. In what follows we will use the fact that if Φ and Ψ are functions such that $A_\Phi A_\Psi$ is a TTO, then for any complex constants c_1, c_2 $A_{\Phi+c_1} A_{\Psi+c_2}$ is also a TTO.

First we prove the sufficiency of both cases. In the trivial case, if either A_Φ or A_Ψ is equal to cI, then $A_\Phi A_\Psi$ is clearly a TTO. In the non-trivial case, if $\alpha = 0$ or ∞, the product is clearly a TTO, so assume $\alpha \in \mathbb{C} \setminus \{0\}$. $A_\Phi = B^\alpha_{\varphi+c_1}$ and $A_\Psi = B^\alpha_{\psi+c_2}$ for some $\varphi, \psi \in K_u^2$ such that $\varphi(0) = \psi(0) = 0$ and $c_1, c_2 \in \mathbb{C}$. It follows from Propositions 3.3.4 that $B^\alpha_{\varphi+c_1} B^\alpha_{\psi+c_2}$ is a TTO if and only if $B^\alpha_\varphi B^\alpha_\psi$ is as well. By the fact that $S_u C = C S_u^*$ and Fact 3.3.6, we have that

$$\begin{aligned}\alpha\left(\varphi \otimes S_u\tilde{\psi}\right) - \alpha\left[\left(S_u\widetilde{S_u\tilde{\varphi}}\right) \otimes S_u\tilde{\psi}\right] &= \alpha\left(\varphi - S_u S_u^*\varphi\right) \otimes S_u\tilde{\psi} \\ &= \left[\left(K_0^u \otimes K_0^u\right)\varphi\right] \otimes \left(\overline{\alpha} S_u\tilde{\psi}\right) \\ &= K_0^u \otimes \left[\overline{\alpha\varphi(0)} S_u\tilde{\psi}\right] = 0\end{aligned}$$

So by Lemma 4.2.1 and the earlier discussion, it follows that the product of the operators in the non-trivial case is a TTO.

In the other direction, suppose $A_\Phi A_\Psi$ is a TTO. By Lemma 4.2.10 it suffices to show that one of A_Φ and A_Ψ is of type α for some α which we can do with Proposition 4.2.9.

There exists $\varphi_i, \psi_i \in K_u^2$ such that we may assume wlog that $\Phi = \varphi_1 + \overline{\varphi_2}$ and that $\Psi = \psi_1 + \overline{\psi_2}$. Then it follows by Lemma 4.2.1 that

$$\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0$$

holds for some Φ_0, Ψ_0 in K_u^2. This can happen in one of five ways:

1. $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = 0$
2. $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = c(K_0^u \otimes K_0^u); c \in \mathbb{C} \setminus \{0\}$
3. $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u; \Phi_0 \neq cK_0^u$
4. $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = K_0^u \otimes \Psi_0; \Psi_0 \neq cK_0^u$
5. $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0, \Phi_0; \Psi_0 \neq cK_0^u$

First consider the case that one of $\varphi_1, \psi_2, S_u\widetilde{\varphi_2}, S_u\widetilde{\psi_1}$ is in $\operatorname{sp}(K_0^u)$. If $S_u\widetilde{\varphi_2}$ (resp. $S_u\widetilde{\psi_1}$) equals cK_0^u, then φ_2 (resp. ψ_1) is also a constant multiple of K_0^u, and thus A_Φ is of holomorphic type (resp. A_Ψ is of antiholomorphic type). Similarly, if φ_1 (resp. ψ_2) equals cK_0^u, then A_Φ is of antiholomorphic type (resp. A_Ψ is of holomorphic type).

In what follows, c and c_i represent complex constants that may change from paragraph to paragraph.

Case 1: We have $\varphi_1 \otimes \psi_2 = (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1})$, which means that ψ_2 and $S_u\widetilde{\psi_1}$ are linearly dependent. Both ψ_2 and $S_u\widetilde{\psi_1}$ are non-zero, so $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1}$ for $\alpha \neq 0$ and it follows that A_Ψ is of type α.

Case 2: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = c(K_0^u \otimes K_0^u); c \neq 0$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. In the latter case, we again get that A_Ψ is of type α for some $\alpha \neq 0$ Assume instead that $\varphi_1 = c_1 S_u\widetilde{\varphi_2}$ for $c_1 \neq 0$. It follows that $S_u\widetilde{\varphi_2} = c_2 K_0^u$ which means A_Φ is of holomorphic type.

Case 3: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u; \Phi_0 \neq cK_0^u$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. In the latter case, we again get that A_Ψ is of type α for some $\alpha \neq 0$. Assume instead that $\varphi_1 = c_1 S_u\widetilde{\varphi_2}$ for $c_1 \neq 0$. Then $S_u\widetilde{\varphi_2} \otimes \left(\overline{c_1}\psi_2 - S_u\widetilde{\psi_1}\right) = \Phi_0 \otimes K_0^u$, so $\overline{c_1}\psi_2 - S_u\widetilde{\psi_1} = c_2 K_0^u$, $c_2 \neq 0$. So by Proposition 4.2.9 A_Ψ is of type α for some α.

Case 4: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = K_0^u \otimes \Psi_0; \Psi_0 \neq cK_0^u$. So either φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent or $S_u\widetilde{\psi_1}$ and ψ_2 are. If φ_1 and $S_u\widetilde{\varphi_2}$ are linearly dependent, then it follows that $\varphi_1 = c_1 K_0^u$ and hence Φ is of type ∞. Otherwise, there exists $\alpha \neq 0$ such that $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1}$ which means A_Ψ is of type α.

Case 5: We have $\varphi_1 \otimes \psi_2 - (S_u\widetilde{\varphi_2}) \otimes (S_u\widetilde{\psi_1}) = \Phi_0 \otimes K_0^u + K_0^u \otimes \Psi_0; \Phi_0, \Psi_0 \neq cK_0^u$. There exists $f \in K_u^2$ such that $f(0) = 0$ and $\langle f, \Phi_0 \rangle = 1$. Then we have

$$\begin{aligned} K_0^u &= \left(\Psi_0 \otimes K_0^u + K_0^u \otimes \Phi_0\right) f \\ &= \left(\psi_2 \otimes \varphi_1\right) f - \left(S_u\widetilde{\psi_1} \otimes S_u\widetilde{\varphi_2}\right) f \\ &= \psi_2 \langle f, \varphi_1 \rangle - S_u\widetilde{\psi_1} \langle f, S_u\widetilde{\varphi_2} \rangle \end{aligned}$$

If $\langle f, \varphi_1 \rangle = 0$, then $cK_0^u = S_u\widetilde{\psi_1}$, and so A_Ψ is of type ∞. Similarly, if $\langle f, S_u\widetilde{\varphi_2} \rangle = 0$, then $cK_0^u = \psi_2$ and A_Ψ is of type 0. So we can assume that $\psi_2 = \overline{\alpha} S_u\widetilde{\psi_1} + cK_0^u$ for some $\alpha \neq 0$. Thus A_Ψ is of type α by Proposition 4.2.9. $\square$

Example 4.2.12. Let $\lambda \in \mathbb{D}$ and consider the rank one TTO $A = \widetilde{K_\lambda^u} \otimes K_\lambda^u$. A simple computation shows that $\left(\widetilde{K_\lambda^u} \otimes K_\lambda^u\right)^2 = u'(\lambda)\widetilde{K_\lambda^u} \otimes K_\lambda^u$ so it follows that

$\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is of type α for some $\alpha \in \mathbb{C}^*$. Furthermore, in the event that $u'(\lambda) = 1$, A is idempotent, and in the event that $u'(\lambda) = 0$, A is nilpotent. Fact 3.3.7 says that the function $u/(z-\lambda)$ is a symbol for A, and since

$$u/(z-\lambda) \stackrel{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)/(z-\lambda) \stackrel{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{zK_\lambda} \stackrel{A}{\equiv} \widetilde{K_\lambda^u} + u(\lambda)\overline{S_u K_\lambda^u}$$

A is of type $u(\lambda)$.

Now instead suppose that $\zeta \in \mathbb{T}$ such that u has an ADC at ζ, and consider $A = K_\zeta^u \otimes K_\zeta^u$. Again it is clear that A^2 is a scalar multiple of A and hence A is of type α for some α. Since A is self-adjoint, it follows that α is unimodular. A simple computation shows that $\widetilde{K_\zeta^u} = \overline{\zeta} u(\zeta) K_\zeta^u$ so

$$S_u \widetilde{K_\zeta^u} = \zeta \widetilde{K_\zeta^u} - u(\zeta) K_0^u = u(\zeta)\left(K_\zeta^u - K_0^u\right)$$

Thus $K_\zeta^u - 1 \stackrel{A}{\equiv} \overline{u(\zeta)} S_u \widetilde{K_\zeta^u}$ and so by Fact 3.3.7 $K_\zeta^u + u(\zeta)\overline{S_u \widetilde{K_\zeta^u}}$ is a symbol for A, which is therefore of type $u(\zeta)$.

Example 4.2.13. Given $\varphi, \psi \in K_u^\infty$, $A_{\varphi(1+\alpha\overline{u})}$ and $A_{\psi(1+\alpha\overline{u})}$ are bounded operators (because their symbols are bounded) and we have that

$$A_{\varphi(1+\alpha\overline{u})} A_{\psi(1+\alpha\overline{u})} = A_{\varphi\psi} + A_{\alpha^2 \overline{u}^2 \varphi\psi} + A_\varphi A_{\overline{\alpha S_u \widetilde{\psi}}} + A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi$$

We know that $A_\varphi A_{\overline{\alpha S_u \widetilde{\psi}}} + A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi$ is a TTO, so we find its symbol using Fact 3.3.1.

$$A_\varphi A_{\overline{\alpha S_u \widetilde{\psi}}} - S_u A_\varphi A_{\overline{\alpha S_u \widetilde{\psi}}} S_u^* = A_\varphi \left(I - S_u S_u^*\right) A_{\overline{\alpha S_u \widetilde{\psi}}} = A_\varphi K_0^u \otimes A_{\overline{\alpha} S_u \widetilde{\psi}} K_0^u = \alpha \left(\varphi \otimes S_u \widetilde{\psi}\right)$$

On the other hand,

$$
\begin{aligned}
&A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi - S_u A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi S_u^* \\
=&A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi - S_u A_{\alpha \overline{S_u \widetilde{\varphi}}} S_u^* S_u A_\psi S_u^* + S_u A_{\alpha \overline{S_u \widetilde{\varphi}}} \left(\widetilde{K_0^u} \otimes \widetilde{K_0^u} \right) A_\psi S_u^* \\
=&A_{\alpha \overline{S_u \widetilde{\varphi}}} A_\psi - \left(A_{\alpha \overline{S_u \widetilde{\varphi}}} - K_0^u \otimes \overline{\alpha} S_u \widetilde{\varphi} \right) (A_\psi - \psi \otimes K_0^u) - \alpha S_u S_u^* \varphi \otimes S_u \widetilde{\psi} \\
=&P_u(\alpha \overline{u} \varphi \psi) \otimes K_0^u + K_0^u \otimes P_u \left(u \overline{\alpha \varphi \left(\psi - \psi(0) \right)} \right) - \alpha \left\langle \overline{u} \varphi \psi, 1 \right\rangle (K_0^u \otimes K_0^u) \\
&- \alpha \varphi \otimes S_u \widetilde{\psi} + \alpha \varphi(0) K_0^u \otimes S_u \widetilde{\psi}
\end{aligned}
$$

The sum of both terms is therefore

$$P_u(\alpha \overline{u} \varphi \psi) \otimes K_0^u + K_0^u \otimes P_u(\overline{\alpha} u \overline{\varphi \psi}) - \alpha \widehat{\overline{u} \varphi \psi}(0) K_0^u \otimes K_0^u$$

which is equal to $A_{\alpha \overline{u} \varphi \psi} - S_u A_{\alpha \overline{u} \varphi \psi} S_u^*$ and so

$$A_{\varphi(1+\alpha\overline{u})} A_{\psi(1+\alpha\overline{u})} = A_{\varphi\psi(1+\alpha\overline{u}+\alpha^2\overline{u}^2)}$$

Now since φ and ψ are in K_u^∞, their product is in H^2. Consider the function $\varphi\psi - P_u(\varphi\psi) = uh$ for some $h \in H^2$. For $g \in H^2$ we have

$$\langle h, ug \rangle = \left\langle uh, u^2 g \right\rangle = \left\langle \varphi\psi, u^2 g \right\rangle = \langle \overline{u} \varphi \overline{u} \psi, g \rangle = 0$$

and so $\varphi\psi = h_1 + u h_2$ for some $h_1, h_2 \in K_u^2$. So

$$\varphi\psi(1+\alpha\overline{u}+\alpha^2\overline{u}^2) = (h_1 + u h_2)(1+\alpha\overline{u}+\alpha^2\overline{u}^2) \overset{A}{\equiv} h_1(1+\alpha\overline{u}) + \alpha h_2(1+\alpha\overline{u})$$

and so $A_{\varphi\psi(1+\alpha\overline{u}+\alpha^2\overline{u}^2)}$ is of type α, and equals $B^\alpha_{h_1+\alpha h_2}$.

If u is a finite Blaschke product, K_u^2 is finite dimensional and equal to K_u^∞. Given $\varphi, \psi \in K_u^2$ and $c_1, c_2 \in \mathbb{C}$, we have then that $B^\alpha_{\varphi+c_1} B^\alpha_{\psi+c_2} = B^\alpha_\varphi B^\alpha_\psi + c_1 B^\alpha_\psi + c_2 B^\alpha_\varphi + c_1 c_2$ is of type α since all the terms on the right hand side are. We therefore have a symbol calculus for the multiplication of TTOs in the case that K_u^2 is finite dimensional.

4.3 Algebras of TTOs

The families $\mathcal{B}^0$ and $\mathcal{B}^\infty$ are actually algebras by Proposition 3.4.2. Example 4.2.13 showed that in certain cases, the product of two TTOs of type α is itself of type α. We will now show that for all $\alpha \in \mathbb{C}^*$, $\mathcal{B}^\alpha$ is an algebra.

Theorem 4.3.1. *Let $\varphi, \psi \in K_u^2$ such that $\varphi(0) = \psi(0) = 0$, let $\alpha \in \mathbb{C}^*$ and let $c_1, c_2 \in \mathbb{C}$. Then $B^\alpha_{\varphi+c_1} B^\alpha_{\psi+c_2}$ is of type α and so $\mathcal{B}^\alpha$ is an algebra.*

Proof.

$$B^\alpha_{\varphi+c_1} B^\alpha_{\psi+c_2} = B^\alpha_\varphi B^\alpha_\psi + c_1 B^\alpha_\psi + c_2 B^\alpha_\varphi + c_1 c_2$$

so it suffices that $B^\alpha_\varphi B^\alpha_\psi$ be a TTO of type α.

As mentioned above, if $\alpha = 0$ or ∞ this follows from Proposition 3.4.2. Suppose then that $\alpha \in \mathbb{C} \setminus \{0\}$. By Proposition 4.2.2 we can find the symbol of the product. We have

$$B^\alpha_\varphi B^\alpha_\psi = A_{\varphi + \alpha \overline{S_u \widetilde{\varphi}}} A_{\psi + \alpha \overline{S_u \widetilde{\psi}}}$$

and since

$$\varphi \otimes \left(\overline{\alpha} S_u \widetilde{\psi} \right) - \left(S_u \widetilde{\overline{\alpha} S_u \widetilde{\varphi}} \right) \otimes S_u \widetilde{\psi} = \alpha \left[(I - S_u S_u^*) \varphi \right] \otimes S_u \widetilde{\psi} = 0$$

the symbol is

$$A_{\varphi+\alpha\overline{S_u\widetilde{\varphi}}}\psi - \langle \psi, \overline{\alpha}S_u\widetilde{\varphi}\rangle K_0^u + \overline{A_{\overline{\psi}+\overline{\alpha}S_u\widetilde{\psi}}\overline{\alpha}S_u\widetilde{\varphi}}$$

Now $S_u\widetilde{K_0^u} \in \mathrm{sp}\,(K_0^u)$ so it suffices to show that

$$\overline{\alpha}S_uC\left(A_{\varphi(1+\alpha\overline{u})}\psi\right) - A_{\overline{\psi}(1+\overline{\alpha}u)}\overline{\alpha}S_u\widetilde{\varphi} = cK_0^u$$

for some $c \in \mathbb{C}$. Because both sides of this equation are in K_u^2 and the reproducing kernels K_λ^u are dense in K_u^2 it suffices to show that the above equation holds pointwise for all $\lambda \in \mathbb{D}$. So let $\lambda \in \mathbb{D}$. Note that $K_0^u(\lambda) = \left(1 - \overline{u(0)}u(\lambda)\right) = \overline{K_\lambda^u(0)}$.

$$\begin{aligned}
\overline{\alpha}S_uC\left(A_{\varphi(1+\alpha\overline{u})}\psi\right)(\lambda) &= \overline{\alpha}\left(S_uA_{\overline{\varphi}(1+\overline{\alpha}u)}\widetilde{\psi}\right)(\lambda)\\
&= \overline{\alpha}\left\langle S_uP_u\left(\left(\overline{\varphi}(1+\overline{\alpha}u)\right)u\overline{z\psi}\right), K_\lambda^u\right\rangle\\
&= \overline{\alpha}\left\langle u\overline{z\varphi\psi}(1+\overline{\alpha}u), S^*K_\lambda^u\right\rangle\\
&= \overline{\alpha}\left\langle u\overline{z\varphi\psi}(1+\overline{\alpha}u), \overline{z}\left(K_\lambda^u - K_\lambda^u(0)\right)\right\rangle\\
&= \overline{\alpha}\left\langle u\overline{\varphi\psi}(1+\overline{\alpha}u), K_\lambda^u - K_\lambda^u(0)\right\rangle\\
&= \left\langle\overline{\psi}(1+\overline{\alpha}u)\overline{\alpha}u\overline{\varphi}, K_\lambda^u\right\rangle - K_0^u(\lambda)\left\langle u\overline{\varphi\psi}(1+\overline{\alpha}u), 1\right\rangle\\
&= \left\langle A_{\overline{\psi}(1+\overline{\alpha}u)}\overline{\alpha}S_u\widetilde{\varphi}, K_\lambda^u\right\rangle - K_0^u(\lambda)\left\langle u\overline{\varphi\psi}(1+\overline{\alpha}u), 1\right\rangle\\
&= \left(A_{\overline{\psi}(1+\overline{\alpha}u)}\overline{\alpha}S_u\widetilde{\varphi}\right)(\lambda) - K_0^u(\lambda)\left\langle u\overline{\varphi\psi}(1+\overline{\alpha}u), 1\right\rangle
\end{aligned}$$

Note that $\left\langle u\overline{\varphi\psi}(1+\overline{\alpha}u), 1\right\rangle$ does not depend on λ and is thus constant, since α, φ and ψ are fixed. □

By Proposition 3.2.2 we see that $\mathcal{B}^\alpha$ is an algebra of commuting operators. If $|\alpha| = 1$ then $\mathcal{B}^\alpha$ is an algebra of commuting normal operators by Proposition 4.2.7.

4.4 Generalized shifts

Definition 4.4.1. Let $\alpha \in \overline{\mathbb{D}}$. Then

$$S_u^\alpha := S_u + \frac{\alpha}{1 - \overline{u(0)}\alpha} K_0^u \otimes \widetilde{K_0^u}$$

Note that $S_u^0 = S_u$.

These are the generalized shift operators and were defined by Sarason in [Sar07]. They are the sum of two TTOs and hence are all TTOs themselves. If α is unimodular, then S_u^α is in fact one of the Clark unitary operators as defined in Section 3.5. The assumption that $|\alpha| \leq 1$ ensures that $1 - \overline{u(0)}\alpha$ is non-zero.

Lemma 4.4.2. *Let $\alpha \in \overline{\mathbb{D}}$. Then S_u^α is of type α. Specifically,*

$$S_u^\alpha = \frac{1}{1 - \overline{u(0)}\alpha} B^\alpha_{S_u K_0^u + \frac{\alpha \overline{u'(0)} K_0^u}{1 - \overline{u(0)}\alpha}}$$

Proof.

$$\begin{aligned}
S_u^\alpha &= S_u + \frac{\alpha}{1 - \overline{u(0)}\alpha} K_0^u \otimes \widetilde{K_0^u} \\
&= A_{S_u K_0^u} + \frac{\alpha}{1 - \overline{u(0)}\alpha} A_{\overline{\frac{\overline{u}}{z}}} \\
&= A_{S_u K_0^u + \frac{\alpha}{1 - \overline{u(0)}\alpha}\left(\overline{\widetilde{K_0^u}} + \overline{u(0)} z\right)} \\
&= A_{S_u K_0^u \left(\frac{1 - \alpha\overline{u(0)} + \alpha\overline{u(0)}}{1 - \alpha\overline{u(0)}}\right) + \frac{\alpha}{1 - \overline{u(0)}\alpha} \overline{\widetilde{K_0^u}}} \\
&= \frac{1}{1 - \overline{u(0)}\alpha} A_{S_u K_0^u + \alpha \overline{\widetilde{K_0^u}}}
\end{aligned}$$

Which is of type α. The symbol of B_φ^α is $\varphi + \alpha\overline{S_u\widetilde{\varphi}}$.

$$\begin{aligned}
S_uC\left(S_uK_0^u + \frac{\alpha\overline{u'(0)}K_0^u}{1-\overline{u(0)}\alpha}\right) &\overset{A}{\equiv} S_uS_u^*\widetilde{K_0^u} + \frac{\overline{\alpha}u'(0)S_u\widetilde{K_0^u}}{1-\overline{\alpha}u(0)} \\
&\overset{A}{\equiv} \widetilde{K_0^u} - u'(0)K_0^u - \frac{\overline{\alpha}u'(0)u(0)K_0^u}{1-\overline{\alpha}u(0)} \\
&\overset{A}{\equiv} \widetilde{K_0^u} - \frac{u'(0)}{1-\overline{\alpha}u(0)}K_0^u
\end{aligned}$$

Therefore the symbol of $B^\alpha_{S_uK_0^u + \frac{\alpha\overline{u'(0)}K_0^u}{1-\overline{u(0)}\alpha}}$ is

$$S_uK_0^u + \frac{\alpha\overline{u'(0)}K_0^u}{1-\overline{u(0)}\alpha} + \alpha\overline{\left(\widetilde{K_0^u} - \frac{u'(0)}{1-\overline{\alpha}u(0)}K_0^u\right)} \overset{A}{\equiv} S_uK_0^u + \alpha\overline{\widetilde{K_0^u}}$$

The result follows. $\square$

Lemma 4.4.3. *Let A be a bounded operator on K_u^2 and let $\alpha \in \overline{\mathbb{D}}$. Then $AS_u^\alpha = S_u^\alpha A$ if and only if A is of type α.*

Proof. If A is of type α, then $AS_u^\alpha = S_u^\alpha A$ by Proposition 3.2.2 and Theorem 4.3.1. To prove the other direction, assume $AS_u^\alpha = S_u^\alpha A$. The first corollary of Theorem 10.1 in [Sar07] implies that A is then a TTO, and hence C-symmetric.

Using the equations

$$AS_u^\alpha = AS_u + \frac{\alpha}{1-\alpha\overline{u(0)}}\left(AK_0^u\right) \otimes \widetilde{K_0^u}$$

and

$$S_u^\alpha A = S_uA + \frac{\alpha}{1-\alpha\overline{u(0)}}K_0^u \otimes \left(A^*\widetilde{K_0^u}\right) = S_uA + \frac{\alpha}{1-\alpha\overline{u(0)}}K_0^u \otimes \left(\widetilde{AK_0^u}\right)$$

we can compute the symbol of A using Fact 3.3.1.

$$\begin{aligned} A - S_u A S_u^* &= A - A S_u S_u^* - \frac{\alpha}{1-\alpha\overline{u(0)}} A K_0^u \otimes S_u \widetilde{K_0^u} + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{A K_0^u} \\ &= A K_0^u \otimes K_0^u + \frac{\overline{u(0)}\alpha}{1-\alpha\overline{u(0)}} A K_0^u \otimes K_0^u + \frac{\alpha}{1-\alpha\overline{u(0)}} K_0^u \otimes S_u \widetilde{A K_0^u} \\ &= \frac{A K_0^u}{1-\alpha\overline{u(0)}} \otimes K_0^u + K_0^u \otimes \overline{\alpha} S_u C\left(\frac{A K_0^u}{1-\alpha\overline{u(0)}}\right) \end{aligned}$$

And so the symbol of A is $\frac{AK_0^u}{1-\alpha\overline{u(0)}} + \alpha\overline{S_u C\left(\frac{AK_0^u}{1-\alpha\overline{u(0)}}\right)}$ which is the symbol of $\frac{1}{1-\alpha\overline{u(0)}} B^{\alpha}_{AK_0^u}$. ☐

Corollary 5. *If A is of type α, $|\alpha| \le 1$, then $A = \frac{1}{1-\alpha\overline{u(0)}} B^{\alpha}_{AK_0^u}$.*

This result yields another (simpler) proof of Theorem 4.3.1. By taking adjoints if necessary, assume that A_Φ and A_Ψ are both of type $\alpha \in \overline{\mathbb{D}}$. Then it follows that S_u^α commutes with both A_Φ and A_Ψ, and thus commutes with their product. Therefore, their product is of type α as well.

Therefore, for any $\alpha \in \mathbb{C}^*$, $\mathcal{B}^\alpha$ is a weakly closed algebra.

Theorem 4.4.4. *Let A_n be TTOs of type $\alpha \in \overline{\mathbb{D}}$ such that A_n converges to A in the weak operator topology. Then A is of type α.*

Proof. Let $f, g \in K_u^2$. Then $\langle S_u^\alpha A f, g\rangle = \langle Af, S_u^{\alpha *} g\rangle = \lim_{n\to\infty} \langle A_n f, S_u^{\alpha *} g\rangle = \lim_{n\to\infty} \langle A_n S_u^\alpha f, g\rangle = \langle A S_u^\alpha f, g\rangle$. ☐

4.5 Bounded interpolation of TTOs

Recall the following result due to Sarason [Sar67]:

Fact 4.5.1. Let A be a bounded operator that commutes with S_u. Then there exists a function $\varphi \in H^\infty$ such that $\|A\| = \|\varphi\|_\infty$ and $A = A_\varphi$.

Hence every bounded operator of type 0 has a bounded symbol. In this section we show that any operator of type $\alpha \in \mathbb{D}$ has a bounded symbol. By taking adjoints, it follows that if α is not unimodular, then any operator of type α has a bounded symbol. In the event that α is unimodular, we will show that there are TTOs of type α with no bounded symbol.

4.5.1 $|\alpha| = 1$

The case of $|\alpha| = 1$ is indirectly dealt with in [Sar07, BCF$^+$09] and we collect those results here. There are TTOs of unimodular type without a bounded symbol under certain conditions. Specifically, in [BCF$^+$09] the following is proven:

Fact 4.5.2. Suppose that u is an inner function with an ADC at $\zeta \in \mathbb{T}$ but such that $K_\zeta^u \notin L^p(\mathbb{T})$ for some $p > 2$. Then $K_\zeta^u \otimes K_\zeta^u$ is a bounded TTO with no bounded symbol.

They also give some conditions on the zeroes of u to ensure that $K_\zeta^u \notin L^p(\mathbb{T})$. Example 4.2.12 shows that $K_\zeta^u \otimes K_\zeta^u$ is of type $u(\zeta)$, and hence it is an example of a TTO of unimodular type without a bounded symbol.

If, however, we weaken what we mean by "bounded symbol" we can find a bounded symbol for any TTO of unimodular type. Specifically, we change the measure with respect to which we take the sup norm of a function.

Let α be unimodular, and fixed for the rest of this section. An operator is of type α if and only if it commutes with S_u^α. S_u^α is unitarily equivalent to M_z on the space $L^2(\mathbb{T}, \mu_\alpha)$ where μ_α is the Clark measure associated with S_u^α. The commutant of M_z is the space of multiplication operators induced by $L^\infty(\mu_\alpha)$ and so by using the unitary equivalence, every operator of type α is equal to $\Phi(S_u^\alpha)$ where $\Phi \in L^\infty(\mu_\alpha)$. In this sense we can think about Φ as a "bounded symbol" for the operator. This gives us a symbol calculus of sorts for operators of type α: given Φ, Ψ bounded μ_α almost everywhere, the product of M_Φ and M_Ψ is clearly $M_{\Phi\Psi}$ where $\Phi\Psi$ is itself bounded μ_α almost everywhere. Hence $\Phi(S_u^\alpha)\Psi(S_u^\alpha) = \Phi\Psi(S_u^\alpha)$.

We can use this symbol calculus to precisely describe the unitary TTOs on a given model space.

Proposition 4.5.3. *Let A be a TTO. Then A is unitary if and only if it is equal to $\Phi(S_u^\alpha)$ for some $\alpha \in \mathbb{T}$ and some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$ such that $|\Phi| = 1$ μ_α almost everywhere. Specifically, any unitary TTO is of unimodular type, and commutes with the Clark unitary operator of the same type.*

Proof. If A is unitary then $AA^* = I$, which means that A and A^* must both be of the same type α. Thus $\alpha = \overline{\alpha}^{-1}$ which implies that α is of unimodular type. So $A = \Phi(S_u^\alpha)$ for some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Then $I = AA^* = \Phi(S_u^\alpha)\overline{\Phi}(S_u^\alpha) = |\Phi|^2(S_u^\alpha)$ which implies that $|\Phi| = 1$ $\mu_\alpha-$almost everywhere. The other direction is obvious. $\square$

4.5.2 $|\alpha| < 1$

Recall that $u_\alpha = \frac{u-\alpha}{1-\bar{\alpha}u}$ for $\alpha \in \mathbb{D}$. In what follows, we will be dealing with TTOs on both K_u^2 and $K_{u_\alpha}^2$. Let A_Φ^u refer to a TTO on K_u^2 and $A_\Phi^{u_\alpha}$ a TTO on $K_{u_\alpha}^2$.

We first consider operators of the form $A^u_{\varphi/(1-\alpha\overline{u})}$ for $\varphi \in H^2$.

Proposition 4.5.4.

1. *$A^u_{\varphi/(1-\alpha\overline{u})} = B_\varphi^\alpha$ for $\varphi \in K_u^2$ and $\alpha \in \mathbb{D}$.*

2. If $\varphi \in H^2$, then $A^u_{\overline{\varphi}/(1-\alpha\overline{u})} = A^u_{\overline{\varphi}}$. Specifically, $A^u_{(1-\alpha\overline{u})^{-1}} = I$.

3. $S^\alpha_u = A^u_{z/(1-\alpha\overline{u})}$.

Proof.

1. Since
$$\frac{1}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty}(\alpha\overline{u})^n$$
we can compute
$$\frac{\varphi}{1-\alpha\overline{u}} = \sum_{n=0}^{\infty}\varphi(\alpha\overline{u})^n$$
But since $\overline{u}\varphi \in \overline{zH^2}$ it follows that $\sum_{n=0}^{\infty}\varphi(\alpha\overline{u})^n \overset{A}{\equiv} \varphi(1+\alpha\overline{u})$ and so $A^u_{\varphi/(1-\alpha\overline{u})} = B^\alpha_\varphi$.

2. $\overline{\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi} + \alpha\overline{u\varphi}/(1-\alpha\overline{u}) \overset{A}{\equiv} \overline{\varphi}$, since $\overline{u\varphi}/(1-\alpha\overline{u}) \in \overline{uH^2}$.

3. First note that $S^\alpha_u = \frac{1}{1-\alpha\overline{u(0)}}\left(B^\alpha_{S_uK^u_0} + \alpha\overline{u'(0)}I\right)$, so it suffices to show that $(1-\alpha\overline{u(0)})A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0}+\alpha\overline{u'(0)}I$. Since $z = S_uK^u_0+uP(\overline{u}z)$, $A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0} + A_{uP(\overline{u}z)/(1-\alpha\overline{u})}$. Now $uP(\overline{u}z)/(1-\alpha\overline{u}) \overset{A}{\equiv} \alpha P(\overline{u}z)/(1-\alpha\overline{u})$, and since $\widetilde{K^u_0} = (u-u(0))\,\overline{z}$, $P(\overline{u}z) = \overline{\widetilde{K^u_0}(0)} - \overline{u(0)}z = \overline{u'(0)} - \overline{u(0)}z$ and so $A^u_{z/(1-\alpha\overline{u})} = B^\alpha_{S_uK^u_0} + \alpha\overline{u'(0)}I + \alpha\overline{u(0)}A^u_{z/(1-\alpha\overline{u})}$. The result follows.

□

Recall $T_\alpha = M_{(1-|\alpha|^2)^{-1/2}(1-\overline{\alpha}u)}$ is an unitary map from $K^2_{u_\alpha}$ onto K^2_u. Note that $T-\alpha^{-1} = M_{(1-|\alpha|^2)^{1/2}(1-\overline{\alpha}u)^{-1}}$.

Lemma 4.5.5. *Let $\varphi \in H^2$ and $\alpha \in \mathbb{D}$. Then $T_\alpha A^{u_\alpha}_\varphi T^{-1}_\alpha = A^u_{\varphi/(1-\alpha\overline{u})}$ and $T_\alpha A^{u_\alpha}_{\overline{\varphi}} T^{-1}_\alpha = A^u_{\overline{\varphi}/(1-\overline{\alpha}u)}$. Therefore $A^{u_\alpha}_\varphi$ and $A^u_{\varphi/(1-\alpha\overline{u})}$ have the same norm, and if $\psi \in H^2$, then $A^u_{\varphi/(1-\alpha\overline{u})} = A^u_{\psi/(1-\alpha\overline{u})}$ if and only if $u_\alpha|(\varphi-\psi)$.*

Proof. It suffices to show that the equalities hold on K_u^∞, so let $f \in K_u^\infty$. Then

$$A^u_{\varphi/(1-\alpha\overline{u})} f = P_u\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) = P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right)$$

On the other hand,

$$\begin{aligned}
T_\alpha A^{u_\alpha}_{\varphi} T_\alpha^{-1} f &= (1-\overline{\alpha}u)\, P_{u_\alpha}\left(\frac{f\varphi}{1-\overline{\alpha}u}\right) \\
&= (1-\overline{\alpha}u)\left[\frac{f\varphi}{1-\overline{\alpha}u} - u_\alpha P\left(\frac{\overline{u_\alpha} f\varphi}{1-\overline{\alpha}u}\right)\right] \\
&= f\varphi - (u-\alpha)P\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\
&= f\varphi + P\left(\frac{\alpha\overline{u}f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right) \\
&= P\left(\frac{f\varphi}{1-\alpha\overline{u}}\right) - uP\left(\frac{\overline{u}f\varphi}{1-\alpha\overline{u}}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
A^u_{\overline{\varphi}/(1-\overline{\alpha}u)} f &= P_u\left(\frac{f\overline{\varphi}}{1-\overline{\alpha}u}\right) \\
&= P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - uP\left(\frac{\overline{u\varphi}f}{1-\overline{\alpha}u}\right) \\
&= (1-\overline{\alpha}u)P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right)
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
T_\alpha A^{u_\alpha}_{\overline{\varphi}} T_\alpha^{-1} f &= (1-\overline{\alpha}u)\, P_{u_\alpha}\left(\frac{f\overline{\varphi}}{1-\overline{\alpha}u}\right) \\
&= (1-\overline{\alpha}u)\left[P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - u_\alpha P\left(\frac{\overline{u_\alpha \varphi} f}{1-\overline{\alpha}u}\right)\right] \\
&= (1-\overline{\alpha}u)\left[P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right) - u_\alpha P\left(\frac{\overline{u\varphi} f}{1-\alpha\overline{u}}\right)\right] \\
&= (1-\overline{\alpha}u) P\left(\frac{\overline{\varphi}f}{1-\overline{\alpha}u}\right)
\end{aligned}$$

□

Theorem 4.5.6. *Let A be an bounded operator on K_u^2 and let $\alpha \in \mathbb{D}$. Then A is of type α if and only if there is a function $\varphi \in H^2$ such that $A = A^u_{\varphi/(1-\alpha\overline{u})}$. In either case, there is a function $\psi \in H^\infty$ such that $\|\psi\|_\infty = \|A\|$ and $A = A^u_{\psi/(1-\alpha\overline{u})}$ and therefore every operator of type α has a bounded symbol. Further, if φ, ψ are in H^∞ then $A^u_{\varphi/(1-\alpha\overline{u})} A^u_{\psi/(1-\alpha\overline{u})} = A^u_{\varphi\psi/(1-\alpha\overline{u})}$.*

Proof. Let $B = T_\alpha^{-1} A T_\alpha$. Then

$$AA^u_{z/(1-\alpha\overline{u})} = A^u_{z/(1-\alpha\overline{u})} A$$

if and only if

$$BA^{u_\alpha}_z = T_\alpha^{-1} A A^u_{z/(1-\alpha\overline{u})} T_\alpha = T_\alpha^{-1} A^u_{z/(1-\alpha\overline{u})} A T_\alpha = A^{u_\alpha}_z B$$

But this is true if and only if $B = A^{u_\alpha}_\varphi$ for some $\varphi \in H^2$ which is true if and only if $A = A^u_{\varphi/(1-\alpha\overline{u})}$ for some $\varphi \in H^2$, hence the first claim holds. To prove the second claim, note that Fact 4.5.1 implies that there is a function $\psi \in H^\infty$ such that $A^{u_\alpha}_\varphi = A^{u_\alpha}_\psi$ and $\|A^{u_\alpha}_\varphi\| = \|\psi\|_\infty$. By Lemma 4.5.5 it therefore follows that

$A = A^u_{\psi/(1-\alpha\overline{u})}$. Since T_α is unitary, $\|A\| = \|\psi\|_\infty$.

To prove the last claim, compute

$$A^u_{\varphi/(1-\alpha\overline{u})} A^u_{\psi/(1-\alpha\overline{u})} = T_\alpha^{-1} A^{u_\alpha}_\varphi A^{u_\alpha}_\psi T_\alpha = T_\alpha^{-1} A^{u_\alpha}_{\varphi\psi} T_\alpha = A^u_{\varphi\psi/(1-\alpha\overline{u})}$$

□

If $|\alpha| > 1$ and A is of type α, then A^* is of type $1/\overline{\alpha} \in \mathbb{D}$, and so the above results can be applied to A^* to get similar results for A. Specifically, A has a bounded symbol. Thus for all α such that $|\alpha| \neq 1$, any operator of type α has a bounded symbol.

Since any operator of type α has a $\mathcal{B}^\alpha$-symbol in K^2_u, we might want to figure out the $\mathcal{B}^\alpha$-symbol of the operator $A_{\varphi/(1-\alpha\overline{u})}$ in the event $\alpha \in \mathbb{D}$. We can achieve this by looking at the decomposition of H^2 induced by the Wandering Subspace lemma by Proposition 3.1.1.

Proposition 4.5.7. *Let $\varphi = \sum_{n=0}^\infty u^n \varphi_n$ where $\varphi_n \in K^2_u$ for all $n \in \mathbb{N}$. Then*

$$A_{\frac{\varphi}{1-\alpha\overline{u}}} = A_{\frac{\sum_{n=0}^\infty \alpha^n \varphi_n}{1-\alpha\overline{u}}}$$

Proof. Let $f, g \in K^\infty_u$. It suffices to show that

$$A_{\frac{\varphi}{1-\alpha\overline{u}}} f = A_{\frac{\sum_{n=0}^\infty \alpha^n \varphi_n}{1-\alpha\overline{u}}} f$$

A simple calculation yields

$$\begin{aligned}\left\langle A_{\frac{\varphi}{1-\alpha\overline{u}}} f, g\right\rangle &= \left\langle \frac{\varphi}{1-\alpha\overline{u}}, \overline{f}g\right\rangle \\ &= \sum_{n=0}^{\infty}\left\langle \frac{u^n\varphi_n}{1-\alpha\overline{u}}, \overline{f}g\right\rangle \\ &= \sum_{n=0}^{\infty}\left\langle \frac{u^n\varphi_n}{1-\alpha\overline{u}} f, g\right\rangle\end{aligned}$$

Since $(1-\alpha\overline{u})^{-1} = \sum_m \alpha^m\overline{u}^m$, for each n we have

$$\begin{aligned}\frac{u^n\varphi_n}{1-\alpha\overline{u}} &\overset{A}{\equiv} u^n\varphi_n \sum_{m=0}^{\infty}\alpha^m\overline{u}^m \\ &\overset{A}{\equiv} \varphi_n \sum_{m=n}^{\infty}\alpha^m\overline{u}^{m-n} \\ &\overset{A}{\equiv} \varphi_n\alpha^n \sum_{m=0}^{\infty}\alpha^m\overline{u}^m \\ &\overset{A}{\equiv} \frac{\alpha^n\varphi_n}{1-\alpha\overline{u}}\end{aligned}$$

and so the conclusion follows. □

4.6 Invertible TTOs of type α and their inverses.

We begin with a theorem that follows from the above.

Theorem 4.6.1. *Let A be an invertible TTO. Then A^{-1} is a TTO if and only if A is of type α for some $\alpha \in \mathbb{C}^*$. As a result, A^{-1} is also of type α.*

Proof. If A^{-1} is a TTO, then both A and A^{-1} are of type α for some $\alpha \in \mathbb{C}^*$ by Theorem 4.2.11 since their product is $I = A_{K_0^u}$. If A is of type α, either $|\alpha| \leq 1$ or A^* is of type $\beta = 1/\overline{\alpha} \leq 1$. In the first case, we have that $AS_u^\alpha = S_u^\alpha A$, so

$A^{-1}S_u^\alpha = A^{-1}S_u^\alpha AA^{-1} = A^{-1}AS_u^\alpha A^{-1} = S_u^\alpha A^{-1}$ and A^{-1} is a TTO of type α by Lemma 4.4.3. In the second case, we have that A^* is an invertible TTO of type β where $|\beta| \leq 1$, so its inverse is a TTO of type β as well. By taking adjoints again, the result follows. $\square$

This raises the question of when a TTO of type α is invertible in the first place. We consider two cases – when $|\alpha| = 1$, and when $|\alpha| < 1$.

4.6.1 $|\alpha| = 1$

We again consider the picture in $L^2(\mathbb{T}, \mu_\alpha)$, where a TTO of type α becomes the multiplication operator M_Φ where $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. It is easy to see precisely when this operator is invertible.

Proposition 4.6.2. *Let $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Then M_Φ is invertible if and only if there exists $\delta > 0$ such that $|\Phi| \geq \delta$ μ_α almost everywhere, and its inverse is $M_{1/\Phi}$.*

For a more concrete example, assume u is a finite Blaschke product of degree n. Then by Fact 3.5.2 it follows that

$$\mu_\alpha = \sum_{j=1}^{n} \frac{\delta_{\zeta_j}}{|u'(\zeta_j)|}$$

where ζ_j are the n distinct zeroes of $u - \alpha$. Thus it follows that M_Φ is invertible if and only if Φ is non-zero on the set where $u = \alpha$.

4.6.2 $|\alpha| < 1$

Fact 3.4.5 gives necessary and sufficient conditions for an operator of holomorphic type to be invertible. The following result is a generalization.

Proposition 4.6.3. *Let $\alpha \in \mathbb{D}$ and let $\varphi \in H^\infty$. Then $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if* $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$

Proof. $A^u_{\varphi/(1-\alpha\overline{u})}$ is invertible if and only if $A^{u_\alpha}_\varphi$ is invertible, which is true if and only if $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) > 0$ by Fact 3.4.5. □

Again we consider the case that u is a finite Blaschke product.

Proposition 4.6.4. *Let $\alpha \in \mathbb{D}$ and $\varphi \in H^\infty$, and let u be a finite Blaschke product. Then $A^{u_\alpha}_\varphi$ is invertible if and only if $\varphi(\zeta) \neq 0$ for all ζ such that $u_\alpha = 0$.*

Proof. From our assumptions, we have that u_α is a finite Blaschke product. Suppose there exists $\zeta \in \mathbb{D}$ such that $u_\alpha(\zeta) = \varphi(\zeta) = 0$. Then clearly $A^{u_\alpha}_\varphi$ is not invertible. If, on the other hand, $\inf_{z\in\mathbb{D}}(|u_\alpha(z)| + |\varphi(z)|) = 0$ then since $|u_\alpha(z)|$ is bounded away from zero near $\mathbb{T}$ it follows that there exists some $\zeta \in \mathbb{D}$ such that $u_\alpha(\zeta) = \varphi(\zeta) = 0$. □

4.7 Eigenvalues and eigenspaces of TTOs of type α

4.7.1 $|\alpha| = 1$

In this section we assume α is in $\mathbb{D}$ and so once again a TTO of type α can be thought of as $\Phi(S^\alpha_u) = V_\alpha M_\Phi$ for some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$. Since V_α is unitary, the eigenvalues of $\Phi(S^\alpha_u)$ are the same as the eigenvalues of M_Φ and the associated eigenspaces are related by V_α.

Proposition 4.7.1. *Let $\{\Phi = \lambda\}$ denote $\{\zeta \in \mathbb{T} : \Phi(\zeta) = \lambda\}$ and let $\{\Phi \neq \lambda\}$ denote its complement. Then the eigenvalues of $\Phi(S^\alpha_u)$ are the $\lambda \in \mathbb{C}$ such that $\mu_\alpha(\{\Phi = \lambda\}) > 0$. The associated eigenspace is $\{V_\alpha f : f \in L^2(\mathbb{T}, \mu_\alpha)$ and $f|_{\{\Phi\neq\lambda\}} = 0\}$.*

Proof. λ is an eigenvalue of M_Φ if and only if Ker $M_{\Phi-\lambda}$ is non-trivial, and the associated eigenspace of λ will be Ker $M_{\Phi-\lambda}$. Ker $M_{\Phi-\lambda}$ consists of functions f whose support is the set $\{\Phi = \lambda\}$, and a non-trivial such f exists if and only if $\mu_\alpha(\{\Phi = \lambda\}) > 0$. By applying V_α to everything, the conclusion follows. $\square$

As an example we consider the case that μ_α is a discrete measure (this includes the case that u is a finite Blaschke product, but it also includes the case $u = \exp\left(\frac{1+z}{1-z}\right)$ [CMR98]). Then we have that

$$\mu_\alpha = \sum_{\zeta \in E_\alpha \setminus E} \frac{\delta_\zeta}{|u'(\zeta)|}$$

where $E_\alpha = \{u = \alpha\}$ and so λ is an eigenvalue of $\Phi(S_u^\alpha)$ if and only if $\Phi(\zeta) = \lambda$ for some $\zeta \in E_\alpha$.

Consider the set $\{K_\zeta^u : \zeta \in E_\alpha \setminus E\}$. For any two distinct $\zeta_1, \zeta_2 \in E_\alpha \setminus E$ we have that $\left\langle K_{\zeta_1}^u, K_{\zeta_2}^u \right\rangle = \frac{1-\overline{u(\zeta_1)}u(\zeta_2)}{1-\overline{\zeta_1}\zeta_2} = 0$, and we also have that $\left\langle K_\zeta^u, K_\zeta^u \right\rangle = \zeta\overline{\alpha}u'(\zeta)$ for all $\zeta \in E_\alpha \setminus E$, and so the K_ζ^u are orthogonal. Furthermore, the spectral theorem says that these vectors are the eigenvectors of S_u^α which is unitary, hence they form an orthonormal basis for K_u^2. This means that the eigenspace of K_u^2 associated with $\Phi(S_u^\alpha)$ is the span of the K_ζ^u such that $\Phi(\zeta) = \lambda$.

For $\zeta \in \mathbb{T}$, $K_\zeta^u \otimes K_\zeta^u$ has one non-zero eigenvalue $\|K_\zeta^u\|^2 = \zeta\overline{u(\zeta)}u'(\zeta)$ with associated eigenvector K_ζ^u. It follows then that

$$\Phi(S_u^\alpha) = \sum_{\zeta \in E_\alpha \setminus E} \frac{\Phi(\zeta)\alpha}{\zeta u'(\zeta)} K_\zeta^u \otimes K_\zeta^u = \sum_{\zeta \in E_\alpha \setminus E} \frac{\Phi(\zeta)\alpha}{\zeta u'(\zeta)} A_{K_\zeta^u(1+\alpha\overline{u})}$$

where the second equality is due to Example 4.2.12.

Suppose now that u is finite rank. Then E_α is a finite point set and $E = \emptyset$, so we have

$$\Phi(S_u^\alpha) = A_{\sum_{\zeta\in E_\alpha} \frac{\Phi(\zeta)\alpha}{\zeta u'(\zeta)} K_\zeta^u (1+\alpha\overline{u})}$$

It is simple to see that $\sum_{\zeta\in E_\alpha} \frac{\Phi(\zeta)\alpha}{\zeta u'(\zeta)} K_\zeta^u(\zeta_j) = \Phi(\eta)$ for all $\eta \in E_\alpha$, and hence $\sum_{\zeta\in E_\alpha} \frac{\Phi(\zeta)\alpha}{\zeta u'(\zeta)} K_\zeta^u = \Phi$ μ_α almost everywhere. In this sense, then, $\Phi(S_u^\alpha) = B_\Phi^\alpha$.

4.7.2 $|\alpha| < 1$

We once more assume $\alpha \in \mathbb{D}$.

Proposition 4.7.2. *Let u be inner, let $\varphi \in H^\infty$, and let ψ_λ be the greatest common inner divisor of u_α and the inner factor of $\varphi - \lambda$ for $\lambda \in \mathbb{C}$. Then the eigenvalues of $A^u_{\varphi/(1-\alpha\overline{u})}$ are the points λ where $\psi_\lambda \neq 1$. The eigenspace associated with λ is*

$$\{\widetilde{f} \ : \ f \in (1-\overline{\alpha}u)K^2_{\psi_\lambda}\}$$

which is a subspace of K_u^2.

Proof. $A^u_{(\varphi-\lambda)/(1-\alpha\overline{u})} = T_\alpha A^{u_\alpha}_{\varphi-\lambda} T_\alpha^{-1}$. It follows that $\ker A^u_{(\varphi-\lambda)/(1-\alpha\overline{u})}$ is equal to $T_\alpha \ker A^{u_\alpha}_{\varphi-\lambda}$. $\ker A^{u_\alpha}_{\varphi-\lambda} = \frac{u_\alpha}{\psi_\lambda} H^2 \ominus uH^2$ and $\langle u\overline{\psi_\lambda} f, uh\rangle = \langle f, \psi_\lambda h\rangle$, hence $u\overline{\psi_\lambda} f \in \ker A^{u_\alpha}_{\varphi-\lambda}$ if and only if $f \in K^2_{\psi_\lambda}$ so we have $\ker A^{u_\alpha}_{\varphi-\lambda} = \frac{u_\alpha}{\psi_\lambda} K^2_{\psi_\lambda}$.

Let $f = u_\alpha \overline{\psi_\lambda} g$ for $g \in K^2_{\psi_\lambda}$. Then

$$\widetilde{f} = u_\alpha \psi_\lambda \overline{z u_\alpha g} = \psi_\lambda \overline{zg}$$

and so $\widetilde{f} \in K^2_{\psi_\lambda}$. Conversely, if $\widetilde{f} \in K^2_{\psi_\lambda}$ then $u_\alpha \overline{zf} = \psi_\lambda \overline{zg}$ for some $g \in K^2_{\psi_\lambda}$ and so we have $f = u_\alpha \overline{\psi_\lambda} g$ and hence $f \in \frac{u_\alpha}{\psi_\lambda} K^2_{\psi_\lambda}$. Hence $\ker A^{u_\alpha}_{\varphi-\lambda} = u_\alpha \overline{z K^2_{\psi_\lambda}}$, and therefore

$$\ker A^u_{(\varphi-\lambda)/(1-\alpha\overline{u})} = (1-\overline{\alpha}u) u_\alpha \overline{z K^2_{\psi_\lambda}} = u\overline{(1-\overline{\alpha}u) z K^2_{\psi_\lambda}}$$

$\square$

Example 4.7.3. Let us apply this to the case where u (and hence u_α) is a Blaschke product. Then ψ_λ is itself a Blaschke product, and $K^2_{\psi_\lambda}$ is spanned by the Hardy space reproducing kernels $K_\zeta^{(j)}$ for all ζ zeroes of ψ_λ and all j less than the multiplicity of the zero at ζ. Note that $(1-\overline{\alpha}u)K_\zeta^{(j)} = \widetilde{K}_\zeta^{u,(j)}$ because $\alpha = u(\zeta)$ for all the zeroes of ψ_λ. Therefore, let $\lambda \in \mathbb{D}$ such that ψ_λ is non-trivial. This implies that for all the zeroes ζ of ψ_λ, $\varphi(\zeta) = \lambda$ and $u(\zeta) = \alpha$. Putting all this together, we have that the eigenvalues of $A^u_{\varphi/(1-\alpha\overline{u})}$ are precisely the points $\varphi(\zeta)$ where $u(\zeta) = \alpha$, and that the associated eigenspace to $\varphi(\zeta)$ is the span of $\widetilde{K_\xi^{u,(j)}}$ for all ξ where $\varphi(z) - \varphi(\zeta)$ and $u(z) - \alpha$ both have a zero (of multiplicity n_1 and n_2 respecitively, and for all $j < n_\xi$, where n_ξ is the smaller of n_1 and n_2.

Appendix A

Other Results

A.1 Self-adjoint TTOs

An ordinary Toeplitz operator T_Φ is self-adjoint if and only if Φ is real-valued. If $\Phi \in L^2(\mathbb{T})$ is real-valued and A_Φ is bounded, then A_Φ is self adjoint. However, $A_{\Phi+u}$ is therefore also self-adjoint, even though $\Phi + u$ isn't real-valued. The following is a close equivalent condition to that in the Hardy space case.

Proposition A.1.1. *Let $\varphi, \psi \in K_u^2$ such that $A = A_{\varphi+\overline{\psi}}$ is bounded. Then A is self-adjoint if and only if $\varphi = \psi + cK_0^u$ for some $c \in \mathbb{R}$. In either case, A has the real-valued symbol $\varphi + \frac{c}{2}K_0^u + \overline{\varphi + \frac{c}{2}K_0^u}$.*

Proof. The condition is clearly sufficient. To prove its necessity, note that $A = A^*$ means that $A_{\varphi-\psi+\overline{\psi-\varphi}} = 0$ which by Proposition 3.3.3 means that $\varphi - \psi = cK_0^u$ for some $c \in \mathbb{R}$. $\square$

If a TTO of type α is self-adjoint, we have that $|\alpha| = 1$ and thus the TTO is $\Phi(S_u^\alpha)$ for some $\Phi \in L^\infty(\mathbb{T}, \mu_\alpha)$ that is real-valued $\mu_\alpha-$almost everywhere.

A.2 Model spaces of dimension 2

A model space is finite dimensional of dimension N if and only if u is a finite Blaschke product with N zeroes, counting multiplicity. The case $N = 1$ is obviously uninteresting, but the case $N = 2$ is weirdly pathological. Here C is the operator that sends $a + bz$ to $\overline{b} + \overline{a}z$. The following result is due to Sarason [Sar07]: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an operator on $K^2_{z^2}$ with respect to the basis $\{1, z\}$, $a, b, c, d \in \mathbb{C}$. With respect to this basis, $C(a, b) = (\overline{b}, \overline{a})$ Then

$$A^* = \begin{bmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{bmatrix} \text{ and } CAC = \begin{bmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{bmatrix}$$

Hence A is a CSO if and only if it is a TTO. Something more is true, however.

Proposition A.2.1. *Let A be a CSO on $K^2_{z^2}$. Then A is of type α for some $\alpha \in \mathbb{C}^*$.*

Proof. A is a TTO by the above discussion, so $A = A_{b\overline{z}+a+cz}$ has the matrix form

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix}$$

If $c = 0$ then A is of type ∞. Otherwise, $b = \alpha c$ for some $\alpha \in \mathbb{C}$, and so A is of type α. □

In fact, in any model space of dimension 2, a CSO is always of type α for some

$\alpha \in \mathbb{C}^*$.

Proposition A.2.2. *Let u be a finite Blaschke product with two zeroes, not necessarily distinct. Then any CSO on K_u^2 is of type α for some $\alpha \in \mathbb{C}^*$.*

Proof. If $u = z^2$ this follows from Proposition A.2.1. Otherwise, let $\lambda \neq 0$ be a zero of u. Fact 3.3.5 says that $S_u \widetilde{K_\lambda^u} = \lambda \widetilde{K_\lambda^u}$ and $S_u \widetilde{K_0^u} = -u(0) K_0^u$. K_0^u and K_λ^u span K_u^2, as do K_0^u and $\widetilde{K_\lambda^u}$. So let A be a CSO, and therefore a TTO. $A = A_{\varphi + \overline{\psi}}$ where $\varphi, \psi \in K_u^2$, and so $\varphi = c_1 K_0^u + a K_\lambda^u$ and $\psi = c_2 K_0^u + b\widetilde{K_\lambda^u}$. If $a = 0$ then A is of type ∞, so assume $a \neq 0$. Then

$$S_u \varphi = -u(0)\overline{c_1} K_0^u + \overline{a}\lambda \widetilde{K_\lambda^u}$$

Let $\alpha = b/\overline{a}\lambda$. Then $\psi - \alpha S_u \widetilde{\varphi} = c K_0^u$ for some $c \in \mathbb{C}$ and hence A is of type α. $\square$

A.3 Sums of TTOs of type α

Recall that $\mathcal{B}^\alpha$ is a $\mathbb{C}$-vector space for all $\alpha \in \mathbb{C}^*$. We also know that any bounded TTO can be written as the sum of two (possibly unbounded) operators, one of holomorphic type and one of antiholomorphic type. We now show that this decomposition extends to any two distinct α_1 and α_2 in $\mathbb{C}^*$.

Proposition A.3.1. *Let A be a bounded TTO, and let $\alpha_1 \neq \alpha_2$ be two points in $\mathbb{C}^*$. Then there exists (possibly unbounded) operators of type α_1 and α_2 such that their sum is bounded and equal to A.*

Proof. We will show that any symbol of a bounded TTO can be decomposed in L^2 into the sum of the symbols of an operator of type α_1 and another operator of type α_2.

First we show that for and $\alpha \in \mathbb{C}$ and bounded TTO A, A has a symbol of the form $f + \alpha\overline{S_u\widetilde{f}} + \overline{g}$ for $f, g \in K_u^2$. A has a symbol of the form $\varphi + \overline{\psi}$ for $\varphi, \psi \in K_u^2$. So we may define $f = \varphi$ and $g = \psi - \overline{\alpha}S_u\widetilde{f}$.

We now show how to write a TTO of holomorphic type $A = A_\varphi$ as the sum of two TTOs of type α_1, α_2 for $\alpha_1 \neq \alpha_2$. Without loss of generality assume $\alpha_1 \in \mathbb{C}$. If $\alpha_2 = \infty$ then we can do this by the above process. Otherwise, we define $\psi_1 = \frac{\alpha_2\varphi}{\alpha_2-\alpha_1}$ and $\psi_2 = \frac{\alpha_1\varphi}{\alpha_1-\alpha_2}$ and then a simple computation shows that $\varphi = \psi_1 + \alpha_1\overline{S_u\widetilde{\psi_1}} + \psi_2 + \alpha_2\overline{S_u\widetilde{\psi_2}}$.

By taking adjoints, we can therefore write any TTO of antiholomorphic type as the sum of a TTO of type α_1 and a TTO of type α_2. This, combined with the first paragraph, completes the proof. $\square$

A.4 Finite rank TTOs

Recall that Fact 3.3.7 establishes that if $\lambda \in \mathbb{D}$, then

$$\widetilde{K_\lambda^u} \otimes K_\lambda^u = A_{\frac{u}{z-\lambda}}$$

and

$$\sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{d^j \widetilde{K_\lambda^u}}{d\lambda^j} \otimes \frac{d^{n-j-1} K_\lambda^u}{d\overline{\lambda}^{n-j-1}} \right) = A_{\frac{(n-1)!u}{(z-\lambda)^n}}$$

Consider $\varphi = \frac{z-\lambda}{1-\overline{\lambda}z}$. Then compute $S_u\widetilde{\varphi} = CS^*\varphi = u\overline{z(\varphi - \varphi(0))/z} = \frac{u(1-|\lambda|^2)}{z-\lambda}$ and so $A_{S_u\widetilde{\varphi}} = (1 - |\lambda|^2)\widetilde{K_\lambda^u} \otimes K_\lambda^u$.

This result generalizes to finite Blaschke products.

Proposition A.4.1. *Let $B(z)$ be a finite Blaschke product. Then $A_{\widetilde{S^*B}}$ has finite* *rank.*

Proof. Let $B(z) = \prod_{j=1}^{n} \left(\frac{z-\lambda_j}{1-\overline{\lambda}_j z}\right)^{\ell_j}$. Then

$$\begin{aligned}
CS^*B(z) &= C\frac{B(z)-B(0)}{z} \\
&= u\overline{(B(z)-B(0))} \\
&= u\left(\prod_{j=1}^{n}\left(\frac{1-\overline{\lambda}_j z}{z-\lambda_j}\right)^{\ell_j} - \overline{B(0)}\right) \\
&= u\frac{\prod_{j=1}^{n}(1-\overline{\lambda}_j z)^{\ell_j} - \overline{B(0)}\prod_{j=1}^{n}(z-\lambda_j)^{\ell_j}}{\prod_{j=1}^{n}(z-\lambda_j)^{\ell_j}} \\
&= u\frac{\prod_{j=1}^{n}(1-\overline{\lambda}_j z)^{\ell_j} - \prod_{j=1}^{n}(|\lambda_j|^2-\overline{\lambda}_j z)^{\ell_j}}{\prod_{j=1}^{n}(z-\lambda_j)^{\ell_j}}
\end{aligned}$$

It suffices to show that the numerator is a polynomial in z of degree at most $\sum_{j=1}^{n} \ell_j - 1$. Then it follows that the fraction can be decomposed into partial fractions and therefore we get

$$\widetilde{S^*B} = \sum_{j=1}^{n}\sum_{k=1}^{\ell_j} \frac{c_{j_k} u}{(z-\lambda_j)^k}$$

and so $A_{\widetilde{S^*B}}$ is a finite sum of finite rank TTOs, and hence a finite rank TTO itself.

Now $\prod_{j=1}^{n}(1-\overline{\lambda}_j z)^{\ell_j} - \prod_{j=1}^{n}(|\lambda_j|^2-\overline{\lambda}_j z)^{\ell_j}$ is clearly a polynomial of degree at most $\sum_{j=1}^{n} \ell_j$ but the coefficient of $z^{\sum_{j=1}^{n} \ell_j}$ is $\overline{B(0)} - \overline{B(0)} = 0$, and hence the claim is proved. $\square$

A.5 Commutants of rank-one TTOs

If two TTOs are of the same type, then they commute. A necessary condition for two TTOs to commute is currently unknown. The following conjecture is based on another result in [BH64].

Conjecture. *Two TTOs commute if and only if they are both of the same type α or if one is a linear transformation of the other.*

This conjecture implies the following weaker conjecture.

Conjecture. *If A is a TTO of type α and B is another TTO, then A and B commute if and only if B is also of type α.*

As a beginning attempt to investigate this second conjecture, consider the rank-one operator $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ for $\lambda \in \mathbb{D}$. Example 4.2.12 establishes that $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ is of type $u(\lambda)$. If the second conjecture were true, then A a TTO would commute with $\widetilde{K_\lambda^u} \otimes K_\lambda^u$ if and only if A were of type $u(\lambda)$.

Proposition A.5.1. *Let A be a CSO on K_u^2, let $\lambda \in \mathbb{D}$ and let $\zeta \in \mathbb{T}$ such that u has an ADC at ζ. Then $A\widetilde{K_\lambda^u} \otimes K_\lambda^u = \widetilde{K_\lambda^u} \otimes K_\lambda^u A$ if and only if $\widetilde{K_\lambda^u}$ is an eigenvector of A. and $AK_\zeta^u \otimes K_\zeta^u = K_\zeta^u \otimes K_\zeta^u A$ if and only if K_ζ^u is an eigenvector of A.*

Proof. $(A\widetilde{K_\lambda^u}) \otimes K_\lambda^u = \widetilde{K_\lambda^u} \otimes (A^* K_\lambda^u) = \widetilde{K_\lambda^u} \otimes (\widetilde{A\widetilde{K_\lambda^u}})$. Thus $A\widetilde{K_\lambda^u} = c\widetilde{K_\lambda^u}$ for some $c \in \mathbb{C}$. The proof of the second statement is similar. □

If the second conjecture is true, then $\widetilde{K_\lambda^u}$ should be an eigenvector of A_Φ (and hence K_λ^u should be an eigenvector of $A_{\overline{\Phi}}$) if and only if A_Φ is of type $u(\lambda)$. This is true at least when $\lambda = 0$.

Proposition A.5.2. *$\widetilde{K_0^u}$ is an eigenvector of A_Φ if and only if A_Φ is of type $u(0)$.*

Proof. Let $\Phi \overset{A}{\equiv} \varphi + \overline{\psi}$ for $\varphi, \psi \in K_u^2$ such that $\varphi(0) = 0$. Then there exists $c \in \mathbb{C}$ such that

$$\begin{aligned} c\widetilde{K_0^u} = A_\Phi \widetilde{K_0^u} &= \widetilde{A_{\overline{\Phi}} K_0^u} \\ &= C\left(P_u(\psi + \overline{\varphi} - \overline{u(0)}u\psi - \overline{u(0)}u\varphi)\right) \\ &= \psi - \overline{u(0)}S_u\widetilde{\varphi} \end{aligned}$$

and the conclusion follows from Proposition 4.2.9. □

More generally, suppose the following were true.

Conjecture. *If A is of type α and A commutes with $\widetilde{K_\lambda^u} \otimes K_\lambda^u$, then A is of type $u(\lambda)$.*

If A were a TTO which commuted with $\widetilde{K_\lambda^u} \otimes K_\lambda^u$, we could decompose it into the sum of the (possibly unbounded) TTOs A_1 and A_2, where A_1 was of type $u(\lambda)$ and A_2 was of type $\alpha \neq u(\lambda)$. It would follow that A_2 were of type $u(\lambda)$, which would imply that A_2 was cI, hence A was of type α.

Appendix B

$u = z^N$

In this appendix, let $u = z^N$ for some fixed $N > 0$. Then K_u^2 is an N dimensional Hilbert space. Specifically, it is the Hilbert space of all polynomials in z of degree at most $N-1$. Let $f(z) = \sum_{k=0}^{N-1} f_k z^k$. Then $\widetilde{f}(z) = \sum_{k=0}^{N-1} \overline{f_{N-1-k}} z^k$. In this setting, TTOs are bounded if and only if they have a bounded symbol, and they have a natural representation as matrices: Let Φ be in $L^\infty(\mathbb{T})$, and let $\hat{\Phi}(k)$ be the k-th Fourier coefficient of Φ. Then

$$A_\Phi = \begin{bmatrix} \hat{\Phi}(0) & \hat{\Phi}(-1) & \hat{\Phi}(-2) & \cdots & \hat{\Phi}(1-N) \\ \hat{\Phi}(1) & \hat{\Phi}(0) & \hat{\Phi}(-1) & \cdots & \hat{\Phi}(2-N) \\ \hat{\Phi}(2) & \hat{\Phi}(1) & \hat{\Phi}(0) & \cdots & \hat{\Phi}(3-N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\Phi}(N-1) & \hat{\Phi}(N-2) & \hat{\Phi}(N-3) & \cdots & \hat{\Phi}(0) \end{bmatrix}$$

with respect to the standard basis $\{z^k\}_{k=0}^{N-1}$. Clearly $\Phi(e^{i\theta}) \overset{A}{\equiv} \sum_{k=1-N}^{N-1} \hat{\Phi}(k) e^{ik\theta}$ which is bounded for any $\Phi \in L^2(\mathbb{T})$. Hence, any $L^2(\mathbb{T})$ function is the symbol of a bounded TTO in K_u^2. Note that TTOs of holomorphic type have lower triangular matrix representations, and TTOs of antiholomorphic type have upper triangular matrix representations. This matrix is constant on its downward diagonals for any Φ. An $n \times n$ matrix which is constant on its downward diagonals is called a Toeplitz matrix.

Let $\alpha \in \mathbb{C}$ and let $\varphi \in K_u^2$. Then

$$\varphi(z) = \sum_{k=0}^{N} \hat{\varphi}(k) z^k$$

and

$$S_u \widetilde{\varphi}(z) = z^N(\varphi(z) - \varphi(0)) = \sum_{k=1}^{N} \overline{\hat{\varphi}(N-k+1)} z^k$$

and so

$$B_\varphi^\alpha = \begin{bmatrix} \hat{\varphi}(0) & \alpha\hat{\varphi}(N) & \alpha\hat{\varphi}(N-1) & \cdots & \alpha\hat{\varphi}(1) \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \alpha\hat{\varphi}(N) & \cdots & \alpha\hat{\varphi}(2) \\ \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) & \cdots & \alpha\hat{\varphi}(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\varphi}(N) & \hat{\varphi}(N-1) & \hat{\varphi}(N-2) & \cdots & \hat{\varphi}(0) \end{bmatrix}$$

with respect to the standard basis, and if $\alpha = \infty$, we have

$$B_\varphi^\alpha = \begin{bmatrix} 0 & \hat{\varphi}(N) & \hat{\varphi}(N-1) & \cdots & \hat{\varphi}(1) \\ 0 & 0 & \hat{\varphi}(N) & \cdots & \hat{\varphi}(2) \\ 0 & 0 & 0 & \cdots & \hat{\varphi}(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

with respect to the standard basis.

Let $\psi \in K_u^2$ as well. Then $B_\varphi^\alpha B_\psi^\alpha$ is a TTO of type α as well, and is equal to B_η^α where

$$\hat{\eta}(\ell) = \sum_{m=0}^{\ell} \hat{\varphi}(m)\hat{\psi}(\ell - m) + \alpha \sum_{n=\ell+1}^{N} \hat{\varphi}(n)\hat{\psi}(N - n + \ell + 1)$$

A clearer picture of what is going on can be found by looking at the product $B_{z^k}^\alpha B_{z^\ell}^\alpha$ for $0 \leq k, \ell < N$. We can use the result in Example 4.2.13 to calculate that the $\mathcal{B}^\alpha$-symbol of the product is $z^{k+\ell}$ if $k + \ell < N$ and $\alpha z^{k+\ell-N}$ otherwise.

Bibliography

[BCF$^+$09] A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timodin. Bounded symbols and reproducing kernel thesis for truncated toeplitz operators. arXiv:0909.0131v1, 2009.

[BH64] A. Brown and P.R. Halmos. Algebraic properties of toeplitz operators. *J. Reine Angew. Math.*, 213:89–102, 1963–1964.

[Cla72] D.N. Clark. One dimensional perturbations of restricted shifts. *J. Analyse Math.*, 25:169–191, 1972.

[CMR98] J.A. Cima, A.L. Matheson, and W.T. Ross. *The Cauchy Transform.* Mathematical Surveys and Monographs, 125. American Mathematical Society, Providence, RI, 1998.

[Cro94] R.B. Crofoot. Multipliers between invariant subspaces of the backward shift. *Pacific J. Math.*, 166(2):225–246, 1994.

[Gar06] S.R. Garcia. Conjugation and clark operators. *Contemporary Mathematics*, 393:67–111, 2006.

[GP06] S.R Garcia and M. Putinar. Complex symmetric operators and applications. *Trans. Amer. Math. Soc.*, 358(3):1285–1315, 2006.

[GP07] S.R Garcia and M. Putinar. Complex symmetric operators and applications ii. *Trans. Amer. Math. Soc.*, 359(8):3913–3931, 2007.

[Hal61] P.R. Halmos. Shifts on hilbert spaces. *J. Reine Agnew. Math.*, 208:102–112, 1961.

[Sar67] D. Sarason. Generalized interpolation in H^∞. *Trans. Amer. Math. Soc*, 127:179–203, 1967.

[Sar94] D. Sarason. *Sub-Hardy Hilbert spaces in the unit disc.* John Wiley and Sons, Inc., New York, 1994.

[Sar07] D. Sarason. Algebraic properties of truncated toeplitz operators. *Operators and Matrices*, 1(4):491–526, 2007.

[Saw09] E.T. Sawyer. *Function theory: interpolation and corona problems.* Fields Institute Monographs, 25. American Mathematical Society, Providence, RI, 2009.