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The Boundary Behavior of Holomorphic Functions

Baili Min
Washington University in St. Louis

WASHINGTON UNIVERSITY

Department of Mathematics

Dissertation Examination Committee:

Steven Krantz, Chair

Carl Bender

Brian Blank

Brody Johnson

John McCarthy

Richard Rochberg

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by

Baili Min

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ABSTRACT

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MIN, Baili

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Professor Steven Krantz, Chairperson

In the theory of several complex variables, the Fatou type problems, the Lindelöf principle, and inner functions have been well studied for strongly pseudoconvex domains. In this thesis, we are going to study more generalized domains, those of finite type. In Chapter 2 we show that there is no Fatou's theorem for approach regions complex tangentially broader than admissible ones, in domains of finite type. In Chapter 3 discussing the Lindelöf principle, we provide some conditions which yield admissible convergence. In Chapter 4 we construct inner functions for a type of domains more general than strongly pseudoconvex ones. Discussion is carried out in $\mathbb{C}^2$.

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夜久喧暂息

池台惟月明

无因驻清景

日出事还生

PREFACE

This thesis presents some results concerning the boundary behavior of holomorphic functions.

There are many topics in this subject, and what I am interested in are the following three: Fatou type problems, the Lindelöf principle, and inner functions.

These root from the classical results in the case of a single complex variable. However, as we move on to the case of several complex variables, the story turns out to be more complicated, and we have to be very careful about the geometry of the boundary.

For the Fatou type problems, it is worthy of paying attention to the shape of the approach regions. In the unit disc for a single variable, the nontangential approach regions are the sharpest regions, but in the unit ball of higher dimension, the nontangential ones, which are cones, are not the sharpest because the admissible approach regions are broader and there is a Fatou's theorem for the admissible regions. The shape of these domains is subtle, nontangential in the complex normal directions but parabolic in the complex tangential direction. And later it is proved that these admissible approach regions are optimal, even for strongly pseudoconvex domains. However, that is not the end of the quest for generalization. After the concept of type of a boundary point is introduced, mathematicians are trying to see if work can be done for domains of finite type. For the Fatou type problems in this case, we already have the definition of admissible approach regions, and have seen that there is also a Fatou's theorem for them. However, we are still not sure if they are optimal. This thesis, in Chapter 2, gives a result showing that for domains for finite type, there is no Fatou's theorem for approach regions tangentially broader than the admissible ones.

The generalization of the Lindelöf principle is more dramatic. As we have the nontangential convergence for the unit disc, we hope to have the admissible convergence for the unit ball, which turns out to be false. Then we begin to wonder what kind of convergence we can have, or under what circumstances we can still have the admissible convergence. There are already answers to both questions, and in Chapter 3 of this thesis we are going to generalize them for domains of finite type. The key is the study of the shape of admissible regions.

The problem of the inner functions is tricky. For a single variable case, this subject has been well explored, and inner functions can be expressed explicitly. But when it comes to several variables, mathematicians even doubted the existence for simple domains such as the unit ball. Eventually inner functions were constructed for the unit ball and strongly pseudoconvex domains. There are quite a few methods and tools developed in this progress, and in Chapter 4 we are going to apply the RW-sequences to construct inner functions for a more general type of domains.

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1. Background

1.1 Complex Tangential Space and Convexity

A domain is a connected open set. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain and $\partial\Omega$ be its boundary. A real-valued, continuously differentiable function defined on a neighborhood U of $\partial\Omega$ is called a defining function if it satisfies

$$\Omega \cap U = \{z \in U : \rho(z) < 0\}$$

and

$$\nabla\rho(z) \neq 0, z \in \partial\Omega.$$

We say that a bounded domain Ω has a C^k boundary if ρ is C^k, that is, ρ being viewed as a function of variables x_1, y_1, $\ldots$, x_n, y_n, the derivatives

$$\frac{\partial^l \rho}{\partial x_1^{m_1} \partial y_1^{t_1} \cdots \partial x_n^{m_n} \partial y_n^{t_n}}$$

exist and are continuous, where $m_1 + t_1 + \cdots + m_n + t_n = l$, $l = 1, 2, \ldots, k$.

If ρ is a smooth function, that is, C^k function for all nonnegative integers k, we say the domain is smooth.

We can think of $z = (z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{C}^n$ as

$$(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n},$$

then we can describe the tangent space $T_P(\partial\Omega)$ at $P \in \partial\Omega$: write $w = (w_1, \cdots, w_n) = (\xi_1 + i\eta_1, \cdots, \xi_n + i\eta_n)$, $w \in T_P(\partial\Omega)$ if and only if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j}\xi_j + \sum_{j=1}^{n} \frac{\partial \rho}{\partial y_j}\eta_j = 0, \tag{1.1}$$

which is the same as

$$2\mathrm{Re}\Big(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j\Big) = 0. \tag{1.2}$$

However, we note that this space is not closed under multiplication by complex numbers. In fact, we are more interested in the space of vectors $w \in \mathbb{C}^n$ that satisfy

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j = 0, \tag{1.3}$$

which is called the complex tangent space to $\partial\Omega$ at P, denoted by $\mathscr{T}_P(\partial\Omega)$.

For example, let us consider the unit ball $B \subset \mathbb{C}^2$. We can equip it with a defining function $\rho(z) = z_1\overline{z}_1 + z_2\overline{z}_2 - 1$. Let $P = (1, 0) \in \partial B$, then $\mathscr{T}_P(\partial B) = \{(0, z_2) : z_2 \in \mathbb{C}\}$.

We have the inner product operation $\langle \cdot, \cdot \rangle$, that is, for $z, w \in \mathbb{C}^n$,

$$\langle z, w \rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n.$$

Let ν_P denote the unit outward normal at P, then $\langle w, \nu_P \rangle = 0$ for $w \in \mathscr{T}_P(\partial\Omega)$.

Consider the linear operator J on $\mathbb{R}^{2n}$ defined by

$$J(x_1, x_2, \cdots, x_{2n-1}, x_{2n}) = (-x_2, x_1, \cdots, -x_{2n}, x_{2n-1}),$$

we can then check that $J : \mathscr{T}_P(\partial\Omega) \to \mathscr{T}_P(\partial\Omega)$ is one to one and onto, $J(\nu_P) \in T_P(\partial\Omega)$ but $J(J(\nu_P)) = -\nu_P \notin T_P(\partial\Omega)$. We call $\mathbb{C}\nu_P$ the complex normal space to $\partial\Omega$ at P.

Now suppose Ω has a C^2 boundary. We say that $P \in \partial\Omega$ is a point of *Levi pseudoconvexity* if the Levi form is positive semi-definite:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geqslant 0$$

for all $w = (w_1, \ldots, w_n) \in \mathscr{T}_P(\partial\Omega)$. If further the Levi form at P is positive definite, we say that P is a point of *strong pseudoconvexity.* For example, the unit ball in $\mathbb{C}^n$, equipped with a defining function $\rho(z) = |z_1|^2 + \cdots + |z_n|^2 - 1$, is strongly pseudoconvex, that is, every boundary point is a point of strongly pseudoconvexity. Another example is the domain $M_2 \subset \mathbb{C}^2$ characterized by the defining function $\rho(z) = |z_1|^2 + |z_2|^4 - 1$, which is weakly pseudoconvex at the boundary points $(e^{i\theta}, 0)$, but strongly pseudoconvex elsewhere.

We must note that the notion of pseudoconvexity is independent of the choice of defining functions (see [18]), so are other notions such as C^k or smooth domains, tangent spaces, etc.

Strongly pseudoconvex domains have been well studied, as we are going to see in the next section.

1.2 Historical Facts I: Fatou's Theorem

We first recall a classical result in his work [12] in 1906 by Fatou. Let D be the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, for $w \in \partial D$ and $\alpha > 1$, define the nontangential approach region Γ_α by

$$\Gamma_\alpha(w) = \{z \in D : |z - w| < \alpha\delta(z)\},$$

where $\delta(z) = \text{dist}(z, \partial D)$ is the ordinary Euclidean distance from the point z to ∂D. Fix $\alpha > 1$. Then, Fatou's theorem states that

Theorem 1.2.1 *Let f be a bounded holomorphic function defined in the unit disc D. Then for almost every $w \in \partial D$, the limit $\lim_{\Gamma_\alpha(w) \ni z \to w} f(z)$ exists.*

In this work, Fatou basically applied the Lebesgue differentiation theorem and Poisson integrals to obtain the result above. Also according to this result, it makes sense to consider the boundary value, as for almost every $w \in \partial D$, we can define $f^*(w) = \lim_{\Gamma_\alpha(w) \ni z \to w} f(z)$ as the boundary value of f. There are several interesting properties about the boundary value of a bounded holomorphic function. As the F. and M. Riesz Uniqueness Theorem (see [28]) says, if $f^*(w) = \lim_{r \to 1-} f(rz) = 0$ for every $w \in E$, where $E \subset \partial D$ is of positive measure, then f is identically 0. However, we need to be careful here and should not drop the condition of being bounded, as shown by Lusin and Privalov in their work [25].

There are other methods and tools. Hardy and Littlewood introduced the Hardy-Littlewood maximal function in [13] in 1930. This method turned out to be very powerful for the study of boundary behavior of functions and had great impact for several complex variables.

That is far from being a complete story for the unit disc. Another fact we must mention is that Littlewood gave a result in his work [22] showing the Fatou's theorem failed for broader approach regions than the nontangential ones:

Theorem 1.2.2 *Let $\gamma = \gamma_0 \subset D \cup \{1\}$ be a simple closed Jordan curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 be the angle θ. Then there exists a bounded holomorphic function $f(z)$ defined in D with the property that, for almost every $\theta \in [0, 2\pi]$, the limit of f along γ_θ does not exist.*

When it comes to several complex variables, a cornerstone is the definition of a holomorphic function. This definition is a little tricky, but we can just think of it as a holomorphic function with respect to every single variable while fixing the others. More details and discussion can be found in [18].

In the quest for generalization of Fatou's theorem to the case of the unit ball in $\mathbb{C}^n$, Korányi discovered a new type of approach region, the "admissible" one. Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball. For $\omega \in \partial B$ and $\alpha > 1$, define the admissible approach region $\mathscr{A}_\alpha(\omega)$ based at ω by

$$\mathscr{A}_\alpha(\omega) = \{z \in B : |1 - \langle z, \omega \rangle| < \alpha\delta(z)\}.$$

Unlike the nontangential one, this type of approach region provides nontangential approach to the base point in complex normal direction but parabolic approacch in the complex tangential directions.

With these admissible approach regions, there is a Fatou's theorem for this case:

Theorem 1.2.3 *Let $f \in H^p(B)$, $0 < p \leqslant \infty$. Then for almost every $\omega \in \partial B$, the limit $\lim_{\mathscr{A}_\alpha(\omega) \ni z \to \omega} f(z)$ exists.*

This result is for functions in the Hardy space $H^p(B)$. Generally speaking,

$$H^p(\Omega) = \{f \text{ holomorphic on } \Omega : \sup_{0<\varepsilon<\varepsilon_0} \int_{\partial\Omega_\varepsilon} |f(\zeta)|^p \, d\sigma_\varepsilon(\zeta) < \infty\},$$

where $\Omega_\epsilon = \{z \in \Omega : \delta(z) > \epsilon\}$, for which we assume that $\partial\Omega_\epsilon$ is C^1, and where $d\sigma_\epsilon$ is the $(2n-1)$-dimensional surface element on $\partial\Omega_\epsilon$.

The proof proceeds by estimating the Poisson-Szegö integral of $g \in L^1(\partial B)$ and the Hardy-Littlewood maximal function $M_2 g(w)$ to obtain a key relation

$$\sup_{z \in \mathscr{A}_\alpha(\omega)} \left| \int_{\partial B} \mathscr{P}(z,\zeta) g(\zeta)\, d\sigma(\zeta) \right| \leqslant C_\alpha M_2 g(w).$$

More technical details can be found in Korányi's original work [17].

This phenomenon was generalized by Stein, who defined the admissible approach region (see [31]) for holomorphic functions in H^p general domains in $\mathbb{C}^n$ as follows:

Suppose that Ω is a domain with C^2 boundary, for $\alpha > 1$, define the admissible approach region based at $P \in \partial\Omega$

$$\mathscr{A}_\alpha(P) = \{z \in \Omega : |\langle z - P, \nu_P \rangle| < \alpha\delta_P(z),\ |z - P|^2 < \alpha\delta_P(z)\},$$

where

$$\delta_P(z) = \min\{\delta(z), \operatorname{dist}(z, T_P(\partial\Omega))\}.$$

And Fatou's theorem in this case is:

Theorem 1.2.4 *Let $0 < p \leqslant \infty$. If $f \in H^p(\Omega)$ then, for almost every $P \in \partial\Omega$, $\lim_{\mathscr{A}_\alpha(P) \ni z \to P} f(z)$ exists.*

It was shown in [14] by Hakim and Sibony that, on the unit ball in $\mathbb{C}^n$, this is the best possible approach region in the sense that there is no Fatou's theorem for approach regions that are complex tangentially broader. In [21], Lempert defined another approach region for pseudoconvex domains in $\mathbb{C}^n$, which is comparable to

Stein's definition to some extent: if the domain is convex, then two types of approach regions are equivalent, but otherwise Lempert's definition provides a larger approach region. His result even fits meromorphic Nevanlinna functions, that is, a meromorphic function f defined in a bounded C^2 domain $\Omega \subset \mathbb{C}^n$ satisfying the following conditions.

Condition I:

$$\sup_{0<\epsilon<\epsilon_0} \int_{\partial\Omega_\epsilon} \log^+ |f(\zeta)| \, d\sigma_\epsilon(\zeta) < \infty$$

where $\log^+ t = \max\{\log t, 0\}$;

Condition II:

$$\int_{\mathscr{P}(f)} \delta(\zeta) \, d\mu(\zeta) < \infty$$

where $\mathscr{P}(f)$ is the set of poles of f and $d\mu$ is the $(2n-2)$-dimensional surface element with the modification that it counts poles of higher order with multiplicity.

How about more general domains? Recently, mathematicians begin to understand more about so called domains of finite type.

1.3 Historical Facts II: The Notion of Finite Type

This concept was introduced by J. J. Kohn when he studied the $\overline{\partial}$ problem (see [16]), which eventually had great impact on the geometry of hypersurfaces in $\mathbb{C}^n$. Among all the work on the finite type conditions, we should mention that by Kohn, Bloom/Graham, Catlin and D'Angelo: they studied the matter in terms of ideals and iterated commutators (see [16], [4], [5], [9] and [10]).

It will take a long discussion to explain the concept of finite type completely. Here we hope to provide some easy description. Let us have a look at $\mathbb{C}^2$. Thinking geometrically, we are interested in the maximal order of contact of complex lines with the boundary of the domain at a boundary point. If a domain is strongly pseudoconvex, for instance, the unit ball, then it is also of finite type, we can check that this maximal order of contact is always 2. However, for some other domains, this order can be bigger. For example, at $(1,0)$ on the boundary of the domain M_2 mentioned previously, it is not strongly pseudoconvex, and the maximal order is 4, still finite. We can also check other boundary points for M_2 and find that all the maximal orders are finite, and in fact either 4 for points $(e^{i\theta},0)$, or 2 for other points.

Thinking analytically, let ρ be a defining function for a domain $\Omega \subset \mathbb{C}^2$ and consider two vector fields:

$$L = -\frac{\partial \rho}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial \rho}{\partial z_1}\frac{\partial}{\partial z_2},$$

and

$$\overline{L} = -\frac{\partial \rho}{\partial \overline{z}_2}\frac{\partial}{\partial \overline{z}_1} + \frac{\partial \rho}{\partial \overline{z}_1}\frac{\partial}{\partial \overline{z}_2}.$$

At $P \in \partial\Omega$, we then keep computing iterated commutators as follows. Degree 1: $[L,\overline{L}]$; degree 2: $[\overline{L},[L,\overline{L}]]$; degree 3: $[L,[\overline{L},[L,\overline{L}]]]$ ...All of them are lying in the real tangent space to $\partial\Omega$ at P. However, we find that at the beginning, these commutators are still within the complex tangent space, but after several steps, there comes a component in the complex normal direction. We are interested in the number m such that the iterated commutator with degree $m-1$ still lies in the complex tangent space but the one with degree m will have a component that jumps to the complex normal direction. For example, for every point in the unit sphere, this number is 2. But for M_2, this number is either 4 for points $(e^{i\theta}, 0)$, or 2 for other points.

As we can see from the examples, those numbers coincide. We call it the type of the point. Actually there is a result in [18] which states that the geometric type and the analytic type mentioned previously are really the same. If the domain is such that for every boundary point, the type is finite, then we say that this domain is of finite type. So this notion is more general than being strongly pseudoconvex. If Ω is strongly pseudoconvex, then it must be of finite type. But the converse may not be true. For example, M_2 is of finite type but not strongly pseudoconvex.

This concept, especially in terms of iterated commutators, will be mentioned and studied in Chapter 2.

In the higher-order case, we can still consider the orders of contact of $(n-1)$-dimensional complex manifolds with the $\partial\Omega$. We can also think analytically upon tangent vector fields and their iterated commutators to define the type of a boundary point. In [4], a theorem by Bloom and Graham shows that they are the same thing.

Back to the Fatou type problems, we now wish to know what approach regions will be like if the domain is of finite type, which means it may not be strongly pseudoconvex. A notable progress is in the joint work [26], where Nagel, Stein, and Wainger defined the admissible approach region for holomorphic functions and obtained a Fatou theorem. Then, in his Ph. D. dissertation [27], Neff showed that the approach region would also work for meromorphic Nevanlinna functions.

So far, however, it is not known whether these approach regions for the finite type case are the best possible. In Chapter 2 we will care about domains of finite type in $\mathbb{C}^2$ and study the admissible approach region of Nagel-Stein-Wainger-Neff type. The main results are Theorem 2.3.1 and Corollary 2.3.1 which assure us that, for other regions broader only in the tangential direction, we can construct a bounded holomorphic function that does not have a limit at the base-points, and consequently there is no Fatou theorem for those broader approach regions: their base-points form a set of positive measure on the boundary. We are going to see this in $\mathbb{C}^2$.

1.4 Historical Facts III: Lindelöf Principle

There is another kind of boundary behavior called the Lindelöf principle. For the theory of a single variable, the classical result is:

Theorem 1.4.1 *Let f be a bounded holomorphic function on the unit disc $D \subseteq \mathbb{C}$. Suppose that the radial limit*

$$\lim_{r \to 1-} f(re^{i\theta}) \equiv \lambda \in \mathbb{C}$$

of f exists at the boundary point $e^{i\theta}$. Then f has nontangential limit λ at $e^{i\theta}$.

For several complex variables, we would like to find an analogous theorem. But what kind of convergence should we have for domains in $\mathbb{C}^n, n \geqslant 2$? Strongly pseudoconvex domains have been well studied; can we generalize results for the case of finite type?

In the case of a single complex variable, the appropriate approach region is the non-tangential one, while in the case of several complex variables, as we have seen, we are more interested in the *admissible* approach regions. So we wonder: is there a Lindelöf principle for domains in $\mathbb{C}^n, n \geqslant 2$ with admissible convergence? Unfortunately, it turns out to be false even for the unit ball as a counterexample in [7] indicates.

However if we consider a sequence $\{z^j\} \subset B$ converging to $P \in \partial B$ such that there exists an $\alpha > 1$ and our $\{z^j\}$ satisfies

$$\frac{\left|1 - \langle z^j, P\rangle\right|}{1 - |z^j|} < \alpha$$

and

$$\limsup_{j\to\infty} \frac{\left|z^j - \langle z^j, P\rangle P\right|^2}{\left|1 - \langle z^j, P\rangle\right|} = 0,$$

we call this *hypoadmissible* convergence, which is asymptotically smaller than any admissible convergence region in the complex tangential directions.

Accordingly, we have the Lindelöf principle:

Theorem 1.4.2 *Let $B \subseteq \mathbb{C}^n$ be the unit ball. Let f be a bounded holomorphic function and fix $P \in \partial B$. If the limit*

$$\lim_{r\to 1-} f(rP) \equiv \lambda \in \mathbb{C}$$

exists, then for any sequence $\{z^{(j)}\}_{j=1}^{\infty} \subseteq B$ that approaches P hypoadmissibly, we have

$$\lim_{j\to\infty} f(z^{(j)}) = \lambda.$$

More details can be found in [6], [7] and [18].

We still wonder how we can have the *admissible* limit, because it has been shown that admissible approach regions are optimal to some extent, for domains strongly pseudoconvex, and even further of finite type, and it is strictly stronger than hypoadmissible convergence.

In Chapter 3 we would like to provide two main results with admissible convergence, dealing with domains in $\mathbb{C}^2$ of finite type. They are Theorem 3.1.1 and Theorem 3.2.2, both based on the study of the shape of the admissible approach regions and the work on strongly pseudoconvex domains.

1.5 Historical Facts IV: Inner Functions

There are other topics on boundary behavior of holomorphic functions. What we are going to do here is the study of inner functions. If Ω is a domain in $\mathbb{C}^n$ and $f : \Omega \to \mathbb{C}$ is a bounded holomorphic function such that for almost every $\zeta \in \partial\Omega$, the radial boundary limit $f^*(\zeta)$ exists and $|f^*(\zeta)| = 1$, then we say that f is an inner function. Here, we do not consider the trivial example of constant functions.

Back to the case of a single variable, this subject has been fully studied, and we know that inner functions are important, such as its central role in the fundamental factorization theorem. Illustrative examples on the unit disc are the Blaschke products with prescribed zeros $\{\alpha_i\}$ that satisfy the Blaschke condition:

$$B(z) = e^{i\theta} z^k \prod \frac{|\alpha_i|}{\alpha_i} \cdot \frac{\alpha_i - z}{1 - \overline{\alpha}_i z}, \quad |c| = 1.$$

Other examples of zero-free functions are

$$G(z) = \exp\Big\{ -\int_{|\zeta|=1} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \Big\}$$

where μ is a positive Borel measure on the unit circle. In fact, every inner function is a product of those two types.

When it comes to the case of several variables, the problem is more complicated. At one time, the existence of such inner functions was doubted, and it was found that, even if an inner function existed, it had some pathological properties, such as being discontinuous at every boundary point of the unit ball in $\mathbb{C}^2$. For more discussion, please see [29] and [17]. Later, a turnaround drew considerable attention: inner functions were constructed for the unit ball in $\mathbb{C}^n$. For more on this, we refer the reader to the work of A. Aleksandrov [1], M. Hakim and N. Sibony [15], and E. Løw [23]. Additionally, in [24], Løw showed that inner functions exist for strongly pseudoconvex domains. Their work uses various methods and tools, including Ryll-Wojtaszczyk polynomials, a method developed by Aleksandrov in [2] as an alternative approach to construct inner functions for the unit ball. Later, W. Rudin wrote a book, [30], on this method, and provided many other applications.

We keep asking ourselves, can we explore more general domains?

In Chapter 4, inspired by Rudin's summary in [30], we present some results on domains that are similar to the unit ball yet more general insomuch as both are complex manifolds and there is a ramified holomorphic map between them. Although we present our results in the context of $\mathbb{C}^2$, our method is generalizable to higher dimensions. Our principal results are Theorem 4.4.2 and Corollary 4.4.1, which establish the existence of inner function on these domains.

2. Approach Regions

The purpose of this chapter is to study one kind of approach region for domains in $\mathbb{C}^2$ of finite type. We first familiarize ourselves with a few standard concepts.

2.1 A Study of Iterated Commutators

The following discussion is carried out in $\mathbb{C}^2$, where the notion of finite type has clear formulations, both geometrically and analytically. This discussion will also shed insight on higher dimensions.

Let Ω be a smoothly bounded domain in $\mathbb{C}^2$ with a defining function ρ. We assume that Ω is of finite type. Suppose $\omega^0 = (\omega_1^0, \omega_2^0) \in \partial\Omega$. Then, in a small neighborhood $V = V_{\omega^0}$ of ω^0, the complex holomorphic tangential vector field has a basis L, $\overline{L}$, where

$$L = -\frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial z_2}, \tag{2.1}$$

and

$$\overline{L} = -\frac{\partial\rho}{\partial \overline{z}_2}\frac{\partial}{\partial \overline{z}_1} + \frac{\partial\rho}{\partial \overline{z}_1}\frac{\partial}{\partial \overline{z}_2}. \tag{2.2}$$

Then we can find a transverse vector field T such that $L, \overline{L}$ and T span the 3-dimensional tangent space to $\partial\Omega$ at any point in V:

$$T = \frac{\partial\rho}{\partial \overline{z}_1}\frac{\partial}{\partial z_1} - \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial \overline{z}_1}. \tag{2.3}$$

A commutator of two vector fields is another vector field of the form

$$[L, M] = LM - ML.$$

We are going to study the iterated commutators which are vector fields still within the 3-dimensional tangent space to $\partial\Omega$. To begin with, we say that L and $\overline{L}$ are of degree 0.

Suppose that $\mathscr{L}_{k-1}$ is an iterated commutator of degree $k-1$, $k \geqslant 1$. Then we can write

$$\mathscr{L}_{k-1} = f_1 L + f_2 \overline{L} + \lambda_{k-1} T, \tag{2.4}$$

or simply $\mathscr{L}_{k-1} \equiv \lambda_{k-1} T \bmod(L, \overline{L})$.

Due to the properties of defining functions, we can normalize coordinates so that the z_1-derivatives of ρ do not vanish.

If $\mathscr{L}_k = [L, \mathscr{L}_{k-1}]$, whose degree we say is k, we can compute that

$$\mathscr{L}_k \equiv \lambda_k T \bmod(L, \overline{L}) \tag{2.5}$$

where λ_k can be expressed explicitly:

$$\begin{aligned}\lambda_k = {} & \frac{\partial \lambda_{k-1}}{\partial z_2}\frac{\partial \rho}{\partial z_1} - \frac{\partial \lambda_{k-1}}{\partial z_1}\frac{\partial \rho}{\partial z_2} + \lambda_{k-1}\frac{\partial^2 \rho}{\partial z_1 \partial z_2} - \lambda_{k-1}\frac{\frac{\partial \rho}{\partial z_2}\frac{\partial^2 \rho}{\partial z_1^2}}{\frac{\partial \rho}{\partial z_1}} \\ & + \frac{f_2}{\frac{\partial \rho}{\partial z_1}\frac{\partial \rho}{\partial \overline{z}_1}}\Big(\frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_1}\frac{\partial \rho}{\partial z_2}\frac{\partial \rho}{\partial \overline{z}_2} + \frac{\partial^2 \rho}{\partial z_2 \partial \overline{z}_2}\frac{\partial \rho}{\partial z_1}\frac{\partial \rho}{\partial \overline{z}_1} \\ & \qquad - \frac{\partial^2 \rho}{\partial z_1 \partial \overline{z}_2}\frac{\partial \rho}{\partial z_2}\frac{\partial \rho}{\partial \overline{z}_1} - \frac{\partial^2 \rho}{\partial z_2 \partial \overline{z}_1}\frac{\partial \rho}{\partial z_1}\frac{\partial \rho}{\partial \overline{z}_2}\Big).\end{aligned} \tag{2.6}$$

We can get similar results for $[\overline{L}, \mathscr{L}_{k-1}]$. This computation shows that, for any iterated commutator of degree k, only $\frac{\partial \rho}{\partial z_1}$ and/or $\frac{\partial \rho}{\partial \overline{z}_1}$ appear in the denominator of the coefficient function of the complex normal vector T.

Let $\mathscr{M}_k$ be the collection of all these linearly independent iterated commutators with degree less or equal to k. Suppose that $\mathscr{L} \in \mathscr{M}_k$, and that $\lambda_{\mathscr{L}}$ is the coefficient function of T in the sense that $\mathscr{L} \equiv \lambda_{\mathscr{L}} T \bmod(L, \overline{L})$. Then we can define $\Lambda_k(z)$ by:

$$\Lambda_k(z) = \sqrt{\sum_{\mathscr{L} \in \mathscr{M}_k} \lambda^2_{\mathscr{L}}(z)}, \tag{2.7}$$

a key function for defining the admissible approach regions.

Remark 1 *By definition, if $\Lambda_{k-1}(\omega^0) \neq 0$, then $\Lambda_k(\omega^0) \neq 0$. Actually, the smallest τ such that $\Lambda_\tau(\omega^0) \neq 0$ is called the type of ω^0. See [10] and [18].*

Remark 2 *Note that we always have $\nabla \rho \neq 0$, since ρ is a defining function. With the assumption that $\frac{\partial \rho}{\partial z_1}(z^0) \neq 0$ and $\frac{\partial \rho}{\partial \bar{z}_1}(z^0) \neq 0$, all $\Lambda_k(z^0) < \infty$, $k \geqslant 2$.*

2.2 Admissible Approach Regions

Let Ω be a domain of finite type in $\mathbb{C}^2$ such that, for all $z \in \Omega$, it is true that $|z| \leqslant 1$. Suppose that $(1,0)$ is on the boundary, that $\frac{\partial \rho}{\partial z_1}$ and $\frac{\partial \rho}{\partial \bar{z}_1}$ do not vanish at $(1,0)$, and the vector $\langle 1, 0 \rangle$ is also a outward normal vector to the boundary at $(1,0)$. Let $U \subset \partial\Omega$ be a neighborhood of $(1,0)$ small enough that, for any $w = (w_1, w_2) \in U$, the vector $\langle 1, 0 \rangle$ is transversal to U at w.

We first would like to see the explicit expression of $\pi(z)$, the Euclidean normal projection of z on the boundary. Suppose $z = (z_1, z_2) \in \Omega$ is sufficiently close to $\partial\Omega$. Then there is a unique point $w = (w_1, w_2) \in \partial\Omega$ determined by

$$\begin{cases} z - w = \lambda \nu_w, \\ \rho(w) = 0, \end{cases} \tag{2.8}$$

where λ is a real number that will also be fixed by this system of equations, and where we recall that ν_w is the unit outward normal unit to $\partial\Omega$ at $w \in \partial\Omega$. Then we say $w = \pi(z)$.

Let τ_z be the type of the point z if $z \in \partial\Omega$, or the type of $\pi(z)$. We also denote the ordinary Euclidean distance of z to $\partial\Omega$ by $\delta(z) = \big|z - \pi(z)\big|$. Let $\tau = \max_{z\in\partial\Omega} \tau_z$. Since we assume that Ω is of finite type, we must have $\tau < \infty$. We denote $\tilde{\tau}$ the type of $(1,0)$. Then of course $\tilde{\tau} \leqslant \tau$.

Define $D(z)$:

$$D(z) = \inf_{2\leqslant k\leqslant\tau} \Big(\frac{\delta(z)}{\Lambda_k\big(\pi(z)\big)}\Big)^{1/k}. \tag{2.9}$$

Define the boundary ball β_2 such that, for $\omega^0 \in \partial\Omega$ and $r > 0$, $\omega \in \beta_2(\omega^0, r)$ if and only if $\omega \in \partial\Omega$ and

$$\begin{cases} |\omega - \omega^0| < r, \\ \Big|R\big(\omega, \omega^0\big)\Big| < \Lambda^r(\omega^0), \end{cases} \tag{2.10}$$

where we use this notation:

$$\Lambda^\theta(\zeta) = \sum_{k=2}^{\tau} \theta^k \Lambda_k(\zeta), \tag{2.11}$$

and where R is a polarization of ρ, that is, $R(z, w)$ is a C^∞ complex-valued function satisfying the following requirements:

$$R(z, z) = \rho(z), \tag{2.12}$$

$$\overline{\partial}_z R(z, w) \text{ vanishes to infinite order on } z = w, \tag{2.13}$$

$$R(z, w) - \overline{R(w, z)} \text{ vanishes to infinite order on } z = w. \tag{2.14}$$

For example, if $\rho(z) = z_1\overline{z}_1 + z_2^2\overline{z}_1^2 - 1$ is a defining function for a domain in $\mathbb{C}^2$, then we can choose one polarization $R(z,w) = z_1\overline{w}_1 + z_2^2\overline{w}_2^2 - 1$.

With these notations, the admissible approach region of Nagel-Stein-Wainger-Neff type is

$$\mathscr{A}_\alpha(1,0) = \big\{z \in \Omega \cap V : \pi(z) \in \beta_2\big((1,0), \alpha D(z)\big)\big\}, \tag{2.15}$$

where $\alpha > 0$. These definitions can be found in [26] and [27].

We see that the definition of the approach region above is equivalent to

$$\begin{cases} \big|\pi(z) - (1,0)\big| < \alpha D(z), \\ \Big|R\big(\pi(z), (1,0)\big)\Big| < \Lambda^{\alpha D(z)}(1,0). \end{cases} \tag{2.16}$$

We recall the relation " $\sim$ ". Suppose that $F_1(t)$ and $F_2(t)$ are two real-valued functions, and there exist two positive constants k_1 and k_2 such that $k_1F_2(t) < F_1(t) < k_2F_2(t)$ for all t, then we write $F_1(t) \sim F_2(t)$. If we only have the second part $F_1(t) < k_2F_2(t)$, we write $F_1(t) \lesssim F_2(t)$.

To see more about the admissible approach region, we need the following lemma:

Lemma 2.2.1 *If $\big|\pi(z) - (1,0)\big| \sim D(z)$, then $\big|z - (1,0)\big| \sim D(z)$.*

Proof From the definition of $D(z)$ and the discussion of the iterated commutators, we know that

$$D(z) = \Big(\frac{\delta(z)}{\Lambda_{\tau_z}\big(\pi(z)\big)}\Big)^{1/\tau_z} \sim \big(\delta(z)\big)^{1/\tau_z}, \text{or } \big(D(z)\big)^{\tau_z} \sim \delta(z), \tag{2.17}$$

where $\tau_z = \tau\big(\pi(z)\big)$ is the type of $\pi(z)$.

We know that, since $\tau_z \geqslant 2$, as $\delta(z) \ll 1$, it is true that $\delta(z) \ll D(z)$.

As a result,

$$\big|z-(1,0)\big| = \big|z-\pi(z)+\pi(z)-(1,0)\big| \sim D(z). \tag{2.18}$$

■

Therefore we know that the following defines an approach region, denoted by $\mathscr{A}(1,0)$, which is comparable to $\mathscr{A}_1(1,0)$:

$$\begin{cases} \big|z-(1,0)\big| < D(z), \\ \Big|R\big(\pi(z),(1,0)\big)\Big| < \Lambda^{D(z)}(1,0). \end{cases} \tag{2.19}$$

2.3 The Best Approach Region

Let $\delta_n(z)$ be the component of $\delta(z)$, the Euclidean distance $\delta(z)$ from the point $z \in \Omega$ to $\pi(z) \in \partial\Omega$, in the real tangent space at $\pi(z)$ but *not* in the complex tangential direction.

Let h_1 and h_2 be two real-valued continuously decreasing functions such that $h_i : (0,1] \to [1,+\infty)$ and $\lim_{x\to 0+} h_i(x) = +\infty$, $i = 1,2$. We may assume that they decrease to 1 very slowly.

Now we consider an approach region in Ω at the point $w \in \partial\Omega$ near $(1,0)$, denoted by $\mathscr{A}_{h_1,h_2}(w)$, defined by the following inequalities:

$$\begin{cases} |z-w| < h_1\big(\delta_n(z)\big)D(z), \\ \Big|R\big(\pi(z),w\big)\Big| < h_2\big(\delta_n(z)\big)\Lambda^{D(z)}(w). \end{cases} \tag{2.20}$$

We can compare $\mathscr{A}_{h_1,h_2}(1,0)$ with $\mathscr{A}(1,0)$ to see how these two regions are related.

First of all, $\mathscr{A}(1,0) \subseteq \mathscr{A}_{h_1,h_2}(1,0)$. If $z \in \mathscr{A}_{h_1,h_2}(1,0) \setminus \mathscr{A}(1,0)$, then $\delta_n(z)$ is very small. This means $\mathscr{A}_{h_1,h_2}(1,0)$ is very similar to $\mathscr{A}(1,0)$, but compared with $\mathscr{A}(1,0)$ it is broader in the complex tangential direction.

Then, the main result of this chapter is: there is no Fatou's theorem for this kind of complex tangentially broader region $\mathscr{A}_{h_1,h_2}$. Therefore the admissible approach regions of Nagel-Stein-Wainger-Neff type are the best possible ones.

To see this, we are going to construct a bounded holomorphic function f that does not have a limit $\mathscr{A}_{h_1,h_2}$-admissibly at *any* point in U. It is inspired by Hakim and Sibony's work in [14].

For each $r > 0$, there exists a set of points $\{\zeta_j\}_{j\in J}$, such that $\{\beta_2(\zeta_j, r^\tau)\}$ is a maximal family of pairwise disjoint balls in U, in the sense that $\{\beta_2(\zeta_j, 2r^\tau)\}$ covers $U \subset \partial\Omega$, because we already know that with the boundary balls β_2, $\partial\Omega$ is a space of homogeneous type. See [26] and [27].

For each $\zeta_j = (\zeta_{j,1}, \zeta_{j,2})$, define

$$V_r(\zeta_j) = \{\zeta \in U : |\zeta - \zeta_j| < K_1 r^\tau, \left|R(\zeta,\zeta_j)\right| < \Lambda^{K_1 r}(\zeta_j)\},$$

where K_1 is a positive constant to be fixed later.

We want to show that:

Lemma 2.3.1 *We can find a positive constant K_1 such that*

$$\bigcup_{j\in J} V_r(\zeta_j) = U.$$

Proof First of all, we realize that we only need to prove that $U \subset \bigcup_{j\in J} V_r(\zeta_j)$, as $V_r(\zeta_j) \subset U$ for each j.

Without loss of generality, we just need to show that there exists a point ζ_j such that

$$(1,0) \in V_r(\zeta_j), \tag{2.21}$$

because, for any other point in the domain U, the same method below shows that it also belongs to $V_r(\zeta_i)$ for some $i \in J$.

Therefore we just need to show that there exists a positive constant K_1 and a point ζ_j which makes the following inequalities true:

$$\begin{cases} \big|(1,0) - z\big| < K_1 r^\tau, \\ \Big|R\big((1,0), z\big)\Big| < \Lambda^{K_1 r}(z). \end{cases} \tag{2.22}$$

But first we are interested in the following inequalities:

$$\begin{cases} \big|(1,0) - z\big| < (K_1 - 2) r^\tau, \\ \Big|R\big((1,0), z\big)\Big| < r^\tau, \end{cases} \tag{2.23}$$

where K_1 is large enough.

We observe that $(1,0)$ satisfies inequalities in (2.23), thus there exists an open neighborhood in U of $(1,0)$ such that any point in this neighborhood also solves the inequalities, that is, we can find a point $w \in U$ such that

$$\begin{cases} \big|(1,0) - w\big| < (K_1 - 2) r^\tau, \\ \Big|R\big((1,0), w\big)\Big| < r^\tau. \end{cases} \tag{2.24}$$

Since $\{\beta_2(\zeta_j, r^\tau)\}$ makes a maximal family in U, there must exist a point ζ_j in the ball $\beta_2(w, 2r^\tau)$ for some $j \in J$. We then wish to check that this ζ_j makes the inequalities in (2.22) valid, by which our goal is achieved.

To see this, we first check an arbitrary point $\zeta \in \beta_2(w, 2r^\tau)$. Immediately by the triangle inequality we know that

$$\begin{aligned}\big|(1,0)-\zeta\big| &\leqslant \big|(1,0)-w\big| + |w-\zeta| \\ &< (K_1-2)r^\tau + 2r^\tau \\ &= K_1 r^\tau. \end{aligned} \tag{2.25}$$

To check the second inequality in (2.22), we first have

$$\Big|R\big((1,0),\zeta\big)\Big| \leqslant \Big|R\big((1,0),w\big)\Big| + \Big|R\big((1,0),w\big) - R\big((1,0),\zeta\big)\Big|. \tag{2.26}$$

Since we already know that $\Big|R\big((1,0),w\big)\Big| < r^\tau$ and

$$\begin{aligned}\Big|R\big((1,0),w\big) - R\big((1,0),\zeta\big)\Big| &< K_2|w-\zeta| \\ &< 2K_2 r^\tau, \end{aligned} \tag{2.27}$$

it is true that

$$\Big|R\big((1,0),\zeta\big)\Big| < K_3 r^\tau. \tag{2.28}$$

By inequalities (2.25) and (2.28), we can choose a positive constant K_1 big enough such that

$$\begin{cases} \big|(1,0)-\zeta\big| < K_1 r^\tau, \\ \Big|R\big((1,0),\zeta\big)\Big| < \Lambda^{K_1 r}(\zeta) = \Lambda_{\tau_\zeta}(\zeta)(K_1 r)^{\tau_\zeta} + \cdots + \Lambda_\tau(\zeta)(K_1 r)^\tau. \end{cases} \tag{2.29}$$

Since there must be one ζ_j in $\beta_2(w, 2r^\tau)$ as argued, this ζ_j then satisfies the inequalities in (2.22), which means that we have $(1,0) \in V_r(\zeta_j)$, and then our statement is proved.

■

For $n \in \mathbb{N}$, $r > 0$ and $\{\zeta_j\}_{j\in J} \subset U$, define

$$g_{n,r}(z) = \sum_{j\in J} \Big(\frac{r^\tau}{R(z,\zeta_j) - r^\tau} \Big)^{2n},$$

and then define $f_n = 1 - \varepsilon_n - g_{n,r}$, where $\varepsilon_n = n^{-1/4}$, and we know that there exists a subsequence $\{\varepsilon_{n_k}\}$ such that $\sum \varepsilon_{n_k} < \infty$.

Lemma 2.3.2 *For any $z \in U$ and $n \in \mathbb{N}$ large enough, $|g_{n,r}(z)| \leqslant 1 + \frac{K_4}{n}$, where K_4 is a positive constant.*

Proof Let $\zeta_0 \in U$ be an arbitrary point and $N_{k,r}$ be the number of balls $\beta_2(\zeta_j, r^\tau)$ that are contained in the ball $\beta_2(\zeta_0, kr^\tau)$. Then we know that

$$N_{k,r} \leqslant K_5 k^t, \tag{2.30}$$

where t is a positive integer and K_5 is a positive constant.

Now fix a point $\zeta \in \Omega$. For any $k \in \mathbb{N}$, define a subfamily of $\{\zeta_j\}_{j\in J}$:

$$J(\zeta, k) = \{\zeta_j : kr^\tau \leqslant \big|R(\zeta,\zeta_j)\big| < (k+1)r^\tau\}.$$

With these preparations, we can estimate $|g_{n,r}|$.

First of all, we notice that if $|R(z,\zeta_j)| \geqslant kr^\tau$, we can get

$$\begin{aligned} \Big|\frac{r^\tau}{R(z,\zeta_j) - r^\tau}\Big|^{2n} &\leqslant \Big|\frac{r^\tau}{\big(\big|R(z,\zeta_j)\big|^2 + r^{2\tau}\big)^{\frac{1}{2}}}\Big|^{2n} \\ &\leqslant \Big|\frac{r^\tau}{\big(k^2 r^{2\tau} + r^{2\tau}\big)^{\frac{1}{2}}}\Big|^{2n} \\ &= \frac{1}{(1+k^2)^n}. \end{aligned} \tag{2.31}$$

It then follows that, for n large enough,

$$\begin{aligned}\big|g_{n,r}(z)\big| &\leqslant 1 + K_6 \sum_{k=1}^{\infty} \frac{k^t}{(1+k^2)^n} \\ &\leq 1 + \frac{K_4}{n}. \end{aligned} \tag{2.32}$$

■

As $\{\zeta_j\}_{j\in J}$ is chosen and f_n is defined, we will be able to see more about the functions f_n.

Lemma 2.3.3 *For each ζ_j, there exists a zero of f_n. Moreover, this zero will approach to ζ_j as n goes to infinity.*

Proof Here we are just going to consider the case for $\zeta_1 = (1, 0)$. This method also applies for other ζ_j.

We introduce two auxiliary functions:

$$\phi_n(z_1) = f_n(z_1, 0)$$

and

$$\psi_n(z_1) = 1 - \varepsilon_n - \Big(\frac{r^\tau}{R\big((z_1,0),(1,0)\big) - r^\tau}\Big)^{2n}.$$

Immediately we know that if $R\big((z_1,0),(1,0)\big) = r^\tau\big(1-(1-\varepsilon_n)^{-\frac{1}{2n}}\big)$, then z_1 is a zero of ψ_n.

We can choose a positive sequence $\{\gamma_n\}$ with $\gamma_n = n^{-4/3}$. On the closed curve of z such that $R\big((z,0),(1,0)\big) - R\big((z_1,0),(1,0)\big) = \gamma_n r^\tau e^{i\theta}$, we estimate that

$$\big|\psi_n(z)\big| = \big|2n\gamma_n e^{i\theta}(1-\varepsilon_n)^{1/2n}\big| + \mathcal{O}(\varepsilon^2). \tag{2.33}$$

On the other hand, we then see that

$$\big|\phi_n(z_1) - \psi_n(z_1)\big| = \Big|\sum_{\zeta_j \neq (1,0)} \Big(\frac{r^\tau}{R(z,\zeta_j) - r^\tau}\Big)^{2n}\Big|. \tag{2.34}$$

If z_1 is close enough to 1, the same argument as in the previous proof indicates that

$$\Big|\sum_{\zeta_j \neq (1,0)} \Big(\frac{r^\tau}{R(z,\zeta_j) - r^\tau}\Big)^{2n}\Big| \leqslant \frac{K_7}{n}. \tag{2.35}$$

Therefore, on this closed curve, we have

$$\big|\phi_n(z_1) - \psi_n(z_1)\big| < |\psi_n(z_1)|, \tag{2.36}$$

and then by Rouché's theorem, we know that ϕ_n also has at least a zero $\omega_{n,r}$ in the region bounded by the closed curve.

According to the construction of the function ϕ_n we then know that f_n has a zero $w_{n,r} = (\omega_{n,r}, 0)$. By checking the argument again, we know that $w_{n,r}$ approaches to $(1,0)$ as n goes to infinity.

For other ζ_j, we define

$$\phi_n(z_1) = f_n(z_1, z_2),$$

and

$$\psi_n(z_1) = 1 - \varepsilon_n - \Big(\frac{r^\tau}{R\big((z_1,z_2),(1,0)\big) - r^\tau}\Big)^{2n},$$

in both of which z_2 is such a complex number that $\pi(z_1, z_2) = \zeta_j$. Then we can do the same argument to show that the claim is true and in this case we have $\pi(w_{n,r}) = \zeta_j$, and the proof is complete. ■

The next key lemma states that if a boundary point is close enough to ζ_j, then the broader approach region based there contains a zero of f_n.

Lemma 2.3.4 *For each n we can choose $r = r_n$ such that, if $w \in V_r(\zeta_j)$, then $\mathscr{A}_{h_1,h_2}(w)$ contains a zero of f_n.*

Proof Again, without loss of generality, we may assume that $\zeta_1 = (1,0)$, and only check this case. For other situations, the same method applies.

Suppose $w_{n,r} = (\omega_{n,r}, 0)$ is the zero of f_n near $(1,0)$ as we had in the previous lemma. So now our task is to verify that if $\big|(1,0) - w\big| < K_1 r^\tau$ and $\Big|R\big(w,(1,0)\big)\Big| < \Lambda^{K_1 r}(1,0)$, we should have

$$\begin{cases} |w_{n,r} - w| < h_1\big(\delta_n(w_{n,r})\big)D(w_{n,r}), \\ \Big|R\big(\pi(w_{n,r}), w\big)\Big| < h_2\big(\delta_n(w_{n,r})\big)\Lambda^{D(w_{n,r})}(w). \end{cases} \tag{2.37}$$

Before starting the work, we need some setup. We claim that there exists a positive constant K_8 such that

$$\delta(w_{n,r}) > K_8 r^\tau. \tag{2.38}$$

If otherwise, we will have $\Big|R\big(\pi(w_{n,r}),(1,0)\big) - R\big(w_{n,r},(1,0)\big)\Big| < K_9 r^\tau$ for any constant K_9. However, this implies that $h(z) = R\big(z,(1,0)\big)$, written as a polynomial of z, does not have terms with degree less or equal to τ other than the constant term. This violates that the maximal type in U is τ.

First of all, we know that

$$|w_{n,r} - w| < \big|(1,0) - w_{n,r}\big| + \big|(1,0) - w\big|. \tag{2.39}$$

On the other hand, we check that

$$\begin{aligned}
h_1\big(\delta_n(w_{n,r})\big)D(w_{n,r}) &> h_1\big(\delta(w_{n,r})\big)D(w_{n,r}) \\
&> K_{10}h_1\big(\delta(w_{n,r})\big)\delta(w_{n,r}) \\
&> \frac{K_{11}}{2}h_1\big(\delta(w_{n,r})\big)\big|(1,0)-w_{n,r}\big| + \frac{K_{11}}{2}h_1\big(\delta(w_{n,r})\big)\delta(w_{n,r}),
\end{aligned} \tag{2.40}$$

because $\delta(w_{n,r}) = \big|(1,0)-w_{n,r}\big|$, as $\pi(w_{n,r}) = (1,0)$. Recall that in proof of the previous lemma we have the result that $\pi(w_{n,r}) = \zeta_j$.

If r is small enough, it is true that

$$\big|(1,0)-w_{n,r}\big| < \big|(1,0)-w_{n,r}\big| \cdot \frac{K_{11}}{2}h_1\big(\delta(w_{n,r})\big) \tag{2.41}$$

and

$$\begin{aligned}
\big|(1,0)-w\big| &< K_1 r^{\tau} \\
&< \frac{K_1}{K_8}\delta(w_{n,r}) \\
&< \frac{K_{11}}{2}h_1\big(\delta(w_{n,r})\big)\delta(w_{n,r}).
\end{aligned} \tag{2.42}$$

These imply that, if r is small enough , we will have

$$|w_{n,r}-w| < h_1\big(\delta_n(w_{n,r})\big)D(w_{n,r}). \tag{2.43}$$

Meanwhile, we have

$$\begin{aligned}
\Big|R\big(\pi(w_{n,r}),w\big)\Big| &\leqslant \Big|R\big(\pi(w_{n,r}),w\big)-R\big(\pi(w_{n,r}),(1,0)\big)\Big| \\
&\quad+\Big|R\big(\pi(w_{n,r}),(1,0)\big)-R\big(w_{n,r},(1,0)\big)\Big|+\Big|R\big(w_{n,r},(1,0)\big)\Big| \\
&< K_{12}\big|w-(1,0)\big|+K_{13}\big|\pi(w_{n,r})-w_{n,r}\big|+\Big|R\big((z_1,0),(1,0)\big)\Big|+\gamma_n r^{\tau} \\
&< K_{14}r^{\tau}+K_{13}\delta(w_{n,r})+\frac{\varepsilon_n}{2n}r^{\tau}+\gamma_n r^{\tau}+o(\frac{\varepsilon_n}{n}) \\
&< K_{15}\delta(w_{n,r}). \qquad (2.44)
\end{aligned}$$

We then consider $h_2\big(\delta_n(w_{n,r})\big)\Lambda^{D(w_{n,r})}(w)$.

By definition, we know that

$$\begin{aligned}
h_2\big(\delta_n(w_{n,r})\big)\Lambda^{D(w_{n,r})}(w) &> K_{16}h_2\big(\delta(w_{n,r})\big)\big(D(w_{n,r})\big)^{\tau_w} \\
&> K_{17}h_2\big(\delta(w_{n,r})\big)\big(\delta(w_{n,r})\big)^{\frac{\tau_w}{\tau_{w_{n,r}}}} \\
&> K_{18}h_2\big(\delta(w_{n,r})\big)\big(\delta(w_{n,r})\big)^{\frac{\tau}{2}}. \qquad (2.45)
\end{aligned}$$

We can then find r so small that this inequality holds:

$$K_{15}\delta(w_{n,r}) < K_{18}h_2\big(\delta(w_{n,r})\big)\big(\delta(w_{n,r})\big)^{\frac{\tau}{2}}. \qquad (2.46)$$

In this way we check that

$$\Big|R\big(\pi(w_{n,r}),w\big)\Big| < h_2\big(\delta_n(w_{n,r})\big)\Lambda^{D(w_{n,r})}(w). \qquad (2.47)$$

Therefore, there exists an $r>0$ such that

$$\begin{cases}
|w_{n,r}-w| < h_1\big(\delta_n(w_{n,r})\big)D(w_{n,r}), \\
\Big|R\big(\pi(w_{n,r}),w\big)\Big| < h_2\big(\delta_n(w_{n,r})\big)\Lambda^{D(w_{n,r})}(w),
\end{cases} \qquad (2.48)$$

that is, $\mathscr{A}_{h_1,h_2}(w)$ contains a zero, $w_{n,r}$, of $f_{n,r}$. ■

Now we are going to construct a bounded holomorphic function f such that, for any $\zeta \in U$, the limit

$$\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} f(z)$$

does not exist.

As we have seen from the proof above, for each n we can choose r_n such that the lemma is true. Then we choose a subsequence $\{\varepsilon_{n_k}\}$ such that $\sum \varepsilon_{n_k} < \infty$. Also for each ζ_j, we can find a zero w_{n_k} for f_{n_k} such that $\{w_{n_k}\}$ converges to ζ_j.

We then build a bounded holomorphic function in Ω:

$$f(z) = \prod_{k=1}^{\infty} \frac{f_{n_k}(z)}{1-(1-\varepsilon_{n_k})g_k(z)},$$

where $g_k = g_{n_k, r_{n_k}}$.

Fix an arbitrary point $\zeta \in U$. We know that, for each n_k, with the corresponding number r_{n_k}, there is a maximal set $\{\zeta_j\}_{j\in J}$ as mentioned, and by the first lemma, there exists a point ζ_j such that $\zeta \in V_{r_{n_k}}(\zeta_j)$. So, by Lemma 2.3.4 we know that a zero of $f_{n_k}(z)$, thus a zero of f, is contained in $\mathscr{A}_{h_1,h_2}(\zeta)$. As k goes to infinity, r_{n_k} converges to 0 and therefore $V_{r_{n_k}}(\zeta)$ is shrinking, making ζ_j converge to ζ. Thus we have a sequence of zeros of f that converges to the point ζ.

Suppose $\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} f(z)$ exists, then so does $\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} |f(z)|$.

By evaluating f along that sequence of zeros we have

$$\liminf_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} |f(z)| = 0, \tag{2.49}$$

and therefore

$$\begin{aligned}
\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} |f(z)| &= \limsup_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} |f(z)| \\
&= \liminf_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} |f(z)| \\
&= 0, \qquad (2.50)
\end{aligned}$$

which means $|f|$, being a subharmonic function, is identically zero, because it reaches its supremum in an interior point. For more facts about subharmonic functions, please refer to [18]. Here we only mention two facts. The first one: If f is holomorphic function of several variables, then $|f|^p$ is subharmonic for all $p > 0$. Second, the maximum principle also applies to subharmonic functions.

So we reach a contradiction, and therefore we know that

$$\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} f(z)$$

does not exist.

To summarize, we have the theorem

Theorem 2.3.1 *On the boundary of* Ω *there exists a boundary neighborhood* U *of* $(1,0)$*, and there exists a bounded holomorphic function* f*, such that for any point* $\zeta \in U$*, the limit*

$$\lim_{\mathscr{A}_{h_1,h_2}(\zeta)\ni z\to\zeta} f(z)$$

does not exist.

Since this U is of positive measure, we immediately have

Corollary 2.3.1 *There is no Fatou's theorem for these broader approach regions* $\mathscr{A}_{h_1,h_2}$.

3. The Lindelöf Principle

In this chapter we are going to study the Lindelöf principle for several complex variables. Again, we suppose that Ω is a domain of finite type in $\mathbb{C}^2$, characterized by a defining function ρ, with $m \geqslant 2$ being the maximal type. Suppose that $(1,0)$ is on the boundary, and its outward normal direction is $\langle 1,0\rangle$. We also suppose that $\frac{\partial\rho}{\partial z_1}$ and $\frac{\partial\rho}{\partial \bar{z}_1}$ do not vanish at $(1,0)$.

3.1 T-approach

There are some interesting results by Krantz in [19], which give admissible convergence. In that paper, however, the work is done for strongly pseudoconvex domains. Here we can generalize them for domains of finite type in $\mathbb{C}^2$.

Before that, let us have a review of the admissible approach regions and explore some of their properties.

3.1.1 Shape of the admissible approach region

Recall that the admissible approach region $\mathscr{A}_\alpha(1,0)$ based at $(1,0)$ is defined by

$$\begin{cases} |\pi(z) - (1,0)| < \alpha D(z), \\ \Big|R\big(\pi(z),(1,0)\big)\Big| < \Lambda^{\alpha D(z)}(1,0). \end{cases} \tag{3.1}$$

We have studied this region in Chapter 2, and in this chapter we are going to see more of it. First write $\pi(z) = (w_1, w_2)$, the Euclidean normal projection of z to the boundary.

By definition, $R\big((w_1, w_2), (1, 0)\big)$ is a holomorphic function, and we can expand it into a formal series around the point $(1, 0)$. Ignoring higher terms, we know that $\mathscr{A}_1(1, 0)$ is comparable to the region defined by

$$\begin{cases} |(w_1, w_2) - (1, 0)| < D(z), \\ \big|c_1(w_1 - 1)^{k_1} + c_2(w_1 - 1)^{k_2} w_2^{k_3} + c_1 w_2^{k_4}\big| < \Lambda^{D(z)}(1, 0), \end{cases} \tag{3.2}$$

where c_i's are complex numbers and k_i's are positive integers.

As we estimate $D(z)$ and $\Lambda^{\alpha D(z)}(1, 0)$ in the previous chapter, we know that this region sits inside, up to comparability, the one defined by

$$\begin{cases} |(w_1, w_2)) - (1, 0)| < \delta(z)^{\frac{1}{m}}, \\ \big|c_1(w_1 - 1)^{k_1} + c_2(w_1 - 1)^{k_2} w_2^{k_3} + c_1 w_2^{k_4}\big| < \delta(z)^{k}, \end{cases} \tag{3.3}$$

where $k > 0$.

By solving the inequalities, we know that $\mathscr{A}_1(1, 0)$ is comparable to the region given by

$$\begin{cases} |(w_1, w_2) - (1, 0)| < \delta(z)^{\frac{1}{m}}, \\ |w_1 - 1| < \delta(z)^{k_5}, \end{cases} \tag{3.4}$$

which is also comparable to, by doing the same analysis as in Lemma 2.2.1 in the previous chapter,

$$\begin{cases} |(z_1, z_2) - (1, 0)| < \delta(z)^{\frac{1}{m}}, \\ |w_1 - 1| < \delta(z)^{k_5}. \end{cases} \tag{3.5}$$

We have

$$|z_1 - 1| \lesssim |z_1 - w_1| + |w_1 - 1| \tag{3.6}$$

and

$$|z_1 - w_1| \sim \delta(z),\ |w_1 - 1| < \delta(z)^{k_5}. \tag{3.7}$$

CASE ONE: $k_5 \geqslant 1$.

In this case we have $|z_1 - 1| \lesssim \delta(z)$. So up to some comparability, the region (3.5) is inside

$$\begin{cases} |(z_1, z_2) - (1, 0)| < \delta(z)^{\frac{1}{m}}, \\ |z_1 - 1| < \delta(z), \end{cases} \tag{3.8}$$

CASE TWO: $k_5 < 1$.

In this case we have $|z_1 - 1| \lesssim \delta(z)^{k_5}$. Therefore we have

$$\begin{cases} |(z_1, z_2) - (1, 0)| < \left(\delta(z)^{k_5}\right)^{\frac{1}{k_5 m}} < \left(\delta(z)^{k_5}\right)^{\frac{1}{m}}, \\ |z_1 - 1| < \delta(z)^{k_5}. \end{cases} \tag{3.9}$$

Therefore, we can have a result about the shape of the admissible approach region:

Proposition 3.1.1 *There is a region $\mathscr{A}$ that is comparable with $\mathscr{A}_1(1,0)$ and is lying inside the region bounded by*

$$|(z_1, z_2) - (1, 0)|^m = |z_1 - 1|.$$

Since admissible convergence is equivalent to $\mathscr{A}$-convergence, we will care more about $\mathscr{A}$ and at the same time obtain results about admissible convergence.

Following Krantz' arguments in [19], we can get some similar ones, to be discussed in the following subsections.

3.1.2 Bounded holomorphic functions

First we define a two-dimensional and totally real region:

$$T = \{(s + i0, t + i0) \in \Omega : s, t \in \mathbb{R}, 0 < s < 1, 0 < |t| < \sqrt[m]{1 - s}\}.$$

Then for $j = 1, 2, \ldots$, we define

$$\Omega_j = \{(z_1, z_2) \in \Omega : 1 - 2^{-j} \leqslant \mathrm{Re} z_1 < 1 - 2^{-j-1}, |\mathrm{Im} z_1| < 2^{-j} \text{ and } |z_2| < \sqrt[m]{2^{-j}}\}.$$

For each Ω_j, the map

$$\varphi_j(z_1, z_2) = (2^{j-j_0}(z_1 - 1) + 1, \sqrt[m]{2^{j-j_0}} z_2)$$

gives a biholomorphic mapping from Ω_j onto a region Ω_{j_0}, where j_0 is a positive integer.

By Proposition (3.1.1), we have

$$\mathscr{A} \subseteq \bigcup_{j=1}^{\infty} \Omega_j = \bigcup_{j=1}^{\infty} \varphi_j^{-1}(\Omega_0). \tag{3.10}$$

Proposition 3.1.2 *Let f be a bounded holomorphic function in Ω. If*

$$\lim_{T \ni z \to (1,0)} f(z) = 0,$$

then

$$\lim_{\mathscr{A} \ni z \to (1,0)} f(z) = 0.$$

Proof First we see that φ_j maps $T \cap \Omega_j$ onto $T \cap \Omega_{j_0}$.

For each j, construct

$$g_j = f \circ \varphi_j^{-1} : \Omega_{j_0} \to \mathbb{C}.$$

These maps are uniformly bounded, so they form a normal family, and therefore we can find g_0, a subsequential limit function.

Note that g_0 vanishes on $T \cap \Omega_{j_0}$, a totally real two-dimensional region. It follows that g_0 vanishes identically.

For any compact set $K \subseteq \Omega_{j_0}$ such that

$$\mathscr{A} \subseteq \bigcup_{j=1}^{\infty} \varphi_j^{-1}(K),$$

we know that $g_j \to 0$ uniformly on K. Therefore f has $\mathscr{A}$-admissible limit 0. ∎

Remark 3 *As seen in the analysis above, the crucial part is* $\mathscr{A} \subseteq \bigcup_{j=1}^{\infty} \Omega_j = \bigcup_{j=1}^{\infty} \varphi_j^{-1}(\Omega_0)$. *If we have more information about types of points near* $(1, 0)$, *we may make* T *sharper.*

For example, if $p > 1$ *is an integer, let us consider the domain* $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2p} < 1\}$. *The maximal type for this domain is* $2p$, *shared by points* $(e^{i\theta}, 0)$, *so, by Proposition 3.1.1 we can say that the admissible approach region based at any boundary point will sit inside another region whose shape we can describe. However, for base points away from those of the form* $(e^{i\theta}, 0)$, *the type is 2, so locally the maximal type is 2, and therefore each of these approach regions can be inscribed in a parabolic one.*

We wish to make this result more general. Define a two-dimensional, totally real manifold

$$\mathscr{T} = \{\big(s + i\rho_1(s,t), t + i\rho_2(s,t)\big) : (s,t) \in T\},$$

where $\rho_i(s,t) : T \to \mathbb{R}$ is a C^2 function, $i = 1, 2$.

Then we have

Proposition 3.1.3 *Let f be a bounded holomorphic function in Ω. If*

$$\lim_{\mathscr{T}\ni z\to(1,0)} f(z) = 0,$$

then

$$\lim_{\mathscr{A}\ni z\to(1,0)} f(z) = 0.$$

Proof Suppose

$$\varphi_j(\mathscr{T}\cap\Omega_j) = \tau_j(T\cap\Omega_{j_0}), \tag{3.11}$$

then each τ_j has bounded derivatives. So we can find a subsequence $\{\tau_{j_k}\}$ that converges uniformly on compacta to τ_0. By the relation (3.11) and the definition of g_j, we know there exists a convergent subsequence $\{g_{j_k}\}$ with the limit g_0.

Let $\mathscr{T}_0$ be the graph of τ_0 over $T\cap\Omega_{j_0}$. We observe that $\mathscr{T}_0$ is a totally real, two-dimensional manifold.

We claim that g_0 vanishes on $\mathscr{T}_0$. It is true because for any $w\in\mathscr{T}_0$, there exists a point $z_{j_k}\in\mathscr{T}\cap\Omega_{j_k}$ such that $\varphi_{j_k}^{-1}(w) = z_{j_k}$. We then know that $\{z_{j_k}\}$ lies in $\mathscr{T}$ and approaches to $(1,0)$ as $k\to\infty$. By the hypothesis we know that

$$\lim_{k\to\infty} f\circ\varphi_{j_k}^{-1}(w) = 0, \tag{3.12}$$

and therefore the claim is verified.

So we can again conclude that $g_0\equiv 0$ and it then follows that f has $\mathscr{A}$-limit 0 at $(1,0)$. ∎

3.1.3 Normal functions

First recall the notion of Kobayashi metric. Let D be the unit disc in $\mathbb{C}$. If $z \in \Omega$ where $\Omega \subset \mathbb{C}^n$ is a domain, and ξ is a vector in $\mathbb{C}^n$, then the infinitesimal form of the Kobayashi metric at z in the direction of ξ is defined to be

$$F_K^{\Omega}(z, \xi) = \inf \left\{ \frac{||\xi||}{||f'(0)||} \right\},$$

where f runs through holomorphic mappings from D to Ω with $f(0) = z$, $f'(0)$ being a positive multiple of ξ.

Thinking of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the Riemann sphere and $\zeta \in \mathbb{C}$ as a tangent vector to $\hat{\mathbb{C}}$ at $w \in \hat{\mathbb{C}}$, then the spherical metric of ζ at w is

$$|\zeta|_{\text{sph},w} = \frac{2||\zeta||}{1 + ||w||^2}.$$

Then we say a holomorphic function $f : \Omega \to \widehat{\mathbb{C}}$ is *normal* if

$$|f'(z) \cdot \xi|_{\text{sph},f(z)} \leqslant C F_F^{\Omega}(z, \xi)$$

for all $z \in \Omega$ and $\xi \in \mathbb{C}^n$.

This is a generalization of the normal functions of a single complex variable. For more details, please refer to [7].

Theorem 3.1.1 *Let f be a normal holomorphic function in Ω. If*

$$\lim_{\mathscr{T} \ni z \to (1,0)} f(z) = 0,$$

then

$$\lim_{\mathscr{A} \ni z \to (1,0)} f(z) = 0.$$

Proof Let D be the unit disc in $\mathbb{C}$. Consider a holomorphic mapping $\psi : D \to \Omega_{j_0}$ with $\psi(0) = p \in \Omega_{j_0}$. We may take ψ to be an extremal function for the Kobayashi metric at the point p.

Define a function $\mu_j : D \to \widehat{\mathbb{C}}$:

$$\mu_j = f \circ \varphi_j^{-1} \circ \psi.$$

It follows that

$$|\mu_j'(0)| \leqslant \left|\nabla f(\varphi_j^{-1}(p))\right| \left|\left(\varphi_j^{-1} \circ \psi\right)'(0)\right|. \tag{3.13}$$

We notice that $\left|\nabla f(\varphi_j^{-1}(p))\right|$ is bounded from the Kobayashi metric on Ω, and $\left|\left(\varphi_j^{-1} \circ \psi\right)'(0)\right|$ is the reciprocal of the Kobayashi metric for Ω_j at $\varphi_j^{-1}(p)$. We also notice that the Kobayashi metric on Ω is smaller than that on Ω_j. Therefore we can see that $|\mu_j'(0)|$ is bounded on compact subset of D, and this bound is independent of j, and the choice of p in a compact subset $K \subseteq D$. By composing with a Möbius transformation we can have a similar estimate for μ_j' at any point of a compact subset of D.

Therefore we can find a normally convergent subsequence $\{\mu_{j_k}'\}$ of μ_j' with the limit function μ_0'. Consequently, $\{\mu_{j_k} = g_{j_k} \circ \psi\}$ is also convergent, and so is $\{g_{j_k}\}$, with the limit function g_0.

As shown in the proof of the previous theorem, we can obtain a totally real, two-dimensional manifold $\mathscr{T}_0$, the graph of τ_0, a subsequential limit of $\{\tau_j\}$. We can further deduce that g_0 vanishes on $\mathscr{T}_0$, then get that $g_0 \equiv 0$ and finally conclude that f has $\mathscr{A}$-limit 0 at $(1, 0)$. ■

3.2 Boundary Approach

Suppose f is a bounded holomorphic function on Ω such that $|f(z)| \leq 1$ for any $z \in \Omega$. For $P \in \partial\Omega$, we define, only for the rest of this chapter, the boundary value $|f(P)|$ to be $\limsup_{\Omega \ni z \to P} |f(z)| \in \mathbb{R} \cup \{\infty\}$.

There are some interesting discoveries in [20] and [7] concerning the boundary curves. The result in [20] of Lehto and Virtanen states that if $\lim_{t\to 1-} |f(\gamma(t))| = 0$ where f is a normal function in $D \subseteq \mathbb{C}$ and $\gamma : [0,1] \to \overline{D}$ is such a curve that $\gamma(1) = P \in \partial D$, then f has non-tangential limit 0 at P. For several complex variables, Cima and Krantz give a similar result for hypoadmissible convergence in [7], which treats complex normal curves. Recall that $\gamma : [0,1] \to \partial\Omega$ is complex normal if $\langle \gamma'(t), \nu_{\gamma(t)} \rangle \neq 0$, all $0 \leqslant t \leqslant 1$. Their theorem states:

Theorem 3.2.1 *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^2 boundary. Let $\gamma : [0,1] \to \partial\Omega$ be a C^2 curve which is complex normal. Let $f : \Omega \to \mathbb{C}$ be a normal and assume $f \in H^p(\Omega)$, $p > 4n$. Suppose that*

$$\lim_{t\to 1-} |f(\gamma(t))| = 0.$$

Then f has hypoadmissible limit 0 at $P \equiv \gamma(1)$.

However, it turns out that there is no way to get an admissible limit. Let us consider this domain

$$M_2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\},$$

which is not strongly pseudoconvex but still of finite type.

Consider the holomorphic function

$$f(z) = f(z_1, z_2) = \frac{z_2^4}{1 - z_1}.$$

It is bounded because $\left|z_2^4/(1-z_1)\right| < \left|(1-|z_1|^2)/(1-|z_1|)\right| = 1 + |z_1| < 2$.

We notice that f has a radial limit 0 at $(1, 0)$. If there exists a complex normal curve terminating at $(1, 0)$, along which f has a limit $\lambda \neq 0$, then according to some other results in [7] by Cima and Krantz, f should have a hypoadmissible limit λ and thus radial limit λ, which gives a contradition. This means, along *any* complex normal curve, if f has a limit, then this limit must be 0. Alternatively, we can just check this curve $\varphi(t) = (e^{it}, 0), 0 \leqslant t \leqslant \frac{\pi}{2}$, and note that it is complex normal and along it f has limit 0.

However, this does not yield the admissible limit 0, as we can find a sequence of points $\{z^{(k)}\}_{k=0}^{\infty}$ with $z^{(k)} = (1 - 2^{-4k}, 2^{-k})$, which are in an admissible approach region, and

$$f(z^{(k)}) = \frac{2^{-4k}}{2^{-4k}} = 1. \tag{3.14}$$

Therefore we have the limit

$$\lim_{k \to \infty} f(z^{(k)}) = 1. \tag{3.15}$$

So we need to apply more restrictive conditions.

Suppose that

$$\lim_{t \to 1^-} f(\varphi(t)) = \ell$$

for any boundary curve $\varphi : [0, 1] \to \partial\Omega$ with $\varphi(1) = (1, 0)$. We hope to find a Lindelöf principle for this case, that is, we wish that f had the admissible limit ℓ at $(1, 0)$.

Since the boundary values of f are defined through nontangential limit and along all curves near $(1,0)$, $|f|$ is defined, we may assume that there exists a neighborhood $W \subset \partial\Omega$ of $(1,0)$ such that

$$\lim_{W \ni \omega \to (1,0)} f(w) = \ell,$$

and may even assume that W is also part of the boundary of another domain V inside Ω that is of C^2 boundary, and f has the nontangential limit at every point in W.

As stated in the previous section, we may just consider the region $\mathscr{A}$ which is comparable with $\mathscr{A}_1(1,0)$. So we hope to get this result:

$$\lim_{\mathscr{A} \ni z \to (1,0)} |f(z) - f(1,0)| = 0.$$

So we begin to estimate $|f(z) - f(1,0)|$.

First we have the triangle inequality

$$\begin{aligned} |f(z) - f(1,0)| \leqslant & \left|f(z) - f((1,0) - \frac{1}{k}\nu)\right| \\ & + \left|f((1,0) - \frac{1}{k}\nu) - f(1,0)\right| \end{aligned} \tag{3.16}$$

for any $k \in \mathbb{N}$, where ν is the outward unit normal vector at $(1,0)$.

We have no need to worry about the second expression because it has the limit 0 when $k \to \infty$, so we hope to have the limit 0 for the first expression. To see this, we use the Poisson integral over ∂V. So the problem turns out to estimate in terms of Poisson kernels.

For any positive ε small enough, define

$$W_\varepsilon = \{P \in W : |P - (1,0)| < \varepsilon^3\},$$

then $\sigma(W_\varepsilon) \sim \varepsilon^9$.

We estimate that

$$\begin{aligned}\left|f(z) - f((1,0) - \frac{1}{k}\nu)\right| &\leqslant \int_{\partial V} \left|P(z,\zeta) - P((1,0) - \frac{1}{k}\nu, \zeta)\right| |f(\zeta)|\, d\sigma(\zeta) \\ &= \int_{\partial V \setminus W_\varepsilon} \left|P(z,\zeta) - P((1,0) - \frac{1}{k}\nu, \zeta)\right| |f(\zeta)|\, d\sigma(\zeta) \quad (*) \\ &+ \int_{W_\varepsilon} \left|P(z,\zeta) - P((1,0) - \frac{1}{k}\nu, \zeta)\right| |f(\zeta)|\, d\sigma(\zeta) \quad (**) \quad (3.17)\end{aligned}$$

We are not worried about (*) because on $\partial V \setminus W_\varepsilon$, as z is approaching to $(1,0)$ and k is tending to ∞, ζ is away from the singularities of the Poisson kernels, and $|f(\zeta)|$ is bounded. Therefore the expression (*) has limit 0.

We know that $P(z,\zeta)$, as in the complex 2-space, equals $\delta(z)/|z-\zeta|^4$ plus an error term, so we want to estimate, for $\zeta \in W_\varepsilon$,

$$\left|\frac{\delta(z)}{|z-\zeta|^4} - \frac{\delta((1,0) - \frac{1}{k}\nu)}{|(1,0) - \frac{1}{k}\nu - \zeta|^4}\right|. \tag{3.18}$$

If $C_1\varepsilon < \delta(z) < C_2\varepsilon$ and $C_3\varepsilon < \frac{1}{k} < C_4\varepsilon$, we have

$$|z-\zeta|^4 \geqslant (\delta(z))^4 > C_1^4\varepsilon^4 \tag{3.19}$$

and

$$\left|(1,0) - \frac{1}{k}\nu\right|^4 = (\frac{1}{k})^4 > C_3^4\varepsilon^4. \tag{3.20}$$

We also need their upper bounds. By the triangle inequality we know that

$$|z-\zeta| \leqslant |z-(1,0)| + |\zeta - (1,0)|. \tag{3.21}$$

Since $z \in \mathscr{A}$, as shown in (3.5), there is the relation $|z-(1,0)| < \delta(z)^{\frac{1}{m}}$.

By the definition of W_ε, we have $|\zeta - (1,0)| < \varepsilon^3$. So we can estimate that

$$|z - \zeta| < C_6 \varepsilon^{\frac{1}{m}}. \tag{3.22}$$

However, we estimate that

$$\big|(1,0) - \frac{1}{k}\nu - \zeta\big| \leqslant \big|(1,0) - \frac{1}{k}\nu - (1,0)\big| + |\zeta - (1,0)| < C_7 \varepsilon. \tag{3.23}$$

Therefore we have

$$\Big|\delta(z)\big|(1,0) - \frac{1}{k}\nu - \zeta\big|^4 - \frac{1}{k}|z-\zeta|^4\Big| < C_8 \cdot \varepsilon \cdot (\varepsilon^{\frac{1}{m}})^4 = C_8 \varepsilon^{1+\frac{4}{m}}. \tag{3.24}$$

Now we can estimate that

$$\begin{aligned}
\Big|\frac{\delta(z)}{|z-\zeta|^4} - \frac{\delta((1,0)-\frac{1}{k}\nu)}{|(1,0)-\frac{1}{k}\nu-\zeta|^4}\Big| &= \Big|\frac{\delta(z)}{|z-\zeta|^4} - \frac{\frac{1}{k}}{|(1,0)-\frac{1}{k}\nu-\zeta|^4}\Big| \\
&= \frac{\Big|\delta(z)|(1,0)-\frac{1}{k}\nu-\zeta|^4 - \frac{1}{k}|z-\zeta|^4\Big|}{|z-\zeta|^4|(1,0)-\frac{1}{k}\nu-\zeta|^4} \\
&\leqslant \frac{C_9}{\varepsilon^8}\Big|\delta(z)\big|(1,0)-\frac{1}{k}\nu-\zeta\big|^4 - \frac{1}{k}|z-\zeta|^4\Big| \\
&< \frac{C_9}{\varepsilon^8} \cdot C_8 \varepsilon^{1+\frac{4}{m}} \\
&= \frac{C_{10}}{\varepsilon^{7-\frac{4}{m}}}.
\end{aligned} \tag{3.25}$$

Therefore, by the boundedness of f on the boundary, we see that

$$\begin{aligned}
\int_{W_\varepsilon} \big|P(z,\zeta) - P((1,0)-\frac{1}{k}\nu,\zeta)\big||f(\zeta)|\,d\sigma(\zeta) &< C_{11}\varepsilon^9 \cdot \frac{C_{10}}{\varepsilon^{7-\frac{4}{m}}} \\
&= C_{12}\varepsilon^{2+\frac{4}{m}} \\
&< C_{12}\varepsilon^2.
\end{aligned} \tag{3.26}$$

This means $|f(z) - f((1,0)-\frac{1}{k}\nu)|$ has limit 0 as z approaches to $(1,0)$ admissibly and k tends to infinity.

Therefore we can establish this result:

Theorem 3.2.2 *Let Ω be a domain in $\mathbb{C}^2$ that is of finite type. Suppose f is a bounded holomorphic function on Ω such that $|f(z)| \leqslant 1$ for any $z \in \Omega$, and*

$$\lim_{t \to 1^-} f(\varphi(t)) = \ell$$

for any boundary curve $\varphi : [0,1] \to \partial\Omega$ with $\varphi(1) = (1,0)$. Then

$$\lim_{\mathscr{A} \ni z \to (1,0)} |f(z) - f(1,0)| = 0.$$

4. Inner Functions

In this chapter we will construct inner functions for a general kind of domain in $\mathbb{C}^n$. Although in the text we are dealing with $\mathbb{C}^2$, the method can be carried out for higher dimensions.

4.1 Integral Formulas

Let B be the unit ball in $\mathbb{C}^2$ and S be its boundary.

Suppose that $f : \mathbb{C}^2 \to \mathbb{C}^2$ holomorphic, and $M \subset \mathbb{C}^2$ is a compact manifold with smooth boundary such that $f(M) = B$, $f(\partial M) = S$.

Also suppose that $f = (f_1, f_2) : \overline{M} \to \overline{B}$ is a finite ramified covering, with N sheets. Let $Z = \{(z, w) \in \overline{M} : \text{either } z \text{ is a zero of } f_1 \text{ or } w \text{ is a zero of } f_2\}$. Denote $Z_1 = Z \cap \partial M$, $Z_2 = f(Z) \cap S$.

We can then think of $\{U_1, \ldots, U_n\}$ as an open cover of $S \setminus Z_2$ such that, for each U_i, $f^{-1}(U_i)$ contains N disjoint components $V_{i1}, \ldots, V_{iN}$ such that each component is biholomorphic to U_i, and $\{V_{i1}, \ldots, V_{iN}\}_i$ covers $\partial M \setminus Z_2$.

Then we choose $\{W_1, \ldots, W_n\}$ such that they are pairwise disjoint, $W_i \subset V_i$, and $\cup_{i=1}^n W_i = S \setminus Z_2$. If we let $\{X_{i1}, \ldots, X_{iN}\} = f^{-1}(W_i)$, then we know that $\cup_{j=1}^N \cup_{i=1}^n X_{ij} = \partial M \setminus Z_1$ and for each k, the intersection of the interiors of X_{ik} and X_{jk}, $i \neq j$, is an empty set.

This case is of interest because it can furnish domains that are more general than strongly pseudocovex ones. For example, for any positive integer p, the map $f_p(z_1, z_2) = (z_1, z_2^{2p})$ relates the domain $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2p} < 1\}$, which is not strongly pseudoconvex but still of finite type, with the maximal type $2p$ for points $(e^{i\theta}, 0)$, to the unit ball. For the notion of finite type, please refer to previous chapters or works such as [5] by D. Catlin.

For $z, w \in \partial M \setminus Z_2$ on the same sheet and sufficiently close to one another, define

$$d_M(z, w) = \sqrt{1 - \left|\langle f(z), f(w)\rangle\right|^2}, \tag{4.1}$$

we can then introduce the open boundary ball $E_M(\omega, r)$, centered at $\omega \in \partial M$, with radius r sufficiently small:

$$E_M(\omega, r) = \{\zeta \in \partial M : d_M(\omega, \zeta) < r\}. \tag{4.2}$$

We wish to introduce a measure over ∂M. The way to do this is to relate it to a measure on S, the boundary of the unit ball in $\mathbb{C}^2$.

Specifically, for $\eta \in S$, put

$$E(\eta, r) = \{\xi \in S : d(\eta, \xi) = \sqrt{1 - \left|\langle \eta, \xi\rangle\right|^2} < r\}, \tag{4.3}$$

and let σ be the unique rotation-invariant probability measure on S.

With the map $f : \partial M \to S$, we immediately have the relations

$$d_M(z, w) = d(f(z), f(w)), \tag{4.4}$$

and

$$f(E_M(\omega, r)) = E(f(\omega), r), \tag{4.5}$$

because $\xi = f(z) \in f(E_M(\omega, r))$ if and only if $z \in E_M(\omega, r)$, which is the same to say that $\sqrt{1 - \left|\langle f(z), f(\omega)\rangle\right|^2} < r$, or equivalently, $\xi = f(z) \in E(f(\omega), r)$.

This inspires us to define a measure σ_M by

$$\sigma_M(E_M(\omega, r)) = \sigma(f(E_M(\omega, r))) = \sigma(E(f(\omega), r)), \tag{4.6}$$

for r sufficiently small.

Although here, σ_M is defined in terms of boundary balls, it also works for open subsets of sheets of $\partial M \setminus Z_2$. We have the relation

$$\sigma_M = \sigma \circ f. \tag{4.7}$$

The study in [29] of $E(\eta, \delta)$ shows that $\sigma(E(\eta, \delta)) = \delta^2$. Consequently, we have

$$\sigma_M(E_M(\omega, r)) = r^2. \tag{4.8}$$

Now we turn our attention to the relation between the integrals over S and those over ∂M.

For $j = 1, 2, \ldots, N$,

$$\bigcup_i^n \int_{X_{ij}} F\big(f(w)\big)\, d\sigma_M(w) = \bigcup_i^n \int_{W_i} F(z)\, d\sigma(z) = \int_{S-Z_2} F(z)\, d\sigma(z). \tag{4.9}$$

Since $\sigma(Z_2) = 0$, we have the equation

$$\bigcup_i^n \int_{X_{ij}} F\big(f(w)\big)\, d\sigma_M(w) = \int_S F(z)\, d\sigma(z). \tag{4.10}$$

Therefore, since $\sigma_M(Z_1) = 0$, we know the relation between these two kinds of integrals is:

$$\begin{aligned}\int_M F\big(f(w)\big)\, d\sigma_M(w) &= \int_{M-Z_1} F\big(f(w)\big)\, d\sigma_M(w) \\ &= N \cdot \Big(\bigcup_i^n \int_{X_{ij}} F\big(f(w)\big)\, d\sigma_M(w)\Big) \\ &= N \int_S F(z)\, d\sigma(z).\end{aligned} \tag{4.11}$$

If $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, where α_1 and α_2 are non-negative integers, we have the following notations: $\alpha! = \alpha_1!\alpha_2!$, $|\alpha| = \alpha_1 + \alpha_2$, and for $z \in \mathbb{C}^2$, $z^\alpha = z^{(\alpha_1,\alpha_2)} = z_1^{\alpha_1} z_2^{\alpha_2}$.

Since, as shown in Rudin's book [29], for $\alpha \neq \beta$,

$$\int_S z^\alpha \overline{z}^\beta\, d\sigma(z) = 0, \tag{4.12}$$

and

$$\int_S z^\alpha \overline{z}^\alpha\, d\sigma(z) = \frac{\alpha!}{(1+|\alpha|)!}, \tag{4.13}$$

we can then obtain similar equations for the case ∂M, which will play important roles in the analysis of RW-sequences in a following section:

$$\int_{\partial M} f(z)^\alpha \overline{f(z)}^\beta\, d\sigma_M(z) = 0, \alpha \neq \beta, \tag{4.14}$$

and

$$\int_{\partial M} f(z)^\alpha \overline{f(z)}^\alpha\, d\sigma_M(z) = \frac{N\alpha!}{(1+|\alpha|)!}. \tag{4.15}$$

4.2 Boundary Balls

Put $L = \sup d_M(z,w)$ and $l = \inf \big(d_M(z,u) + d_M(u,w)\big)$, where $z, w, u \in \partial M$ and they are on the same sheet. We notice that $L < \infty$ and $l > 0$ are constants, and for any $z, w, u \in \partial M$ on the same sheet, we have

$$d_M(z,w) \leqslant L \leqslant \frac{L}{l}\big(d_M(z,u) + d_M(u,w)\big), \tag{4.16}$$

which means we have a triangle inequality, that is, there exists a positive constant C_1 such that

$$d_M(z,w) \leqslant C_1(d_M(z,u) + d_M(u,w)). \tag{4.17}$$

With the help of this general result, or, more simply, from the definition of the area of boundary balls as shown in (4.8), along with what we established in (4.5), we can check that:

1. $0 < \sigma_M(E_M(\omega,r)) < \infty$;

2. There exists $C_2 > 0$ such that $\sigma_M(E_M(\omega,2r)) \leqslant C_2\sigma_M(E_M(\omega,r))$;

3. There exists $C_3 > 0$ such that if $E_M(\omega,r) \cap E_M(\zeta,s) = \emptyset$ and $r \leqslant s$, then $E_M(\omega,r) \subseteq E_M(\zeta,C_3 s)$.

These properties show that ∂M is actually a space of homogeneous type (see [8]), which gives us a geometric result:

Theorem 4.2.1 *For $r > 0$, there exists a maximal set $\{\omega_1,\ldots,\omega_K\} \subset \partial M$ with respect to having the balls $E_M(\omega_j,r)$ pairwise disjoint but $\cup_{j=1}^{K} E_M(\omega_j,2r)$ cover ∂M.*

As a consequence of this, we have

$$C_4 = \sigma_M(\partial M) \leqslant K\sigma_M(E_M(\omega_j,2r)) = K(2r)^2 = 4Kr^2, \tag{4.18}$$

which implies that

$$K \geqslant \frac{C_4}{4r^2}. \tag{4.19}$$

4.3 Construction of f-Polynomials

Let $r_1, \ldots, r_K$ be Rademacher functions:

$$r_j(t) = \operatorname{sgn} \sin(2^j \pi t), \ \ t \in [0, 1],$$

and define

$$Q_t \circ f(z) = \sum_{j=1}^{K} r_j(t) \langle f(z), f(\omega_j) \rangle^k, \tag{4.20}$$

where $k \in \mathbb{N}$.

We first notice that $Q_t \circ f$ is a polynomial of $f_1(z)$ and $f_2(z)$, and $Q_t(\lambda f_1, \lambda f_2) = \lambda^k Q_t \circ f$.

Now, take $r = 1/\sqrt{k}$. We wish to find bounds for $|Q_t \circ f|$.

4.3.1 Lower Bounds

We first calculate that

$$\begin{aligned} \big|\langle f(\zeta), f(\omega_j)\rangle\big|^{2k} &= \big(f_1(\zeta)\overline{f_1(\omega_j)} + f_2(\zeta)\overline{f_2(\omega_j)}\big)^k \big(\overline{f_1(\zeta)} f_1(\omega_j) + \overline{f_2(\zeta)} f_2(\omega_j)\big)^k \\ &= \sum_{0 \leqslant i,s \leqslant k} D_{i,s} f(\zeta)^{(i,k-i)} \overline{f(\zeta)}^{(s,k-s)}, \end{aligned} \tag{4.21}$$

where

$$D_{i,s} = \binom{k}{i}^2 f(\omega)^{(s,k-s)} \overline{f(\omega)}^{(i,k-i)}. \tag{4.22}$$

We note that $|f_1(\omega)|^2 + |f_2(\omega)|^2 = 1$ for $\omega \in \partial M$. Using (4.14) and (4.15), we have the equations:

$$\begin{aligned}
\int_{\partial M} \left|\langle f(\zeta), f(\omega_j)\rangle\right|^{2k} d\sigma_M(\zeta) &= \int_{\partial M} \sum_{0\leqslant i,s\leqslant k} D_{i,s} f(\zeta)^{(i,k-i)}\overline{f(\zeta)}^{(s,k-s)}\, d\sigma_M(\zeta) \qquad (4.23)\\
&= \sum_{i=0}^{k} D_{i,i} \int_{\partial M} f(\zeta)^{(i,k-i)}\overline{f(\zeta)}^{(i,k-i)}\, d\sigma_M(\zeta)\\
&= \sum_{i=0}^{k} \binom{k}{i}^2 f(\omega)^{(i,k-i)}\overline{f(\omega)}^{(i,k-i)} \cdot \frac{Ni!(k-i)!}{(1+k)!}\\
&= \frac{N}{1+k} \sum_{i=0}^{k} \binom{k}{i} f(\omega)^{(i,k-i)}\overline{f(\omega)}^{(i,k-i)}\\
&= \frac{N}{1+k} \sum_{i=0}^{k} \binom{k}{i} (|f_1(\omega)|^2)^i (|f_2(\omega)|^2)^{k-i}\\
&= \frac{N}{1+k} (|f_1(\omega)|^2 + |f_2(\omega)|^2)^k\\
&= \frac{N}{1+k}.
\end{aligned}$$

Therefore, by the definition of $Q_t \circ f$, we have estimations:

$$\begin{aligned}
\int_0^1 dt \int_{\partial M} |Q_t \circ f(\zeta)|^2\, d\sigma_M(\zeta) &= \sum_{j=1}^{K} \int_{\partial M} \left|\langle f(\zeta), f(\omega_j)\rangle\right|^{2k} d\sigma_M(\zeta) \qquad (4.24)\\
&= \frac{KN}{(1+k)}\\
&\geqslant \frac{C_4 N}{4(1+k)r^2}\\
&> C_5.
\end{aligned}$$

This implies that there exists $t \in [0,1]$ such that

$$\int_{\partial M} |Q_t \circ f(\zeta)|^2\, d\sigma_M(\zeta) > C_5. \qquad (4.25)$$

Our focus will be on this $Q_t \circ f$.

4.3.2 Upper Bounds

Fixing $\zeta \in \partial M$, for $m = 0, 1, 2, \ldots$, define the set H_m by

$$H_m = \{\omega_j : mr \leqslant d_M(\zeta, \omega_j) < (m+1)r\}.$$

If $\omega_j \in H_m$, we have

$$\big|\langle f(\zeta), f(\omega_j)\rangle\big|^2 \leqslant 1 - m^2 r^2, \tag{4.26}$$

and $\omega_j \in E_M(\zeta, (m+1)r)$, which implies that $E_M(\omega_j, r) \subset E_M(\zeta, C_1(m+2)r)$, because for any $\omega \in E_M(\omega_j, r)$, we have

$$\begin{aligned} d_M(\zeta, \omega) &\leqslant C_1\big(d_M(\zeta, \omega_j) + d_M(\omega_j, \omega)\big) \\ &\leqslant C_1\big((m+1)r + r\big) \\ &= C_1(m+2)r \end{aligned} \tag{4.27}$$

and therefore

$$\sigma_M(E_M(\omega_j, r)) \cdot \#H_m \leqslant \sigma_M(E_M(\zeta, (m+2)r)), \tag{4.28}$$

where $\#H_m$ denotes the cardinality of H_m. According to the estimation (4.8),

$$\#H_m \leqslant C_1^2 (m+2)^2. \tag{4.29}$$

Thus we have estimations

$$
\begin{aligned}
|Q_t \circ f(\zeta)| &\leqslant \sum_{j=1}^{K} \left|\langle f(\zeta), f(\omega_j)\rangle\right|^k \\
&= \sum_{m=0}^{\infty} \sum_{\omega_j \in H_m} \left|\langle f(\zeta), f(\omega_j)\rangle\right|^k \\
&\leqslant C_1^2 \sum_{m=0}^{\infty} (m+2)^2 (1 - m^2 r^2)^{\frac{k}{2}} \\
&< C_1^2 \sum_{m=0}^{\infty} (m+2)^2 e^{-\frac{k}{2} m^2 r^2} \\
&< C_1^2 \sum_{m=0}^{\infty} (m+2)^2 e^{-\frac{m^2}{2}}.
\end{aligned} \tag{4.30}
$$

The last series is convergent. Using Σ to denote the sum, we therefore have

$$
\left|\frac{Q_t \circ f(\zeta)}{\Sigma}\right| < 1. \tag{4.31}
$$

4.3.3 RW-Sequences

According to the results above, we define, for $\zeta \in \partial M$,

$$
W_k \circ f(\zeta) = \frac{Q_t \circ f(\zeta)}{\Sigma}, \tag{4.32}
$$

which is a polynomial of $f_1(z)$ and $f_2(z)$. Moreover, $W_k\big(\lambda f_1(z), \lambda f_2(z)\big) = \lambda^k W_k \circ f$. We call this a *homogeneous f-polynomial of degree k*. This leads us to the following theorem.

Theorem 4.3.1 *There exists a positive constant c such that, for $k \in \mathbb{N}$ and $W_k \circ f$ as defined, we have*

1. *$W_k \circ f$ is a homogeneous f-polynomial of degree k,*

2. $|W_k \circ f(\zeta)| \leqslant 1$, *and*

3. $\int_{\partial M} |W_k \circ f|^2 \, d\sigma \geqslant c$.

Letting $\mathscr{U}$ be a compact subgroup of $U(2)$, we have that

$$\int_{\mathscr{U}} |W_t \circ U \circ f|^2 \, dU = \int_{\partial M} |W_t \circ f|^2 \, d\sigma \geqslant c. \tag{4.33}$$

If μ is a positive Borel measure on ∂M, we have

$$\int_{\mathscr{U}} dU \int_{\partial M} |W_t \circ U \circ f|^2 \, d\mu \geqslant c \int_{\partial M} d\mu, \tag{4.34}$$

and therefore we can find U_k such that

$$\int_{\partial M} |W_t \circ U_k \circ f|^2 \, d\mu \geqslant c \int_{\partial M} d\mu. \tag{4.35}$$

Note that the results in the theorem remain true if we consider $W_t \circ U_k \circ f$ instead of $W_t \circ f$. We add another property for this f-polynomial:

Proposition 4.3.1 *There exists a positive constant c such that, for $k \in \mathbb{N}$ and $W_k \circ f$ as defined, we have*

1. *$W_k \circ f$ is an homogeneous f-polynomial of degree k,*

2. $|W_k \circ f(\zeta)| \leqslant 1$,

3. $\int_{\partial M} |W_k \circ f|^2 \, d\sigma \geqslant c$, *and*

4. $\int_{\partial M} |W_k \circ f|^2 \, d\mu \geqslant c \int_{\partial M} d\mu$, *if μ is a positive Borel measure.*

4.4 Inner Functions

Any holomorphic function can be written as a series of homogeneous polynomials:

$$h(z) = \sum_{k=0}^{\infty} h_k(z) \tag{4.36}$$

Therefore $h \circ f$ can be written as a series of f-homogeneous polynomials

$$h \circ f(z) = \sum_{k=0}^{\infty} h_k \circ f(z) \tag{4.37}$$

Let E be a set of nonnegative integers, an (E, f)-polynomial is a finite sum of the form $\sum_{k \in E} F_k$, where F_k is a f-homogeneous polynomial of degree k. If we take k range from 0 to ∞, we call this series an (E, f)-function.

If, additionally, E is such that there are such integers a_m ($m = 1, 2, 3, \ldots$) that E contains $j + a_m$ for $j = 1, 2, \ldots, m$, we say that E is an LI-set, which means E contains arbitrarily long intervals of consecutive integers. A quick result is that the removal of any finite subset of an LI-set still gives an LI-set.

Proposition 4.4.1 *Suppose that $\varphi \in C(\overline{B})$, E is an LI-set, and for $k = 1, 2, \ldots,$ f_k is an f-homogeneous polynomial of degree k, with $|f_k| \leqslant 1$ on $\overline{M}$.*

Then there is a sequence $\{k_i\}$, and there are (E, f)-polynomials F_i such that

$$\lim_{i \to \infty} |F_i(z) - f_{k_i}(z)\varphi\big(f(z)\big)| = 0 \tag{4.38}$$

uniformly on $\overline{M}$.

The following gives a proof of this proposition, which is analogous to Rudin's in [30], with a few necessary modifications. First of all, we need to give a Cauchy integral formula for our case.

Using the change of variables $z = f(w)$ and applying the known Cauchy integral over S, we have

$$F(f(w)) = \frac{1}{N} \int_{\partial M} \frac{F(f(\zeta))\, d\sigma_M(\zeta)}{\big(1 - \langle f(w), f(\zeta)\rangle\big)^2}. \tag{4.39}$$

We may assume that $\varphi(z)$ is actually a polynomial of $z_1, z_2, \overline{z}_1$ and $\overline{z}_2$, and we first consider the monomial case $\varphi(z) = \psi(z) = z^{(\alpha_1,\alpha_2)}\overline{z}^{(\beta_1,\beta_2)}$.

We then define

$$P_k = \frac{1}{N}\int_{\partial M} \frac{f_k(\zeta)\varphi\big(f(\zeta)\big)\, d\sigma_M(\zeta)}{\big(1-\langle f(z), f(\zeta)\rangle\big)^2}. \tag{4.40}$$

Note that $f_k(\zeta)$ is a finite sum of terms of the form $f(\zeta)^{(i,k-i)}$. We expand $(1-\langle f(z), f(\zeta)\rangle)^{-2}$, and due to (4.14), we are only interested in the following non-vanishing integral, with the integer t to be fixed:

$$\int_{\partial M} f(\zeta)^{(\alpha_1,\alpha_2)}\overline{f(\zeta)}^{(\beta_1,\beta_2)} f(\zeta)^{(i,k-i)}\Big(f_1(z_1)\overline{f_1(\zeta)} + f_2(z_2)\overline{f_2(\zeta_2)}\Big)^t\, d\sigma_M(\zeta). \tag{4.41}$$

We compute

$$\Big(f_1(z)\overline{f_1(\zeta)} + f_2(z)\overline{f_2(\zeta)}\Big)^t = \sum_{j=0}^{t}\binom{t}{j} f(z)^{(j,t-j)}\overline{f(\zeta)}^{(j,t-j)}. \tag{4.42}$$

Therefore, in order for the integral to be nonzero, we must have relations

$$\begin{cases} \alpha_1 + i = \beta_1 + j, \\ \alpha_2 + k - i = \beta_2 + t - j, \end{cases} \tag{4.43}$$

from which we obtain

$$t = k + (\alpha_1+\alpha_2) - (\beta_1+\beta_2), \tag{4.44}$$

and this fixes t.

So we can conclude that P_k defined in (4.40) is actually an f-homogeneous polynomial of degree $k + |\alpha| - |\beta|$.

Taking into consideration that $\varphi \circ f(z) = \sum C_{\alpha\beta} f(z)^{\alpha}\overline{f(z)}^{\beta}$, and that E is an LI-set, we can check that P_k is an (E, f)-polynomial for infinitely many k.

To check for uniform convergence, we first note that

$$P_k(z) - f_k(z)\varphi\big(f(z)\big) = \Xi_k(z) = \frac{1}{N}\int_{\partial M} \frac{f_k(\zeta)\Big(\varphi\big(f(\zeta)\big) - \varphi\big(f(z)\big)\Big)\, d\sigma_M(\zeta)}{\big(1 - \langle f(z), f(\zeta)\rangle\big)^2} \tag{4.45}$$

for $z \in \overline{M}, \zeta \in \partial M - \{z\}$.

Then we can check that

$$\lim_{k\to\infty} \Xi_k(z) = 0. \tag{4.46}$$

Noting that $\{\Xi_k\}$ is equicontinuous completes our verification of the proposition.

Now, suppose that φ is a positive LSC function on $\overline{B}$, that is, $\limsup_{\overline{B}\ni\xi\to\xi_0} \varphi(\xi) \geqslant \varphi(\xi_0)$ for any $\xi_0 \in \overline{B}$. We also suppose that $\varphi \circ f \in L^2(\mu)$. In fact, we may simply assume that $\varphi \in C(\overline{B})$ because we can use increasing sequences of positive continuous functions to approximate LSC φ from below.

According to the result in Proposition 4.3.1, for *any* k, we can find f-homogeneous polynomials $W_k \circ f$ such that $|W_k \circ f| \leqslant 1$ and

$$\int_{\partial M} |(W_k \circ f)(\varphi \circ f)|^2\, d\mu \geqslant c \int_{\partial M} (\varphi \circ f)^2\, d\mu. \tag{4.47}$$

As the constant c in Proposition 4.3.1 is fixed, we can choose another positive constant C sufficiently small, such that $C(2-C) < c/2$. Then applying to Proposition 4.4.1, we can find an (E, f)-polynomial F and *one* $W_k \circ f$ with the relation

$$|F - (W_k \circ f)(\varphi \circ f)| < C\varphi \circ f, \tag{4.48}$$

since $C\varphi \circ f > 0$.

Then it comes to two cases. If $|F| \leqslant |(W_k \circ f)(\varphi \circ f)|$, then we have $|(W_k \circ f)(\varphi \circ f)| - |F| < C\varphi \circ f$, which indicates that

$$|(W_k \circ f)(\varphi \circ f)| - C\varphi \circ f < |F| \leqslant |(W_k \circ f)(\varphi \circ f)| \leqslant \varphi \circ f < (1 + C)\varphi \circ f, \tag{4.49}$$

because $|W_k \circ f| \leqslant 1$; if instead, $|F| > |(W_k \circ f)(\varphi \circ f)|$, it follows that $|F| - |(W_k \circ f)(\varphi \circ f)| < C\varphi \circ f$, and we have

$$|(W_k \circ f)(\varphi \circ f)| - C\varphi \circ f < |(W_k \circ f)(\varphi \circ f)| < |F| < (1 + C)\varphi \circ f. \tag{4.50}$$

Therefore, in either case, we always have

$$|(W_k \circ f)(\varphi \circ f)| - C\varphi \circ f < |F| < (1 + C)\varphi \circ f. \tag{4.51}$$

Now, let $P = \frac{1}{1+C}F$. From the inequality (4.51), we immediately have the result

$$\begin{aligned} |P| &= \frac{1}{1 + C}|F| \\ &< \frac{1}{1 + C}(1 + C)\varphi \circ f = \varphi \circ f. \end{aligned} \tag{4.52}$$

Additionally, we estimate that, as $|(W_k \circ f)| \leqslant 1$,

$$\begin{aligned} |F|^2 &> |(W_k \circ f)(\varphi \circ f)|^2 - 2C|(W_k \circ f)(\varphi \circ f)|\varphi \circ f + C^2(\varphi \circ f)^2 \\ &\geqslant |(W_k \circ f)(\varphi \circ f)|^2 - (2C - C^2)(\varphi \circ f)^2. \end{aligned} \tag{4.53}$$

This gives us:

$$\begin{aligned} \int_{\partial M} |P|^2 &= \frac{1}{(1 + C)^2} \int_{\partial M} |F|^2 \\ &> \frac{1}{(1 + C)^2} \int_{\partial M} \big(|(W_k \circ f)(\varphi \circ f)|^2 - (2C - C^2)(\varphi \circ f)^2\big) \\ &> \frac{1}{2} \int_{\partial M} \big(|(W_k \circ f)(\varphi \circ f)|^2 - (2C - C^2)(\varphi \circ f)^2\big) \\ &\geqslant \frac{1}{2}\big(c - C(2 - C)\big) \int_{\partial M} (\varphi \circ f)^2 \\ &= \frac{c}{4} \int_{\partial M} (\varphi \circ f)^2, \end{aligned} \tag{4.54}$$

with our assumptions $0 < C < 1$ and $C(2 - C) < c/2$.

To summarize, we re-state the result in a theorem:

Theorem 4.4.1 *If φ is a positive LSC function on $\overline{B}$ and $\varphi \circ f \in L^2(\mu)$, we can find an (E, f)-polynomial P such that $|P(z)| < \varphi\big(f(z)\big)$ and*

$$\int_{\partial M} |P|^2 \, d\mu > \frac{c}{4} \int_{\partial M} (\varphi \circ f)^2 \, d\mu,$$

where c is the same positive constant that appears in Proposition 4.3.1.

We can now verify the existence of inner functions.

Theorem 4.4.2 *Suppose $\varphi > 0$ on S, $\varphi \circ f \in LSC \cap L^2(\sigma_M)$, and E is an LI-set. Then there is a nonconstant E-function $F \in H^2(M)$ such that for almost every $\zeta \in \partial M$, with respect to the measure σ_M, we have*

$$|F^*(\zeta)| = \varphi \circ f(\zeta). \tag{4.55}$$

Proof We start with an (E, f)-polynomial P_0 satisfying $|P_0| < \varphi \circ f$, and denote the set of integers which are degrees of monomials in P_0 by E_0. For example, we may take $P_0 = 0$.

Letting $Q_0 = P_0$, then on ∂M we have

$$|Q_0| < \varphi \circ f. \tag{4.56}$$

Thus, according to Theorem 4.4.1 we can construct an $(E \setminus E_0, f)$-polynomial P_1 such that $|P_1| < \varphi \circ f - |Q_0|$ on ∂M, and

$$\int_{\partial M} |P_1|^2 \, d\sigma_M > \frac{c}{4} \int_{\partial M} (\varphi \circ f - |Q_0|)^2 \, d\sigma_M. \tag{4.57}$$

Denote the set of integers which are degrees of monomials in P_1 by E_1, and note that $E_0 \cap E_1 = \emptyset$ and therefore, P_0 and P_1 are orthogonal to each other. We can still regard P_1 as an (E, f)-polynomial.

Now, suppose we have found pairwise orthogonal (E, f)-polynomials $P_0, P_1, \dots, P_N$ such that

$$|Q_N| < \varphi \circ f \tag{4.58}$$

and

$$\int_{\partial M} |P_N|^2 \, d\sigma_M > \frac{c}{4} \int_{\partial M} (\varphi \circ f - |Q_{N-1}|)^2 \, d\sigma_M, \tag{4.59}$$

where we define $Q_j = \sum_{i=0}^{j} P_i$.

Then, by Theorem 4.4.1 we can find an $\big(E \setminus \cup_{i=0}^{N} E_i, f\big)$-polynomial P_{N+1}, which can be also regarded as an (E, f)-polynomial, satisfying

$$|P_{N+1}| < \varphi \circ f - |Q_N|, \tag{4.60}$$

and

$$\int_{\partial M} |P_{N+1}|^2 \, d\sigma_M > \frac{c}{4} \int_{\partial M} (\varphi \circ f - |Q_N|)^2 \, d\sigma_M. \tag{4.61}$$

We note that P_{N+1} is orthogonal to $P_1, \dots, P_N$. By definition of Q_{N+1} and the inequality (4.60), we see that

$$|Q_{N+1}| \leqslant |Q_N| + |P_{N+1}| < |Q_N| + \varphi \circ f - |Q_N| = \varphi \circ f. \tag{4.62}$$

In short, we start with an (E, f)-polynomial P_0 and then construct Q_0. If we already have pairwise orthogonal (E, f)-polynomials $P_1, \dots, P_N$, with $Q_N = P_0 + \cdots + P_N$, satisfying

$$|Q_N| < \varphi \circ f \tag{4.63}$$

and

$$\int_{\partial M} |P_N|^2 \, d\sigma_M > \frac{c}{4} \int_{\partial M} (\varphi \circ f - |Q_{N-1}|)^2 \, d\sigma_M, \tag{4.64}$$

then we can inductively construct an (E, f)-polynomial P_{N+1} orthogonal to $P_i, 0 \leqslant i \leqslant N$, and $Q_{N+1} = Q_N + P_{N+1}$, such that

$$|Q_{N+1}| < \varphi \circ f \tag{4.65}$$

and

$$\int_{\partial M} |P_{N+1}|^2 \, d\sigma_M > \frac{c}{4} \int_{\partial M} (\varphi \circ f - |Q_N|)^2 \, d\sigma_M. \tag{4.66}$$

Next, we notice that for any N, because of the orthogonality of $\{P_i\}$,

$$\int_{\partial M} |Q_N|^2 \, d\sigma_M = \int_{\partial M} \Big| \sum_{i=0}^{N} P_i \Big|^2 \, d\sigma_M = \sum_{i=0}^{N} \int_{\partial M} |P_i|^2 \, d\sigma_M. \tag{4.67}$$

On the other hand, by the inequality (4.63), we have

$$\int_{\partial M} |Q_N|^2 \, d\sigma_M < \int_{\partial M} (\varphi \circ f)^2 \, d\sigma_M. \tag{4.68}$$

Therefore we have the relation

$$\sum_{i=0}^{\infty} \int_{\partial M} |P_i|^2 \, d\sigma_M \leqslant \int_{\partial M} (\varphi \circ f)^2 \, d\sigma_M, \tag{4.69}$$

and it makes sense to define $F = \sum_{i=0}^{\infty} P_i$, because this series converges and furthermore we know that $F \in H^2(M)$.

According to (4.67), we can deduce that $|Q_N| \to |F^*|$ in $L^2(\sigma_M)$. However, by (4.64), whose left-hand side goes to 0 as $N \to \infty$, we have $|Q_N| \to \varphi \circ f$ in $L^2(\sigma_M)$. Thus, we can conclude that $|F^*(\zeta)| = \varphi \circ f(\zeta)$ for almost every $\zeta \in \partial M$, and the theorem is then proved. ■

We are finally ready to state the main result of the existence of inner functions, simply a special case of Theorem 4.4.2, taking $\varphi \equiv 1$:

Corollary 4.4.1 *Inner functions exist for the domain M.*

As a reminder, $M \subset \mathbb{C}^2$ is such a compact domain with smooth boundary that there exists a ramified holomorphic mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ with $f(M) = B$ and $f(\partial M) = S$. This assumption provides room for domains more general than strongly pseudoconvex ones, such as $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2p} < 1\}$ which we have already seen.

5. Concluding Remarks

Before ending the thesis, we would like to give several remarks about potential work to do in the future.

In Chapter 2 we are discussing the problem in $\mathbb{C}^2$ and obtain the result that admissible approach regions for domains of finite type are optimal and there is no Fatou's theorem for approach regions that are complex tangentially broader than admissible ones. We are wondering if this method could also be applied to explore domains of finite type in complex space of higher dimension, because we will have to be more careful about the definition of admissible approach regions and analyzing inequalities.

That's the same problem for Chapter 3. In this chapter we give several results regarding the admissible convergence as supplements to the Lindelöf principle. If we wish to move on to domains of finite type in $\mathbb{C}^n$, $n > 2$, we will first need to figure out the shape of admissible approach regions and consider applying the same method.

In Chapter 4 our result shows the existence of inner functions a more general type of domains. Although we can think of this for such domains by just composing with the mapping f, we provide more work than that such as set up integral formulas and construct RW-sequences for our case. Actually, if we just consider a specific domain, for example, M_2 as mentioned, we may approach by setting up different measures at the beginning instead of making compositions. But the idea is still the same. This also provides insight for some domains of finite type. However, not all domains of finite type can be related to the unit ball in such a way. Therefore, it is still unknown whether for all domains of finite type inner functions exist. There are other methods and tools we may take advantage of. We may also take into consideration that the weakly pseudoconvex points form a set of measure zero on the boundary of domains of finite type, which is a result from the work of D. Catlin in [5].

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