Weyl Type Theorems for Unbounded Non-normal Operators

The theory of operators is a branch of mathematics that focuses on bounded linear operators but which includes closed operators and unbounded operators. The subject of operator theory and its most important part, *the spectral theory*, came into focus rapidly after 1900. A major event was the appearance of *Fredholms theory of integral equations*, which arose as a new approach to the Dirichlet problem.

In 1909, H. Weyl (*Über beschränkte quadatische Formen, deren Differenz Vollstetig ist*) examined the spectra of all compact perturbations of a self adjoint operator on a Hilbert space and found that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. A bounded linear operator satisfying this property is said to satisfy **Weyl's theorem**.

Further, in 2002, M. Berkani (*Index of B-Fredholm operators and generalization of a Weyl's theorem*) proved that if T is a bounded normal operator acting on a Hilbert space H, then σ_{BW}(T) = σ(T) \E(T), where E(T) is the set of all isolated eigenvalues of T, which gives the generalization of the Weyl's theorem. He also proved this generalized version of classical Weyl's theorem for bounded hyponormal operators (*Generalized Weyl's theorem and hyponormal operators*).

Following Weyl and Berkani, various variants of Weyl's theorem, generally known as the Weyl-type theorems, have been introduced with much attention to an approximate point version called a-Weyl's theorem. Study of other generalizations began in 2003 that resulted in the Browder's theorem, a-Browder's theorem, generalized a-Weyl's theorem, property (w), property(ab), etc. This study, however, was limited to the classes of bounded operators.

Most applications of linear operators use unbounded linear operators which are closed or atleast have closed linear extensions. Our objective was to define and study non-normal classes of unbounded operators on a Hilbert space and study various Weyl-type theorems for those classes of operators. Some of the classes studied include the class of hyponormal operators and class-$\mathcal{A}$ operators. We have proved the following results:

"If T is an unbounded hyponormal operator or an unbounded class-$\mathcal{A}$ operator, then

(i) p(T - λI) $\leqslant 1$ for every $\lambda \in \mathbb{C}$,

(ii) λ is an isolated point of $\sigma(T)$ iff λ is a simple pole of the resolvent of T.

(iii) σ(T) = σ_w(T) $\cup$ isoσ_o(T) = σ_w(T) $\cup$ π_o(T), where isoσ_o(T) is the set of all isolated spectral points of finite multiplicity and π_o(T) is the set of poles of finite multiplicity.

(iv) σ(T) = σ_{BW}(T) $\cup$ isoσ(T) = σ_{BW}(T) $\cup$ π(T).""

As a consequence of the results proved, the following equivalences, between several variants, were established:

"If T is an unbounded hyponormal operator or an unbounded class-$\mathcal{A}$ operator, then

(i) property (w) is equivalent to Weyl's Theorem,

(ii) property (b) is equivalent to Browder's Theorem,

(iii) Weyl's Theorem is equivalent to Browder's Theorem

(iv) generalized Weyl's Theorem is equivalent to generalized Browder's Theorem

(v) property (aw) is equivalent to a-Weyl's Theorem,

(vi) property (ab) is equivalent to a-Browder's Theorem.