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The Nonexistence of Shearlet-Like Scaling Multifunctions that Satisfy Certain Minimally Desirable Properties and Characterizations of the Reproducing Properties of the Integer Lattice Translations of a Countable Collection of Square Integrable Functions

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THE NONEXISTENCE OF SHEARLET-LIKE SCALING MULTIFUNCTIONS THAT SATISFY CERTAIN MINIMALLY DESIRABLE PROPERTIES

AND

CHARACTERIZATIONS OF THE REPRODUCING PROPERTIES OF THE INTEGER LATTICE TRANSLATIONS OF A COUNTABLE COLLECTION OF SQUARE INTEGRABLE FUNCTIONS

by

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Abstract

In Chapter 1, we introduce three varieties of reproducing systems—Bessel systems, frames, and Riesz bases—within the Hilbert space context and prove a number of elementary results, including qualitative characterizations of each and several results regarding the combination and partitioning of reproducing systems.

In Chapter 2, we characterize when the integer lattice translations of a countable collection of square integrable functions forms a Bessel system, a frame, and a Riesz basis.

In Chapter 3, we introduce composite wavelet systems and generalize several well-known classical wavelet system results—including those regarding pointwise values of the Fourier transform of the wavelet and scaling function and those regarding dependencies on the multiresolution analysis defining properties—to the composite case. Two corollaries of these results are the nonexistence of composite scaling multifunctions of Haar-type, when the composite dilation group is infinite, and the nonexistence of classical multiwavelets, when the dilation matrix is integral and has determinant 1 in

absolute value.

There is a well-known connection, via the Fourier transform, between smoothness and integral polynomial decay. In Chapter 4, we prove several generalized versions of this result in which smoothness and integral polynomial decay are replaced with Hölder continuity and fractional polynomial decay; logarithmic continuity and logarithmic decay; iterated Hölder continuity and multivariable fractional polynomial decay.

In Chapter 5, we prove the nonexistence of shearlet-like scaling multifunctions that satisfy a minimal amount of decay and either a minimal amount of regularity or one of two "finite type" conditions.

In Chapter 6, we indicate a number of interesting questions that arise from the reproducing system characterizations of Chapter 2 and the scaling multifunction nonexistence results of Chapters 3 and 5.

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Dedication

I dedicate this dissertation to my wife Annie, who, in addition to many wonderful qualities, happens to be the perfect wife for a mathematics graduate student. Annie does vastly more than her share of the cooking, cleaning, and other household duties, thereby providing me with both the large amount of time and the relative lack of responsibility and preoccupation I find necessary to do mathematical research. When I become totally absorbed in a mathematical problem, she more than admirably tolerates my 12 to 14 hour workdays, my fickle moods, and my reply of "I'm thinking" to most attempts to engage me in conversation. When and if I do succeed, she is appropriately congratulatory and proud. Annie's presence provides a healthy and productive balance to my life without which total absorbtion in mathematics would often threaten to become unhealthy and counterproductive obsession. I truly appreciate all the many things she does for me.

Contents

Notation

- We represent elements of the time domain, $\mathbb{R}^n$, by column vectors x and elements of the frequency domain, $\hat{\mathbb{R}}^n$, by row vectors ξ.
- $\hat{\mathbb{Z}}^n$ denotes the collection of all $1 \times n$ row vectors with integer entries.
- $\mathbb{N}$ and $\mathbb{Z}^+$ denote the collections $\{p \in \mathbb{Z} : p \geq 0\}$ and $\{p \in \mathbb{Z} : p \geq 1\}$, respectively.
- We define the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\hat{\mathbb{R}}^n)$ for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi\imath\xi x}\, dx.$$

- A measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is said to be $\mathbb{Z}^n$-periodic if for each $k \in \mathbb{Z}^n$ we have that

$$f(x + k) = f(x)$$

  for almost every (a.e.) x; $\hat{\mathbb{Z}}^n$-periodic functions are defined similarly.

- For $1 \leq p < \infty$, we define $L^p(\mathbb{T}^n)$ to be the collection of all $\mathbb{Z}^n$-periodic functions satisfying
$$\int_{[0,1]^n} |f(x)|^p \, dx < \infty.$$

- We define $L^\infty(\mathbb{T}^n)$ to be the collection of all $\mathbb{Z}^n$-periodic functions that are essentially bounded.

- For $f \in L^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, we define the translation operator $T_y : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by
$$T_y f(x) = f(x - y).$$

- For $f \in L^2(\mathbb{R}^n)$ and $c \in GL_n(\mathbb{R})$, we define the dilation operator $D_c : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by
$$D_c f(x) = |\det c|^{-1/2} f(c^{-1}x).$$

- $C(\mathbb{R}^n)$ denotes the collection of all continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$.

- $C_0(\mathbb{R}^n)$ denotes the collection of all functions in $C(\mathbb{R}^n)$ that vanish at infinity.

- For each positive integer p, $C^p(\mathbb{R}^n)$ denotes the collection of all functions in $C(\mathbb{R}^n)$ whose partial derivatives up to order p exist and are continuous.

- $C^\infty(\mathbb{R}^n)$ denotes the collection of all functions that belong to $C^p(\mathbb{R}^n)$, for each positive integer p.

- The Schwartz class of $\mathbb{R}^n$, denoted by $\mathcal{S}(\mathbb{R}^n)$, consists of all functions that belong to $C^\infty(\mathbb{R}^n)$ and that, together with all their derivatives, vanish at ∞ faster than any power of $\|x\|$.

- The spaces $L^p(\hat{\mathbb{T}}^n)$, $C(\hat{\mathbb{R}}^n)$, $C_0(\hat{\mathbb{R}}^n)$, $C^p(\hat{\mathbb{R}}^n)$, and $\mathcal{S}(\hat{\mathbb{R}}^n)$ are defined similarly.

Chapter 1

Reproducing Systems in Hilbert Spaces

This chapter contains the definitions and results regarding reproducing systems within the general Hilbert space context that will be needed in subsequent chapters. Section 1 includes some basic facts regarding unconditional convergence of sums in Banach spaces. In section 2, we introduce three varieties of Hilbert space reproducing systems—Bessel systems, frames, and Riesz bases—and prove several elementary results. In particular, we obtain illuminating qualitative characterizations of each of the three. In section 3, we offer several results regarding the circumstances under which, first, two reproducing systems of a certain type can be combined to form a reproducing system of the same type, and, second, a subcollection of a reproducing system of a certain type forms a reproducing system of the same type. In particular, we obtain interesting results regarding the circumstances under which two frames (Riesz bases) may be combined to form a new frame (Riesz basis).

We enumerate definitions, theorems, etc. as indicated by the following example: Definition 1.2 refers to the second definition in the first chapter.

1.1 Unconditional Convergence of Sums in Banach Spaces

In this section, we define unconditional convergence of sums and other relevant Banach space terminology and prove several elementary facts thereabout. We have the following definition:

Definition 1.1. *Suppose that $\mathcal{X}$ is a Banach space, that I is a nonempty countable indexing set, and that $\{x_i : i \in I\} \subset \mathcal{X}$.*

(i) We say that $\sum_{i\in I} x_i$ converges unconditionally to $x \in \mathcal{X}$ if for every $\epsilon > 0$ there corresponds a finite subset $F = F(\epsilon)$ of I such that for any finite subset F' of I satisfying $F' \supset F$ we have

$$\Big\| \sum_{i\in F'} x_i - x \Big\| \leq \epsilon.$$

(ii) We say that $\sum_{i\in I} x_i$ is unconditionally Cauchy if for every $\epsilon > 0$ there corresponds a finite subset $F = F(\epsilon)$ of I such that for any finite subsets

F', F'' of I satisfying $F', F'' \supset F$ we have

$$\Big\| \sum_{i \in F'} x_i - \sum_{i \in F''} x_i \Big\| \leq \epsilon.$$

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, we define $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ to be the collection of all bounded linear operators mapping $\mathcal{X}$ into $\mathcal{Y}$. We write $\mathcal{B}(\mathcal{X})$ in place of $\mathcal{B}(\mathcal{X}, \mathcal{X})$. For $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, we denote the operator norm of Λ by $\|\Lambda\|$; that is,

$$\|\Lambda\| = \sup\{\|\Lambda x\| : x \in \mathcal{X}, \|x\| = 1\}.$$

Let $\mathcal{X}$, I, and $\{x_i : i \in I\}$ be as in Definition 1.1. We make the following observations:

(i) If $\sum_{i \in I} x_i$ converges unconditionally to $x \in \mathcal{X}$, then for any collection $\{I_p\}_{p=1}^{\infty}$ of finite subsets of I satisfying $I_1 \subset I_2 \subset I_3 \subset \ldots$ and $\bigcup_{p=1}^{\infty} I_p = I$ we have

$$\lim_{p \to \infty} \sum_{i \in I_p} x_i = x.$$

In particular, x is unique.

(ii) Since $\mathcal{X}$ is complete, it follows that $\sum_{i \in I} x_i$ is unconditionally Cauchy if and only if $\sum_{i \in I} x_i$ converges unconditionally.

(iii) $\sum_{i \in I} x_i$ is unconditionally Cauchy if and only if for every $\epsilon > 0$ there corresponds a finite subset $F = F(\epsilon)$ of I such that for any finite subset

F' of I satisfying $F' \cap F = \emptyset$ we have

$$\Big\| \sum_{i \in F'} x_i \Big\| \leq \epsilon.$$

(iv) Suppose that $\mathcal{Y}$ is a Banach space and that $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. If $\sum_{i \in I} x_i$ converges unconditionally to $x \in \mathcal{X}$, then $\sum_{i \in I} \Lambda x_i$ converges unconditionally to Λx.

We have the following lemma, which validates certain rearrangements of unconditionally convergent sums.

Lemma 1.1. *Let $\mathcal{X}$, I, and $\{x_i : i \in I\}$ be as in Definition 1.1. Assume that $\sum_{i \in I} x_i$ converges unconditionally to $x \in \mathcal{X}$.*

(i) If J is a nonempty subset of I, then $\sum_{j \in J} x_j$ converges unconditionally.

(ii) If $\{I_1, I_2, I_3, \ldots, I_N\}$ is a finite collection of nonempty disjoint subsets of I for which $I = \bigcup_{p=1}^{N} I_p$, then for each p, $\sum_{i \in I_p} x_i$ converges unconditionally, and we have

$$\sum_{p=1}^{N} \sum_{i \in I_p} x_i = x.$$

(iii) If $\{I_1, I_2, I_3, \ldots\}$ is a collection of nonempty disjoint subsets of I for which $I = \bigcup_{p=1}^{\infty} I_p$, then for each p, $\sum_{i \in I_p} x_i$ converges unconditionally, and $\sum_{p=1}^{\infty} \sum_{i \in I_p} x_i$ converges unconditionally to x.

(iv) If I_1 and I_2 are countable indexing sets and if $I = I_1 \times I_2$, then for each

$i_1 \in I_1$, $\sum_{i_2 \in I_2} x_{(i_1,i_2)}$ *converges unconditionally, and*

$$\sum_{i_1 \in I_1} \sum_{i_2 \in I_2} x_{(i_1,i_2)}$$

converges unconditionally to x*.*

Proof. Part (i) is easily verified. The proof of part (ii) is similar to (but easier than) the proof of part (iii); part (iv) follows from parts (ii) and (iii). We therefore only prove part (iii).

Suppose that $\{I_1, I_2, I_3, \ldots\}$ is a collection of nonempty disjoint subsets of I for which $I = \bigcup_{p=1}^{\infty} I_p$. It follows from part (i) of this lemma that for each p, $\sum_{i \in I_p} x_i$ converges unconditionally to some element $y_p \in \mathcal{X}$. Let $\epsilon > 0$ and choose a finite subset F of I for which $\|\sum_{i \in F'} x_i - x\| \le \epsilon/2$, for all finite subsets F' of I satisfying $F' \supset F$. Choose N for which $F \subset \bigcup_{p=1}^{N} I_p$. Fix a finite subset $\mathcal{P}$ of $\mathbb{Z}^+$ satisfying $\mathcal{P} \supset \{1, \ldots, N\}$. For each $q \in \mathbb{Z}^+$ choose a collection $\{F_p^q : p \in \mathcal{P}\}$ of finite subsets satisfying

(i) $F_p^q \subset I_p$, for all $p \in \mathcal{P}$;

(ii) $F \subset \bigcup_{p \in \mathcal{P}} F_p^q$, for all $q = 1, 2, 3, \ldots$;

(iii) $F_p^1 \subset F_p^2 \subset F_p^3 \subset \ldots$, for all $p \in \mathcal{P}$;

(iv) $\bigcup_{q=1}^{\infty} F_p^q = I_p$, for all $p \in \mathcal{P}$.

For $q \in \mathbb{Z}^+$, we have

$$\begin{aligned}
\|\sum_{p\in\mathcal{P}} y_p - x\| &= \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i + \sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i - x\| \\
&\leq \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i\| + \|\sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i - x\| \\
&\leq \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i\| + \epsilon/2,
\end{aligned}$$

where we have used that F is a subset of the disjoint union $\bigcup_{p\in\mathcal{P}} F_p^q$. Using that

$$\begin{aligned}
\lim_{q\to\infty} \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}}\sum_{i\in F_p^q} x_i\| &= \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}} \lim_{q\to\infty}\sum_{i\in F_p^q} x_i\| \\
&= \|\sum_{p\in\mathcal{P}} y_p - \sum_{p\in\mathcal{P}} y_p\| \\
&= 0,
\end{aligned}$$

it follows that $\|\sum_{p\in\mathcal{P}} y_p - x\| \leq \epsilon$, which completes the proof of part (iii). $\square$

1.2 Elementary Properties of Reproducing Systems

If $\mathcal{H}$ is a Hilbert space and if $\{f_i : i \in I\}$ is an orthonormal (ON) basis for $\mathcal{H}$, then it is well-known that

$$f = \sum_{i \in I} \langle f, f_i \rangle f_i \tag{1.1}$$

is valid for each $f \in \mathcal{H}$. In the sequel, we shall be very interested in collections $\{f_i : i \in I\} \subset \mathcal{H}$ that are not necessarily ON bases but nevertheless satisfy a reconstruction property similar to (1.1). We have the following definition:

Definition 1.2. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$, and that C and D are constants satisfying $0 < C \leq D < \infty$.*

(i) $\{f_i : i \in I\}$ is said to be a Bessel system with constant D if

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2,$$

for all $f \in \mathcal{H}$;

(ii) $\{f_i : i \in I\}$ is said to be a frame for $\mathcal{H}$ with constants $C \leq D$ if

$$C\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2,$$

for all $f \in \mathcal{H}$;

(iii) $\{f_i : i \in I\}$ *is said to be a Riesz basis for $\mathcal{H}$ with constants $C \leq D$ if its linear span is dense in $\mathcal{H}$ and if for each finite subset F of I and each collection $\{\alpha_i : i \in F\} \subset \mathbb{C}$ we have*

$$C \sum_{i \in F} |\alpha_i|^2 \leq \| \sum_{i \in F} \alpha_i f_i \|^2 \leq D \sum_{i \in F} |\alpha_i|^2.$$

A frame with constants $1 \leq 1$ is said to be a Parseval frame. Clearly, a Riesz basis with constants $1 \leq 1$ is an ON basis. Bessel systems, frames, and Riesz bases will be referred to, in general, as reproducing systems.

In this section, we prove a number of basic results regarding Bessel systems, frames, and Riesz bases and the relationship between the three. In particular, we obtain interesting characterizations of the Bessel, frame, and Riesz basis properties.

1.2.1 The Bessel Property

This section includes a result clarifying the relationship between the Bessel and Riesz basis properties (Proposition 1.1) and an interesting characterization of the Bessel property (Proposition 1.2).

Some Preliminary Results

We have the following easy result regarding Riesz bases, which is needed in the proof of Proposition 1.1:

Lemma 1.2. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$, and that $0 < C \le D < \infty$.*

(i) If

$$\Big\| \sum_{i \in F} \alpha_i f_i \Big\|^2 \le D \sum_{i \in F} |\alpha_i|^2,$$

for all finite subsets F of I and all $\{\alpha_i : i \in I\} \subset \mathbb{C}$, then for each $\{\alpha_i\}_{i \in I} \in l^2(I)$ the sum $\sum_{i \in I} \alpha_i f_i$ converges unconditionally and we have

$$\Big\| \sum_{i \in I} \alpha_i f_i \Big\|^2 \le D \sum_{i \in I} |\alpha_i|^2.$$

(ii) If $\{f_i : i \in I\}$ is a Riesz basis for $\mathcal{H}$ with constants $C \le D$, then for each $\{\alpha_i\}_{i \in I} \in l^2(I)$ the sum $\sum_{i \in I} \alpha_i f_i$ converges unconditionally and we have

$$C \sum_{i \in I} |\alpha_i|^2 \le \Big\| \sum_{i \in I} \alpha_i f_i \Big\|^2 \le D \sum_{i \in I} |\alpha_i|^2.$$

Moreover, for each $f \in \mathcal{H}$ there exists a unique sequence $\{\alpha_i\}_{i \in I} \in l^2(I)$ such that $f = \sum_{i \in I} \alpha_i f_i$.

We need the following two results in the proof of Proposition 1.2. The first is the Closed Graph Theorem (see, for instance, section 12 of chapter 3 of [1]). The second is a rather interesting general measure-theoretic result.

Theorem 1.1 (The Closed Graph Theorem). *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and that $\Lambda : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear transformation that satisfies the following condition:*

Whenever $x_n \to x$ in $\mathcal{X}$ and $\Lambda x_n \to y$ in $\mathcal{Y}$, then $\Lambda(x) = y$.

Then Λ is bounded.

Let (X, μ) be a measure space. Recall that X is said to be σ-finite if there exists a sequence $\{X_l\}_{l=1}^{\infty}$ of measurable subsets of X satisfying $\mu(X_l) < \infty$ (for all l) and $X = \cup_{l=1}^{\infty} X_l$. Suppose that $f : X \longrightarrow \mathbb{C}$ is a measurable function. For $1 \leq p < \infty$, we define

$$\|f\|_p = \begin{cases} \left(\int_X |f|^p\right)^{1/p}, & \text{if } \int_X |f|^p < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

and we define

$$\|f\|_\infty = \sup\{\alpha \geq 0 : \mu(\{x \in X : |f(x)| \geq \alpha\}) > 0\}.$$

When context makes the index p clear, we write $\|f\|$ in place of $\|f\|_p$. For $1 \leq p \leq \infty$, $L^p(X)$ will denote the Banach space of all measurable functions $f : X \longrightarrow \mathbb{C}$ satisfying $\|f\|_p < \infty$. We have the following result:

Lemma 1.3. *Let (X, μ) be a measure space and let p and q satisfy $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f : X \longrightarrow \mathbb{C}$ is a measurable function satisfying $\|f\|_p = \infty$.*

(i) If $p = 1$, then there exists a function $g \in L^q(X) = L^\infty(X)$ such that $f(x)\overline{g(x)} \geq 0$ (for almost every (a.e.) x) and $\int_X f\overline{g} = \infty$.

(ii) If X is σ-finite and if $1 < p \leq \infty$, then there exists $g \in L^q(X)$ such that $f(x)\overline{g(x)} \geq 0$ (for a.e. x) and $\int_X f\overline{g} = \infty$.

(iii) If the hypothesis "X is σ-finite" is omitted from the statement of (ii), then the assertion can fail for each $1 < p \leq \infty$.

Proof. If $p = 1$, it is easy to see that the function $g : X \longrightarrow \mathbb{C}$ defined by

$$g(x) = \begin{cases} \frac{f(x)}{|f(x)|}, & \text{if } f(x) \neq 0; \\ 0, & \text{if } f(x) = 0. \end{cases}$$

belongs to $L^\infty(X)$ and satisfies $f(x)\overline{g(x)} = |f(x)| \geq 0$ (a.e. x) and

$$\int_X f\overline{g} = \int_X |f| = \infty.$$

This proves (i).

To prove (ii), suppose that X is σ-finite and that $1 < p \leq \infty$. If $p = \infty$, making use of the σ-finite property of X, choose a sequence $\{X_l\}_{l=1}^\infty$ of disjoint measurable subsets of X satisfying

(i) $0 < \mu(X_l) < \infty$, for all l;

(ii) $|f(x)| \geq l^2$, for a.e. $x \in X_l$.

It is straightforward to verify that the function $g : X \longrightarrow \mathbb{C}$ defined by

$$g(x) = \begin{cases} \frac{1}{l^2}\frac{1}{\mu(X_l)}\frac{f(x)}{|f(x)|}, & \text{if } x \in X_l, \text{ for some } l; \\ 0, & \text{otherwise.} \end{cases}$$

belongs to $L^1(X)$ and satisfies the desired result.

Now suppose that $1 < p < \infty$. Making use of the σ-finite property of X, choose a sequence $\{X_l\}_{l=1}^{\infty}$ of disjoint measurable subsets of X such that for each l the following is satisfied:

(i) $\mu(X_l) < \infty$;

(ii) $f(x) \neq 0$, for a.e. $x \in X_l$;

(iii) f is bounded on X_l;

(iv) $\int_{X_l} |f|^p \geq 1$.

Define the function $g : X \longrightarrow \mathbb{C}$ by

$$g(x) = \begin{cases} \frac{1}{l}\frac{1}{\alpha_l} f(x)|f(x)|^{p-2}, & \text{if } x \in X_l, \text{ for some } l; \\ 0, & \text{otherwise.} \end{cases}$$

where

$$\alpha_l = \left(\int_{X_l} |f(x)|^{(p-1)q} \right)^{1/q} = \left(\int_{X_l} |f(x)|^p \right)^{1/q},$$

the latter equality holding since $(p-1)q = p$. Clearly, $f(x)\overline{g(x)} \geq 0$, for a.e.

x. First note that

$$\int_X |g(x)|^q = \sum_{l=1}^{\infty} \frac{1}{l^q} \frac{1}{\alpha_l^q} \int_{X_l} |f(x)|^{(p-1)q} = \sum_{l=1}^{\infty} \frac{1}{l^q} < \infty,$$

since $1 < q < \infty$, implying that $g \in L^q(X)$. Second, note that

$$\begin{aligned}
\int_X f(x)\overline{g(x)} &= \sum_{l=1}^{\infty} \frac{1}{l\alpha_l} \int_{X_l} |f(x)|^p \\
&= \sum_{l=1}^{\infty} \frac{1}{l} \left(\int_{X_l} |f(x)|^p \right)^{1-1/q} \\
&\geq \sum_{l=1}^{\infty} \frac{1}{l} = \infty,
\end{aligned}$$

proving part (ii).

We now prove part (iii). For $l \in \mathbb{Z}^+$, write $Y_l = \{0,1\}$ and consider the infinite product $Y = \prod_{l=1}^{\infty} Y_l$. For each l, let $\pi_l : Y \longrightarrow Y_l$ denote projection onto the l^{th} coordinate. Let $\mathcal{S}$ denote the collection of all subsets of Y. Consider the measure space $(Y, \mathcal{S}, \nu)$, where ν is defined for $E \in \mathcal{S}$ by $\nu(E) = \infty$ if $\pi_l x = 1$, for some $x \in E$ and some l and $\nu(E) = 0$ otherwise.

If $g : Y \longrightarrow \mathbb{C}$ is any function (note that g is necessarily measurable), it is easy to see that $\int_Y |g| < \infty$ if and only if $g(x) = 0$, for almost every x. Thus, for $q \in [1, \infty)$, it follows that the space $L^q(Y)$ consists solely of the zero function. Therefore, to prove (iii), it suffices to exhibit a function $f : Y \longrightarrow \mathbb{C}$ such that $\|f\|_p = \infty$, for all $1 < p \leq \infty$. This is simple to do:

For each l, let $x_l \in Y$ be the element satisfying

$$\pi_{l'} x_l = \begin{cases} 1, & \text{if } l' = l; \\ 0, & \text{otherwise.} \end{cases}$$

and consider the function $f : Y \longrightarrow \mathbb{C}$ defined by

$$f(x) = \begin{cases} l, & \text{if } x = x_l,\ l = 1, 2, 3, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f satisfies the desired property. $\square$

The Bessel and Riesz Basis Properties

If $\mathcal{H}$ is a Hilbert space and if $\{f_i : i \in I\}$ forms a frame for $\mathcal{H}$ with constants $C \leq D$, then $\{f_i : i \in I\}$ clearly is a Bessel system with constant D. Below is the analog of this result for Riesz bases.

Proposition 1.1. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$, and that $0 < D < \infty$. Then, $\{f_i : i \in I\}$ is Bessel with constant D if and only if*

$$\Big\| \sum_{i \in F} \alpha_i f_i \Big\|^2 \leq D \sum_{i \in F} |\alpha_i|^2, \tag{1.2}$$

for all finite subsets F of I and all $\{\alpha_i : i \in I\} \subset \mathbb{C}$.

Proof. If $\{f_i : i \in I\}$ is Bessel with constant D, then (1.3) shows that (1.2) is satisfied for all finite subsets F of I and all $\{\alpha_i : i \in I\} \subset \mathbb{C}$.

Suppose now that (1.2) is satisfied for all finite subsets F of I and all $\{\alpha_i : i \in I\} \subset \mathbb{C}$. Let $f \in \mathcal{H}$. Using part (i) of Lemma 1.2 and Proposition 1.2, we see that $\sum_{i\in I} |\langle f, f_i\rangle|^2 < \infty$. Using again part (i) of Lemma 1.2 we have

$$\begin{aligned}
\sum_{i\in I} |\langle f, f_i\rangle|^2 &= \sum_{i\in I} \langle f, f_i\rangle \overline{\langle f, f_i\rangle} \\
&= \langle f, \sum_{i\in I} \langle f, f_i\rangle f_i\rangle \\
&\le \|f\| \Big\| \sum_{i\in I} \langle f, f_i\rangle f_i \Big\| \\
&\le \|f\| D^{1/2} \Big(\sum_{i\in I} |\langle f, f_i\rangle|^2 \Big)^{1/2},
\end{aligned}$$

implying that

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \le D\|f\|^2.$$

This verifies that $\{f_i : i \in I\}$ is Bessel with constant D and thus completes the proof. □

A Characterization of the Bessel Property

We have the following interesting characterization of the Bessel property:

Proposition 1.2. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, and that $\{f_i : i \in I\} \subset \mathcal{H}$. Then the following are equivalent:*

(i) $\{f_i : i \in I\}$ is a Bessel system;

(ii) $\sum_{i\in I}\alpha_i f_i$ *converges unconditionally, for each* $\{\alpha_i\}_{i\in I}\in l^2(I)$*;*

(iii) $\sum_{i\in I}|\langle f, f_i\rangle|^2<\infty$*, for all* $f\in\mathcal{H}$*.*

We note that the equivalence of (i) and (ii) in the above proposition provide the following qualitative characterization of the Bessel property:

The Bessel property is the weakest condition that can be imposed on a countable collection $\{f_i : i\in I\}$ in a Hilbert space $\mathcal{H}$ which ensures that $\sum_{i\in I}\alpha_i f_i$ converges unconditionally, for each $\{\alpha_i\}_{i\in I}\in l^2(I)$.

We now prove Proposition 1.2.

Proof of Proposition 1.2. Suppose first that $\{f_i : i\in I\}$ is a Bessel system with constant D and that $\{\alpha_i\}_{i\in I}\in l^2(I)$. If F is any finite subset of I, we use the Schwarz inequality and obtain

$$\begin{aligned}\Big\|\sum_{i\in F}\alpha_i f_i\Big\| &= \sup_{\|f\|\le 1}\Big|\langle\sum_{i\in F}\alpha_i f_i, f\rangle\Big| \\ &\le \sup_{\|f\|\le 1}\sum_{i\in F}|\alpha_i||\langle f_i, f\rangle| \\ &\le \sup_{\|f\|\le 1}\Big(\sum_{i\in F}|\alpha_i|^2\Big)^{1/2}\Big(\sum_{i\in F}|\langle f_i, f\rangle|^2\Big)^{1/2} \qquad (1.3) \\ &\le \sup_{\|f\|\le 1}\Big(\sum_{i\in F}|\alpha_i|^2\Big)^{1/2}D^{1/2}\|f\| \\ &\le D^{1/2}\Big(\sum_{i\in F}|\alpha_i|^2\Big)^{1/2},\end{aligned}$$

from which it follows easily that $\sum_{i\in I}\alpha_i f_i$ converges unconditionally, verifying that property (i) implies property (ii).

Suppose next that property (ii) holds and let $f \in \mathcal{H}$. For any $\{\alpha_i\}_{i\in I} \in l^2(I)$, using that $\sum_{i\in I} \alpha_i f_i$ converges unconditionally, that $g \mapsto \langle g, f\rangle$ is a continuous linear functional on $\mathcal{H}$, and hence that

$$\langle \sum_{i\in I} \alpha_i f_i, f\rangle = \sum_{i\in I} \alpha_i \langle f_i, f\rangle,$$

it follows that $\sum_{i\in I} \alpha_i \langle f_i, f\rangle$ converges unconditionally. By the contrapositive of part (ii) of Lemma 1.3, it follows that $\sum_{i\in I} |\langle f, f_i\rangle|^2 < \infty$. This verifies that property (ii) implies property (iii).

Finally, suppose that property (iii) holds. We may then define the linear map $\Lambda : \mathcal{H} \longrightarrow l^2(I)$ by $\Lambda(f) = \{\langle f, f_i\rangle\}_{i\in I}$. It follows immediately from the Closed Graph Theorem that Λ is bounded. We thus have

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 = \|\Lambda f\|^2 \leq \|\Lambda\|^2 \|f\|^2,$$

implying that $\{f_i : i \in I\}$ is Bessel with constant $\|\Lambda\|^2$. This verifies that property (iii) implies property (i) and thus completes the proof. □

1.2.2 The Frame Property

In this subsection, we introduce the dual frame concept and prove an interesting result regarding the frame property (Proposition 1.3). As a corollary of these, we obtain an interesting qualitative characterization of the frame property.

Some Necessary Tools

We will need the following two results in the proof of Proposition 1.3. The first is the Open Mapping Theorem (see, for instance, section 12 of chapter 3 of [1]) and the second is an easy corollary thereof.

Theorem 1.2 (The Open Mapping Theorem). *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and if $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is surjective, then Λ is open.*

Lemma 1.4. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and that $\Lambda : \mathcal{X} \longrightarrow \mathcal{Y}$ is linear and open. Then there exists $C = C(\Lambda) > 0$ such that for each $y \in \mathcal{Y}$ we can find $x \in \mathcal{X}$ satisfying*

$$C\|x\|^2 \leq \|y\|^2 \quad \textit{and} \quad \Lambda x = y.$$

Dual Frames

We now introduce the dual frame concept. Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, and that $\{f_i : i \in I\}$ forms a frame for $\mathcal{H}$ with constants $C \leq D$. A Bessel system $\{g_i : i \in I\} \subset \mathcal{H}$ is said to be a dual frame to $\{f_i : i \in I\}$ if, for all $f \in \mathcal{H}$, we have

$$f = \sum_{i \in I} \langle f, g_i \rangle f_i.$$

Note that, by Proposition 1.2, the convergence of the above sum is necessarily unconditional. We make the following comments regarding dual frames:

(i) If $\{g_i : i \in I\} \subset \mathcal{H}$ is a dual frame to $\{f_i : i \in I\}$, then $\{g_i : i \in I\}$ also forms a frame for $\mathcal{H}$ with lower frame constant D^{-1}. The upper frame constant of $\{g_i : i \in I\}$ must be $\geq C^{-1}$ (provided C is optimal) but need not equal C^{-1}.

(ii) If $\{g_i : i \in I\} \subset \mathcal{H}$ is a dual frame to $\{f_i : i \in I\}$, then $\{f_i : i \in I\}$ is a dual frame to $\{g_i : i \in I\}$.

(iii) The Bessel assumption in the dual frame defintion is not redundant. More precisely, for every Hilbert space $\mathbb{H}$, we can find a frame $\{g_i : i \in I\}$ for $\mathbb{H}$, and a non-Bessel collection $\{h_i : i \in I\} \subset \mathcal{H}$ such that, for each $f \in \mathbb{H}$, we have

$$f = \sum_{i \in I} \langle f, h_i \rangle g_i,$$

where the above sum converges unconditionally.

Dual frames, in fact, always exist. We sketch the canonical construction; for the omitted details, see section 1 of chapter 8 of [6]. Consider $S \in \mathcal{B}(\mathcal{H})$ defined by

$$Sf = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

It can be shown that S is invertible. For each $i \in I$, define $\widetilde{f_i} = S^{-1} f_i$. The collection $\{\widetilde{f_i} : i \in I\}$ enjoys the following properties:

(i) $\{\widetilde{f_i} : i \in I\}$ is a dual frame to $\{f_i : i \in I\}$;

(ii) $\{\widetilde{f_i} : i \in I\}$ forms a frame for $\mathcal{H}$ with constants $C^{-1} \leq D^{-1}$;

(iii) if $\{f_i : i \in I\}$ forms a Parseval frame for $\mathcal{H}$, then $\widetilde{f_i} = f_i$, for all i.

To distinguish $\{\widetilde{f_i} : i \in I\}$ from other dual frames of $\{f_i : i \in I\}$, we call the former the canonical dual frame of $\{f_i : i \in I\}$.

A Characterization of the Frame Property

We have the following result:

Proposition 1.3. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$ is Bessel with constant D, and that $C > 0$. If for each $f \in \mathcal{H}$ we can find a sequence $\{\alpha_i\}_{i \in I} \in l^2(I)$ satisfying*

$$f = \sum_{i \in I} \alpha_i f_i,$$

then $\{f_i : i \in I\}$ forms a frame for $\mathcal{H}$. Moreover, if the sequence $\{\alpha_i\}_{i \in I}$ can always be chosen to satisfy

$$C \sum_{i \in I} |\alpha_i|^2 \leq \|f\|^2,$$

then $\{f_i : i \in I\}$ has frame constants $C \leq D$.

We note that the above discussion on dual frames and Propositions 1.2 and 1.3 imply the following qualitative characterization of the frame property:

The frame property is the weakest condition that can be imposed on a countable collection $\{f_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ which ensures that

$$\sum_{i \in I} \alpha_i f_i \tag{1.4}$$

converges unconditionally in $\mathcal{H}$, for all $\{\alpha_i\}_{i\in I} \in l^2(I)$ and that each $f \in \mathcal{H}$ can be written in the form (1.4), for some $\{\alpha_i\}_{i\in I} \in l^2(I)$.

We now prove Proposition 1.3.

Proof of Proposition 1.3. Using Proposition 1.2 and that $\{f_i : i \in I\}$ is Bessel, we may define the linear map $\Lambda : l^2(I) \longrightarrow \mathcal{H}$ by

$$\Lambda\{\alpha_i\}_{i\in I} = \sum_{i\in I} \alpha_i f_i.$$

By assumption, Λ is onto, and it follows from part (i) of Lemma 1.2 and Proposition 1.1 that Λ is bounded. Applying the Open Mapping Theorem and Lemma 1.4, we see that the following condition is satisfied:

There exists $C > 0$ such that for every $f \in \mathcal{H}$ we can find a sequence $\{\alpha_i\}_{i\in I} \in l^2(I)$ satisfying

$$C \sum_{i\in I} |\alpha_i|^2 \leq \|f\|^2 \quad \text{and} \quad f = \sum_{i\in I} \alpha_i f_i.$$

We now show that this condition implies that $\{f_i : i \in I\}$ forms a frame for $\mathcal{H}$ with frame constants $C \leq D$. Let $f \in \mathcal{H}$ be nonzero and write $f = \sum_{i\in I} \alpha_i f_i$ where $\{\alpha_i\}_{i\in I} \in l^2(I)$ satisfies $C \sum_{i\in I} |\alpha_i|^2 \leq \|f\|^2$. By the

Schwarz inequality, we have

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \geq \Big|\sum_{i\in I} \langle f, f_i\rangle \overline{\beta_i}\Big|,$$

for all $\{\beta_i\}_{i\in I} \in l^2(I)$ satisfying $\sum_{i\in I} |\beta_i|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2$. By substituting $\{\beta_i\}_{i\in I} \in l^2(I)$ defined by

$$\beta_i = \frac{(\sum_{j\in I} |\langle f, f_j\rangle|^2)^{1/2}}{(\sum_{j\in I} |\alpha_j|^2)^{1/2}} \alpha_i$$

into this inequality and using Proposition 1.2, we obtain

$$\begin{aligned}
\sum_{i\in I} |\langle f, f_i\rangle|^2 \geq \sum_{i\in I} \langle f, f_i\rangle \overline{\beta_i} &= \frac{(\sum_{j\in I} |\langle f, f_j\rangle|^2)^{1/2}}{(\sum_{j\in I} |\alpha_j|^2)^{1/2}} \sum_{i\in I} \langle f, f_i\rangle \overline{\alpha_i} \\
&= \frac{(\sum_{j\in I} |\langle f, f_j\rangle|^2)^{1/2}}{(\sum_{j\in I} |\alpha_j|^2)^{1/2}} \langle f, \sum_{i\in I} \alpha_i f_i\rangle \\
&= \frac{(\sum_{j\in I} |\langle f, f_j\rangle|^2)^{1/2}}{(\sum_{j\in I} |\alpha_j|^2)^{1/2}} \|f\|^2 \\
&\geq \frac{(\sum_{j\in I} |\langle f, f_j\rangle|^2)^{1/2}}{(\frac{1}{C}\|f\|^2)^{1/2}} \|f\|^2 \\
&= C^{1/2} \Big(\sum_{i\in I} |\langle f, f_i\rangle|^2\Big)^{1/2} \|f\|.
\end{aligned}$$

Since $f \neq 0$ and since the linear span of $\{f_i : i \in I\}$ is dense in $\mathcal{H}$, it follows that $\sum_{i\in I} |\langle f, f_i\rangle|^2 \neq 0$. We thus conclude that

$$C\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

Since $\{f_i : i \in I\}$ clearly has upper frame constant D, this completes the proof. $\square$

1.2.3 The Riesz Basis Property

In this subsection, we characterize the Riesz basis property (Proposition 1.4). We require the following definition in the formulation of Proposition 1.4:

Definition 1.3. *Suppose that $\mathcal{H}$ is a Hilbert space and that I is a countable indexing set. A Bessel system $\{f_i : i \in I\} \subset \mathcal{H}$ is said to be l^2-linearly independent if the following condition holds:*

If $\{\alpha_i\}_{i \in I} \in l^2(I)$ and if $\sum_{i \in I} \alpha_i f_i = 0$, then $\alpha_i = 0$, for all i.

We have the following result:

Proposition 1.4. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$, and that $0 < C \leq D < \infty$. Then $\{f_i : i \in I\}$ is a Riesz basis for $\mathcal{H}$ with constants $C \leq D$ if and only if it is a frame for $\mathcal{H}$ with constants $C \leq D$ and is l^2-linearly independent.*

We note that Proposition 1.4 and the qualitative characterization of the frame property (following the statement of Proposition 1.3) imply the following qualitative characterization of the Riesz basis property:

The Riesz basis property is the weakest condition that can be imposed on a countable collection $\{f_i : i \in I\}$ in a Hilbert space $\mathcal{H}$ which ensures that

$$\sum_{i \in I} \alpha_i f_i \tag{1.5}$$

converges unconditionally in $\mathcal{H}$, for all $\{\alpha_i\}_{i \in I} \in l^2(I)$ and that each $f \in \mathcal{H}$ can be written in the form (1.5), for a unique $\{\alpha_i\}_{i \in I} \in l^2(I)$.

We now prove Proposition 1.4.

Proof of Proposition 1.4. Suppose that $\{f_i : i \in I\}$ is a Riesz basis for $\mathcal{H}$ with constants $C \leq D$. It then follows from Proposition 1.1 that $\{f_i : i \in I\}$ is Bessel with constant D. Thus, part (ii) of Lemma 1.2 and Proposition 1.3 together imply that $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with constants $C \leq D$. Moreover, using again part (ii) of Lemma 1.2, we see that $\{f_i : i \in I\}$ must be l^2-linearly independent.

Conversely, suppose that $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with constants $C \leq D$ and is l^2-linearly independent. It is clear that the linear span of $\{f_i : i \in I\}$ is dense in $\mathcal{H}$. For, if $f \in \mathcal{H}$ is orthogonal to the collection $\{f_i : i \in I\}$, then it follows that

$$\|f\|^2 \leq \frac{1}{C} \sum_{i \in I} |\langle f, f_i \rangle|^2 = 0,$$

implying that $f = 0$. Let $\{\widetilde{f}_i : i \in I\}$ be the canonical dual frame to $\{f_i : i \in I\}$ (see the discussion immediately following the proof of Propositon 1.1). Consider $\{\alpha_i\}_{i \in I} \in l^2(I)$ and $f = \sum_{i \in I} \alpha_i f_i \in \mathcal{H}$. Using the dual frame property, we may also write

$$f = \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i.$$

The l^2-linear independence of $\{f_i : i \in I\}$ implies that $\langle f, \widetilde{f}_i \rangle = \alpha_i$, for all i. Using that $\{\widetilde{f}_i : i \in I\}$ has frame constants $\frac{1}{D} \le \frac{1}{C}$, we obtain

$$\sum_{i \in I} |\alpha_i|^2 = \sum_{i \in I} |\langle f, \widetilde{f}_i \rangle|^2 \le \frac{1}{C} \|f\|^2 = \frac{1}{C} \Big\| \sum_{i \in I} \alpha_i f_i \Big\|^2$$

and

$$\sum_{i \in I} |\alpha_i|^2 = \sum_{i \in I} |\langle f, \widetilde{f}_i \rangle|^2 \ge \frac{1}{D} \|f\|^2 = \frac{1}{D} \Big\| \sum_{i \in I} \alpha_i f_i \Big\|^2,$$

verifying that $\{f_i : i \in I\}$ is a Riesz basis for $\mathcal{H}$ with constants $C \le D$. This completes the proof. □

1.3 New Reproducing Systems from Old

In this section, we offer several results regarding the circumstances under which, first, two reproducing systems of a certain type can be combined to form a reproducing system of the same type, and, second, a subcollection of a reproducing system of a certain type forms a reproducing system of the same type.

1.3.1 Combining Reproducing Systems

Suppose that $\mathcal{H}$ is a Hilbert space, that I and J are countable indexing sets, and that $\{f_i : i \in I\}$ and $\{g_j : j \in I\}$ are subsets of $\mathcal{H}$. The following two statements are quite clear:

(i) If $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are Bessel systems with respective constants D_1 and D_2, then $\{f_i, g_j : i \in I, j \in J\}$ is a Bessel system with constant $D_1 + D_2$.

(ii) Assume that $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ form ON bases for their respective closed spans V and W. Then $\{f_i, g_j : i \in I, j \in J\}$ forms an ON basis for its closed span if and only if $V \perp W$.

The corresponding statements for frames (Theorem 1.4) and Riesz bases (Corollary 1.1), which we prove in this subsection, turn out to be very interesting and suprisingly somewhat difficult.

Some Preliminary Results

We need the following two results in the proof of Theorem 1.4. The first, the Inverse Mapping Theorem, is an easy corollary of the Open Mapping Theorem. The second is a result on weak convergence in Hilbert spaces. For its proof, see section 4 of chapter 5 of [1].

Theorem 1.3 (The Inverse Mapping Theorem)**.** *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and if $\Lambda \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is bijective, then $\Lambda^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.*

Lemma 1.5. *Suppose that $\mathcal{H}$ is a Hilbert space and that $\{x_p\}_{p=1}^{\infty}$ is a sequence in $\mathcal{H}$. If the sequence $\{\langle x_p, y\rangle\}_{p=1}^{\infty}$ converges for each $y \in \mathcal{H}$, then there exists $x \in \mathcal{H}$ such that $\langle x_p, y\rangle \to \langle x, y\rangle$, for each $y \in \mathcal{H}$.*

Combining Frames

We have the following result regarding the circumstances under which two frames can be combined to form a new frame:

Theorem 1.4. *Suppose that $\mathcal{H}$ is a Hilbert space and that V and W are closed subspaces of $\mathcal{H}$ for which $V + W$ is dense in $\mathcal{H}$. Then the following properties are equivalent:*

(i) *$V + W$ is closed; that is, $V + W = \mathcal{H}$.*

(ii) *There is some $\epsilon = \epsilon(V, W) > 0$ such that for all $x \in \mathcal{H}$ we have*

$$\epsilon\|x\|^2 \leq \|P_V x\|^2 + \|P_W x\|^2,$$

where P_V and P_W denote the orthogonal projections of $\mathcal{H}$ onto V and W, respectively.

(iii) *Let I and J be countable indexing sets. Whenever the collections $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ form frames for V and W, respectively, then the collection $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for $\mathcal{H}$.*

(iv) *Let I and J be countable indexing sets. There exists a frame $\{f_i : i \in I\}$*

for V and a frame $\{g_j : i \in J\}$ for W such that the collection $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for $\mathcal{H}$.

Proof. We first show that (i) implies (ii). Suppose that (i) holds. Define addition and scalar multiplication on $V \times W$ componentwise, and for $(v,w) \in V \times W$, define $\|(v,w)\| = \sqrt{\|v\|^2 + \|w\|^2}$. It is straightforward to verify that, with these operations, $V \times W$ is a Banach space.

Consider the subspace Z of $V \times W$ given by $Z = \{(P_V x, P_W x) : x \in \mathcal{H}\}$. We claim that Z is closed. To see this, let $\{(P_V x_p, P_W x_p)\}_{p=1}^{\infty}$ be a sequence in Z that converges to some point $(y_1, y_2) \in V \times W$. By assumption, if $y \in \mathcal{H}$, we may write $y = v + w$, for some $v \in V$ and $w \in W$. We then have

$$\begin{aligned}
\langle x_p, y \rangle &= \langle x_p, v \rangle + \langle x_p, w \rangle \\
&= \langle P_V x_p, v \rangle + \langle P_W x_p, w \rangle \\
&\to \langle y_1, v \rangle + \langle y_2, w \rangle,
\end{aligned}$$

as $p \to \infty$. By Lemma 1.5, there is some $x \in \mathcal{H}$ such that $\langle x_p, y \rangle \to \langle x, y \rangle$, for each $y \in \mathcal{H}$. Using that P_V is self-adjoint, for any $y \in \mathcal{H}$ we have

$$\langle y_1, y \rangle = \lim_{p \to \infty} \langle P_V x_p, y \rangle = \lim_{p \to \infty} \langle x_p, P_V y \rangle = \langle x, P_V y \rangle = \langle P_V x, y \rangle,$$

from which it follows that $P_V x = y_1$. A similar argument shows that $P_W x = y_2$. It follows that Z is closed and thus a Banach space in its own right.

Consider the map $\Lambda : \mathcal{H} \longrightarrow Z$ defined by $\Lambda x = (P_V x, P_W x)$, which is easily seen to be a surjective element of $\mathcal{B}(\mathcal{H}, Z)$. For $x \in \mathcal{H}$, write $x = v+w$, where $v \in V$ and $w \in W$. If $\Lambda x = 0$, then x is orthogonal to both V and W and we find

$$\|x\|^2 = \langle x, x\rangle = \langle x, v\rangle + \langle x, w\rangle = 0 + 0 = 0,$$

implying that $x = 0$ and hence that Λ is injective. It thus follows from Theorem 1.3 that Λ^{-1} is bounded. Therefore, for every $x \in \mathcal{H}$, we have

$$\|x\| = \|\Lambda^{-1}\Lambda x\| \leq \|\Lambda^{-1}\|\|\Lambda x\| = \|\Lambda^{-1}\|\sqrt{\|P_V x\|^2 + \|P_W x\|^2},$$

which verifies that (ii) is satisfied with $\epsilon = \|\Lambda^{-1}\|^{-2}$.

We next show that (ii) implies (iii). To do this, suppose (ii) holds for some $\epsilon > 0$ and let I and J be countable indexing sets. Assume that $\{f_i : i \in I\}$ and $\{g_j : i \in J\}$ form frames for V and W, respectively, with respective frame constants $C_1 \leq D_1$ and $C_2 \leq D_2$. Write $C = \min\{C_1, C_2\}$ and let $f \in \mathcal{H}$. We then have

$$\begin{aligned}\sum_{i\in I} |\langle f, f_i\rangle|^2 + \sum_{j\in J} |\langle f, g_i\rangle|^2 &= \sum_{i\in I} |\langle P_V f, f_i\rangle|^2 + \sum_{j\in J} |\langle P_W f, g_i\rangle|^2 \\ &\geq C_1\|P_V f\|^2 + C_2\|P_W f\|^2 \\ &\geq C(\|P_V f\|^2 + \|P_W f\|^2) \\ &\geq C\epsilon\|f\|^2.\end{aligned}$$

Since $\{f_i, g_j : i \in I, j \in J\}$ is clearly Bessel with constant $D_1 + D_2$, it follows that $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for $\mathcal{H}$.

To see that property (iii) implies property (iv), simply choose ON bases (and, in particular, frames) $\{f_i : i \in I\}$ and $\{g_j : i \in J\}$ for V and W, respectively. If property (iii) holds, then the collection $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for $\mathcal{H}$.

Lastly, we show that (iv) implies (i). Suppose that (iv) holds. That is, let I and J be countable indexing sets and assume that $\{f_i : i \in I\}$ is a frame for V, that $\{g_j : i \in J\}$ is a frame for W, and that $\{f_i, g_j : i \in I, j \in J\}$ is a frame for $\mathcal{H}$. If $f \in \mathcal{H}$, using part (ii) of Lemma 1.1 and the dual frame to $\{f_i, g_j : i \in I, j \in J\}$, it follows that we can find two sequences $\{\alpha_i\}_{i\in I} \in l^2(I)$ and $\{\beta_j\}_{j\in J} \in l^2(J)$ such that

$$f = \sum_{i\in I} \alpha_i f_i + \sum_{j\in J} \beta_j g_j,$$

where each sum converges unconditionally, implying that $f \in V + W$. This shows that (i) is satisfied and thus completes the proof. $\square$

Combining Riesz Bases

We have the following corollary of Theorem 1.4 which reveals the circumstances under which two Riesz bases can be combined to form a new Riesz basis:

Corollary 1.1. *Suppose that $\mathcal{H}$ is a Hilbert space, that V and W are closed subspaces of $\mathcal{H}$, and that I and J are countable indexing sets. Assume that*

$\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ form Riesz bases for V and W, respectively. Then the collection

$$\{f_i, g_j : i \in I, j \in J\}$$

forms a Riesz basis for its closed span if and only if $V + W$ is closed and $V \cap W = \{0\}$.

Proof. First note that, by Proposition 1.4, both $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ form frames for V and W, respectively, and both $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are l^2-linearly independent.

Now, suppose that $\{f_i, g_j : i \in I, j \in J\}$ forms a Riesz basis for its closed span. Then, by Proposition 1.4, $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for its closed span and is l^2-linearly independent. Thus, by Theorem 1.4 and the discussion in the preceding paragraph, it follows that $V + W$ is closed. If $f \in V \cap W$, we can find sequences $\{\alpha_i\}_{i \in I} \in l^2(I)$ and $\{\beta_j\}_{j \in J} \in l^2(J)$ for which

$$f = \sum_{i \in I} \alpha_i f_i = \sum_{j \in J} \beta_j g_j,$$

implying that

$$\sum_{i \in I} \alpha_i f_i - \sum_{j \in J} \beta_j g_j = 0.$$

Since $\{f_i, g_j : i \in I, j \in J\}$ is l^2-linearly independent, it follows that $\alpha_i = \beta_j = 0$, for all i and j. In particular, $f = 0$; this shows that $V \cap W = 0$.

Conversely, suppose that $V + W$ is closed and that $V \cap W = \{0\}$. In conjunction with Theorem 1.4 and the first paragraph of this proof, the former implies that $\{f_i, g_j : i \in I, j \in J\}$ forms a frame for its closed span. Assume that $\{\alpha_i\}_{i\in I} \in l^2(I)$ and $\{\beta_j\}_{j\in J} \in l^2(J)$ satisfy

$$\sum_{i\in I} \alpha_i f_i + \sum_{j\in J} \beta_j g_j = 0.$$

Then,

$$\sum_{i\in I} \alpha_i f_i = -\sum_{j\in J} \beta_j g_j \in V \cap W,$$

implying (since $V \cap W = \{0\}$) that

$$\sum_{i\in I} \alpha_i f_i = -\sum_{j\in J} \beta_j g_j = 0.$$

Since both $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are l^2-linearly independent, this implies that $\alpha_i = \beta_j = 0$, for all i and j. It follows that $\{f_i, g_j : i \in I, j \in J\}$ is l^2-linearly independent. Therefore, by Proposition 1.4, $\{f_i, g_j : i \in I, j \in J\}$ forms a Riesz basis for its closed span. $\square$

An Example

In Section 2.3, we give an example of two functions φ, ψ in the Schwartz class of $\mathbb{R}$ that satisfy the following:

(i) $\{T_k\varphi : k \in \mathbb{Z}\}$ and $\{T_k\psi : k \in \mathbb{Z}\}$ both form ON bases for their respective closed spans V and W;

(ii) $V \cap W = \{0\}$;

(iii) $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ does not form a frame for its closed span.

In conjunction with Theorem 1.4 and Corollary 1.1, we see that $V+W$ cannot be closed.

1.3.2 Separating Reproducing Systems

Suppose that $\mathcal{H}$ is a Hilbert space, that I and J are countable indexing sets, and that $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are subsets of $\mathcal{H}$. The following assertions are all straightforward to verify:

(i) If $\{f_i, g_j : i \in I, j \in J\}$ is Bessel with constant D, then $\{f_i : i \in I\}$ is Bessel with constant D.

(ii) If $\{f_i, g_j : i \in I, j \in J\}$ is an ON basis for $\mathcal{H}$, then $\{f_i : i \in I\}$ forms an ON basis for its closed span.

(iii) If $\{f_i, g_j : i \in I, j \in J\}$ is a Riesz basis for $\mathcal{H}$ with constants $C \leq D$, then $\{f_i : i \in I\}$ is a Riesz basis for its closed span with constants $C \leq D$.

Below is an analogous statement for frames, which is not quite so clear.

Proposition 1.5. *Suppose that $\mathcal{H}$ is a Hilbert space, that I and J are countable indexing sets, and that $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$ are subsets of $\mathcal{H}$. Let V and W denote the closed linear spans of $\{f_i : i \in I\}$ and $\{g_j : j \in J\}$, respectively. If $\{f_i, g_j : i \in I, j \in J\}$ is a frame for $\mathcal{H}$ with constants $C \leq D$ and if $V \cap W = \{0\}$, then $\{f_i : i \in I\}$ is a frame for V with constants $C \leq D$.*

Proof of Proposition 1.5. Suppose that $\{f_i, g_j : i \in I, j \in J\}$ is a frame for $\mathcal{H}$ with constants $C \leq D$ and that $V \cap W = \{0\}$. Clearly, $\{f_i : i \in I\}$ is Bessel with constant D. Let $\{\widetilde{f}_i, \widetilde{g}_j : i \in I, j \in J\}$ be the canonical dual frame to $\{f_i, g_j : i \in I, j \in J\}$ and let $f \in V$. Then we may write

$$f = \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i + \sum_{j \in J} \langle f, \widetilde{g}_j \rangle g_j,$$

or

$$f - \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i = \sum_{j \in J} \langle f, \widetilde{g}_j \rangle g_j,$$

implying that $f - \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i \in V \cap W$ and hence that $f = \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i$. Moreover, since $\{\widetilde{f}_i, \widetilde{g}_j : i \in I, j \in J\}$ has upper frame constant $\frac{1}{C}$, we have that

$$\sum_{i \in I} |\langle f, \widetilde{f}_i \rangle|^2 \leq \sum_{i \in I} |\langle f, \widetilde{f}_i \rangle|^2 + \sum_{i \in J} |\langle f, \widetilde{g}_j \rangle|^2 \leq \frac{1}{C} \|f\|^2.$$

It thus follows from Proposition 1.3 that $\{f_i : i \in I\}$ is a frame for V with constants $C \leq D$. $\square$

An Example

In Section 2.3, we give an example of two functions φ and ψ in the Schwartz class of $\mathbb{R}$ which satisfy the following:

(i) $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ forms a Parseval frame for its closed span;

(ii) Neither $\{T_k\varphi : k \in \mathbb{Z}\}$ nor $\{T_k\psi : k \in \mathbb{Z}\}$ forms a frame for its closed span;

(iii) Denoting the closed spans of $\{T_k\varphi : k \in \mathbb{Z}\}$ and $\{T_k\psi : k \in \mathbb{Z}\}$ by V and W, respectively, we have $V \cap W^\perp \neq \{0\} \neq W \cap V^\perp$.

In particular, we see that the assumption that $V \cap W = \{0\}$ (or, at least, some additional assumption) in Proposition 1.5 is necessary.

Chapter 2

Shift Invariant Spaces

A shift invariant (SI) space is a closed subspace V of $L^2(\mathbb{R}^n)$ that satisfies $T_k V \subset V$, for all $k \in \mathbb{Z}^n$. Such spaces play an important role in many areas of mathematics, particularly in the theory and applications of wavelets. In section 1, we introduce the bracket product, an extremely useful tool in the study of SI spaces. Clearly, a very natural choice of reproducing system for an SI space V is one of the form

$$\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}, \tag{2.1}$$

where $\{\varphi_i : i \in I\} \subset V$ and I is a countable indexing set. If $|I| = 1$ ($|I|$ denotes the cardinality of I), we say that (2.1) is singly generated; otherwise, we say that (2.1) is multiply generated. When $|I| = 1$, it is well-known that essentially every reproducing property of (2.1) is characterized relatively simply in terms of the bracket product. In section 2, we show that several reproducing properties of (2.1) (for any countable indexing set I) are characterized in terms of operator inequalities involving matrices of bracket products. These are the main results of this chapter. In section 3, we use the above mentioned characterizations to give two interesting examples of SI spaces.

2.1 The Bracket Product

As mentioned above, a very useful concept in studying SI spaces is the bracket product. In this section, we define the bracket product and prove several elementary results.

Let $f, g \in L^2(\hat{\mathbb{R}}^n)$. The bracket product of f and g, denoted by $[f, g]$, is defined as follows: For any cube C of side length 1 we have

$$\begin{aligned}
\int_C \sum_{k \in \hat{\mathbb{Z}}^n} |f(\xi + k)|^2 &= \sum_{k \in \hat{\mathbb{Z}}^n} \int_C |f(\xi + k)|^2 \\
&= \sum_{k \in \hat{\mathbb{Z}}^n} \int_{C+k} |f(\xi)|^2 \\
&= \int_{\hat{\mathbb{R}}^n} |f(\xi)|^2 \\
&< \infty,
\end{aligned}$$

implying, in particular, that

$$\sum_{k\in\hat{\mathbb{Z}}^n} |f(\xi+k)|^2 < \infty,$$

for a.e. ξ. Using the Schwarz inequality and the result of the above computation, for a.e. ξ we have

$$\sum_{k\in\hat{\mathbb{Z}}^n} |f(\xi+k)g(\xi+k)| \leq \Big(\sum_{k\in\hat{\mathbb{Z}}^n} |f(\xi+k)|^2\Big)^{1/2}\Big(\sum_{k\in\hat{\mathbb{Z}}^n} |g(\xi+k)|^2\Big)^{1/2} < \infty,$$

implying that $\sum_{k\in\hat{\mathbb{Z}}^n} f(\xi+k)\overline{g(\xi+k)}$ converges absolutely. For such ξ, denote its value by $[f,g](\xi)$, thereby defining the function $[f,g]$ a.e. From the above computations, it is clear that $[f,g] \in L^1(\mathbb{T}^n)$. We have the following basic result:

Lemma 2.1. *Let $f, g, h \in L^2(\mathbb{R}^n)$ and let $\{\varphi_i : i \in I\} \subset L^2(\mathbb{R}^n)$, where I is a countable indexing set. Denote the closed linear span of the collection $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ by V.*

(i) If $\{\alpha_k\}_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$ and if $\{T_kf : k \in \mathbb{Z}^n\}$ is Bessel, then

$$\Big(\sum_{k\in\mathbb{Z}^n} \alpha_k T_k f\Big)^{\widehat{}} = m\hat{f},$$

where $m \in L^2(\hat{\mathbb{T}}^n)$ is given by $m(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_k e^{-2\pi\imath\xi k}$.

(ii) If the collections $\{T_kf : k \in \mathbb{Z}^n\}$ and $\{T_kg : k \in \mathbb{Z}^n\}$ are both Bessel,

then $[\hat{h}, \hat{f}] \in L^2(\hat{\mathbb{T}}^n)$ *and*

$$\Big(\sum_{k \in \mathbb{Z}^n} \langle h, T_k f \rangle T_k g \Big)^{\widehat{}} = [\hat{h}, \hat{f}]\hat{g}.$$

(iii) $\{T_k f : k \in \mathbb{Z}^n\} \perp \{T_k g : k \in \mathbb{Z}^n\}$ *if and only if* $[\hat{f}, \hat{g}](\xi) = 0$, *for a.e.* ξ.

(iv) If $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *is Bessel with constant* D, *then*

$$\sum_{i \in I} \big\| [\hat{f}, \widehat{\varphi}_i] \big\|^2 \leq D \|f\|^2.$$

(v) If $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *is a Bessel system and if* $\{m_i : i \in I\} \subset L^2(\hat{\mathbb{T}}^n)$ *satisfies* $\sum_{i \in I} \|m_i\|^2 < \infty$, *then* $\sum_{i \in I} m_i \widehat{\varphi}_i$ *converges unconditionally in* $\widehat{V}$.

(vi) Suppose $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *forms a frame for* V *with canonical dual frame* $\{\widetilde{T_k \varphi_i} : k \in \mathbb{Z}^n, i \in I\}$. *Then for each* $k \in \mathbb{Z}^n$ *and* $i \in I$ *we have* $\widetilde{T_k \varphi_i} = T_k \widetilde{\varphi}_i$.

(vii) Suppose $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *forms a frame for* V *with dual frame* $\{T_k \psi_i : k \in \mathbb{Z}^n, i \in I\}$ *and let* $f \in V$. *Then*

$$\hat{f} = \sum_{i \in I} [\hat{f}, \widehat{\psi}_i] \widehat{\varphi}_i,$$

with unconditional convergence in $\widehat{V}$.

Proof. To prove (i) suppose that the collection $\{T_k f : k \in \mathbb{Z}^n\}$ is Bessel and that $\{\alpha_k\}_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n)$. Define $m \in L^2(\mathbb{T}^n)$ by $m(\xi) = \sum_{k\in\mathbb{Z}^n} \alpha_k e^{-2\pi\imath\xi k}$. Using Proposition 1.2, we obtain

$$\begin{aligned}
\Big(\sum_{k\in\mathbb{Z}^n} \alpha_k T_k f\Big)\widehat{} &= \sum_{k\in\mathbb{Z}^n} \alpha_k \widehat{T_k f} \\
&= \sum_{k\in\mathbb{Z}^n} \alpha_k e^{-2\pi\imath\cdot k} \hat{f} \\
&= \Big(\sum_{k\in\mathbb{Z}^n} \alpha_k e^{-2\pi\imath\cdot k}\Big) \hat{f} \\
&= m\hat{f},
\end{aligned}$$

which proves (i).

To prove (ii), suppose that the collections $\{T_k f : k \in \mathbb{Z}^n\}$ and $\{T_k g : k \in \mathbb{Z}^n\}$ are both Bessel. We have

$$\begin{aligned}
\int_{[0,1]^n} [\hat{h}, \hat{f}](\xi) e^{2\pi\imath\xi k}\, d\xi &= \int_{[0,1]^n} \sum_{l\in\hat{\mathbb{Z}}^n} \hat{h}(\xi + l)\overline{\hat{f}(\xi + l)} e^{2\pi\imath\xi k}\, d\xi \\
&= \sum_{l\in\hat{\mathbb{Z}}^n} \int_{[0,1]^n} \hat{h}(\xi + l)\overline{\hat{f}(\xi + l)} e^{2\pi\imath\xi k}\, d\xi \\
&= \sum_{l\in\hat{\mathbb{Z}}^n} \int_{[0,1]^n + l} \hat{h}(\xi)\overline{\hat{f}(\xi)} e^{2\pi\imath\xi k}\, d\xi \\
&= \int_{\hat{\mathbb{R}}^n} \hat{h}(\xi)\overline{\hat{f}(\xi)} e^{2\pi\imath\xi k}\, d\xi \\
&= \int_{\hat{\mathbb{R}}^n} \hat{h}(\xi)\overline{\widehat{T_k f}(\xi)}\, d\xi \\
&= \langle \hat{h}, \widehat{T_k f} \rangle \\
&= \langle h, T_k f \rangle,
\end{aligned}$$

where the switch in summation in the second equality is easily verified. Thus, since $\sum_{k\in\mathbb{Z}^n} |\langle h, T_k f\rangle|^2 < \infty$, we have

$$[\hat{h}, \hat{f}] = \sum_{k\in\hat{\mathbb{Z}}^n} \langle h, T_k f\rangle e^{-2\pi\imath\cdot k} \in L^2(\hat{\mathbb{T}}^n);$$

(ii) now follows from (i).

To prove (iii), first note that $\{T_k f : k \in \mathbb{Z}^n\} \perp \{T_k g : k \in \mathbb{Z}^n\}$ if and only if $\langle f, T_k g\rangle = 0$, for all $k \in \mathbb{Z}^n$. By the calculation in the proof of part (ii) of this lemma, we know that

$$\int_{[0,1]^n} [\hat{f}, \hat{g}](\xi) e^{2\pi\imath\xi k}\, d\xi = \langle f, T_k g\rangle; \tag{2.2}$$

(iii) now follows.

To prove (iv), suppose that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is Bessel with constant D. It follows from (2.2) that

$$\sum_{i\in I} \left\|[\hat{f}, \widehat{\varphi}_i]\right\|^2 = \sum_{i\in I}\sum_{k\in\mathbb{Z}^n} |\langle f, T_k\varphi_i\rangle|^2 \le D\|f\|^2.$$

To prove (v), suppose that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a Bessel system and that $\{m_i : i \in I\} \subset L^2(\hat{\mathbb{T}}^n)$ satisfies $\sum_{i\in I} \|m_i\|^2 < \infty$. For each i, write $m_i(\xi) = \sum_{k\in\hat{\mathbb{Z}}^n} \alpha_k^i e^{-2\pi\imath\xi k}$. Since

$$\sum_{i\in I}\sum_{k\in\hat{\mathbb{Z}}^n} |\alpha_k^i|^2 = \sum_{i\in I} \|m_i\|^2 < \infty,$$

it follows that

$$\sum_{i\in I, k\in\hat{\mathbb{Z}}^n} \alpha_k^i T_k\varphi_i$$

converges unconditionally in V. Thus, using part (iv) of Lemma 1.1 and part (i) of this lemma, we have

$$\begin{aligned}\Big(\sum_{i\in I, k\in\hat{\mathbb{Z}}^n} \alpha_k^i T_k\varphi_i\Big)^{\widehat{}} &= \Big(\sum_{i\in I}\sum_{k\in\hat{\mathbb{Z}}^n} \alpha_k^i T_k\varphi_i\Big)^{\widehat{}} \\ &= \sum_{i\in I}\Big(\sum_{k\in\hat{\mathbb{Z}}^n} \alpha_k^i T_k\varphi_i\Big)^{\widehat{}} \\ &= \sum_{i\in I} m_i\widehat{\varphi}_i,\end{aligned}$$

from which (v) follows.

To prove (vi), suppose $\{T_k\varphi_i : k\in\mathbb{Z}^n, i\in I\}$ forms a frame for V with canonical dual frame $\{\widetilde{T_k\varphi_i} : k\in\mathbb{Z}^n, i\in I\}$; that is, with $S : V \longrightarrow V$ defined by

$$Sf = \sum_{i\in I, k\in\mathbb{Z}^n} \langle f, T_k\varphi_i\rangle T_k\varphi_i,$$

we have $\widetilde{T_k\varphi_i} = S^{-1}T_k\varphi_i$. For $l \in \mathbb{Z}^n$ and $f \in V$ we have

$$\begin{aligned}
T_lST_{-l}f &= T_l\Big(\sum_{i\in I,k\in\mathbb{Z}^n} \langle T_{-l}f, T_k\varphi_i\rangle T_k\varphi_i\Big) \\
&= T_l\Big(\sum_{i\in I,k\in\mathbb{Z}^n} \langle f, T_{l+k}\varphi_i\rangle T_k\varphi_i\Big) \\
&= \sum_{i\in I,k\in\mathbb{Z}^n} \langle f, T_{l+k}\varphi_i\rangle T_{l+k}\varphi_i \\
&= Sf,
\end{aligned}$$

from which it follows that for each $i \in I$ and $k \in \mathbb{Z}^n$ we have $\widetilde{T_k\varphi_i} = T_k\widetilde{\varphi}_i$.

To prove (vii), suppose $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a frame for V with dual frame $\{T_k\psi_i : k \in \mathbb{Z}^n, i \in I\}$ and let $f \in V$. Using the dual frame reconstruction property, of part (ii) of this lemma, and part (iv) of Lemma 1.1 we obtain

$$\begin{aligned}
\hat{f} &= \Big(\sum_{i\in I,k\in\mathbb{Z}^n} \langle f, T_k\psi_i\rangle T_k\varphi_i\Big)^{\widehat{}} \\
&= \Big(\sum_{i\in I}\sum_{k\in\mathbb{Z}^n} \langle f, T_k\psi_i\rangle T_k\varphi_i\Big)^{\widehat{}} \\
&= \sum_{i\in I}\Big(\sum_{k\in\mathbb{Z}^n} \langle f, T_k\psi_i\rangle T_k\varphi_i\Big)^{\widehat{}} \\
&= \sum_{i\in I} [\hat{f}, \widehat{\psi_i}]\widehat{\varphi}_i.
\end{aligned}$$

$\square$

2.2 Characterizations of Reproducing Systems

As mentioned above, essentially every reproducing property of the collection (2.1) (with $|I| = 1$) is characterized in terms of the bracket product. The following theorem (see [5]) lists four such characterizations:

Theorem 2.1. *Let $\varphi \in L^2(\mathbb{R}^n)$. Denote the closed span of the collection $\{T_k\varphi : k \in \mathbb{Z}^n\}$ by V.*

(i) $f \in V$ if and only if $\hat{f} = m\hat{\varphi}$, for some measurable function $m : [0,1]^n \longrightarrow \mathbb{C}$ satisfying

$$\int_{[0,1]^n} |m(\xi)|^2 [\hat{\varphi}, \hat{\varphi}](\xi)\, d\xi < \infty.$$

(ii) $\{T_k\varphi : k \in \mathbb{Z}^n\}$ forms a Bessel system with constant D if and only if $[\hat{\varphi}, \hat{\varphi}](\xi) \leq D$, for a.e. ξ.

(iii) $\{T_k\varphi : k \in \mathbb{Z}^n\}$ forms a frame for V with constants $C \leq D$ if and only if for a.e. ξ either

$$[\hat{\varphi}, \hat{\varphi}](\xi) = 0 \quad \text{or} \quad C \leq [\hat{\varphi}, \hat{\varphi}](\xi) \leq D.$$

(iv) $\{T_k\varphi : k \in \mathbb{Z}^n\}$ forms a Riesz basis for V with constants $C \leq D$ if and only if $C \leq [\hat{\varphi}, \hat{\varphi}](\xi) \leq D$, for a.e. ξ.

The three main results of this chapter (Theorems 2.4, 2.5, and 2.6) generalize parts (ii), (iii), and (iv) of Theorem 2.1 in a very interesting manner to collections of the form (2.1), where I is a general countable indexing set. We state and prove these results in subsection 2.2.2.

2.2.1 Some Necessary Tools

In this subsection, we develop the various tools necessary in the proofs of Theorems 2.4, 2.5, and 2.6.

A Density Result

The proof of the following lemma is very similar to the proof of Lemma 1.10 in chapter 7 of [6]. We include it here for the sake of completeness.

Lemma 2.2. *Suppose that $\mathcal{H}$ is a Hilbert space, that I is a countable indexing set, that $\{f_i : i \in I\} \subset \mathcal{H}$, and that $0 < C \leq D < \infty$.*

(i) If

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2, \tag{2.3}$$

for all f in some dense subset of $\mathcal{H}$, then $\{f_i : i \in I\}$ is Bessel with constant D.

(ii) If

$$C\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2, \tag{2.4}$$

for all f in some dense subset of $\mathcal{H}$, then $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with constants $C \leq D$.

Proof. To prove (i), suppose that inequality (2.3) is satisfied for all elements in some dense subset $\mathcal{D}$ of $\mathcal{H}$ and let $f \in \mathcal{H}$. Choose a sequence $\{g_p\}_{p=1}^{\infty}$ in $\mathcal{D}$ converging to f. Using the Bessel property within $\mathcal{D}$, for any finite subset F of I we obtain

$$\sum_{i \in F} |\langle f, f_i \rangle|^2 = \lim_{p \to \infty} \sum_{i \in F} |\langle g_p, f_i \rangle|^2 \leq \lim_{p \to \infty} D\|g_p\|^2 = D\|f\|^2,$$

from which it follows that $\sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2$. This proves (i).

To prove (ii), suppose that inequality (2.4) is satisfied for all elements in some dense subset $\mathcal{D}$ of $\mathcal{H}$. It follows from part (i) that $\{f_i : i \in I\}$ is Bessel with constant D. Let $f \in \mathcal{H}$ and let $\epsilon > 0$. Choose $g \in \mathcal{D}$ satisfying

$$\|g - f\| \leq \frac{C^{1/2}}{D^{1/2}} \epsilon \leq \epsilon.$$

Using the frame property within $\mathcal{D}$, the Bessel property within $\mathcal{H}$, and

Minkowski's inequality in $l^2(I)$, we obtain

$$\begin{aligned}
\|f\| - 2\epsilon &\leq \|g\| - \epsilon \\
&\leq \|g\| - \frac{D^{1/2}}{C^{1/2}}\|g - f\| \\
&\leq \frac{1}{C^{1/2}}\Big(\sum_{i\in I} |\langle g, f_i\rangle|^2\Big)^{1/2} - \frac{D^{1/2}}{C^{1/2}}\frac{1}{D^{1/2}}\Big(\sum_{i\in I} |\langle g - f, f_i\rangle|^2\Big)^{1/2} \\
&= \frac{1}{C^{1/2}}\Big(\sum_{i\in I} |\langle g, f_i\rangle|^2\Big)^{1/2} - \frac{1}{C^{1/2}}\Big(\sum_{i\in I} |\langle g - f, f_i\rangle|^2\Big)^{1/2} \\
&\leq \frac{1}{C^{1/2}}\Big(\sum_{i\in I} |\langle f, f_i\rangle|^2\Big)^{1/2},
\end{aligned}$$

implying that $C\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2$. Part (ii) now follows. $\square$

Positive Operators on a Hilbert Space

Theorems 2.4, 2.5, and 2.6 will be formulated in terms of operator inequalities. If $\mathcal{H}$ is a Hilbert space and if $S, T \in \mathcal{B}(\mathcal{H})$, we say that T is positive if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$ (note that some authors call such a T "positive semidefinite"). In this case, we write $0 \leq T$. We write $S \leq T$ if $T - S$ is positive.

We have the following two results. For a proof of the first, see chapter 12 (page 314) of [10].

Theorem 2.2. *If $\mathcal{H}$ is a Hilbert space and if $T \in \mathcal{B}(\mathcal{H})$ is positive, then there exists a unique positive $S \in \mathcal{B}(\mathcal{H})$ satisfying $S^2 = T$.*

Lemma 2.3. *Let $\mathcal{H}$ be a Hilbert space and assume that $T \in \mathcal{B}(\mathcal{H})$ is positive. Then the following are equivalent:*

(i) $\|T\| \leq D$*;*

(ii) $T \leq D$*;*

(iii) $T^2 \leq DT$*.*

Proof. If $\|T\| \leq D$, then making use of the Schwarz inequality we obtain

$$\langle Tx, x\rangle \leq \|Tx\|\|x\| \leq D\|x\|^2 = D\langle x, x\rangle,$$

for all $x \in \mathcal{H}$. Thus, $T \leq D$, verifiying that (i) implies (ii).

If $T \leq D$, in conjunction with Theorem 2.2, let $S \in \mathcal{B}(\mathcal{H})$ be the unique positive square root of T. Using that positive operators are self-adjoint, we obtain

$$\begin{aligned}\langle T^2x, x\rangle = \langle S^4x, x\rangle &= \langle S^3x, Sx\rangle \\ &= \langle TSx, Sx\rangle \\ &\leq D\langle Sx, Sx\rangle = D\langle S^2x, x\rangle = D\langle Tx, x\rangle,\end{aligned}$$

for all $x \in \mathcal{H}$. Thus, $T^2 \leq DT$, verifying that (ii) implies (iii).

If $T^2 \leq DT$, then for all $x \in \mathcal{H}$ we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^2x, x\rangle \leq D\langle Tx, x\rangle \leq D\|Tx\|\|x\|,$$

implying that $\|Tx\| \leq D\|x\|$ and hence that $\|T\| \leq D$. This verfies that (iii) implies (i) and thus completes the proof. $\square$

Two Lemmas

The proofs of Theorems 2.4, 2.5, and 2.6 all involve similar techniques. To avoid repetition, we have isolated the common arguments into Lemmas 2.4 and 2.5 below.

We require the Lebesgue Differentiation Theorem (see chapter 8 of [11]) in the proof of Lemma 2.4. A measurable function $f : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$ is said to be locally integrable if

$$\int_K |f(\xi)|\, d\xi < \infty,$$

for all compact subsets K of $\hat{\mathbb{R}}^n$. If $\eta \in \hat{\mathbb{R}}^n$, a sequence $\{E_p\}_{p=1}^{\infty}$ of Borel subsets of $\hat{\mathbb{R}}^n$ is said to shrink nicely to η if there exists an $\alpha > 0$ and a sequence $\{r_p\}_{p=1}^{\infty}$ of positive numbers satisfying the following:

(i) $r_p \to 0$, as $p \to \infty$;

(ii) $E_p \subset B(\eta, r_p) = \{\xi \in \hat{\mathbb{R}}^n : \|\xi - \eta\| < r_p\}$, for all p;

(iii) $|E_p| \geq \alpha |B(\eta, r_p)|$, for all p, where $|\cdot|$ denotes Lebesgue measure.

We have the following result:

Theorem 2.3 (The Lebesgue Differentiation Theorem)**.** *Suppose that the function $f : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$ is locally integrable. Define the Lebesgue set, L_f, of f*

to be the collection of all $\eta \in \hat{\mathbb{R}}^n$ *such that*

$$\lim_{p\to\infty} \frac{1}{|E_p|} \int_{E_p} |f(\xi) - f(\eta)|\, d\xi = 0,$$

for all sequences $\{E_p\}_{p=1}^{\infty}$ *shrinking nicely to* η*. Then a.e.* η *belongs to* L_f*.*

Note that if $f : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$ is locally integrable and if $\eta \in L_f$, then in particular we have

$$\lim_{p\to\infty} \frac{1}{|E_p|} \int_{E_p} f(\xi)\, d\xi = f(\eta),$$

for all sequences $\{E_p\}_{p=1}^{\infty}$ shrinking nicely to η. We now state and prove Lemmas 2.4 and 2.5.

Lemma 2.4. *Suppose that* $\{\varphi_i : i \in I\}$ *is a collection of functions in* $L^2(\mathbb{R}^n)$*, where either* $I = \{1, \ldots, N\}$ *for some* $N \in \mathbb{Z}^+$ *or* $I = \mathbb{Z}^+$*. Let* V *denote the closed linear span of the collection* $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$*. For* $\xi \in \hat{\mathbb{R}}^n$*, let* $P(\xi)$ *denote the (bi-finite or bi-infinite) matrix*

$$\begin{pmatrix} [\widehat{\varphi_1}, \widehat{\varphi_1}](\xi) & [\widehat{\varphi_2}, \widehat{\varphi_1}](\xi) & [\widehat{\varphi_3}, \widehat{\varphi_1}](\xi) & \cdots \\ [\widehat{\varphi_1}, \widehat{\varphi_2}](\xi) & [\widehat{\varphi_2}, \widehat{\varphi_2}](\xi) & [\widehat{\varphi_3}, \widehat{\varphi_2}](\xi) & \cdots \\ [\widehat{\varphi_1}, \widehat{\varphi_3}](\xi) & [\widehat{\varphi_2}, \widehat{\varphi_3}](\xi) & [\widehat{\varphi_3}, \widehat{\varphi_3}](\xi) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Then, we can find a subset E *of* $\hat{\mathbb{R}}^n$ *of full measure such that for each* $\eta \in E$ *every entry of* $P(\eta)$ *is a well-defined complex number and the following are satisfied:*

For any $M \in I$ and any $h \in l^2(I)$ of the form

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_M \\ 0 \\ \vdots \end{pmatrix},$$

we can find a sequence of functions $\{f_p\}_{p=1}^{\infty}$ in V that satisfies the following:

(i) for each p, $\hat{f}_p$ is of the form $\sum_{j=1}^{M} m_j^p \widehat{\varphi_j}$, where $m_j^p \in L^\infty(\hat{\mathbb{T}}^n)$, for all p and j;

(ii) $\lim_{p\to\infty} \|f_p\|^2 = \langle P(\eta)h, h\rangle$;

(iii) if, for each p and j, we write $m_j^p = \sum_{k\in\mathbb{Z}^n} \alpha_{jk}^p e^{-2\pi\imath\cdot k}$, then

$$\sum_{j=1}^{M} \sum_{k\in\mathbb{Z}^n} |\alpha_{jk}^p|^2 = \langle h, h\rangle.$$

Moreover, if for each $i' \in I$ we have $\int_{[0,1]^n} \sum_{i\in I} |[\widehat{\varphi}_i, \widehat{\varphi_{i'}}](\xi)|^2 < \infty$, then every entry of $P(\eta)^2$ is a well-defined complex number and we have

(iv) $\lim_{p\to\infty} \sum_{i\in I} \sum_{k\in\mathbb{Z}^n} |\langle f_p, T_k\varphi_i\rangle|^2 = \langle P^2(\eta)h, h\rangle$.

Proof. In conjunction with part (i) of Theorem 2.1, first consider $f \in V$ of the form $\hat{f} = \sum_{j=1}^{M} m_j \widehat{\varphi_j}$, where $M \in I$ and $m_j \in L^\infty(\mathbb{T}^n)$, for each j. Define

$m : \hat{\mathbb{R}}^n \longrightarrow l^2(I)$ by

$$m(\xi) = \begin{pmatrix} m_1(\xi) \\ \vdots \\ m_M(\xi) \\ 0 \\ \vdots \end{pmatrix}.$$

Let $C \subset \hat{\mathbb{R}}^n$ be any cube of side length 1. First note that

$$\begin{aligned} \|f\|^2 = \|\hat{f}\|^2 &= \int_{\hat{\mathbb{R}}^n} \Big| \sum_{j=1}^{M} m_j(\xi)\widehat{\varphi_j}(\xi)\Big|^2 \, d\xi \\ &= \int_{\hat{\mathbb{R}}^n} \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}\widehat{\varphi_j}(\xi)\overline{\widehat{\varphi_{j'}}(\xi)} \, d\xi \\ &= \sum_{l\in\mathbb{Z}^n} \int_{C+l} \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}\widehat{\varphi_j}(\xi)\overline{\widehat{\varphi_{j'}}(\xi)} \, d\xi \qquad (2.5) \\ &= \sum_{l\in\mathbb{Z}^n} \int_{C} \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}\widehat{\varphi_j}(\xi+l)\overline{\widehat{\varphi_{j'}}(\xi+l)} \, d\xi \\ &= \int_{C} \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}[\widehat{\varphi_j}, \widehat{\varphi_{j'}}](\xi) \, d\xi, \end{aligned}$$

where the switch in order of summation in the final equality is easily verified.

Also note that for all i and k we have

$$\begin{aligned}
\langle f, T_k\varphi_i\rangle &= \langle \hat{f}, e^{-2\pi\imath\cdot k}\widehat{\varphi_i}\rangle \\
&= \langle \sum_{j=1}^{M} m_j\widehat{\varphi_j}, e^{-2\pi\imath\cdot k}\widehat{\varphi_i}\rangle \\
&= \int_{\hat{\mathbb{R}}^n} \sum_{j=1}^{M} m_j(\xi)\widehat{\varphi_j}(\xi)\overline{\widehat{\varphi_i}(\xi)}e^{2\pi\imath\xi k}\,d\xi \\
&= \sum_{l\in\hat{\mathbb{Z}}^n} \int_{C+l} \sum_{j=1}^{M} m_j(\xi)\widehat{\varphi_j}(\xi)\overline{\widehat{\varphi_i}(\xi)}e^{2\pi\imath\xi k}\,d\xi \qquad (2.6) \\
&= \sum_{l\in\hat{\mathbb{Z}}^n} \int_{C} \sum_{j=1}^{M} m_j(\xi)\widehat{\varphi_j}(\xi+l)\overline{\widehat{\varphi_i}(\xi+l)}e^{2\pi\imath\xi k}\,d\xi \\
&= \int_{C} \sum_{j=1}^{M} m_j(\xi) \sum_{l\in\hat{\mathbb{Z}}^n} \widehat{\varphi_j}(\xi+l)\overline{\widehat{\varphi_i}(\xi+l)}e^{2\pi\imath\xi k}\,d\xi \\
&= \int_{C} \Big(\sum_{j=1}^{M} m_j(\xi)[\widehat{\varphi_j}, \widehat{\varphi_i}](\xi)\Big)e^{2\pi\imath\xi k}\,d\xi.
\end{aligned}$$

Using (2.6) and that the collection $\{e^{2\pi\imath\cdot k} : k \in \mathbb{Z}\}$ forms an ON basis for

$L^2(C)$, we have

$$\sum_{k\in\mathbb{Z}^n} |\langle f, T_k\varphi_i\rangle|^2 = \sum_{k\in\mathbb{Z}^n} \Big| \int_C \Big(\sum_{j=1}^{M} m_j(\xi)[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi) \Big) e^{2\pi\imath\xi k}\, d\xi \Big|^2$$
$$= \int_C \Big| \sum_{j=1}^{M} m_j(\xi)[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi) \Big|^2$$
$$= \int_C \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi)\overline{[\widehat{\varphi_{j'}}, \widehat{\varphi}_i](\xi)} \tag{2.7}$$
$$= \int_C \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi)[\widehat{\varphi}_i, \widehat{\varphi_{j'}}](\xi),$$

where the above quantity may be infinite. However, if for all $i' \in I$ we have $\int_{[0,1]^n} \sum_{i\in I} |[\widehat{\varphi}_i, \widehat{\varphi_{i'}}](\xi)|^2 < \infty$, then clearly we also have

$$\int_C \sum_{i\in I} |[\widehat{\varphi}_i, \widehat{\varphi_{i'}}](\xi)|^2 < \infty,$$

for all i', and it follows that

$$\sum_{i\in I}\sum_{k\in\mathbb{Z}^n} |\langle f, T_k\varphi_i\rangle|^2$$
$$= \sum_{i\in I} \int_C \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi)[\widehat{\varphi}_i, \widehat{\varphi_{j'}}](\xi)\, d\xi \tag{2.8}$$
$$= \int_C \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)} \sum_{i\in I} [\widehat{\varphi_j}, \widehat{\varphi}_i](\xi)[\widehat{\varphi}_i, \widehat{\varphi_{j'}}](\xi)\, d\xi,$$

where the last equality is easily verified.

In conjunction with the Lebesgue Differentiation Theorem, choose a measurable subset E of $\hat{\mathbb{R}}^n$ that satisfies the following:

(i) E has full measure;

(ii) $\sum_{k\in\hat{\mathbb{Z}}^n} |\widehat{\varphi}_i(\xi+k)|^2 < \infty$, for all ξ in E and all $i \in I$;

(iii) E is contained in the Lebesgue set of the functions $[\widehat{\varphi}_i, \widehat{\varphi_{i'}}](\xi)$, for all $i, i' \in I$.

Fix $\eta \in E$, $M \in I$, and $h \in l^2(I)$ of the form

$$h = \begin{pmatrix} h_1 \\ \vdots \\ h_M \\ 0 \\ \vdots \end{pmatrix}. \tag{2.9}$$

Note that every entry of $P(\eta)$ is a well-defined complex number. Let $\{F_p\}_{p=1}^{\infty}$ be a sequence of measurable sets each contained in some cube C of side length 1 that shrink nicely to η. Denote the periodic extension of $|F_p|^{-1/2}\chi_{F_p}$ to $\hat{\mathbb{R}}^n$ by m^p, where χ_{F_p} denotes the characteristic function of F_p. For each p, define the collection

$$\{m_1^p, \ldots, m_M^p\} \subset L^2(\hat{\mathbb{T}}^n)$$

by $m_q^p = h_q m^p$ and define $f_p \in V$ by

$$\hat{f}_p = \sum_{j=1}^{M} m_j^p \widehat{\varphi_j}.$$

Clearly the sequence $\{f_p\}_{p=1}^{\infty}$ satisfies property (i) in the statement of this lemma.

Using equality (2.5), for each p we obtain

$$\begin{aligned}\|f_p\|^2 &= \int_C \sum_{j,j'=1}^{M} m_j^p(\xi)\overline{m_{j'}^p(\xi)}[\widehat{\varphi_j}, \widehat{\varphi_{j'}}](\xi)\, d\xi \\ &= \sum_{j,j'=1}^{M} h_j \overline{h_{j'}} \frac{1}{|F_p|} \int_{F_p} [\widehat{\varphi_j}, \widehat{\varphi_{j'}}](\xi)\, d\xi.\end{aligned}$$

Letting $p \to \infty$ in the above equality, we obtain

$$\lim_{p\to\infty} \|f_p\|^2 = \sum_{j,j'=1}^{M} h_j \overline{h_{j'}} [\widehat{\varphi^j}, \widehat{\varphi^{j'}}](\eta) = \langle P(\eta)h, h\rangle,$$

where the last equality is easily seen to be true. This verifies property (ii) in the statement of this lemma.

To verify property (iii) in the statement of this lemma, for each p and j write

$$m_j^p = \sum_{k\in\mathbb{Z}^n} \alpha_{jk}^p e^{-2\pi\imath\cdot k}.$$

We then have

$$
\begin{aligned}
\sum_{j=1}^{M} \sum_{k\in\mathbb{Z}^n} |\alpha_{jk}^p|^2 &= \sum_{j=1}^{M} \|m_j^p\|^2 \\
&= \sum_{j=1}^{M} \int_C |m_j^p(\xi)|^2 \, d\xi \\
&= \sum_{j=1}^{M} \frac{1}{|F_p|} \int_{F_p} |h_j|^2 \, d\xi = \sum_{j=1}^{M} |h_j|^2 = \langle h, h\rangle.
\end{aligned}
\tag{2.10}
$$

Now, suppose that for all $i' \in I$ we have

$$\int_{[0,1]^n} \sum_{i\in I} |[\varphi_i, \varphi_{i'}](\xi)|^2 < \infty.$$

It follows that we may assume that E satisfies the two additional properties:

(iv) $\sum_{i\in I} |[\varphi_i, \varphi_{i'}](\xi)|^2 < \infty$, for all $\xi \in E$ and all $i' \in I$;

(v) E lies within the Lebesgue set of the functions

$$\sum_{i\in I} [\varphi_{i'}, \varphi_i](\xi)[\varphi_i, \varphi_{i''}](\xi),$$

for all $i', i'' \in I$.

It is then clear that every entry of $P(\eta)^2$ is a well-defined complex number.

Using (2.8), for each p we have

$$\begin{aligned}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n}|\langle f_p, T_k\varphi_i\rangle|^2 &= \int_C \sum_{j,j'=1}^{M} m_j^p(\xi)\overline{m_{j'}^p(\xi)}\sum_{i\in I}[\widehat{\varphi_j},\widehat{\varphi}_i](\xi)[\widehat{\varphi}_i,\widehat{\varphi_{j'}}](\xi)\,d\xi\\ &= \sum_{j,j'=1}^{M} h_j\overline{h_{j'}}\frac{1}{|F_p|}\int_{F_p}\sum_{i\in I}[\widehat{\varphi_j},\widehat{\varphi}_i](\xi)[\widehat{\varphi}_i,\widehat{\varphi_{j'}}](\xi)\,d\xi.\end{aligned}$$

Letting $p\to\infty$ in the above equality, we obtain

$$\begin{aligned}\lim_{p\to\infty}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n}|\langle f_p, T_k\varphi_i\rangle|^2 &= \sum_{j,j'=1}^{M} h_j\overline{h_{j'}}\sum_{i\in I}[\widehat{\varphi_j},\widehat{\varphi}_i](\eta)[\widehat{\varphi}_i,\widehat{\varphi_{j'}}](\eta)\,d\xi\\ &= \langle P^2(\eta)h, h\rangle,\end{aligned}$$

where, again, the last equality is straightforward to verify. This verifies property (iv) in the statement of this lemma and thus completes the proof.

□

Lemma 2.5. *Let I, $\{\varphi_i : i\in I\}$, V, and P be as in Lemma 2.4 and let M, f, $\{m_j : j=1,\dots,M\}$, and m be defined as in the beginning of the proof of Lemma 2.4. For each j, write*

$$m_j = \sum_{k\in\mathbb{Z}^n}\alpha_{jk}e^{-2\pi\imath\cdot k}.$$

Then,

(i) $\|f\|^2 = \int_{[0,1]^n}\langle P(\xi)m(\xi), m(\xi)\rangle\,d\xi$;

(ii) $\sum_{j=1}^{M}\sum_{k\in\mathbb{Z}^n}|\alpha_{jk}|^2 = \int_{[0,1]^n}\langle m(\xi), m(\xi)\rangle\,d\xi$;

(iii) if, for all $i' \in I$, we have $\int_{[0,1]^n} \sum_{i\in I} |[\widehat{\varphi_i}, \widehat{\varphi_{i'}}](\xi)|^2 < \infty$, then every entry of P^2 belongs to $L^1(\mathbb{T}^n)$ and we have

$$\sum_{i\in I} \sum_{k\in\mathbb{Z}^n} \left|\langle f, T_k\varphi_i\rangle\right|^2 = \int_{[0,1]^n} \langle P^2(\xi)m(\xi), m(\xi)\rangle \, d\xi.$$

Proof. Using (2.5) with $C = [0,1]^n$, we obtain

$$\begin{aligned}
\|f\|^2 &= \int_{[0,1]^n} \sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}[\widehat{\varphi_j}, \widehat{\varphi_{j'}}](\xi)\, d\xi \\
&= \int_{[0,1]^n} \langle P(\xi)m(\xi), m(\xi)\rangle \, d\xi,
\end{aligned}$$

where the last equality is easily seen to be true. This proves part (i).

To prove part (ii), note

$$\begin{aligned}
\sum_{j=1}^{M} \sum_{k\in\mathbb{Z}^n} |\alpha_{jk}|^2 &= \sum_{j=1}^{M} \|m_j\|^2 \\
&= \sum_{j=1}^{M} \int_{[0,1]^n} |m_j(\xi)|^2 \, d\xi \\
&= \int_{[0,1]^n} \sum_{j=1}^{M} |m_j(\xi)|^2 \, d\xi = \int_{[0,1]^n} \langle m(\xi), m(\xi)\rangle \, d\xi.
\end{aligned}$$

To prove part (iii), suppose that for all $i' \in I$ we have

$$\int_{[0,1]^n} \sum_{i\in I} |[\varphi_i, \varphi_{i'}](\xi)|^2 < \infty.$$

It then follows that for each j and j' the function

$$\sum_{i\in I}[\widehat{\varphi_j},\widehat{\varphi}_i](\xi)[\widehat{\varphi}_i,\widehat{\varphi_{j'}}](\xi)$$

belongs to $L^1(\mathbb{T}^n)$; these are the entries of P^2. Using (2.8), we obtain

$$\begin{aligned}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n}\left|\langle f,T_k\varphi_i\rangle\right|^2 &= \int_{[0,1]^n}\sum_{j,j'=1}^{M} m_j(\xi)\overline{m_{j'}(\xi)}\sum_{i\in I}[\widehat{\varphi_j},\widehat{\varphi}_i](\xi)[\widehat{\varphi}_i,\widehat{\varphi_{j'}}](\xi)\,d\xi\\ &= \int_{[0,1]^n}\langle P^2(\xi)m(\xi),m(\xi)\rangle\,d\xi,\end{aligned}$$

where the last equality is straigtforward to verify. $\square$

2.2.2 Main Results

The following three theorems—which generalize parts (ii), (iii), and (iv) of Theorem 2.1 to collections of the form (2.1), where I is a general countable indexing set—are the main results of this chapter. Although we only prove these results for indexing sets I of form $I=\{1,\dots,N\}$ for some $N\in\mathbb{Z}^+$ or $I=\mathbb{Z}^+$, it should be clear that the analogous results for the general countable indexing set I follow from this special case.

Theorem 2.4. *Let I, $\{\varphi_i : i\in I\}$, V, and P be as in Lemma 2.4. If the collection $\{T_k\varphi_i : k\in\mathbb{Z}^n, i\in I\}$ is a Bessel system with constant D, then for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$, is positive, and satisfies each of the following conditions:*

(i) $\|P(\xi)\| \leq D$*;*

(ii) $P(\xi) \leq D$*;*

(iii) $P(\xi)^2 \leq DP(\xi)$*.*

Conversely, if, for almost every ξ*,* $P(\xi)$ *belongs to* $\mathcal{B}(l^2(I))$ *and satisfies one of the of the above three conditions, then* $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *is a Bessel system with constant* D*.*

Proof of Theorem 2.4. Suppose that the collection $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is a Bessel system with constant D. For $i' \in I$, by letting $f = \varphi_{i'}$ in (2.7) (with $C = [0,1]^n$) and using the Bessel property we obtain

$$\begin{aligned} \int_{[0,1]^n} \sum_{i\in I} \left|[\widehat{\varphi_{i'}}, \widehat{\varphi}_i](\xi)\right|^2 d\xi &= \sum_{i\in I} \int_{[0,1]^n} \left|[\widehat{\varphi_{i'}}, \widehat{\varphi}_i](\xi)\right|^2 d\xi \\ &= \sum_{i\in I} \sum_{k\in\mathbb{Z}^n} \left|\langle \varphi_{i'}, T_k\varphi_i\rangle\right|^2 \\ &\leq D\|\varphi_{i'}\|^2 < \infty. \end{aligned} \tag{2.11}$$

Choose E as in Lemma 2.4. Fix $\eta \in E$, $M \in I$, and let $h \in l^2(I)$ be as in (2.9). Choose a sequence of functions $\{f_p\}_{p=1}^{\infty}$ as in Lemma 2.4. Using properties (ii) and (iv) of the same lemma and the Bessel property yields

$$\begin{aligned} \langle P^2(\eta)h, h\rangle &= \lim_{p\to\infty} \sum_{i\in I} \sum_{k\in\mathbb{Z}^n} |\langle f_p, T_k\varphi_i\rangle|^2 \\ &\leq \lim_{p\to\infty} D\|f_p\|^2 = D\langle P(\eta)h, h\rangle. \end{aligned}$$

Although we do not yet know that $P(\eta) \in \mathcal{B}(l^2(I))$, it is clear that $P(\eta)h \in l^2(I)$ and that the relation

$$\langle P(\eta)h, P(\eta)h\rangle = \langle P^2(\eta)h, h\rangle$$

is valid (since $P(\eta)$ is "self-adjoint"). We thus have

$$\begin{aligned} \|P(\eta)h\|^2 &= \langle P(\eta)h, P(\eta)h\rangle \\ &= \langle P^2(\eta)h, h\rangle \\ &\leq D\langle P(\eta)h, h\rangle \leq D\|P(\eta)h\|\|h\|, \end{aligned} \tag{2.12}$$

implying that $\|P(\eta)h\| \leq D\|h\|$. Since h of form (2.9) constitute a dense subspace of $l^2(I)$ it follows that $P(\eta) \in \mathcal{B}(l^2(I))$ and that $\|P(\eta)\| \leq D$. Using again that h of form (2.9) constitute a dense subspace of $l^2(I)$, that the inner product is bi-continuous, and that

$$\langle P(\eta)h, h\rangle = \lim_{p\to\infty} \|f_p\| \geq 0,$$

it follows that $P(\eta)$ is positive. In conjunction with Lemma 2.3, this completes the proof of the forward implication.

We now prove the converse. Suppose that for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies one of the three properties in the statement of this proposition. Using Lemma 2.4 (particularly, property (ii)), that h of form (2.9) constitute a dense subspace of $l^2(I)$, and that the inner product is bi-continuous it follows that $P(\xi)$ is positive almost everywhere. It then follows from Lemma 2.3 that for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies all three properties in the statement of this proposition. In particular, we have that $\|P(\xi)\| \leq D$ and hence that

$$\sum_{i\in\mathbb{Z}^+} \left|[\widehat{\varphi_j}, \widehat{\varphi}_i](\xi)\right|^2 = \|P(\xi)e_j\|^2 \leq D^2\|e_j\|^2 = D^2,$$

for almost every ξ and for all $j \in I$, where e_j is the j^{th} canonical basis vector of $l^2(I)$.

Let M, f, $\{m_j : j = 1, \ldots, M\}$, and m be defined as in the beginning of the proof of Lemma 2.4. Making use of Lemma 2.5 (particularly parts (i) and (iii)), we obtain

$$\begin{aligned}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n} |\langle f, T_k\varphi_i\rangle|^2 &= \int_{[0,1]^n} \langle P^2(\xi)m(\xi), m(\xi)\rangle \, d\xi \\ &\leq D\int_{[0,1]^n} \langle P(\xi)m(\xi), m(\xi)\rangle \, d\xi \\ &= D\|f\|^2.\end{aligned}$$

It thus follows from part (i) of Lemma 2.2 that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is Bessel with constant D. This completes the proof. □

Theorem 2.5. *Let I, $\{\varphi_i : i \in I\}$, V, and P be as in Lemma 2.4. The collection $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a frame for V with constants $C \leq D$*

if and only if for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies

$$CP(\xi) \leq P(\xi)^2 \leq DP(\xi).$$

Proof of Theorem 2.5. Suppose that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a frame for V with constants $C \leq D$. Then $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is clearly Bessel with constant D. Using (2.11), we see that

$$\int_{[0,1]^n} \sum_{i \in I} \left|[\widehat{\varphi_{i'}}, \widehat{\varphi}_i](\xi)\right|^2 d\xi < \infty,$$

for all i'. Choose E as in Lemma 2.4. It follows from the proof of Theorem 2.4 that, for every $\xi \in E$, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies $P(\xi)^2 \leq DP(\xi)$. Fix $\eta \in E$, $M \in I$, and $h \in l^2(I)$ as in (2.9), and choose a sequence of functions $\{f_p\}_{p=1}^{\infty}$ as in Lemma 2.4. Using parts (ii) and (iv) of the same lemma and the frame property, we obtain

$$\begin{aligned}\langle P^2(\eta)h, h\rangle &= \lim_{p\to\infty} \sum_{i \in I} \sum_{k \in \mathbb{Z}^n} \left|\langle f_p, T_k\varphi_i\rangle\right|^2 \\ &\geq \lim_{p\to\infty} C\|f_p\|^2 = C\langle P(\eta)h, h\rangle.\end{aligned}$$

Using that $P(\eta) \in \mathcal{B}(l^2(I))$, that h of form (2.9) constitute a dense subspace of $l^2(I)$, and that the inner product is bi-continuous, it follows that $CP(\eta) \leq P(\eta)^2$. This completes the proof of the forward implication.

To prove the reverse implication, suppose that, for almost every ξ, $P(\xi)$

belongs to $\mathcal{B}(l^2(I))$ and satisfies

$$CP(\xi) \leq P(\xi)^2 \leq DP(\xi).$$

By Theorem 2.4, it follows that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is Bessel with constant D. Using (2.11), we see that

$$\int_{[0,1]^n} \sum_{i \in I} |[\widehat{\varphi_{i'}}, \widehat{\varphi}_i](\xi)|^2 \, d\xi < \infty,$$

for all i'. Let M, f, $\{m_j : j = 1, \ldots, M\}$, and m be as defined in the beginning of the proof of Lemma 2.4. Using parts (i) and (iii) of Lemma 2.5, we obtain

$$\begin{aligned}
\sum_{i \in I} \sum_{k \in \mathbb{Z}^n} |\langle f, T_k\varphi_i \rangle|^2 &= \int_{[0,1]^n} \langle P^2(\xi)m(\xi), m(\xi) \rangle \, d\xi \\
&\geq C \int_{[0,1]^n} \langle P(\xi)m(\xi), m(\xi) \rangle \, d\xi \\
&= C\|f\|^2.
\end{aligned}$$

It thus follows from Lemma 2.2 that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a frame for V with constants $C \leq D$. This completes the proof. $\square$

Theorem 2.6. *Let I, $\{\varphi_i : i \in I\}$, V, and P be as in Lemma 2.4. The collection $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in \mathbb{Z}^+\}$ forms a Riesz basis for V with constants*

$C \leq D$ if and only if for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies

$$C \leq P(\xi) \leq D.$$

Proof of Theorem 2.6. Suppose that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ forms a Riesz basis for V with constants $C \leq D$. It then follows from Theorem 1.4 that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is Bessel with constant D. Using (2.11), we see that

$$\int_{[0,1]^n} \sum_{i\in I} \left|[\widehat{\varphi_{i'}}, \widehat{\varphi}_i](\xi)\right|^2 d\xi < \infty,$$

for all i'. Choose E as in Lemma 2.4. It follows from the proof of Theorem 2.4 that, for every $\xi \in E$, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies $P(\xi) \leq D$. Fix $\eta \in E$, $M \in I$, and $h \in l^2(I)$ of form (2.9). Let the sequence $\{f_p\}_{p=1}^{\infty}$ and the collections

$$\{m_j^p : j = 1, \ldots, M\}_{p=1}^{\infty} \quad \text{and} \quad \{\alpha_{jk}^p : j = 1, \ldots, M, k \in \mathbb{Z}^n\}_{p=1}^{\infty}$$

be as in the statement of Lemma 2.4. Using Lemmas 1.1, 1.2, and 2.1, it follows that

$$f_p = \sum_{j\in\mathcal{M}, k\in\mathbb{Z}^n} \alpha_{jk}^p T_k\varphi_j,$$

with unconditional convergence, where $\mathcal{M} = \{1, \ldots, M\}$. Using again Lemma 2.4

and Lemma 1.2, we obtain

$$\begin{aligned}\langle P(\eta)h, h\rangle &= \lim_{p\to\infty} \|f_p\|^2 \\ &= \lim_{p\to\infty} \Big\| \sum_{j\in\mathcal{M}, k\in\mathbb{Z}^n} \alpha_{jk}^p T_k \varphi_j \Big\|^2 \\ &\geq \lim_{p\to\infty} C \sum_{j=1}^{M} \sum_{k\in\mathbb{Z}^n} |\alpha_{jk}^p|^2 = C\langle h, h\rangle.\end{aligned}$$

Using that $P(\eta) \in \mathcal{B}(l^2(I))$, that h of form (2.9) constitute a dense subspace of $l^2(I)$, and that the inner product is bi-continuous it follows that $C \leq P(\eta)$. This completes the proof of the forward implication.

To prove the reverse implication, suppose that, for almost every ξ, $P(\xi)$ belongs to $\mathcal{B}(l^2(I))$ and satisfies

$$C \leq P(\xi) \leq D.$$

It then follows from Theorem 2.4 that $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is Bessel with constant D. Proposition 1.1 thus implies that

$$\Big\| \sum_{i\in I, k\in K} \alpha_{ik} T_k \varphi_i \Big\|^2 \leq D \sum_{i\in F} \sum_{k\in K} |\alpha_{ik}|^2,$$

for all finite subsets F of I and K of $\mathbb{Z}^n$ and all $\{\alpha_i : i \in I\} \subset \mathbb{C}$. Let M, f, $\{m_j : j = 1, \ldots, M\}$, and m be as defined in the beginning of the proof of Lemma 2.4 and let

$$\{\alpha_{jk} : j = 1, \ldots, M, k \in \mathbb{Z}^n\}$$

be as in the statement of Lemma 2.5. Using Lemmas 1.1 and 2.1 and Proposition 1.2, it follows that

$$f = \sum_{j \in \mathcal{M}, k \in \mathbb{Z}^n} \alpha_{jk} T_k \varphi_j,$$

with unconditional convergence, where $\mathcal{M} = \{1, \ldots, M\}$. Using parts (i) and (ii) of Lemma 2.5, we obtain

$$\begin{aligned} C \sum_{j=1}^{M} \sum_{k \in \mathbb{Z}^n} |\alpha_{jk}|^2 &= \int_{[0,1]^n} C \langle m(\xi), m(\xi) \rangle \, d\xi \\ &\leq \int_{[0,1]^n} \langle P(\xi) m(\xi), m(\xi) \rangle \, d\xi \\ &= \|f\|^2 \\ &= \Big\| \sum_{j \in \mathcal{M}, k \in \mathbb{Z}^n} \alpha_{jk} T_k \varphi_j \Big\|^2. \end{aligned}$$

It follows that $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}$ is a Riesz basis for V with constants $C \leq D$. This completes the proof. □

If we "deperiodize" the result of Theorem 2.4, we obtain the following corollary, which we will use in the next chapter:

Corollary 2.1. *Let I, $\{\varphi_i : i \in I\}$, V, and P be as in Lemma 2.4. Suppose* *that* $\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in I\}$ *is Bessel with constant* D*. For* $\xi \in \hat{\mathbb{R}}^n$*, define*

$$\hat{\Phi}(\xi) = \begin{pmatrix} \widehat{\varphi_1}(\xi) \\ \widehat{\varphi_2}(\xi) \\ \widehat{\varphi_3}(\xi) \\ \vdots \end{pmatrix}.$$

Then, for almost every ξ*, the collection* $\{\hat{\Phi}(\xi+k) : k \in \hat{\mathbb{Z}}^n\}$ *is Bessel in* $l^2(I)$ *with constant* D*. That is, for almost every* ξ*, we have*

$$\sum_{k\in\hat{\mathbb{Z}}^n} \Big| \sum_{i=1}^{\infty} x_i \widehat{\varphi}_i(\xi+k) \Big|^2 \leq D \sum_{i=1}^{\infty} |x_i|^2,$$

for all $\{x_i\}_{i=1}^{\infty} \in l^2(I)$*.*

Proof. Let $M \in I$ and define $x \in l^2(I)$ by

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_M \\ 0 \\ \vdots \end{pmatrix}.$$

Using Theorem 2.4, we obtain

$$\begin{aligned}
\sum_{k\in\hat{\mathbb{Z}}^n} |\langle \hat{\Phi}(\xi+k), x\rangle|^2 &= \sum_{k\in\hat{\mathbb{Z}}^n} \Big| \sum_{i=1}^{M} \overline{x_i}\widehat{\varphi}_i(\xi+k)\Big|^2 \\
&= \sum_{k\in\hat{\mathbb{Z}}^n} \sum_{i,i'=1}^{M} \overline{x_i}x_{i'}\widehat{\varphi}_i(\xi+k)\overline{\widehat{\varphi_{i'}}(\xi+k)} \\
&= \sum_{i,i'=1}^{M} \overline{x_i}x_{i'} \sum_{k\in\hat{\mathbb{Z}}^n} \widehat{\varphi}_i(\xi+k)\overline{\widehat{\varphi_{i'}}(\xi+k)} \\
&= \sum_{i,i'=1}^{M} \overline{x_i}x_{i'}[\widehat{\varphi}_i, \widehat{\varphi_{i'}}](\xi) \\
&= \langle P(\xi)\overline{x}, \overline{x}\rangle \le D\|x\|^2,
\end{aligned}$$

for a.e. ξ. The desired result now follows from Lemma 2.2. $\square$

2.3 Two Examples

We now utilize Theorems 2.5 and 2.6 to give the two examples that were promised after Corollary 1.1 and 1.5.

Example 2.1.

Our first example is that of two functions $\varphi, \psi : \mathbb{R} \longrightarrow \mathbb{C}$ that belong to the Schwartz class of $\mathbb{R}$ and satisfy the following:

(i) $\{T_k\varphi : k \in \mathbb{Z}\}$ and $\{T_k\psi : k \in \mathbb{Z}\}$ both form orthonormal bases for their respective closed spans, V and W;

(ii) $V \cap W = \{0\}$;

(iii) $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ does not form a frame for its closed span.

It is interesting to compare this example with the results of Theorem 1.4 and Corollary 1.1. In particular, we see that $V + W$ cannot be closed.

Choose real valued functions $\alpha, \beta \in C(\mathbb{R})$ satisfying the following:

(i) α is supported on $[0, 3]$;

(ii) $|\alpha(\xi)|^2 + |\alpha(\xi + 1)|^2 + |\alpha(\xi + 2)|^2 = 1$, for all $\xi \in [0, 1]$;

(iii) $\alpha(\xi) > 0$, for all $\xi \in (2, 3)$;

(iv) β is supported on $[0, 2]$;

(v) $|\beta(\xi)|^2 + |\beta(\xi + 1)|^2 = 1$, for all $\xi \in [0, 1]$;

(vi) $\alpha(1) = \beta(1) = 1$.

Define φ and ψ in $L^2(\mathbb{R})$ by $\hat{\varphi} = \alpha$ and $\hat{\psi} = \beta$. For $\xi \in [0, 1]$, simple calculations show that

(i) $[\hat{\varphi}, \hat{\varphi}](\xi) = 1$;

(ii) $[\hat{\psi}, \hat{\psi}](\xi) = 1$;

(iii) $[\hat{\varphi}, \hat{\psi}](\xi) = \alpha(\xi)\beta(\xi) + \alpha(\xi + 1)\beta(\xi + 1)$.

By Theorem 2.1, we see that $\{T_k\varphi : k \in \mathbb{Z}\}$ and $\{T_k\psi : k \in \mathbb{Z}\}$ both form orthonormal bases for their respective closed spans, which we'll denote by V and W.

If $f \in V \cap W$, then by part (i) of Theorem 2.1 there exist measurable 1-periodic functions s and t such that $\hat{f} = s\hat{\varphi} = t\hat{\psi}$. For $\xi \in (2,3)$, we have

$$s(\xi)\alpha(\xi) = s(\xi)\hat{\varphi}(\xi) = t(\xi)\hat{\psi}(\xi) = t(\xi)\beta(\xi) = 0,$$

implying that $s(\xi) = 0$ (since $\alpha(\xi) \neq 0$). It follows that s (and hence f) is 0 almost everywhere. We conclude that $V \cap W = \{0\}$.

For $\xi \in [0,1]$, write

$$\gamma(\xi) = [\hat{\varphi}, \hat{\psi}](\xi) = \alpha(\xi)\beta(\xi) + \alpha(\xi+1)\beta(\xi+1).$$

Using the notation of Theorem 2.4, we have that

$$P(\xi) = \begin{pmatrix} [\widehat{\varphi}, \widehat{\varphi}](\xi) & [\widehat{\psi}, \widehat{\varphi}](\xi) \\ [\widehat{\varphi}, \widehat{\psi}](\xi) & [\widehat{\psi}, \widehat{\psi}](\xi) \end{pmatrix} = \begin{pmatrix} 1 & \gamma(\xi) \\ \gamma(\xi) & 1 \end{pmatrix},$$

and hence that

$$P^2(\xi) = \begin{pmatrix} \gamma^2(\xi) + 1 & 2\gamma(\xi) \\ 2\gamma(\xi) & \gamma^2(\xi) + 1 \end{pmatrix}.$$

For $\xi \in [0,1]$, one calculates that

$$\left\langle P(\xi) \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = 2(1 - \gamma(\xi)) \tag{2.13}$$

and

$$\left\langle P^2(\xi)\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\rangle = 2(1-\gamma(\xi))^2. \tag{2.14}$$

Using the properties of α and β and the Schwarz inequality, for $\xi \in (0,1)$ we have

$$\begin{aligned}\gamma(\xi)^2 &\leq (|\alpha(\xi)|^2 + |\alpha(\xi+1)|^2)(|\beta(\xi)|^2 + |\beta(\xi+1)|^2)\\ &= |\alpha(\xi)|^2 + |\alpha(\xi+1)|^2 \\ &= 1 - |\alpha(\xi+2)|^2 < 1.\end{aligned} \tag{2.15}$$

Since $\gamma \in C([0,1])$ and $\gamma(0) = 1$, it now follows from (2.13), (2.14), (2.15), and Theorem 2.5 that the collection $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ cannot form a frame for its closed span.

Finally, note that if we choose α and β to be in $C^\infty(\mathbb{R})$, then φ and ψ belong to the Schwartz class.

Example 2.2.

Our second example is that of two functions $\varphi, \psi : \mathbb{R} \longrightarrow \mathbb{C}$ that belong to the Schwartz class of $\mathbb{R}$ and satisfy the following:

(i) $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ forms a Parseval frame for its closed span;

(ii) Neither $\{T_k\varphi : k \in \mathbb{Z}\}$ nor $\{T_k\psi : k \in \mathbb{Z}\}$ forms a frame for its closed span;

(iii) Denoting the closed linear spans of $\{T_k\varphi : k \in \mathbb{Z}\}$ and $\{T_k\psi : k \in \mathbb{Z}\}$ by V and W, respectively, we have

$$V \cap W^{\perp} \neq \{0\} \neq W \cap V^{\perp}.$$

It is interesting to compare this example with the result of Proposition 1.5.

Choose $\theta \in C(\mathbb{R})$ satisfying the following:

(i) θ is supported on $[0, 2]$;

(ii) $\theta(\xi) > 0$, for all $\xi \in (0, 2)$;

(iii) $|\theta(\xi)|^2 + |\theta(\xi + 1)|^2 = 1$, for all $\xi \in [0, 1]$.

Also, choose $m_1, m_2 \in L^2(\mathbb{T}) \cap C(\mathbb{R})$ satisfying the following:

(i) $m_1(\xi) = 0$, for all $\xi \in [1/5, 2/5]$;

(ii) $m_2(\xi) = 0$, for all $\xi \in [3/5, 4/5]$;

(iii) $|m_1(\xi)|^2 + |m_2(\xi)|^2 = 1$, for all $\xi \in [0, 1]$.

Define φ and ψ in $L^2(\mathbb{R})$ by $\hat{\varphi} = m_1\theta$ and $\hat{\psi} = m_2\theta$. For $\xi \in [0, 1]$, simple calculations show that

(i) $[\hat{\varphi}, \hat{\varphi}](\xi) = |m_1(\xi)|^2$;

(ii) $[\hat{\psi}, \hat{\psi}](\xi) = |m_2(\xi)|^2$;

(iii) $[\hat{\varphi}, \hat{\psi}](\xi) = m_1(\xi)\overline{m_2(\xi)}$.

Using the notation of Theorem 2.4, we have

$$P(\xi) = \begin{pmatrix} [\widehat{\varphi}, \widehat{\varphi}](\xi) & [\widehat{\psi}, \widehat{\varphi}](\xi) \\ [\widehat{\varphi}, \widehat{\psi}](\xi) & [\widehat{\psi}, \widehat{\psi}](\xi) \end{pmatrix} = \begin{pmatrix} |m_1(\xi)|^2 & \overline{m_1(\xi)} m_2(\xi) \\ m_1(\xi)\overline{m_2(\xi)} & |m_2(\xi)|^2 \end{pmatrix}.$$

By performing a simple computation and using property (iii) of m_1 and m_2, it follows easily that $P(\xi)^2 = P(\xi)$, for all ξ. It thus follows by Theorem 2.5 that $\{T_k\varphi, T_k\psi : k \in \mathbb{Z}\}$ forms a Parseval frame for its closed span.

Since m_1 and m_2 are continuous, it follows from part (iii) of Theorem 2.1 that neither $\{T_k\varphi : k \in \mathbb{Z}\}$ nor $\{T_k\psi : k \in \mathbb{Z}\}$ forms a frame for its closed span.

Denote the closed span of $\{T_k\varphi : k \in \mathbb{Z}\}$ by V and the closed span of $\{T_k\psi : k \in \mathbb{Z}\}$ by W. Define the function $s \in L^\infty(\mathbb{T})$ to be the periodic extension of

$$s(\xi) = \begin{cases} \frac{\overline{m_1(\xi)}}{\theta(\xi)}, & \text{if } \xi \in [3/5, 4/5]; \\ 0, & \text{if } \xi \in [0, 3/5) \cup (4/5, 1]. \end{cases}$$

to $\mathbb{R}$. By part (i) of Theorem 2.1, $s\hat{\varphi} \in V$ and for $\xi \in [3/5, 4/5]$ we have

$$s(\xi)\hat{\varphi}(\xi) = \frac{\overline{m_1(\xi)}}{\theta(\xi)}\hat{\varphi}(\xi) = \frac{\overline{m_1(\xi)}}{\theta(\xi)} m_1(\xi)\theta(\xi) = |m_1(\xi)|^2 = 1.$$

Note also that the support of $s\hat{\varphi}$ is contained in $[3/5, 4/5] \cup [8/5, 9/5]$. Appealing again to part (i) of Theorem 2.1, we see that if $f \in W$, then

$$f = t\hat{\psi} = tm_2\theta$$

for some measurable 1-periodic function t, and therefore f must vanish almost everywhere on $[3/5, 4/5] \cup [8/5, 9/5]$. We thus conclude that $s\hat{\varphi} \in V \cap W^{\perp}$. This shows that $V \cap W^{\perp} \neq \{0\}$. A similar argument shows that $W \cap V^{\perp} \neq \{0\}$.

Finally, note that if we choose θ, m_1, and m_2 to be in $C^{\infty}(\mathbb{R})$, then φ and ψ belong to the Schwartz class.

Chapter 3

Classical and Composite Wavelet Systems

Wavelets, MRAs, and scaling functions will play a prominent role in this and subsequent chapters. In general terms, if $\mathcal{C}$ is a countable subset of $GL_n(\mathbb{R})$, then a countable collection

$$\{\psi_l : l \in L\} \subset L^2(\mathbb{R}^n)$$

is said to be a wavelet if

$$\{D_c T_k \psi_l : c \in \mathcal{C}, k \in \mathbb{Z}^n, l \in L\}$$

forms a reproducing system (e.g., frame, Riesz basis, etc.) for $L^2(\mathbb{R}^n)$.

In section 1, we introduce both classical and composite wavelet systems and give several examples. In section 2, we generalize to composite wavelet systems several well-known classical wavelet system results regarding pointwise values of the Fourier transform of the wavelet and scaling function. An interesting corollary of these results will be the nonexistence of aB-scaling multifunctions of Haar-type when B is infinite. It is widely known that dependencies exist among the defining properties of a 2-MRA. In section 3, we show that these dependencies are retained in the defining properties of an aB-MRA. An interesting corollary of these dependency results is the nonexistence of a-multiwavelets, for all $a \in \widetilde{SL_n}(\mathbb{Z})$.

3.1 Definitions and Examples

In this section, we introduce and define 2-wavelet systems, a-wavelet systems, and composite wavelet systems and give examples of each.

3.1.1 2-Wavelet Systems

We begin this subsection by defining 2-wavelets and 2-MRAs. We then discuss the relationship between the two and give a sketch of their use in applications.

Definition 3.1. *A function $\psi \in L^2(\mathbb{R})$ is said to be a 2-wavelet if the collection*

$$\{D_2^j T_k \psi : j, k \in \mathbb{Z}\}$$

forms an ON basis for $L^2(\mathbb{R})$. *Note that*

$$D_2^j T_k \psi(x) = 2^{-j/2}\psi(2^{-j}x - k).$$

A very important concept that is intimately related to 2-wavelets is the following:

Definition 3.2. *A sequence* $\{V_j\}_{j\in\mathbb{Z}}$ *of closed subspaces of* $L^2(\mathbb{R})$ *is said to be a* 2*-multiresolution analysis (*2*-MRA) if the following conditions hold:*

(i) $V_j \subset V_{j+1}$, *for all* j;

(ii) $V_j = D_2^{-j} V_0$, *for all* j;

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$;

(v) There is a function $\varphi \in V_0$ *such that the collection*

$$\{T_k \varphi : k \in \mathbb{Z}\}$$

forms an ON basis for V_0.

In this case, we say that φ *is a* 2*-scaling function for the given MRA.*

The Relationship Between 2-Wavelets and 2-MRAs

We give a brief description of the relationship between 2-wavelets and 2-MRAs. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a 2-MRA with scaling function φ. Since

$1/2\varphi(\cdot/2) \in V_0$, we may write

$$\frac{1}{2}\varphi(x/2) = \sum_{k\in\mathbb{Z}} \alpha_k \varphi(x+k), \tag{3.1}$$

with unconditional convergence in $L^2(\mathbb{R})$, where

$$\alpha_k = \frac{1}{2}\int_{\mathbb{R}} \varphi(x/2)\overline{\varphi(x+k)}\, dx. \tag{3.2}$$

Applying the Fourier transform to both sides of the above equality, we obtain

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi),$$

where $m_0 \in L^2(\mathbb{T})$ (the so-called low pass filter) is given by

$$m_0(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{2\pi\imath\xi k}. \tag{3.3}$$

Define $W_0 = V_0^{\perp} \cap V_1$, and, for each $j \neq 0$, define $W_j = D_2^{-j} W_0$. It follows easily from the 2-MRA properties that

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}). \tag{3.4}$$

It is simple to obtain (from φ) a function $\psi \in W_0$ such that

$$\{T_k\psi : k \in \mathbb{Z}\} \text{ forms an ON basis for } W_0. \tag{3.5}$$

For instance, it can be shown that the function $\psi \in L^2(\mathbb{R})$ defined by

$$\hat{\psi}(2\xi) = e^{2\pi\imath\xi} m_0(\xi + 1/2)\hat{\varphi}(\xi), \tag{3.6}$$

for a.e. ξ or, equivalently,

$$\frac{1}{2}\psi(x/2) = \sum_{k\in\mathbb{Z}} (-1)^{k+1}\overline{\alpha_{1-k}}\varphi(x+k), \tag{3.7}$$

for a.e. x satisfies (3.5) (see section 2 of chapter 2 of [6]). It is immediate from (3.4) and (3.5) that the collection

$$\{D_2^j T_k \psi : j, k \in \mathbb{Z}\}$$

forms an ON basis for $L^2(\mathbb{R})$; i.e., that ψ is a 2-wavelet. In general, when a 2-wavelet ψ arises from a 2-MRA $\{V_j\}_{j\in\mathbb{Z}}$ with scaling function φ in this fashion, we say that ψ, $\{V_j\}_{j\in\mathbb{Z}}$, and φ are associated, and we call the collection ψ, $\{V_j\}_{j\in\mathbb{Z}}$, φ a 2-MRA wavelet system. We have the following example:

Example 3.1.

Define $\varphi = \chi_{[0,1]}$ and $\psi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. It is straightforward to verify that φ is a 2-scaling function for an MRA and that ψ is an associated wavelet. This simplest and most well-known of 2-MRA wavelet systems is called the Haar system; ψ and φ are called the Haar wavelet and Haar scaling function, respectively.

Applications of 2-MRA Wavelet Systems

2-MRA wavelet systems have been used very successfully in a variety of applications. This success is due, in large part, to the plentiful existence of compactly supported and smooth 2-scaling functions. Specifically, we have the following remarkable result, due to I. Daubechies:

Theorem 3.1. *For each $p \in \mathbb{Z}^+$, there exists a 2-scaling function φ that is compactly supported and belongs to $C^p(\mathbb{R})$.*

In the following two paragraphs we give a rough sketch of how 2-MRA wavelet systems are often used in applications and indicate why the compact support and smoothness of the scaling function play an important role.

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a 2-MRA with scaling function φ and associated wavelet ψ given by (3.7). As above, define $W_0 = V_0^\perp \cap V_1$, and, for each $j \neq 0$, define $W_j = D_2^{-j} W_0$. Suppose that $f \in L^2(\mathbb{R})$ is a "signal" we wish to store in compressed form. Choose $J \in \mathbb{Z}^+$ large enough so that $f \approx f_J$, where f_J is the orthogonal projection of f onto V_J. Since $V_1 = W_0 \oplus V_0$ and since the operator D_2 is unitary, it follows that $V_j = W_{j-1} \oplus V_{j-1}$, for each j. We thus have

$$\begin{aligned} V_J = W_{J-1} \oplus V_{J-1} &= W_{J-1} \oplus W_{J-2} \oplus V_{J-2} \\ &= \dots \\ &= W_{J-1} \oplus W_{J-2} \oplus \cdots \oplus W_0 \oplus V_0. \end{aligned}$$

We thus may write

$$f \approx f_J = g_{J-1} + g_{J-2} + \cdots + g_0 + f_0, \tag{3.8}$$

where $f_0 \in V_0$ and, for each j, the function $g_j \in W_j$ can be thought of as the component of f_J (or the approximate component of f) at the j^{th} scale. At first sight, it seems that we have gone in the opposite direction of compressing f—we have replaced the approximation f_J with the sum of $J+1$ functions. However, it is often the case that many of the functions g_j are small enough (either in whole or part) so that if they are discarded (either wholly or partially) from (3.8), the resulting sum (which contains significantly less data than f_J) still remains a very good approximation to f. This abbreviated sum is then stored.

The above outlined scheme of using the 2-MRA wavelet system $\{V_j\}_{j\in\mathbb{Z}}$, φ, and ψ to decompose the signal f into different scales, discarding the scales at which f is small (either partially or wholly), and then storing what remains can be a very effective method for compressing data. However, for this process to be efficiently implemented, it is almost always necessary that the 2-scaling function φ satisfies a certain amount of decay and regularity. For decay, it is usually very desirable that φ be compactly supported. In this case, it is clear that the sequence $\{\alpha_k\}_{k\in\mathbb{Z}}$ given by (3.2) is finitely supported, implying that both (3.1) and (3.7) are finite sums, a crucial property. For regularity, it is usually desirable that φ satisfies some degree of smoothness or, more generally, Hölder continuity.

3.1.2 a-Wavelet Systems

The above concepts (2-wavelet, 2-MRA, 2-scaling function, etc.) all extend very naturally to higher dimensions. Essentially, the operators

$$\{D_2^j : j \in \mathbb{Z}\} \quad \text{and} \quad \{T_k : k \in \mathbb{Z}\}$$

are replaced with the operators

$$\{D_a^j : j \in \mathbb{Z}\} \quad \text{and} \quad \{T_k : k \in \mathbb{Z}^n\}, \tag{3.9}$$

where a belongs to $GL_n(\mathbb{Z})$ and is usually taken to be expanding. More precisely, we have the following two definitions:

Definition 3.3. *Let $a \in GL_n(\mathbb{Z})$ and let $L \in \mathbb{Z}^+$. We say that*

$$\{\psi_l : l = 1, \ldots, L\} \subset L^2(\mathbb{R}^n)$$

is an a-multiwavelet if the collection

$$\{D_a^j T_k \psi_l : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \ldots, L\}$$

forms an ON basis for $L^2(\mathbb{R}^n)$.

Definition 3.4. *Let $a \in GL_n(\mathbb{Z})$. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is said to be an a-multiresolution analysis (a-MRA) if the following conditions hold:*

(i) $V_j \subset V_{j+1}$, *for all* j;

(ii) $V_j = D_a^{-j} V_0$, *for all* j;

(iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iv) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$;

(v) There is a function $\varphi \in V_0$ *such that*

$$\{T_k\varphi : k \in \mathbb{Z}^n\}$$

forms an ON basis for V_0.

In this case, we say that the fucntion φ is an a-scaling function for the given MRA.

The relationship between a-multiwavelets and a-MRAs, the concept of a low pass filter, the meaning of an associated a-wavelet, a-MRA, and a-scaling function, the use of a-MRA multiwavelet systems in applications, and the importance of compact support and smoothness are all similar to their analogs in the 2-wavelet case.

3.1.3 Composite Wavelet Systems

As indicated above, 2-MRA wavelet systems have been used very successfully in a variety of applications. Given how naturally one-dimensional 2-wavelet concepts extend to higher dimensional a-wavelet concepts, one would expect a-MRA wavelet systems to also be very useful in applications. In many instances, this is very much the case. However, there are some important applications where a-wavelet systems do not perform as well as would be expected. This deficiency arises from the fact that the geometry of $\mathbb{R}^n$ $(n \geq 2)$ is considerably more complex than the geometry of $\mathbb{R}$. In particular, in dimensions two and higher, there is a nontrivial directional component that is simply not captured by the operators (3.9) employed by a-wavelet systems.

In an attempt to create higher-dimensional wavelet systems with directional sensitivity, the composite wavelet systems were introduced in [3]. Composite wavelet systems, in addition to employing the operators in (3.9), use operators of the form $\{D_b : b \in B\}$, where B is a countable subset of

$$\widetilde{SL_n}(\mathbb{R}) = \{c \in GL_n(\mathbb{R}) : |\det c| = 1\}.$$

The following two definitions are adapted from [3]:

Definition 3.5. *Let $a \in GL_n(\mathbb{R})$, let B be a countable subset of $\widetilde{SL_n}(\mathbb{R})$, and let L be a countable indexing set. We say that $\{\psi_l : l \in L\} \subset L^2(\mathbb{R}^n)$ is an aB-multiwavelet (or composite multiwavelet if we do not wish to specify* *a and B) if the collection*

$$\{D_a^j D_b T_k \psi_l : j \in \mathbb{Z}, b \in B, k \in \mathbb{Z}^n, l \in L\}$$

forms a frame for $L^2(\mathbb{R}^n)$*. Note that*

$$D_a^j D_b T_k \psi_l(x) = |\det a|^{-j/2} \psi_l(b^{-1} a^{-j} x - k).$$

Definition 3.6. *Let a and B be as in Definition 3.5 and let I be a countable indexing set. A sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is said to be an aB-multiresolution analysis (aB-MRA or composite MRA) if the following conditions hold:*

(i) $V_j \subset V_{j+1}$*, for all* j*;*

(ii) $V_j = D_a^{-j} V_0$*, for all* j*;*

(iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$*;*

(iv) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$*;*

(v) There is a collection $\{\varphi_i : i \in I\} \subset V_0$ *such that*

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\}$$

forms a frame for V_0*.*

In this case, we say that the collection $\{\varphi_i : i \in I\}$ is an aB-scaling multifunction (or composite scaling multifunction) for the given MRA.

We will refer to a-wavelet systems (in particular, 2-wavelet systems) as classical wavelet systems. Although we will make no formal definitions, the meaning of a Parseval/Riesz/ON aB-multiwavelet/aB-scaling multifunction should be clear. Also, as before, the relationship between aB-multiwavelets and aB-MRAs, the concept of a low pass filter, the meaning of an associated aB-multiwavelet, aB-MRA, and aB-scaling multifunction, the use of aB-MRA multiwavelet systems in applications, and the importance of compact support and smoothness are all similar to their analogs in the 2-wavelet case.

Two Examples

In Definitions 3.5 and 3.6, the set B may either be finite or infinite. We offer the following two examples of aB-MRA wavelet systems to illustrate each situation:

Example 3.2.

The following example is borrowed from [7] and is a higher dimensional composite analog of the Haar wavelet system of Example 1. Let a be the quincunx matrix

$$a = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and let B be the group of symmetries of the square

$$\left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $T_1, T_2 \subset \mathbb{R}^2$ be the triangles with vertices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Define

$$\varphi = 2\sqrt{2}(\chi_{T_1} + \chi_{T_2}) \quad \text{and} \quad \psi = 2\sqrt{2}(\chi_{T_1} - \chi_{T_2}).$$

Although we will verify none of the details, it is not hard to show that φ is an ON aB-scaling function and that ψ is an associated ON aB-wavelet (see [7]).

Example 3.3.

The following example (and notation) is borrowed from [3]. Let a be the matrix

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and let B be the so-called shear group

$$\left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Define the sets $I_0^+, I_1^+, I_2^+, I_3^+ \subset \hat{\mathbb{R}}^2$ by

$$\begin{aligned}
I_0^+ &= \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 0 \le \xi_1 \le 1, 0 \le \xi_2 \le \xi_1\}, \\
I_1^+ &= \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 \le \xi_1 \le 2, 0 \le \xi_2 \le 1/2\}, \\
I_2^+ &= \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 \le \xi_1 \le 2, 1/2 \le \xi_2 \le 1\}, \\
I_3^+ &= \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 1 \le \xi_1 \le 2, 1 \le \xi_2 \le \xi_1\}.
\end{aligned}$$

For $l = 0, 1, 2, 3$, define $I_l^- = -I_l^+$ and $I_l = I_l^+ \cup I_l^-$. Define the functions $\varphi, \psi_1, \psi_2, \psi_3 \in L^2(\mathbb{R}^2)$ by

$$\hat{\varphi} = \chi_{I_0} \quad \text{and} \quad \widehat{\psi_l} = \chi_{I_l}.$$

Although we will verify none of the details, it is not hard to see that φ is an ON aB-scaling function and that $\{\psi_1, \psi_2, \psi_3\}$ is an associated ON aB-multiwavelet (see [3]).

3.2 Pointwise Values of the Fourier Transform

There are several well-known results regarding the value at integers of the Fourier transform of a 2-wavelet and 2-scaling function. In this section, we generalize these results to composite wavelet systems. A very interesting corollary will be the nonexistence of aB-scaling multifunctions of Haar-type, for all countably infinite $B \subset \widetilde{SL_n}(\mathbb{R})$ and all $a \in GL_n(\mathbb{R})$.

3.2.1 Preliminary Results

Before we state and prove the above mentioned results, we need the results of this subsection. The following lemma, which we require in the proof of Theorem 3.2 below, is an easy consequence of part (iii) of Theorem 2.1.

Lemma 3.1. *Suppose that E is a measurable subset of $\hat{\mathbb{R}}^n$. The collection $\{e^{2\pi\imath\cdot k} : k \in \mathbb{Z}^n\}$ forms a Parseval frame for $L^2(E)$ if and only if $|E \cap (E + k)| = 0$, for all $k \in \hat{\mathbb{Z}}^n \setminus \{0\}$.*

We note that the below result is very similar to Proposition 4.1 of [4].

Theorem 3.2. *Let $\mathcal{C}$ be a countable subset of $GL_n(\mathbb{R})$ and let I be a countable indexing set. If the collection*

$$\{D_c T_k \varphi_i : c \in \mathcal{C}, k \in \mathbb{Z}^n, i \in I\} \subset L^2(\mathbb{R}^n)$$

forms a Bessel system with constant D, then

$$\sum_{c\in\mathcal{C}}\sum_{i\in I}|\widehat{\varphi}_i(\xi c)|^2 \le D,$$

for a.e. ξ.

Proof of Theorem 3.2. Let $\mathcal{F}$ be any finite subset of $\mathcal{C}$ and let E be any measurable subset of $\hat{\mathbb{R}}^n$ satisfying

$$|(Ec+k)\cap Ec| = 0, \text{ for all } k \in \hat{\mathbb{Z}}^n \setminus \{0\} \text{ and all } c \in \mathcal{F}. \tag{3.10}$$

Define $\theta \in L^2(\mathbb{R}^n)$ by $\hat{\theta} = \chi_E$. A calculation shows that

$$(D_cT_k\varphi_i)\widehat{\ }(\xi) = |\det c|^{1/2}\widehat{\varphi}_i(\xi c)e^{-2\pi\imath\xi ck}. \tag{3.11}$$

For $c \in \mathcal{F}$, using a change of variable, we obtain

$$\begin{aligned}\langle D_cT_k\varphi_i,\theta\rangle &= \langle (D_cT_k\varphi_i)\widehat{\ },\hat{\theta}\rangle \\ &= |\det c|^{1/2}\int_E \widehat{\varphi}_i(\xi c)e^{-2\pi\imath\xi ck}\,d\xi \\ &= |\det c|^{-1/2}\int_{Ec}\widehat{\varphi}_i(\xi)e^{-2\pi\imath\xi k}\,d\xi.\end{aligned} \tag{3.12}$$

Using the above equality, Lemma 3.1, and another change of variable gives us

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n} \left|\langle D_c T_k \varphi_i, \theta\rangle\right|^2 &= |\det c|^{-1} \sum_{k\in\mathbb{Z}^n} \left| \int_{Ec} \widehat{\varphi}_i(\xi) e^{-2\pi\imath\xi k}\, d\xi \right|^2 \\
&= |\det c|^{-1} \int_{Ec} |\widehat{\varphi}_i(\xi)|^2\, d\xi \qquad (3.13) \\
&= \int_E |\widehat{\varphi}_i(\xi c)|^2\, d\xi.
\end{aligned}$$

Using the above equality and the Bessel property, we have

$$\begin{aligned}
\int_E \sum_{c\in\mathcal{F}} \sum_{i\in I} |\widehat{\varphi}_i(\xi c)|^2\, d\xi &= \sum_{c\in\mathcal{F}} \sum_{i\in I} \int_E |\widehat{\varphi}_i(\xi c)|^2\, d\xi \\
&= \sum_{c\in\mathcal{F}} \sum_{i\in I} \sum_{k\in\mathbb{Z}^n} \left|\langle D_c T_k \varphi_i, \theta\rangle\right|^2 \\
&\le \sum_{c\in\mathcal{C}} \sum_{i\in I} \sum_{k\in\mathbb{Z}^n} \left|\langle D_c T_k \varphi_i, \theta\rangle\right|^2 \\
&\le D\|\theta\|^2 = D|E|.
\end{aligned}$$

Letting E range over all measurable subsets satisfying (3.10), it follows easily from the above equality that

$$\sum_{c\in\mathcal{F}} \sum_{i\in I} |\widehat{\varphi}_i(\xi c)|^2 \le D,$$

for a.e. ξ. Since $\mathcal{F}$ was an arbitrary finite subset of $\mathcal{C}$, it follows that

$$\sum_{c\in\mathcal{C}} \sum_{i\in I} |\widehat{\varphi}_i(\xi c)|^2 \le D,$$

for a.e. ξ. $\square$

If we assume some continuity of the functions $\{\widehat{\varphi}_i : i \in I\}$, we obtain the following corollary of Theorem 3.2:

Corollary 3.1. *Let $\mathcal{C}$ and I be defined as in Theorem 3.2, let $\eta \in \hat{\mathbb{R}}^n$, and let $|\mathcal{C}|$ denote the cardinality of $\mathcal{C}$. Assume that*

$$\{D_c T_k \varphi_i : c \in \mathcal{C}, k \in \mathbb{Z}^n, i \in I\} \subset L^2(\mathbb{R}^n)$$

forms a Bessel system with constant D.

(i) If $|\widehat{\varphi}_i|$ is continuous at ηc, for all i and all c, then

$$\sum_{c \in \mathcal{C}} \sum_{i \in I} |\widehat{\varphi}_i(\eta c)|^2 \leq D.$$

(ii) If η is fixed by $\mathcal{C}$ (i.e., $\eta c = \eta$, for all c) and if $|\widehat{\varphi}_i|$ is continuous at η, for each i, then

$$\sum_{i \in I} |\widehat{\varphi}_i(\eta)|^2 \leq \frac{D}{|\mathcal{C}|},$$

if $|\mathcal{C}| < \infty$, and $\widehat{\varphi}_i(\eta) = 0$, for all i, if $|\mathcal{C}| = \infty$.

(iii) If $|\widehat{\varphi}_i|$ is continuous at 0, for each i, then

$$\sum_{i \in I} |\widehat{\varphi}_i(0)|^2 \leq \frac{D}{|\mathcal{C}|},$$

if $|\mathcal{C}| < \infty$, and $\widehat{\varphi}_i(0) = 0$, for all i, if $|\mathcal{C}| = \infty$.

Proof. Since parts (ii) and (iii) follow from part (i), we shall only prove the latter. Suppose that $|\widehat{\varphi}_i|$ is continuous at ηc, for all i and all c. Let $\mathcal{F}$ be any finite subset of $\mathcal{C}$ and let F be any finite subset of I. It then follows that the function

$$\xi \mapsto \sum_{c\in\mathcal{F}} \sum_{i\in F} |\widehat{\varphi}_i(\xi c)|^2$$

is continuous at η. In conjunction with Theorem 3.2, choose a sequence $\{\xi_p\}_{p=1}^{\infty}$ converging to η which satisfies

$$\sum_{c\in\mathcal{C}} \sum_{i\in I} |\widehat{\varphi}_i(\xi_p c)|^2 \leq D,$$

and thus, in particular,

$$\sum_{c\in\mathcal{F}} \sum_{i\in F} |\widehat{\varphi}_i(\xi_p c)|^2 \leq D,$$

for each p. Using the above inequality, we obtain

$$\sum_{c\in\mathcal{F}} \sum_{i\in F} |\widehat{\varphi}_i(\eta c)|^2 = \lim_{p\to\infty} \sum_{c\in\mathcal{F}} \sum_{i\in F} |\widehat{\varphi}_i(\xi_p c)|^2 \leq D.$$

Since $\mathcal{F}$ and F were arbitrary finite subsets of $\mathcal{C}$ and I, respectively, the desired conclusion now follows. $\square$

3.2.2 The Fourier Transform at Zero

In this subsection, we generalize to the case of composite wavelets two well-known results regarding the value of the Fourier transform of a 2-wavelet and 2-scaling function at zero.

The Wavelet

If ψ is a 2-wavelet and if $|\hat{\psi}|$ is continuous at 0, then it is well-known that $\hat{\psi}(0) = 0$. The below Corollary, which is an immediate consequence of Corollary 3.1, generalizes this result to aB-multiwavelets.

Corollary 3.2. *Let $a \in GL_n(\mathbb{R})$, let B be a countable subset of $\widetilde{SL_n}(\mathbb{R})$, and let L be a countable indexing set. If $\{\psi_l : l \in L\}$ is an aB-multiwavelet and if*

$$\left|\{a^j b : j \in \mathbb{Z}, b \in B\}\right| = \infty$$

(in partciular, if $|\det a| \neq 1$), then $\widehat{\psi_l}(0) = 0$, for each l such that $|\widehat{\psi_l}|$ is continuous at 0.

The Scaling Function

If φ is a 2-scaling function and if $|\hat{\varphi}|$ is continuous at 0, then it is well-known that $|\hat{\varphi}(0)| = 1$. Corollary 3.3 below, which is an immediate consequence of Corollary 3.1, partially extends this result to aB-scaling multifunctions:

Corollary 3.3. *Let $a \in GL_n(\mathbb{R})$, let B be a countable subset of $\widetilde{SL_n}(\mathbb{R})$, and let I be a countable indexing set. Suppose that $\{\varphi_i : i \in I\}$ is an aB-scaling* *multifunction with upper frame constant D.*

(i) If $|B| < \infty$ and if $|\widehat{\varphi}_i|$ is continuous at 0 (for each i), then

$$\sum_{i \in I} |\widehat{\varphi}_i(0)|^2 \leq \frac{D}{|B|}. \tag{3.14}$$

(ii) If B is infinite, then $\widehat{\varphi}_i(0) = 0$, for each i such that $|\widehat{\varphi}_i|$ is continuous at 0.

Part (ii) above is somewhat surprising considering the intimate relationship, in the 2-wavelet case, between the nonzero value of $|\hat{\varphi}|$ at 0 and the density (property (iv) of Definition 3.2) of the associated MRA system. When B is finite, part (i) above only partially generalizes the above quoted 2-scaling function result to aB-scaling functions. If we strengthen our assumptions on the matrix a, we can complete the generalization by obtaining a nontrivial lower estimate to the sum in (3.14). In the formulation of this result (Theorem 3.3 below), we will require the following terminology:

If $c \in GL_n(\mathbb{R})$ and $\lambda \in \mathbb{C}$, λ is said to be a left eigenvalue of c if there is some $z \in \mathbb{C}^n$ with $z \neq 0$ such that $cz = \lambda z$; right eigenvalues are defined similarly. A matrix $c \in GL_n(\mathbb{R})$ is said to be left expanding if $|\lambda| > 1$, for all left eigenvalues λ of c; right expanding is defined similarly. The following result is taken from [4].

Lemma 3.2. *A matrix $c \in GL_n(\mathbb{R})$ is left expanding if and only if there exist constants k and γ with $0 < k \leq 1 < \gamma < \infty$ such that for all $j \in \mathbb{N}$ and all*

$x \in \mathbb{R}^n$ we have

$$\|c^j x\| \geq k\gamma^j \|x\|$$

or, equivalently,

$$\|c^{-j} x\| \leq k^{-1}\gamma^{-j}\|x\|.$$

It is clear that an analogous characterization holds for matrices that are right expanding. Using these two results, it follows easily that a matrix $c \in GL_n(\mathbb{R})$ is left expanding if and only if it is right expanding. When $c \in GL_n(\mathbb{R})$ satisfies either (and hence both) of these conditions we will simply say that c is expanding. We have the following result:

Theorem 3.3. *Let $a \in GL_n(\mathbb{R})$ be expanding, let B be a finite subset of $\widetilde{SL_n}(\mathbb{R})$, and let I be a finite indexing set. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed subspaces of $L^2(\mathbb{R}^n)$ and let $\{\varphi_i : i \in I\}$ be a subset of $L^2(\mathbb{R}^n)$. Suppose that*

- *the sequence $\{V_j\}_{j\in\mathbb{Z}}$ and the collection $\{\varphi_i : i \in I\}$ satisfy properties (i), (ii), (iv), and (v) in the definition of an aB-MRA, where (v) is satisfied with frame constants $C \leq D$;*
- *$|\widehat{\varphi}_i|$ is continuous at 0, for each i.*

Then

$$\frac{C}{|B|} \leq \sum_{i\in I} |\widehat{\varphi}_i(0)|^2 \leq \frac{D}{|B|}.$$

Proof. It follows from part (iii) of Corollary 3.1 that

$$\sum_{i\in I} |\widehat{\varphi}_i(0)|^2 \leq \frac{D}{|B|}.$$

We proceed as in the proof of Theorem 3.2. Let E be any bounded measurable subset of $\hat{\mathbb{R}}^n$ with $|E| = 1$ and define $\theta \in L^2(\mathbb{R}^n)$ by $\hat{\theta} = \chi_E$. Using the computations (3.11) and (3.12), for all b, k, and i and all $j \geq 0$, we have

$$\langle D_a^{-j} D_b T_k \varphi_i, \theta\rangle = |\det a|^{j/2} \int_{Ea^{-j}b} \widehat{\varphi}_i(\xi) e^{-2\pi\imath\xi k}\, d\xi.$$

Since a is expanding and since B is finite, it follows from Lemma 3.2 that, for large enough j, we have $Ea^{-j}b \subset [-1/2, 1/2]^n$, for all b. Thus, using Lemma 3.1 and the computation (3.13), we obtain

$$\sum_{k\in\mathbb{Z}^n} \left|\langle D_a^{-j} D_b T_k \varphi_i, \theta\rangle\right|^2 = |\det a|^j \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2\, d\xi, \tag{3.15}$$

for all b and i, when j is large enough. For these j, let P_j denote the orthogonal projection of $L^2(\mathbb{R}^n)$ onto V_j. Since

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\}$$

forms a frame for V_0 with constants $C \leq D$ and since the operator D_a is unitary, it follows that

$$\{D_a^{-j} D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\}$$

forms a frame for V_j with constants $C \leq D$. Thus, using (3.15), we have

$$\begin{aligned}
C\|P_j\theta\|^2 &\leq \sum_{b\in B}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n} \left|\langle D_a^{-j} D_b T_k \varphi_i, P_j\theta\rangle\right|^2 \\
&= \sum_{b\in B}\sum_{i\in I}\sum_{k\in\mathbb{Z}^n} \left|\langle D_a^{-j} D_b T_k \varphi_i, \theta\rangle\right|^2 \\
&= \sum_{b\in B}\sum_{i\in I} |\det a|^j \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2 \, d\xi \\
&= \sum_{b\in B}\sum_{i\in I} \frac{1}{|Ea^{-j}b|} \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2 \, d\xi,
\end{aligned}$$

for all large enough j. Using properties (i), (ii), and (iv) in the aB-MRA definition, the continuity of each $|\widehat{\varphi}_i|$ at 0, and that a is expanding, we obtain

$$\begin{aligned}
C = C\|\theta\|^2 &= \lim_{j\to\infty} C\|P_j\theta\|^2 \\
&\leq \lim_{j\to\infty} \sum_{b\in B}\sum_{i\in I} \frac{1}{|Ea^{-j}b|} \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2 \, d\xi \\
&= \sum_{b\in B}\sum_{i\in I} \lim_{j\to\infty} \frac{1}{|Ea^{-j}b|} \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2 \, d\xi \\
&= \sum_{b\in B}\sum_{i\in I} |\widehat{\varphi}_i(0)|^2 = |B| \sum_{i\in I} |\widehat{\varphi}_i(0)|^2.
\end{aligned}$$

This completes the proof. □

3.2.3 The Fourier Transform at Nonzero Integers

In this subsection, we generalize to the case of composite wavelets two well-known results regarding the value of the Fourier transform of a 2-wavelet and 2-scaling function at nonzero integers.

The Scaling Function

If φ is a 2-scaling function and if $\hat{\varphi}$ satisfies certain regularity assumptions, then it follows that $\hat{\varphi}(k) = 0$, for all $k \in \mathbb{Z} \setminus \{0\}$. The below proposition extends this result to aB scaling functions, when B is finite.

Proposition 3.1. *Let $a \in GL_n(\mathbb{R})$ be expanding and let B be a finite subset of*

$$\widetilde{SL_n}(\mathbb{Z}) = \{c \in GL_n(\mathbb{Z}) : |\det c| = 1\}.$$

Suppose that φ is an aB-scaling function (with frame constants $C \leq D$) such that $\hat{\varphi}$ is continuous at each $k \in \hat{\mathbb{Z}}^n$. Then

$$\sum_{k \in \hat{\mathbb{Z}}^n \setminus \{0\}} \Big| \sum_{b \in B} \hat{\varphi}(kb) \Big|^2 \leq (D - C)|B|.$$

In particular, if $C = D$, then $\sum_{b \in B} \hat{\varphi}(kb) = 0$ for all $k \in \hat{\mathbb{Z}}^n \setminus \{0\}$.

Proof. Since $B \subset \widetilde{SL_n}(\mathbb{Z})$, the collection

$$\{D_b T_k \varphi : b \in B, k \in \mathbb{Z}^n\}$$

may be written as

$$\{T_k D_b \varphi : b \in B, k \in \mathbb{Z}^n\}.$$

Since this collection is, in particular, Bessel with constant D, it follows from Corollary 2.1 that

$$\begin{aligned}\sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}\hat{\varphi}((\xi+k)b)\Big|^2 &= \sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}1\cdot\hat{\varphi}((\xi+k)b)\Big|^2\\ &\le D\sum_{b\in B}1^2 = D|B|,\end{aligned}$$

for a.e. ξ. Since $\hat{\varphi}$ is continuous at each $k \in \hat{\mathbb{Z}}^n$, it follows easily that

$$\sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}\hat{\varphi}(kb)\Big|^2 = \sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}\hat{\varphi}((0+k)b)\Big|^2 \le D|B|.$$

Using this and Theorem 3.3, we obtain

$$\begin{aligned}\sum_{k\in\hat{\mathbb{Z}}^n\setminus\{0\}}\Big|\sum_{b\in B}\hat{\varphi}(kb)\Big|^2 &= \sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}\hat{\varphi}(kb)\Big|^2 - \Big|\sum_{b\in B}\hat{\varphi}(0b)\Big|^2\\ &= \sum_{k\in\hat{\mathbb{Z}}^n}\Big|\sum_{b\in B}\hat{\varphi}(kb)\Big|^2 - |B|^2|\hat{\varphi}(0)|^2\\ &\le D|B| - |B|^2\frac{C}{|B|}\\ &= (D-C)|B|,\end{aligned}$$

which completes the proof. □

It is natural to wonder if a version of Proposition 3.1 holds for a-scaling multifunctions. For instance, if $a \in GL_n(\mathbb{R})$ is expanding, if I is a finite indexing set, and if $\{\varphi_i : i \in I\}$ is an ON a-scaling multifunction such that, for each i, $\widehat{\varphi_i}$ is continuous at each $k \in \hat{\mathbb{Z}}^n$, does it then follow that

$$\sum_{i \in I} \widehat{\varphi_i}(k) = 0,$$

for all $k \in \hat{\mathbb{Z}}^n \setminus \{0\}$? The answer to this question is "no". To see why, assume that the answer is "yes" and that $\{\varphi_i : i \in I\}$ is such an ON a-scaling multifunction. Then,

$$\{\alpha_i \varphi_i : i \in I\} \tag{3.16}$$

is also such an ON a-scaling function, for any collection $\{\alpha_i : i \in I\} \subset \mathbb{C}$ satisfying $|\alpha_i| = 1$, for all i. Applying the assumed result to appropriate collections of the form (3.16), it follows that

$$\widehat{\varphi_i}(k) = 0, \text{ for all } k \in \hat{\mathbb{Z}}^n \setminus \{0\} \text{ and all } i. \tag{3.17}$$

Thus, to obtain a contradiction, it suffices to exhibit an ON a-scaling multifunction of the above sort that does not satisfy (3.17). It will be clear from the discussion of section 3.2.5 that the aB-scaling function of Example 3.2 is an example of such an a-scaling multifunction.

The Wavelet

If ψ is a 2-MRA wavelet with associated 2-scaling function φ and if $\hat{\psi}$ and $\hat{\varphi}$ satisfy certain regularity assumptions, then it follows that $\hat{\psi}(2k) = 0$, for all $k \in \mathbb{Z}$. The corollary below extends this result to aB-multiwavelets, when B is finite.

Corollary 3.4. *Let $a \in GL_n(\mathbb{R})$ be expanding, let B be a finite subgroup of $\widetilde{SL_n}(\mathbb{Z})$, and let L be a countable indexing set. Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is an aB-MRA with Parseval frame scaling function φ such that $\hat{\varphi}$ is continuous at each $k \in \hat{\mathbb{Z}}^n$. Suppose that $\{\psi_l : l \in L\}$ is an associated aB-multiwavelet. If, for fixed l, we have that*

(i) $\widehat{\psi_l}$ is continuous at ka, for all $k \in \hat{\mathbb{Z}}^n$, and

(ii) $[\widehat{\psi_l}(\cdot a), \hat{\varphi}(\cdot b)]$ is continuous at 0, for all b,

then

$$\sum_{b\in B} \widehat{\psi_l}(kba) = 0,$$

for all $k \in \hat{\mathbb{Z}}^n$. In particular, if a normalizes B (i.e., if $aBa^{-1} \subset B$), then

$$\sum_{b\in B} \widehat{\psi_l}(kb) = 0,$$

for all $k \in \hat{\mathbb{Z}}^n a$.

Proof. Since the second assertion follows from the first, we shall only prove the latter. Assume that we have fixed an l such that conditions (i) and (ii) in the statement of this corollary hold. First note that by Corollary 3.2, we have

$$\sum_{b\in B}\widehat{\psi}_l(0ba) = |B|\widehat{\psi}_l(0) = 0.$$

Since $D_a\psi_l \in V_0$, using part (vi) of Lemma 2.1, it follows that

$$\begin{aligned}
|\det a|^{1/2}\widehat{\psi}_l(\xi a) &= \widehat{D_a\psi_l}(\xi) \\
&= \sum_{b\in B}[\widehat{D_a\psi_l}, \widehat{D_b\varphi}](\xi)\widehat{D_b\varphi}(\xi) \\
&= |\det a|^{1/2}\sum_{b\in B}[\widehat{\psi}_l(\cdot a), \hat{\varphi}(\cdot b)](\xi)\hat{\varphi}(\xi b),
\end{aligned}$$

for a.e. ξ. By the various continuity assumptions, it follows that the above equality must hold for all $k \in \hat{\mathbb{Z}}^n$. Thus, using Proposition 3.1 and that B is a group, for any $k \in \hat{\mathbb{Z}}^n \setminus \{0\}$ we have

$$\begin{aligned}
\sum_{b\in B}\widehat{\psi}_l(kba) &= \sum_{b\in B}\sum_{b'\in B}[\widehat{\psi}_l(\cdot a), \hat{\varphi}(\cdot b')](kb)\hat{\varphi}(kbb') \\
&= \sum_{b\in B}\sum_{b'\in B}[\widehat{\psi}_l(\cdot a), \hat{\varphi}(\cdot b')](0)\hat{\varphi}(kbb') \\
&= \sum_{b'\in B}[\widehat{\psi}_l(\cdot a), \hat{\varphi}(\cdot b')](0)\sum_{b\in B}\hat{\varphi}(kbb') = 0.
\end{aligned}$$

This completes the proof. $\square$

3.2.4 The Nonexistence of aB-Scaling Multifunctions of Haar-Type When B is Infinite

We have the following interesting corollary of Corollary 3.1:

Corollary 3.5. *Let $\mathcal{C}$ be a countably infinite subset of $GL_n(\mathbb{R})$.*

(i) If φ is a nonzero element of $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that satisfies $\varphi(x) \geq 0$, for a.e. x, then the collection $\{D_c T_k \varphi : c \in \mathcal{C}, k \in \mathbb{Z}^n\}$ cannot form a Bessel system.

(ii) If E is a measurable subset of $\mathbb{R}^n$ with finite positive measure, then the collection $\{D_c T_k \chi_E : c \in \mathcal{C}, k \in \mathbb{Z}^n\}$ cannot form a Bessel system.

Proof. Since part (ii) follows from part (i), we only prove the latter. Suppose that φ belongs to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and satisfies $\varphi(x) \geq 0$, for a.e. x, and assume that the collection $\{D_c T_k \varphi : c \in \mathcal{C}, k \in \mathbb{Z}^n\}$ is a Bessel system. Since $\varphi \in L^1(\mathbb{R}^n)$, $\hat{\varphi}$ is continuous. Thus, using part (iii) of Corollary 3.1, we have

$$0 = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)\, dx.$$

Since φ is nonnegative, the above equality implies that $\varphi(x) = 0$, for a.e. x. This completes the proof. □

The scaling functions of Examples 3.1 and 3.2 are clearly related. We formalize this relation with the following definition:

Definition 3.7. *Let a, B, and I be as in Corollary 3.2. We say an aB-scaling multifunction*

$$\{\varphi_i : i \in I\} \subset L^2(\mathbb{R}^n)$$

is of Haar-type if, for each i, φ_i is of the form $\alpha_i \chi_{E_i}$, for some $\alpha_i > 0$ and some measurable subset E_i of $\mathbb{R}^n$.

Example 3.2 shows that aB scaling functions of Haar-type exist when B is nontrivial and finite. When B is infinite, however, we have the following striking result, which is an immediate consequence of Corollary 3.5.

Corollary 3.6. *Let a and B be as in Corollary 3.2 and let I be a countable indexing set; assume that $|B| = \infty$.*

(i) There does not exist an aB-scaling multifunction $\{\varphi_i : i \in I\}$ such that, for each i, $\varphi_i \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\varphi_i(x) \geq 0$, for a.e. x.

(ii) There does not exist an aB-scaling multifunction of Haar-type.

Define

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Recall that, in Example 3.3, an aB-scaling function φ of minimally supported frequency (MSF) type is constructed; that is, $\hat{\varphi} = \chi_E$, for some measurable subset E of $\hat{\mathbb{R}}^2$. It is interesting to compare this example with part (ii) of Corollary 3.6 (which implies that no aB-scaling multifunctions of Haar-type exist).

3.2.5 An Example

We now verify some of the results of this section within the context of a specific example. Let a, B, T_1, T_2, φ, and ψ be as in Example 3.2. Note that a is expanding and that a normalizes B. Using that $|T_1| = |T_2| = 1/16$, we obtain

$$\begin{aligned}\hat{\psi}(0) = \int_{\mathbb{R}^2} \psi(x)\, dx &= 2\sqrt{2}\Big(\int_{T_1} 1\, dx - \int_{T_2} 1\, dx\Big) \\ &= 2\sqrt{2}(|T_1| - |T_2|) = 0\end{aligned}$$

and

$$\begin{aligned}\hat{\varphi}(0) = \int_{\mathbb{R}^2} \varphi(x)\, dx &= 2\sqrt{2}\Big(\int_{T_1} 1\, dx + \int_{T_2} 1\, dx\Big) \\ &= 2\sqrt{2}(|T_1| + |T_2|) \\ &= \frac{2\sqrt{2}}{8} = \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{|B|}},\end{aligned}$$

verifying Corollary 3.2 and Theorem 3.3. We now examine the results of Proposition 3.1 and Corollary 3.4 in this context. Calculations show that

$$\hat{\varphi}(\xi_1, \xi_2) = \begin{cases} \frac{\sqrt{2}(1-e^{-\pi\imath\xi_2})}{2\pi^2\xi_2^2} + \frac{\sqrt{2}}{2\pi\imath\xi_2}, & \text{if } \xi_1 = 0, \xi_2 \neq 0; \\ \frac{\sqrt{2}(e^{-\pi\imath\xi_1}-1)}{2\pi^2\xi_1^2} + \frac{-\sqrt{2}e^{-\pi\imath\xi_1}}{2\pi\imath\xi_1}, & \text{if } \xi_1 \neq 0, \xi_2 = 0. \end{cases}$$

and that

$$\hat{\psi}(\xi_1, \xi_2) = \begin{cases} \frac{\sqrt{2}(3-4e^{-\pi\imath\xi_2/2}+e^{-\pi\imath\xi_2})}{2\pi^2\xi_2^2} + \frac{\sqrt{2}}{2\pi\imath\xi_2}, & \text{if } \xi_1 = 0, \xi_2 \neq 0; \\ \frac{\sqrt{2}(4e^{-\pi\imath\xi_1/2}-3e^{-\pi\imath\xi_1}-1)}{2\pi^2\xi_1^2} + \frac{-\sqrt{2}e^{-\pi\imath\xi_1}}{2\pi\imath\xi_1}, & \text{if } \xi_1 \neq 0, \xi_2 = 0. \end{cases}$$

We have

$$\begin{aligned} \sum_{b\in B} \hat{\varphi}((1,0)b) =& 2\hat{\varphi}((1,0)) + 2\hat{\varphi}((-1,0)) + 2\hat{\varphi}((0,1)) + 2\hat{\varphi}((0,-1)) \\ =& 2\Big(-\frac{\sqrt{2}}{\pi^2} + \frac{\sqrt{2}}{2\pi\imath}\Big) + 2\Big(-\frac{\sqrt{2}}{\pi^2} - \frac{\sqrt{2}}{2\pi\imath}\Big) \\ &+ 2\Big(\frac{\sqrt{2}}{\pi^2} + \frac{\sqrt{2}}{2\pi\imath}\Big) + 2\Big(\frac{\sqrt{2}}{\pi^2} - \frac{\sqrt{2}}{2\pi\imath}\Big) \\ =& 0, \end{aligned}$$

which is in accordance with Proposition 3.1 and completes the discussion began after its proof. Note that

$$(2,0) = (0,1)a^2 \in \hat{\mathbb{Z}}^2 a.$$

We have

$$\begin{aligned}\sum_{b\in B}\hat{\psi}((2,0)b) =&2\hat{\psi}((2,0)) + 2\hat{\psi}((-2,0)) + 2\hat{\psi}((0,2)) + 2\hat{\psi}((0,-2))\\ =&2\Big(-\frac{\sqrt{2}}{\pi^2}-\frac{\sqrt{2}}{4\pi\imath}\Big)+2\Big(-\frac{\sqrt{2}}{\pi^2}+\frac{\sqrt{2}}{4\pi\imath}\Big)\\ &+2\Big(\frac{\sqrt{2}}{\pi^2}+\frac{\sqrt{2}}{4\pi\imath}\Big)+2\Big(\frac{\sqrt{2}}{\pi^2}-\frac{\sqrt{2}}{4\pi\imath}\Big)\\ =&0,\end{aligned}$$

which is in accordance with the last assertion made in Corollary 3.4.

3.3 Dependencies in the MRA Definition

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ is a collection of closed subspaces of $L^2(\mathbb{R})$ and that $\varphi \in L^2(\mathbb{R})$. The following dependencies in the the 2-MRA definition (Definition 3.2) are well-known:

(I) If $\{V_j\}_{j\in\mathbb{Z}}$ and φ satisfy properties (i), (ii), and (v) in the 2-MRA definition, then

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

(II) If $\{V_j\}_{j\in\mathbb{Z}}$ and φ satisfy properties (i), (ii), and (v) in the 2-MRA definition and if $|\hat{\varphi}|$ is continuous and nonzero at 0, then

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Theorems 3.4 and 3.5 below extend these results to aB-MRAs. A very interesting corollary of Theorem 3.4 is the nonexistence of a-multiwavelets, for all $a \in \widetilde{SL_n}(\mathbb{Z})$.

3.3.1 Dependency I

Before we state and prove Theorem 3.4 (which extends dependency I to aB-MRAs) we need the following definition and lemma:

Definition 3.8. *Let $a \in GL_n(\mathbb{Z})$ and let B be a subset of $\widetilde{SL_n}(\mathbb{Z})$. We say the pair (a, B) is admissible if there exists a countable collection $\mathcal{E}$ of measurable subsets of $\hat{\mathbb{R}}^n$ satisfying the following:*

(i) $\bigcup\{E \in \mathcal{E}\} = \mathbb{R}^n$ *(in measure);*

(ii) $|Ea^j b_1 \cap Ea^j b_2| = 0$, *for all* $j \in \mathbb{Z}^+$, *all* $b_1 \neq b_2$ *in* B, *and all* $E \in \mathcal{E}$;

(iii) for each fixed $E \in \mathcal{E}$ *and positive* r *and* ϵ, *we have*

$$\left|\Big(\bigcup_{b\in B} Ea^j b\Big) \cap \{\xi \in \hat{\mathbb{R}}^n : \|\xi\| \leq r\}\right| \leq \epsilon,$$

for some j.

We make the following observations regarding admissibility:

(i) The assumption "a normalizes B" significantly simplifies the admissibility criterion: If $a \in GL_n(\mathbb{Z})$, if B is a subset of $\widetilde{SL_n}(\mathbb{Z})$, and if

a normalizes B, then (a, B) is admissible if there exists a countable collection $\mathcal{E}$ of measurable subsets of $\hat{\mathbb{R}}^n$ satisfying the following:

(a) $\bigcup\{E \in \mathcal{E}\} = \mathbb{R}^n$ (in measure);

(b) $|Eb_1 \cap Eb_2| = 0$, for all $b_1 \neq b_2$ in B and all $E \in \mathcal{E}$;

(c) for each fixed $E \in \mathcal{E}$ and positive r and ϵ, we have

$$\Big|\Big(\bigcup_{b \in B} Eb\Big)a^j \cap \{\xi \in \hat{\mathbb{R}}^n : \|\xi\| \leq r\}\Big| \leq \epsilon,$$

for some j.

(ii) If $a \in GL_n(\mathbb{Z})$ is expanding, if B is a finite subset of $\widetilde{SL_n}(\mathbb{Z})$, and if a normalizes B, then (a, B) is admissible.

(iii) If

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\},$$

then (a, B) is admissible.

(iv) Let $b \in \widetilde{SL_n}(\mathbb{Z})$ and write $B = \{b^j : j \in \mathbb{Z}\}$. Writing b in its Jordan canonical form and using that

$$\{(e^{2\pi \imath l\theta_1}, \ldots, e^{2\pi \imath l\theta_m}) : l \in \mathbb{Z}\}$$

is uniformly distributed (and, in particular, topologically dense) in the set

$$\{(z_1, \ldots, z_m) \in \hat{\mathbb{C}}^m : |z_1| = \cdots = |z_m| = 1\}$$

(when $1, \theta_1, \ldots, \theta_m$ are rationally independent), one can show that (a, B) is admissible, for any expanding matrix $a \in GL_n(\mathbb{Z})$ that normalizes B. In particular, (a, B) is admissible, where $a \in GL_n(\mathbb{Z})$ is the diagonal matrix with all diagonal entries equal to 2.

(v) If B is the group

$$SL_2(\mathbb{Z}) = \{b \in GL_2(\mathbb{Z}) : \det b = 1\},$$

then (a, B) is not admissible for any $a \in GL_2(\mathbb{Z})$. This follows from the fact that for $(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2$ with $\xi_2 \neq 0$ and $\xi_1/\xi_2 \notin \mathbb{Q}$, 0 is in the closure of the orbit $\{\xi b : b \in B\}$.

The proof of the following lemma is straightforward and is omitted:

Lemma 3.3. *If $c \in GL_n(\mathbb{Z})$ and if $g \in L^1(\mathbb{T}^n)$, then*

$$\int_{[0,1]^n c} g(\xi)\, d\xi = |\det c| \int_{[0,1]^n} g(\xi)\, d\xi.$$

We have the following result, which extends Dependency I to aB-MRAs:

Theorem 3.4. *Let $a \in GL_n(\mathbb{Z})$, let B be a subset of $\widetilde{SL_n}(\mathbb{Z})$, and let I be a* *finite indexing set. Suppose that (a, B) is admissible. If the collection*

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\} \subset L^2(\mathbb{R}^n)$$

forms a frame for its closed linear span V, then

$$\bigcap_{j \in \mathbb{Z}^+} D_a^j V = \{0\}.$$

Proof of Theorem 3.4. In order to obtain a contradiction, suppose that

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\} \subset L^2(\mathbb{R}^n)$$

forms a frame for its closed linear span V with constants $C \leq D$ and that there exists $f \in \bigcap_{j \in \mathbb{Z}^+} D_a^j V$ with $\|f\| = 1$. Fix $j \in \mathbb{Z}^+$. Since $D_a^{-j} f \in V$, it follows from parts (iv) and (vi) of Lemma 2.1 that

$$|\det a|^{-j/2} \hat{f}(\xi a^{-j}) = \widehat{D_a^{-j} f}(\xi) = \sum_{i \in I, b \in B} m_{ib}^j(\xi) \widehat{\varphi}_i(\xi b),$$

with unconditional convergence in $L^2(\mathbb{R}^n)$, where the collection

$$\{m_{ib}^j(\xi) : i \in I, b \in B\} \subset L^2(\hat{\mathbb{T}}^n)$$

satisfies

$$\sum_{i \in I} \sum_{b \in B} \|m_{ib}^j\|^2 \leq C^{-1} \|f\|^2 = C^{-1}.$$

Using both the Schwarz inequality and that convergent sequences in $L^2(\mathbb{R}^n)$ contain subsequences that converge pointwise a.e., it follows that

$$|\hat{f}(\xi a^{-j})| \le |\det a|^{j/2}\Big(\sum_{i\in I, b\in B} |m_{ib}^j(\xi)|^2\Big)^{1/2}\Big(\sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\Big)^{1/2},$$

for a.e. ξ.

Choose $\mathcal{E}$ as in Definition 3.8. We may refine $\mathcal{E}$ so that for each $E \in \mathcal{E}$, E is contained in some cube K of the form $K = [0,1]^n + k$, for some $k \in \hat{\mathbb{Z}}^n$. Fix such an $E \in \mathcal{E}$ and a cube K with $E \subset K$. Using a change of variable, the Schwarz inequality, and Lemma 3.3, we obtain

$$\begin{aligned}
\int_E |\hat{f}(\xi)|\,d\xi &= |\det a|^{-j} \int_{Ea^j} |\hat{f}(\xi a^{-j})|\,d\xi \\
&\le |\det a|^{-j/2} \int_{Ea^j} \Big(\sum_{i\in I, b\in B} |m_{ib}^j(\xi)|^2\Big)^{1/2}\Big(\sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\Big)^{1/2} d\xi \\
&\le |\det a|^{-j/2}\bigg(\int_{Ea^j} \sum_{i\in I, b\in B} |m_{ib}^j(\xi)|^2\bigg)^{1/2}\bigg(\int_{Ea^j} \sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\,d\xi\bigg)^{1/2} \\
&\le \bigg(|\det a|^{-j} \int_{Ka^j} \sum_{i\in I, b\in B} |m_{ib}^j(\xi)|^2\bigg)^{1/2}\bigg(\int_{Ea^j} \sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\,d\xi\bigg)^{1/2} \\
&= \bigg(\int_K \sum_{i\in I, b\in B} |m_{ib}^j(\xi)|^2\bigg)^{1/2}\bigg(\int_{Ea^j} \sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\,d\xi\bigg)^{1/2} \\
&\le C^{-1/2}\bigg(\int_{Ea^j} \sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2\,d\xi\bigg)^{1/2}.
\end{aligned}$$

Using a change of variable and admissibility property (ii), we obtain

$$\begin{aligned}\int_{Ea^j} \sum_{i\in I, b\in B} |\widehat{\varphi}_i(\xi b)|^2 \, d\xi &= \sum_{b\in B} \int_{Ea^j} \sum_{i\in I} |\widehat{\varphi}_i(\xi b)|^2 \, d\xi \\ &= \sum_{b\in B} \int_{Ea^j b} \sum_{i\in I} |\widehat{\varphi}_i(\xi)|^2 \, d\xi \\ &= \int_{\bigcup_{b\in B} Ea^j b} \sum_{i\in I} |\widehat{\varphi}_i(\xi)|^2 \, d\xi.\end{aligned}$$

Varying j and using admissibility property (iii), it follows from the above two calculations that

$$\int_E |\hat{f}(\xi)| \, d\xi = 0.$$

By admissibility property (i), it follows that $\hat{f}(\xi) = 0$, for a.e. ξ, a contradiction. This completes the proof. $\square$

We make the following comments regarding Theorem 3.4:

(i) If $a \in GL_n(\mathbb{Z})$ is expanding, if B is a finite subset of $\widetilde{SL_n}(\mathbb{Z})$, and if I is a finite indexing set, then we may omit the assumption "(a, B) is admissible" from the statement of Theorem 3.4, since, in this case, the collection

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\} \subset L^2(\mathbb{R}^n)$$

can be rewritten as

$$\{T_k D_b \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\}.$$

(ii) It follows from the existence of Gabor systems that the assumption "I is finite" in the statement of Theorem 3.4 is necessary.

We have the following interesting corollary of Theorem 3.4:

Corollary 3.7. *Let B be a subset of $\widetilde{SL_n}(\mathbb{Z})$, let I be a finite indexing set, and let*

$$\{\varphi_i : i \in I\} \subset L^2(\mathbb{R}^n).$$

If there exists a matrix $a \in GL_n(\mathbb{Z})$ for which (a, B) is admissible, then the collection

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i \in I\}$$

cannot form a frame for $L^2(\mathbb{R}^n)$.

Using comment (iv) following Definition 3.8 and the above corollary, we obtain the following very interesting result regarding the nonexistence of certain a-multiwavelets:

Corollary 3.8. *There does not exist an a-multiwavelet $\{\psi_1, \ldots, \psi_L\}$ ($L \in \mathbb{Z}^+$) for any $a \in \widetilde{SL_n}(\mathbb{Z})$.*

3.3.2 Dependency II

We have the following result, which extends Dependency II to aB-MRAs:

Theorem 3.5. *Let $a \in GL_n(\mathbb{Z})$ be expanding, let B be a subset of $\widetilde{SL_n}(\mathbb{Z})$, and let I be a countable indexing set. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed* *subspaces of* $L^2(\mathbb{R}^n)$ *and let*

$$\{\varphi_i : i \in I\} \subset L^2(\mathbb{R}^n).$$

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ *and* $\{\varphi_i : i \in I\}$ *satisfy properties (i), (ii), and (v) in the the aB-MRA definition. If there exists a measurable set* E *such that*

(i) $\sum_{i\in I} |\widehat{\varphi}_i(\xi)|^2 > 0$, *for all* $\xi \in E$ *and*

(ii) $\bigcup_{j\in\mathbb{Z}} \bigcup_{b\in B} Eb^{-1}a^j = \hat{\mathbb{R}}^n$ *(in measure),*

then $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$*. In particular, if, for some* i, $|\widehat{\varphi}_i|$ *is continuous and nonzero at* 0*, then* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$.

Proof. Write $V = \bigcup_{j\in\mathbb{Z}} V_j$. We claim that $\overline{V}$ is translation invariant, i.e., that $T_y V \subset V$, for all $y \in \mathbb{R}^n$. If $f \in V$, $p \in \mathbb{Z}^+$, and $l \in \mathbb{Z}^n$, it follows that we may write f in the form

$$f = \sum_{b\in B, k\in\mathbb{Z}^n, i\in I} \alpha_{bki} D_a^{-j} D_b T_k \varphi_i,$$

with unconditional convergence in $L^2(\mathbb{R}^n)$, for some $j \geq p$ and some sequence $\{\alpha_{bki}\} \in l^2(B \times \mathbb{Z}^n \times I)$. Thus,

$$\begin{aligned} T_{a^{-p}l} f &= \sum_{b\in B, k\in\mathbb{Z}^n, i\in I} \alpha_{bki} T_{a^{-p}l} D_a^{-j} D_b T_k \varphi_i \\ &= \sum_{b\in B, k\in\mathbb{Z}^n, i\in I} \alpha_{bki} D_a^{-j} D_b T_{b^{-1}a^{j-p}l+k} \varphi_i \in V_j \subset V, \end{aligned}$$

since $b^{-1}a^{j-p}l \in \mathbb{Z}^n$. We have shown that $T_{a^{-p}l}V \subset V$. Since $T_{a^{-p}l}$ is unitary, it follows that $T_{a^{-p}l}\overline{V} \subset \overline{V}$. Since a is expanding, it follows from Lemma 3.2 that the collection

$$D = \{a^{-p}l : p \in \mathbb{Z}^+, l \in \mathbb{Z}^n\}$$

is dense in $\mathbb{R}^n$. Now, if $f \in \overline{V}$ and $y \in \mathbb{R}^n$, choose a sequence $\{y_q\}_{q=1}^{\infty}$ in D converging to y. The above argument implies that $T_{y_q}f \in \overline{V}$, for each q. Since $T_{y_q}f \to T_y f$ in $L^2(\mathbb{R}^n)$, it follows that $T_y f \in \overline{V}$.

To show that $\overline{V} = L^2(\mathbb{R}^n)$, let $g \in \overline{V}^{\perp}$. For any $f \in \overline{V}$, the function $\hat{f}\,\overline{\hat{g}}$ belongs to $L^1(\hat{\mathbb{R}}^n)$. For $y \in \mathbb{R}^n$, using the Plancherel theorem and the translation invariance of $\overline{V}$, we calculate

$$\begin{aligned}
(\hat{f}\,\overline{\hat{g}})^{\vee}(y) &= \int_{\hat{\mathbb{R}}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}e^{2\pi\imath\xi y}\,d\xi \\
&= \langle e^{2\pi\imath\cdot y}\hat{f}, \hat{g}\rangle \\
&= \langle \widehat{T_{-y}f}, \hat{g}\rangle \\
&= \langle T_{-y}f, g\rangle = 0.
\end{aligned}$$

It follows that $\hat{f}(\xi)\overline{\hat{g}(\xi)} = 0$, for a.e. ξ. For each i, j, and b, setting $f = D_a^j D_b \varphi_i$, we see that

$$\widehat{\varphi}_i(\xi a^j b)\overline{g(\xi)} = 0,$$

for a.e. ξ. By assumptions (i) and (ii) in the statement of this proposition, it follows that $g(\xi) = 0$, for a.e. ξ. This completes the proof. □

Chapter 4

Decay, Regularity, and the Fourier Transform

For $p = 1, \ldots, n$, x_p and ξ_p denote the p^{th} coordinate functions in $\mathbb{R}^n$ and $\hat{\mathbb{R}}^n$, respectively. We write ∂_p in place of $\frac{\partial}{\partial x_p}$ (or $\frac{\partial}{\partial \xi_p}$). The results of the following lemma are well-known (see, for instance, Theorem 8.22 of [2]):

Lemma 4.1. *We have the following:*

(i) If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\hat{\mathbb{R}}^n)$.

(ii) If $f, x_p f \in L^1(\mathbb{R}^n)$, then, for all ξ, $\partial_p \hat{f}(\xi)$ exists and is given by

$$\partial_p \hat{f}(\xi) = (-2\pi\imath x_p f)\widehat{\ }(\xi).$$

In particular, $\partial_p \hat{f} \in C_0(\hat{\mathbb{R}}^n)$.

(iii) *If* $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ *and* $\partial_p f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$*, then*

$$(\partial_p f)\hat{}(\xi) = 2\pi\imath\xi_p \hat{f}(\xi),$$

for all ξ. *In particular,* $\xi_p \hat{f} \in C_0(\hat{\mathbb{R}}^n)$.

In this chapter, we state and prove several generalized versions of Lemma 4.1 in which smoothness and integral polynomial decay are replaced with Hölder continuity and fractional polynomial decay (section 1); logarithmic continuity and logarithmic decay (section 2); iterated Hölder continuity and multivariable fractional polynomial decay (section 3). These results will be needed in the following chapter.

4.1 Hölder Continuity and Fractional Polynomial Decay

In this section, we state and prove an analog of Lemma 4.1 in which smoothness and integral polynomial decay are replaced with Hölder continuity and fractional polynomial decay. We have the following definition:

Definition 4.1. *Let* $e_1, \ldots, e_n$ *and* $\hat{e}_1, \ldots, \hat{e}_n$ *be the canonical basis vectors of* $\mathbb{R}^n$ *and* $\hat{\mathbb{R}}^n$*, respectively. Let* $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ *be measurable and let* $0 < \alpha \leq 1$.

(i) f *is said to be Hölder continuous in the direction* e_p *with exponent* α *if*

$f \in L^\infty(\mathbb{R}^n)$ and if there exists $0 \leq M < \infty$ such that

$$\frac{\|f(x+te_p) - f(x)\|_\infty}{|t|^\alpha} \leq M, \tag{4.1}$$

for all $t \neq 0$.

(ii) f is said to be locally Hölder continuous in the direction e_p with exponent α if there exists $0 \leq M < \infty$ such that (4.1) holds for all $0 < |t| \leq 1$. Note that we are not requiring that $f \in L^\infty(\mathbb{R}^n)$.

(iii) f is said to be strongly Hölder continuous in the direction e_p with exponent α if $f \in L^\infty(\mathbb{R}^n)$ and if

$$\frac{\|f(x+te_p) - f(x)\|_\infty}{|t|^\alpha} \leq M(t), \tag{4.2}$$

for all $t \neq 0$, where the function M is bounded and satisfies $M(t) \to 0$, as $t \to 0$.

(iv) f is said to be strongly locally Hölder continuous in the direction e_p with exponent α if (4.2) holds for all $0 < |t| \leq 1$, where the function M is bounded and satisfies $M(t) \to 0$, as $t \to 0$. Again, note that we are not requiring that $f \in L^\infty(\mathbb{R}^n)$.

(v) f is said to be L^1-Hölder continuous in the direction e_p with exponent

α if $f \in L^1(\mathbb{R}^n)$ and if there exists $0 \leq M < \infty$ such that

$$\int_{\mathbb{R}^n} \frac{|f(x+te_p) - f(x)|}{|t|^\alpha}\, dx \leq M,$$

for all $t \neq 0$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable and let $0 < \alpha \leq 1$. We make the following comments regarding Definition 4.1:

(i) If f is (locally) Hölder continuous in the direction e_p with exponent α, then f is strongly (locally) Hölder continuous in the direction e_p with exponent β, for all $0 < \beta < \alpha$.

(ii) If f is everywhere defined, if $\partial_p f(x)$ exists for all x, and if there exists $0 \leq M < \infty$ such that $|\partial_p f(x)| \leq M$ (for all x), then f is locally Hölder continuous in the direction e_p with exponent 1. If, in addition, $f \in L^\infty(\mathbb{R}^n)$, then f is Hölder continuous in the direction e_p with exponent 1.

(iii) If f is L^1-Hölder continuous in the direction e_p with exponent α, then f is L^1-Hölder continuous in the direction e_p with exponent β, for all $0 < \beta < \alpha$.

(iv) If $f \in L^1(\mathbb{R}^n)$ is everywhere defined, if $\partial_p f(x)$ exists for all x, and if $\partial_p f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then f is L^1-Hölder continuous in the direction e_p with exponent 1.

(v) (local) Hölder continuity, strong (local) Hölder continuity, and L^1-Hölder continuity are defined similarly for functions $f : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$; the analogs of properties (i) through (iv) above hold in this context as well.

We have the following version of Lemma 4.1:

Lemma 4.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable and let $0 < \alpha \leq 1$.*

(i) If $|x_p|^\alpha f \in L^1(\mathbb{R}^n)$, then there exists $0 \leq M < \infty$ such that

$$\int_{\mathbb{R}^n} \frac{|e^{-2\pi\imath t x_p} - 1|}{|t|^\alpha} |f(x)|\, dx \leq M, \tag{4.3}$$

for all $t \neq 0$. If, in addition, $\alpha < 1$, then the left hand side of (4.3) approaches 0 as $t \to 0$.

(ii) If $f, |x_p|^\alpha f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is Hölder continuous in the direction $\hat{e}_p$ with exponent α. If, in addition, $\alpha < 1$, then $\hat{f}$ is strongly Hölder continuous in the direction $\hat{e}_p$ with exponent α.

(iii) If $f \in L^1(\mathbb{R}^n)$, then for $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_p \neq 0$ we have

$$|\xi_p|^\alpha |\hat{f}(\xi)| \leq M_p^\alpha(f) \left(\frac{1}{2\xi_p} \right),$$

where, for $t \neq 0$,

$$M_p^\alpha(f)(t) = \int_{\mathbb{R}^n} \frac{|f(x + te_p) - f(x)|}{|t|^\alpha}\, dx.$$

(iv) If f is L^1-Hölder continuous in the direction e_p with exponent α, then $\hat{f}, |\xi_p|^\alpha \hat{f} \in L^\infty(\mathbb{R}^n)$.

Proof. To prove (i), suppose that $|x_p|^\alpha f \in L^1(\mathbb{R}^n)$. Using that $|e^{ix}-1| \leq |x|$, it follows easily that $|e^{ix}-1| \leq 2|x|^\alpha$, for all $x \in \mathbb{R}$. Thus, for a.e. x we have

$$\frac{|e^{-2\pi\imath t x_p}-1|}{|t|^\alpha}|f(x)| \leq \frac{2|2\pi\imath t x_p|^\alpha}{|t|^\alpha}|f(x)| = 2^{\alpha+1}\pi^\alpha|x_p|^\alpha|f(x)| \tag{4.4}$$

for all $t \neq 0$, which implies (4.3). If $\alpha < 1$, then for a.e. x we have

$$\begin{aligned}\frac{|e^{-2\pi\imath t x_p}-1|}{|t|^\alpha}|f(x)| &= |t|^{1-\alpha}\frac{|e^{-2\pi\imath t x_p}-1|}{|t|}|f(x)| \\ &\leq |t|^{1-\alpha}\frac{2\pi|t||x_p|}{|t|}|f(x)| \\ &= |t|^{1-\alpha}2\pi|x_p||f(x)| \\ &\to 0,\end{aligned} \tag{4.5}$$

as $t \to 0$. Using (4.4), (4.5), and the Dominated Convergence Theorem (see Theorem 2.24 of [2]), we obtain

$$\int_{\mathbb{R}^n} \frac{|e^{-2\pi\imath t x_p}-1|}{|t|^\alpha}|f(x)|\,dx \to 0,$$

as $t \to 0$. This proves (i).

To prove (ii), suppose that $f, |x_p|^\alpha f \in L^1(\mathbb{R}^n)$. It follows from part (i) of

Lemma 4.1 that $\hat{f} \in L^\infty(\hat{\mathbb{R}}^n)$. Since

$$\frac{\hat{f}(\xi + t\hat{e}_p) - \hat{f}(\xi)}{|t|^\alpha} = \left(\frac{e^{-2\pi \imath t x_p} - 1}{|t|^\alpha} f(x)\right)^\wedge (\xi),$$

it follows that

$$\frac{|\hat{f}(\xi + t\hat{e}_p) - \hat{f}(\xi)|}{|t|^\alpha} \leq \int_{\mathbb{R}^n} \frac{|e^{-2\pi \imath t x_p} - 1|}{|t|^\alpha} |f(x)|\, dx,$$

for all ξ and all $t \neq 0$; (ii) now follows from (i).

To prove (iii), suppose that $f \in L^1(\mathbb{R}^n)$. Since

$$\frac{e^{2\pi \imath t \xi_p} - 1}{|t|^\alpha} \hat{f}(\xi) = \left(\frac{f(x + te_p) - f(x)}{|t|^\alpha}\right)^\wedge (\xi),$$

it follows that

$$\frac{|e^{2\pi \imath t \xi_p} - 1|}{|t|^\alpha} |\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} \frac{|f(x + te_p) - f(x)|}{|t|^\alpha}\, dx,$$

for all ξ and all $t \neq 0$. Denote the right hand side of the above inequality by $M_p^\alpha(f)(t)$. For $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_p \neq 0$, substituting $t = (2\xi_p)^{-1}$ into the above inequality yields

$$2^{\alpha+1} |\xi_p|^\alpha |\hat{f}(\xi)| = \frac{|e^{2\pi \imath (2\xi_p)^{-1} \xi_p} - 1|}{|(2\xi_p)^{-1}|^\alpha} |\hat{f}(\xi)| \leq M_p^\alpha(f) \left(\frac{1}{2\xi_p}\right),$$

which proves (iii).

Part (iv) follows immediately from part (iii) of this lemma and part (i) of Lemma 4.1. $\square$

4.2 Logarithmic Continuity and Logarithmic Decay

In this section, we state and prove an analog of Lemma 4.1 (Lemma 4.4) in which smoothness and integral polynomial decay are replaced with logarithmic continuity and logarithmic decay. We have the following definition:

Definition 4.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable and let $\alpha > 0$.*

(i) f is said to be logarithmically continuous in the direction e_p with exponent α if $f \in L^\infty(\mathbb{R}^n)$ and if there exists $0 \leq M < \infty$ such that

$$\big|\log|t|\big|^\alpha \big\|f(x+te_p) - f(x)\big\|_\infty \leq M,$$

for all $0 < |t| < 1$.

(ii) f is said to be strongly logarithmically continuous in the direction e_p with exponent α if $f \in L^\infty(\mathbb{R}^n)$ and if

$$\big|\log|t|\big|^\alpha \big\|f(x+te_p) - f(x)\big\|_\infty \leq M(t),$$

for all $0 < |t| < 1$, where the function M is bounded and satisfies $M(t) \to 0$, as $t \to 0$.

(iii) f is said to be L^1-logarithmically continuous in the direction e_p with exponent α if $f \in L^1(\mathbb{R}^n)$ and if there exists $0 \leq M < \infty$ such that

$$\int_{\mathbb{R}^n} \big| \log |t| \big|^{\alpha} \big| f(x + te_p) - f(x) \big| \, dx \leq M,$$

for all $0 < |t| < 1$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable. We make the following comments regarding Definition 4.2:

(i) Let $\alpha > 0$. If f is logarithmically continuous (L^1-logarithmically continuous) in the direction e_p with exponent α, then f is strongly logarithmically continuous (L^1-logarithmically continuous) in the direction e_p with exponent β, for all $0 < \beta < \alpha$.

(ii) Let $0 < \alpha \leq 1$. If f is Hölder continuous (L^1-Hölder continuous) in the direction e_p with exponent α, then f is strongly logarithmically continuous (L^1-logarithmically continuous) in the direction e_p with exponent β, for all $0 < \beta < \infty$.

(iii) Logarithmic continuity, strong logarithmic continuity, and L^1-logarithmic continuity are defined similarly for functions $f : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$, and the analogous versions of properties (i) and (ii) above hold in this context as well.

We need the following result in the proof of Lemma 4.4:

Lemma 4.3. *We have the following:*

(i) If $\alpha > 0$*, then there exists* $0 < M' = M'(\alpha) < \infty$ *such that*

$$t^\alpha \log(1/t) \leq M' \frac{\log(x+2)}{x^\alpha}, \tag{4.6}$$

for all $x, t \in \mathbb{R}$ *satisfying* $x > 0$ *and* $0 < t \leq \min\{1, x^{-1}\}$*.*

(ii) If $\alpha > 0$*, then there exists* $0 < M = M(\alpha) < \infty$ *such that*

$$\left|e^{itx} - 1\right| \leq M \left(\frac{\log(|x|+2)}{\left|\log|t|\right|} \right)^\alpha,$$

for all $x, t \in \mathbb{R}$ *satisfying* $0 < |t| < 1$*.*

Proof. To prove part (i), let $\alpha > 0$ and choose $M' \in [1, \infty)$ satisfying

$$e^{-1}\alpha^{-1} \leq M' \frac{\log(x+2)}{x^\alpha}, \tag{4.7}$$

for all $0 < x \leq e^{\alpha^{-1}}$. Consider $f \in C[0,1] \cap C^\infty(0,1)$ defined by

$$f(t) = \begin{cases} t^\alpha \log(1/t), & \text{if } 0 < t \leq 1; \\ 0, & \text{if } t = 0. \end{cases}$$

For $0 < t < 1$, a calculation shows that

$$f'(t) = t^{\alpha-1}(\alpha \log(1/t) - 1).$$

Using the mean value theorem (see Theorem 5.10 of [12]), it follows from the above equality that f is increasing on $[0, e^{-\alpha^{-1}}]$ and decreasing on $[e^{-\alpha^{-1}}, 1]$. Let $x, t \in \mathbb{R}$ be such that $x > 0$ and $0 < t \leq \min\{1, x^{-1}\}$. If $x^{-1} \geq e^{-\alpha^{-1}}$, then $x \leq e^{\alpha^{-1}}$ and using (4.7) we obtain

$$t^\alpha \log(1/t) = f(t) \leq f(e^{-\alpha^{-1}}) = e^{-1}\alpha^{-1} \leq M' \frac{\log(x+2)}{x^\alpha}.$$

If $x^{-1} \leq e^{-\alpha^{-1}}$, then

$$t^\alpha \log(1/t) = f(t) \leq f(x^{-1}) = x^{-\alpha} \log(x) \leq M' \frac{\log(x+2)}{x^\alpha}.$$

This proves (i).

To prove (ii), let $\alpha > 0$. Choose $M' = M'(\alpha^{-1})$ as guaranteed by part (i) of this lemma and set $M = \max\{2, M'\}$. Rearranging (4.6), we obtain

$$tx \leq M \left(\frac{\log(x+2)}{\log(1/t)} \right)^\alpha, \tag{4.8}$$

for all $x, t \in \mathbb{R}$ satisfying $x > 0$ and $0 < t \leq \min\{1, x^{-1}\}$.

Let $x, t \in \mathbb{R}$ be such that $0 < |t| < 1$. We may assume that $x \neq 0$. If $|xt| \geq 1$, then $\log(|x|) + \log(|t|) = \log(|xt|) \geq 0$, implying that

$$\log(|x| + 2) \geq \log(|x|) \geq -\log(|t|) = \big|\log|t|\big|.$$

We thus obtain

$$|e^{itx}-1| \le |e^{itx}|+|1| = 2 \le M\left(\frac{\log(|x|+2)}{|\log|t||}\right)^{\alpha}.$$

If $|xt|\le 1$, then we may substitute $|t|$ and $|x|$ into (4.8) to obtain

$$|e^{itx}-1| \le |t||x| \le M\left(\frac{\log(|x|+2)}{\log(1/|t|)}\right)^{\alpha} = M\left(\frac{\log(|x|+2)}{\big|\log|t|\big|}\right)^{\alpha}.$$

This proves (ii). □

We have the following version of Lemma 4.1:

Lemma 4.4. *Let $f:\mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable and let $\alpha>0$.*

(i) If $f, \big(\log(|x_p|+1)\big)^{\alpha} f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is strongly logarithmically continuous in the direction $\hat{e}_p$ with exponent α.

(ii) If f is L^1-logarithmically continuous in the direction e_p with exponent α, then $\hat{f}, \big(\log(|x_p|+1)\big)^{\alpha}\hat{f} \in L^{\infty}(\mathbb{R}^n)$.

Proof of Lemma 4.4. To prove (i), suppose that $f, \big(\log(|x_p|+1)\big)^{\alpha} f \in L^1(\mathbb{R}^n)$. It follows from part (i) of Lemma 4.1 that $\hat{f} \in L^{\infty}(\hat{\mathbb{R}}^n)$. Choose $M(\alpha)$ and $M(\alpha+1)$ as guaranteed by part (ii) of Lemma 4.3 and set $M = \max\{M(\alpha), M(\alpha+1)\}$. Since

$$\big|\log|t|\big|^{\alpha}\big(\hat{f}(\xi+t\hat{e}_p)-\hat{f}(\xi)\big) = \Big(\big|\log|t|\big|^{\alpha}\big(e^{-2\pi\imath t x_p}-1\big)f(x)\Big)^{\widehat{}}(\xi),$$

it follows that

$$\big|\log|t|\big|^{\alpha}\big|\hat{f}(\xi+t\hat{e}_p)-\hat{f}(\xi)\big| \leq \int_{\mathbb{R}^n}\big|\log|t|\big|^{\alpha}\big|e^{-2\pi\imath t x_p}-1\big|\big|f(x)\big|\,dx, \tag{4.9}$$

for all ξ and all $0<|t|<1$. Using part (ii) of Lemma 4.3, for a.e. x we obtain

$$\begin{aligned}\big|\log|t|\big|^{\alpha}\big|e^{-2\pi\imath t x_p}-1\big|\big|f(x)\big| &\leq \big|\log|t|\big|^{\alpha}M\left(\frac{\log(2\pi|x_p|+2)}{\big|\log|t|\big|}\right)^{\alpha}\big|f(x)\big| \\ &= M\big(\log(2\pi|x_p|+2)\big)^{\alpha}\big|f(x)\big|, \end{aligned} \tag{4.10}$$

for all $0<|t|<1$. Moreover, it follows easily from our assumptions that the right hand side of the above inequality belongs to $L^1(\mathbb{R}^n)$. Using again part (ii) of Lemma 4.3, for a.e. x we have

$$\begin{aligned}&\big|\log|t|\big|^{\alpha}\big|e^{-2\pi\imath t x_p}-1\big|\big|f(x)\big| \\ &\qquad\leq \big|\log|t|\big|^{\alpha}M\left(\frac{\log(2\pi|x_p|+2)}{\big|\log|t|\big|}\right)^{\alpha+1}\big|f(x)\big| \\ &\qquad= M\frac{\big(\log(2\pi|x_p|+2)\big)^{\alpha+1}}{\big|\log|t|\big|}\big|f(x)\big| \to 0,\end{aligned}$$

as $t\to 0$. Using (4.9), (4.10), the above calculation, and the Dominated Convergence Theorem, it follows that $\hat{f}$ is strongly logarithmically continuous in the direction $\hat{e}_p$ with exponent α.

To prove (ii), suppose that f is L^1-logarithmically continuous in the direction e_p with exponent α. It follows from part (i) of Lemma 4.1 that $\hat{f} \in L^\infty(\hat{\mathbb{R}}^n)$. Choose M as in part (iii) of Definition 4.2. Since

$$\big|\log|t|\big|^\alpha\big(e^{2\pi\imath t\xi_p}-1\big)\hat{f}(\xi) = \Big(\big|\log|t|\big|^\alpha\big(f(x+te_p)-f(x)\big)\Big)\widehat{\ }(\xi),$$

it follows that

$$\big|\log|t|\big|^\alpha\big|e^{2\pi\imath t\xi_p}-1\big|\big|\hat{f}(\xi)\big| \le \int_{\mathbb{R}^n}\big|\log|t|\big|^\alpha\big|f(x+te_p)-f(x)\big|\,dx \le M$$

for all ξ and all $0<|t|<1$. For $\xi=(\xi_1,\dots,\xi_n)$ with $|\xi_p|\ge 1$, substituting $t=(2\xi_p)^{-1}$ into the above inequality yields

$$\begin{aligned}
\big(\log(|\xi_p|+1)\big)^\alpha\big|\hat{f}(\xi)\big| &\le \big(\log(|\xi_p|+|\xi_p|)\big)^\alpha\big|\hat{f}(\xi)\big| \\
&\le 2\big(\log(2|\xi_p|)\big)^\alpha\big|\hat{f}(\xi)\big| \\
&= 2\Big|\log\big|(2\xi_p)^{-1}\big|\Big|^\alpha\big|\hat{f}(\xi)\big| \\
&= \Big|\log\big|(2\xi_p)^{-1}\big|\Big|^\alpha\big|e^{2\pi\imath(2\xi_p)^{-1}\xi_p}-1\big|\big|\hat{f}(\xi)\big| \\
&\le M.
\end{aligned}$$

It follows that $\big(\log(|x_p|+1)\big)^\alpha\hat{f} \in L^\infty(\mathbb{R}^n)$. This proves (ii). $\square$

4.3 Iterated Hölder Continuity and Multivariable Fractional Polynomial Decay

In this section, we state and prove an analog of Lemma 4.1 in which smoothness and integral polynomial decay are replaced with iterated Hölder continuity and multivariable fractional polynomial decay.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be measurable and let $y \in \mathbb{R}^n$. Define $\Delta_y f : \mathbb{R}^n \longrightarrow \mathbb{C}$ by

$$\Delta_y f(x) = f(x + y) - f(x).$$

For $i \in \{1, \ldots, n\}$ and $t \in \mathbb{R}$, write $\Delta_i(t) = \Delta_{te_i}$. Note that Δ_y belongs to $\mathcal{B}\big(L^p(\mathbb{R}^n)\big)$ $(p = 1, \infty)$. We make the following observations:

(i) If $z \in \mathbb{R}^n$, then $\Delta_y \Delta_z f = \Delta_z \Delta_y f$. More generally, if $y_1, \ldots, y_m \in \mathbb{R}^n$, the operators $\Delta_{y_1}, \ldots, \Delta_{y_m}$ may be applied to f in any order with the same result.

Let $m \in \mathbb{Z}^+$ and let $i_1, \ldots, i_m \in \{1, \ldots, n\}$.

(ii) Let $0 < \beta_j \leq \alpha_j \leq 1$ $(j = 1, \ldots, m)$ and let $p \in \{1, \infty\}$. Suppose that $\|f\|_p < \infty$ and that there exists a constant K' such that for all $k = 1, \ldots, m$ and all $j_1 < \cdots < j_k$ belonging to $\{1, \ldots, m\}$ we have

$$\|\Delta_{i_{j_1}}(t_{j_1}) \ldots \Delta_{i_{j_k}}(t_{j_k}) f\|_p \leq K' |t_{j_1}|^{\alpha_{j_1}} \ldots |t_{j_k}|^{\alpha_{j_k}},$$

for all $t_{j_1}, \ldots, t_{j_k} \in \mathbb{R}$. Then

$$\|\Delta_{i_1}(t_1) \ldots \Delta_{i_m}(t_m) f\|_p \leq K_1(t_1) \ldots K_m(t_m) |t_1|^{\beta_1} \ldots |t_m|^{\beta_m},$$

for all $t_1, \ldots, t_m \in \mathbb{R}$, where, for each j, the function K_j is bounded and, if $\beta_j < \alpha_j$, satisfies $K_j(t) \to 0$ as $t \to 0$.

Let $0 < \alpha \leq 1$ and assume $m \geq 2$. Suppose that f is everywhere defined and that $\partial_{i_j} \ldots \partial_{i_m} f(x)$ exists for all x (for each $j = 2, \ldots, m$).

(iii) If there exists a constant K' such that

$$|\Delta_{i_1}(t_1) \partial_{i_2} \ldots \partial_{i_m} f(x)| \leq K' |t_1|^{\alpha},$$

for all $t_1 \in \mathbb{R}$ and all x, then there is a constant K such that

$$|\Delta_{i_1}(t_1) \ldots \Delta_{i_m}(t_m) f(x)| \leq K |t_1|^{\alpha} |t_2| \ldots |t_m|,$$

for all x and all $t_1, \ldots, t_m \in \mathbb{R}$. In particular,

$$\|\Delta_{i_1}(t_1) \ldots \Delta_{i_m}(t_m) f\|_{\infty} \leq K |t_1|^{\alpha} |t_2| \ldots |t_m|,$$

for all $t_1, \ldots, t_m \in \mathbb{R}$.

(iv) If $\partial_{i_j} \ldots \partial_{i_m} f$ is continuous (for each $j = 2, \ldots, m$) and if there exists a constant K' such that

$$\|\Delta_{i_1}(t_1)\partial_{i_2}\dots\partial_{i_m}f\|_1 \le K'|t_1|^\alpha,$$

for all $t_1 \in \mathbb{R}$, then there is a constant K such that

$$\|\Delta_{i_1}(t_1)\dots\Delta_{i_m}(t_m)f\|_1 \le K|t_1|^\alpha|t_2|\dots|t_m|,$$

for all $t_1,\dots,t_m \in \mathbb{R}$.

(v) For $\xi \in \hat{\mathbb{R}}^n$, the operator Δ_ξ is defined similarly, and the analogs of properties (i) through (iv) above hold in this situation as well.

If $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, write $f = f_1 + f_2$ with $f_p \in L^p(\mathbb{R}^n)$ $(p = 1, 2)$ and define $\hat{f} = \hat{f}_1 + \hat{f}_2$. It is simple to check that this definition results in a well-defined extension of the Fourier transform to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. We have the following version of Lemma 4.1 (and extension of Lemma 4.2):

Lemma 4.5. *Let $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, let $i_1,\dots,i_m \in \{1,\dots,n\}$, and let $\alpha_1,\dots,\alpha_m \in (0,1]$.*

(i) If $|x_{i_1}|^{\alpha_1}\dots|x_{i_m}|^{\alpha_m}f \in L^1(\mathbb{R}^n)$, then

$$\|\Delta_{i_1}(t_1)\dots\Delta_{i_m}(t_m)\hat{f}\|_\infty \le K(t_1,\dots,t_m)|t_1|^{\alpha_1}\dots|t_m|^{\alpha_m},$$

for all $t_1,\dots,t_m \in \mathbb{R}$, where the function K is bounded and, for each j

such that $\alpha_j < 1$, satisfies

$$\sup\{K(t_1, \ldots, t_m) : t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m \in \mathbb{R}\} \to 0,$$

as $t_j \to 0$.

(ii) If there exists a constant K such that

$$\|\Delta_{i_1}(t_1) \ldots \Delta_{i_m}(t_m) f\|_1 \leq K|t_1|^{\alpha_1} \ldots |t_m|^{\alpha_m}$$

for all $t_1, \ldots, t_m \in \mathbb{R}$, then $|\xi_{i_1}|^{\alpha_1} \ldots |\xi_{i_m}|^{\alpha_m} \hat{f} \in L^\infty(\hat{\mathbb{R}}^n)$.

Proof. If $g : \mathbb{R}^n \longrightarrow \mathbb{C}$ and $h : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$ are measurable and if $y \in \mathbb{R}^n$ and $\eta \in \hat{\mathbb{R}}^n$, define $M_\eta g : \mathbb{R}^n \longrightarrow \mathbb{C}$ by

$$M_\eta g(x) = \left(e^{-2\pi\imath\eta x} - 1\right) f(x);$$

$M_y h : \hat{\mathbb{R}}^n \longrightarrow \mathbb{C}$ is defined similarly. If $g \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, it is straightforward to verify that

$$\Delta_\eta \hat{g} = (M_\eta g)\widehat{\ } \quad \text{and} \quad (\Delta_y g)\widehat{\ } = M_{-y}\hat{g},$$

where, for $g', h' \in C_0(\hat{\mathbb{R}}^n) + L^2(\hat{\mathbb{R}}^n)$, we write $g' = h'$ if and only if $g'(\xi) = h'(\xi)$ for a.e. ξ. If $y_1, \ldots, y_m \in \mathbb{R}^n$ and $\eta_1, \ldots, \eta_m \in \hat{\mathbb{R}}^n$, it follows that

$$\Delta_{\eta_1} \ldots \Delta_{\eta_m} \hat{g} = (M_{\eta_1} \ldots M_{\eta_m} g)\widehat{\ } \tag{4.11}$$

and that

$$(\Delta_{y_1} \dots \Delta_{y_m} g)\widehat{\ } = M_{-y_1} \dots M_{-y_m} \hat{g}. \tag{4.12}$$

We now prove (i). Define the function K by

$$K(t_1, \dots, t_m) = \int_{\mathbb{R}^n} \frac{|e^{-2\pi\imath t_1 x_{i_1}} - 1| \dots |e^{-2\pi\imath t_m x_{i_m}} - 1|}{|t_1|^{\alpha_1} \dots |t_m|^{\alpha_m}} |f(x)| \, dx$$

if $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$ and $K(t_1, \dots, t_m) = 0$ otherwise. If $t_1, \dots, t_m \in \mathbb{R}$, using (4.11) and that

$$|\hat{g}(\eta)| \leq \int_{\mathbb{R}^n} |g(x)| \, dx, \tag{4.13}$$

for all $g \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ and a.e. η, we obtain

$$\begin{aligned}
|\Delta_{i_1}(t_1) \dots \Delta_{i_m}(t_m) \hat{f}(\xi)| &= |\Delta_{t_1 \hat{e}_{i_1}} \dots \Delta_{t_m \hat{e}_{i_m}} \hat{f}(\xi)| \\
&= \left| \left(M_{t_1 \hat{e}_{i_1}} \dots M_{t_m \hat{e}_{i_m}} f \right)\widehat{\ }(\xi) \right| \\
&\leq \int_{\mathbb{R}^n} \left| M_{t_1 \hat{e}_{i_1}} \dots M_{t_m \hat{e}_{i_m}} f(x) \right| dx \\
&= \int_{\mathbb{R}^n} |e^{-2\pi\imath t_1 x_{i_1}} - 1| \dots |e^{-2\pi\imath t_m x_{i_m}} - 1| |f(x)| \, dx \\
&= K(t_1, \dots, t_m) |t_1|^{\alpha_1} \dots |t_m|^{\alpha_m},
\end{aligned}$$

for a.e. ξ, and thus

$$\|\Delta_{i_1}(t_1) \dots \Delta_{i_m}(t_m) \hat{f}\|_\infty \leq K(t_1, \dots, t_m) |t_1|^{\alpha_1} \dots |t_m|^{\alpha_m}.$$

If $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$ and $j \in \{1, \dots, m\}$, using that

$$|e^{ix} - 1| \le 2|x|^\alpha \tag{4.14}$$

(for all $\alpha \in (0,1]$ and all $x \in \mathbb{R}$) we obtain

$$\begin{aligned}
\Big(\prod_{k=1}^{m} \frac{|e^{-2\pi\imath t_k x_{i_k}} - 1|}{|t_k|^{\alpha_k}}\Big)|f(x)| &= \Big(\prod_{k\neq j} \frac{|e^{-2\pi\imath t_k x_{i_k}} - 1|}{|t_k|^{\alpha_k}}\Big)\frac{|e^{-2\pi\imath t_j x_{i_j}} - 1|}{|t_j|^{\alpha_j}}|f(x)| \\
&\le \Big(\prod_{k\neq j} \frac{2|2\pi t_k x_{i_k}|^{\alpha_k}}{|t_k|^{\alpha_k}}\Big)\frac{|e^{-2\pi\imath t_j x_{i_j}} - 1|}{|t_j|^{\alpha_j}}|f(x)| \\
&= \Big(\prod_{k\neq j} 2|2\pi x_{i_k}|^{\alpha_k}\Big)\frac{|e^{-2\pi\imath t_j x_{i_j}} - 1|}{|t_j|^{\alpha_j}}|f(x)| \qquad (4.15) \\
&\le \Big(\prod_{k\neq j} 2|2\pi x_{i_k}|^{\alpha_k}\Big)\frac{2|2\pi t_j x_{i_j}|^{\alpha_j}}{|t_j|^{\alpha_j}}|f(x)| \\
&= 2^m (2\pi)^{\alpha_1+\cdots+\alpha_m}|x_{i_1}|^{\alpha_1}\dots|x_{i_m}|^{\alpha_m}|f(x)|,
\end{aligned}$$

for a.e. x.

If $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$, using (4.15) we obtain

$$\begin{aligned}
K(t_1, \dots, t_m) &= \int_{\mathbb{R}^n} \frac{|e^{-2\pi\imath t_1 x_{i_1}} - 1|\dots|e^{-2\pi\imath t_m x_{i_m}} - 1|}{|t_1|^{\alpha_1}\dots|t_m|^{\alpha_m}}|f(x)|\,dx \\
&\le 2^m (2\pi)^{\alpha_1+\cdots+\alpha_m}\int_{\mathbb{R}^n} |x_{i_1}|^{\alpha_1}\dots|x_{i_m}|^{\alpha_m}|f(x)| < \infty.
\end{aligned}$$

It follows that K is bounded.

If $\alpha_j < 1$ and $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$, using (4.15) we obtain

$$\begin{aligned} K(t_1, \dots, t_m) &= \int_{\mathbb{R}^n} \frac{|e^{-2\pi\imath t_1 x_{i_1}} - 1| \dots |e^{-2\pi\imath t_m x_{i_m}} - 1|}{|t_1|^{\alpha_1} \dots |t_m|^{\alpha_m}} |f(x)|\, dx \\ &\leq \int_{\mathbb{R}^n} \Big(\prod_{k \neq j} 2|2\pi x_{i_k}|^{\alpha_k} \Big) \frac{|e^{-2\pi\imath t_j x_{i_j}} - 1|}{|t_j|^{\alpha_j}} |f(x)|\, dx. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} &\sup\{K(t_1, \dots, t_m) : t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_m \in \mathbb{R}\} \\ &\qquad\qquad \leq \int_{\mathbb{R}^n} \Big(\prod_{k \neq j} 2|2\pi x_{i_k}|^{\alpha_k} \Big) \frac{|e^{-2\pi\imath t_j x_{i_j}} - 1|}{|t_j|^{\alpha_j}} |f(x)|\, dx, \end{aligned}$$

for each $t_j \neq 0$. Using (4.14), (4.15), and the Dominated Convergence Theorem, it follows that the right hand side of the above inequality approaches 0 as $t_j \to 0$. This proves (i).

We now prove (ii). Choose $\epsilon > 0$ such that

$$|e^{2\pi\imath x} - 1| \geq \epsilon, \tag{4.16}$$

for all $1/3 \leq |x| \leq 2/3$.

Fix $t_1, \dots, t_m \in \mathbb{R} \setminus \{0\}$ such that $t_j = t_k$ if $i_j = i_k$. Using (4.12) and (4.13)

we obtain

$$
\begin{aligned}
|e^{2\pi\imath t_1\xi_{i_1}} - 1| \dots |e^{2\pi\imath t_m\xi_{i_m}} - 1||\hat{f}(\xi)| \\
&= |M_{-t_1e_{i_1}} \dots M_{-t_me_{i_m}}\hat{f}(\xi)| \\
&= \left|(\Delta_{t_1e_{i_1}} \dots \Delta_{t_me_{i_m}} f)\widehat{\ }(\xi)\right| \qquad (4.17) \\
&\leq \|\Delta_{t_1e_{i_1}} \dots \Delta_{t_me_{i_m}} f\|_1 \\
&= \|\Delta_{i_1}(t_1) \dots \Delta_{i_m}(t_m) f\|_1 \leq K|t_1|^{\alpha_1} \dots |t_m|^{\alpha_m},
\end{aligned}
$$

for a.e. ξ. Let $E(t_1, \dots, t_m)$ denote the collection of all $\xi \in \hat{\mathbb{R}}^n$ satisfying $1/3 \leq |t_j\xi_{i_j}| \leq 2/3$ (for all j). Using (4.16) and (4.17), for a.e. $\xi \in E(t_1, \dots, t_m)$ we obtain

$$
\begin{aligned}
\epsilon^m|\hat{f}(\xi)| &\leq |e^{2\pi\imath t_1\xi_{i_1}} - 1| \dots |e^{2\pi\imath t_m\xi_{i_m}} - 1||\hat{f}(\xi)| \\
&\leq K|t_1|^{\alpha_1} \dots |t_m|^{\alpha_m} \\
&\leq K\left(\frac{2}{3|\xi_{i_1}|}\right)^{\alpha_1} \dots \left(\frac{2}{3|\xi_{i_m}|}\right)^{\alpha_m},
\end{aligned}
$$

or, equivalently,

$$
|\xi_{i_1}|^{\alpha_1} \dots |\xi_{i_m}|^{\alpha_m}|\hat{f}(\xi)| \leq \frac{K}{\epsilon^m}\left(\frac{2}{3}\right)^{\alpha_1+\dots+\alpha_m}.
$$

Since $\hat{\mathbb{R}}^n$ can be obtained (up to a set of measure zero) as an appropriate countable union of the sets $E(t_1, \dots, t_m)$, it follows from the above inequality that

$$\big\| |\xi_{i_1}|^{\alpha_1} \dots |\xi_{i_m}|^{\alpha_m} \hat{f} \big\|_\infty \leq \frac{K}{\epsilon^m} \left(\frac{2}{3} \right)^{\alpha_1 + \dots + \alpha_m}.$$

This proves (ii). $\square$

Of course, one can also study the L^p-boundedness properties ($p = 1, \infty$) of the operators $\Delta_{i_1}(t_1) \dots \Delta_{i_m}(t_m)$ in terms of products of the form

$$\frac{1}{\big| \log |t_1| \big|^{\alpha_1}} \cdots \frac{1}{\big| \log |t_m| \big|^{\alpha_m}},$$

where $i_1, \dots, i_m \in \{1, \dots, n\}$, $0 < |t_1|, \dots, |t_m| < 1$, and

$$0 < \alpha_1, \dots, \alpha_m < \infty.$$

Comments (i) and (ii) following Definition 4.2, Lemma 4.4, etc. can all be formulated within this more general context. However, we do not require these results and therefore do not state or prove them formally.

Chapter 5

The Nonexistence of Shearlet Scaling Multifunctions

Let
$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

B is said to be a shear group (due to its action on $\mathbb{R}^2$ (or $\hat{\mathbb{R}}^2$)). We call an aB-multiwavelet, an aB-MRA, and an aB-scaling multifunction a multishearlet, a shearlet MRA, and a shearlet scaling multifunction, respectively. There are numerous results indicating that shearlet systems (and their variants) have the potential for great success in multi-dimensional applications: Continuous variants of shearlet systems have been shown to be very effective (indeed, to outperform the continuous variants of classical wavelet systems) in edge analysis. Discrete variants of shearlet systems are known to be essentially optimal at representing a certain subclass of $L^2(\mathbb{R}^2)$ (see section 5 of [8] for these and other results).

As in the case of classical wavelet systems, a very useful tool for developing fast algorithmic implementations of a multishearlet $\{\psi_1, \ldots, \psi_L\}$ would be an associated shearlet MRA $\{V_j : j \in \mathbb{Z}\}$ along with an associated "suitable" shearlet scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ (i.e., one that satisfies a certain amount of decay, regularity, and/or other desirable properties).

In section 1 of this chapter, we prove results regarding the nonexistence of shearlet scaling multifunctions that satisfy a minimal amount of decay and regularity. In section 2, we prove results regarding the nonexistence of shearlet scaling multifunctions that satisfy a minimal amount of decay and one of two "finite type" conditions. Combining the results of this chapter with those of Corollary 3.6 (which implies the nonexistence of shearlet scaling multifunctions of Haar-type), it will follow that essentially no "suitable" shearlet scaling multifunctions exist.

5.1 Nonexistence Results Regarding Decay and Regularity

The two main results of this section (Theorem 5.1 and Corollary 5.1) regard the nonexistence of a large collection of shearlet-like aB-scaling multifunctions that satisfy a minimal amount of decay and regularity.

5.1.1 Some Preliminary Results

Before we state and prove Theorem 5.1 and Corollary 5.1, we need the two lemmas and proposition contained in this subsection.

Lemma 5.1. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha > 1$ and $\beta \neq 0$. Then*

$$\sum_{l=-\infty}^{\infty} \frac{1}{(1+|\gamma + l\beta|)^{\alpha}} \leq 2 \sum_{l=0}^{\infty} \frac{1}{(1+l|\beta|)^{\alpha}}.$$

Proof. We may assume that γ lies between 0 and β. Using this assumption, it follows that

$$|l\beta| \leq |\gamma| + |l\beta| = |\gamma + l\beta|,$$

for $l = 0, 1, 2, \ldots$ and that

$$\begin{aligned}
|l\beta| = -l|\beta| &\leq -l|\beta| + |\beta| - |\gamma| \\
&= -(l-1)|\beta| - |\gamma| \\
&= |(l-1)\beta| - |\gamma| = |\gamma + (l-1)\beta|,
\end{aligned}$$

for $l = \ldots, -2, -1, 0$. We thus obtain

$$\begin{aligned}
\sum_{l=-\infty}^{\infty} \frac{1}{(1+|\gamma+l\beta|)^{\alpha}} &= \sum_{l=-\infty}^{-1} \frac{1}{(1+|\gamma+l\beta|)^{\alpha}} + \sum_{l=0}^{\infty} \frac{1}{(1+|\gamma+l\beta|)^{\alpha}} \\
&= \sum_{l=-\infty}^{0} \frac{1}{(1+|\gamma+(l-1)\beta|)^{\alpha}} + \sum_{l=0}^{\infty} \frac{1}{(1+|\gamma+l\beta|)^{\alpha}} \\
&\leq \sum_{l=-\infty}^{0} \frac{1}{(1+|l\beta|)^{\alpha}} + \sum_{l=0}^{\infty} \frac{1}{(1+|l\beta|)^{\alpha}} \\
&= 2\sum_{l=0}^{\infty} \frac{1}{(1+l|\beta|)^{\alpha}}.
\end{aligned}$$

□

The below proposition is the major ingredient in the proof of Theorem 5.1.

Proposition 5.1. *Let $z > 0$, let B_0 be an infinite subset of*

$$B = \left\{ b(l) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\},$$

and let $\varphi \in L^2(\mathbb{R}^2)$. Suppose that

(i) the collection $\{D_b T_k \varphi : b \in B_0, k \in \mathbb{Z}^2\}$ forms a Bessel system;

(ii) $|\hat{\varphi}|$ is continuous;

(iii) there exists $\alpha > 1/2$ such that $|\hat{\varphi}|$ and $|\xi_2|^{\alpha}|\hat{\varphi}|$ are strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent $1/2$.

Then, for (ξ_1, ξ_2) *with* $0 < |\xi_1| \leq \min\{1, (2z)^{-1}\}$ *we have*

$$\sum_{b \in B} |\hat{\varphi}((\xi_1, \xi_2)b)|^2 \leq M(\xi_1),$$

where the function M *is bounded, measurable, and satisfies* $M(t) \to 0$, *as* $t \to 0$.

Proof. Note that

$$(\xi_1, \xi_2)b(l) = (\xi_1, \xi_2 + lz\xi_1),$$

for each l and each (ξ_1, ξ_2). In particular, B_0 fixes the ξ_2-axis. Thus, using assumptions (i) and (ii) and Corollary 3.1, we see that $\hat{\varphi}(0, \xi_2) = 0$, for all ξ_2. Hence, using assumptions (ii) and (iii), we obtain

$$|\hat{\varphi}(\xi_1, \xi_2)| \leq M_1(\xi_1)|\xi_1|^{1/2} \text{ and } |\xi_2|^\alpha |\hat{\varphi}(\xi_1, \xi_2)| \leq M_2(\xi_1)|\xi_1|^{1/2}, \tag{5.1}$$

for all (ξ_1, ξ_2) with $|\xi_1| \leq 1$, where, for $p = 1, 2$, the function M_p is bounded, measurable, and satisfies $M_p(t) \to 0$, as $t \to 0$. Using (5.1) and that there exists a constant K such that

$$(1 + x)^\alpha \leq K(1 + x^\alpha),$$

for all $x \geq 0$, it follows that

$$|\hat{\varphi}(\xi_1, \xi_2)| \leq \frac{M(\xi_1)|\xi_1|^{1/2}}{(1 + |\xi_2|)^\alpha}, \tag{5.2}$$

for all (ξ_1, ξ_2) with $|\xi_1| \le 1$, where the function M is bounded, measurable, and satisfies $M(t) \to 0$, as $t \to 0$.

Fix (ξ_1, ξ_2) with $0 < |\xi_1| \le \min\{1, (2z)^{-1}\}$. Using (5.2) and Lemma 5.1, we obtain

$$\begin{aligned}
\sum_{b \in B} |\hat{\varphi}((\xi_1, \xi_2)b)|^2 &= \sum_{l=-\infty}^{\infty} |\hat{\varphi}((\xi_1, \xi_2)b(l))|^2 \\
&= \sum_{l=-\infty}^{\infty} |\hat{\varphi}((\xi_1, \xi_2 + lz\xi_1))|^2 \\
&\le \sum_{l=-\infty}^{\infty} \frac{M(\xi_1)^2 |\xi_1|}{(1 + |\xi_2 + lz\xi_1|)^{2\alpha}} \\
&= M(\xi_1)^2 |\xi_1| \sum_{l=-\infty}^{\infty} \frac{1}{(1 + |\xi_2 + lz\xi_1|)^{2\alpha}} \\
&\le 2M(\xi_1)^2 |\xi_1| \sum_{l=0}^{\infty} \frac{1}{(1 + lz|\xi_1|)^{2\alpha}}.
\end{aligned} \tag{5.3}$$

Using integral estimation and a change of variable, we obtain

$$\begin{aligned}
\sum_{l=0}^{\infty} \frac{1}{(1 + lz|\xi_1|)^{2\alpha}} &\le \int_{-1}^{\infty} \frac{dx}{(1 + xz|\xi_1|)^{2\alpha}} \\
&= \frac{1}{z|\xi_1|} \int_{1-z|\xi_1|}^{\infty} \frac{dx}{x^{2\alpha}} \\
&\le \frac{1}{z|\xi_1|} \int_{1/2}^{\infty} \frac{dx}{x^{2\alpha}} = \frac{1}{z|\xi_1|} \frac{2^{2\alpha-1}}{2\alpha - 1}.
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) gives the desired result. □

We need the following lemma in the proof of Corollary 5.1:

Lemma 5.2. *Let $\varphi \in L^2(\mathbb{R}^2)$ and consider the following properties:*

(i) *$\partial_2\varphi(x)$ exists for all x, $\partial_2\varphi \in C(\mathbb{R}^2)$, and $\varphi, |x_1|^\alpha\varphi, |x_1|^\alpha\partial_2\varphi \in L^1(\mathbb{R}^2)$, for some $1/2 \le \alpha \le 1$.*

(ii) *$\varphi \in L^1(\mathbb{R}^2)$ and $|x_1|^\alpha\varphi$ is L^1-Hölder continuous in the direction e_2 with exponent β, for some $1/2 < \alpha, \beta \le 1$.*

(iii) *φ is compactly supported and Hölder continuous in the direction e_2 with exponent β, for some $1/2 < \beta \le 1$.*

(iv) *$\hat{\varphi}$ is continuous and there exists $1/2 < \beta \le 1$ such that $\hat{\varphi}$ and $|\xi_2|^\beta\hat{\varphi}$ are strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent $1/2$.*

Each of properties (i), (ii), and (iii) imply property (iv).

Proof. Since (iii) implies (ii), we need only verify that (i) implies (iv) and that (ii) implies (iv).

If $\varphi \in L^1(\mathbb{R}^2)$ and if $1/2 \le \alpha \le 1$ and $1/2 < \beta \le 1$, then using the notation and result of part (iii) of Lemma 4.2 yields

$$\begin{aligned}\frac{\big||\xi_2|^\beta\hat{\varphi}(\xi + t\hat{e}_1) - |\xi_2|^\beta\hat{\varphi}(\xi)\big|}{|t|^\alpha} &= |\xi_2|^\beta\frac{|\hat{\varphi}(\xi + t\hat{e}_1) - \hat{\varphi}(\xi)|}{|t|^\alpha} \\ &= |\xi_2|^\beta\left|\left(\frac{e^{-2\pi\imath t x_1} - 1}{|t|^\alpha}\varphi(x)\right)^{\widehat{}}(\xi)\right| \qquad (5.5) \\ &\le M_2^\beta\left(\frac{e^{-2\pi\imath t x_1} - 1}{|t|^\alpha}\varphi(x)\right)\left(\frac{1}{2\xi_2}\right),\end{aligned}$$

for all $\xi = (\xi_1, \xi_2)$ with $\xi_2 \ne 0$ and all $t \ne 0$. For $s \ne 0$, a simple calculation shows that

$$M_2^\beta\left(\frac{e^{-2\pi\imath t x_1}-1}{|t|^\alpha}\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)(s)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath t x_1}-1|}{|t|^\alpha}\frac{1}{|s|^\beta}\left|\varphi\begin{pmatrix}x_1\\x_2+s\end{pmatrix}-\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|dx_1dx_2. \quad (5.6)$$

Suppose that φ satisfies property (i), for some $1/2 \le \alpha \le 1$. It follows from part (i) of Lemma 4.1 and part (ii) of Lemma 4.2 that $\hat{\varphi}$ is both continuous and strongly locally Hölder continuous in the direction e_1 with exponent $1/2$. Using the continuity of $\partial_2\varphi$ and the Fundamental Theorem of Calculus, for $s > 0$ and fixed x_1, x_2 we obtain

$$\begin{aligned}\left|\varphi\begin{pmatrix}x_1\\x_2+s\end{pmatrix}-\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right| &= \left|\int_{x_2}^{x_2+s}\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}dr\right|\\ &\le \int_{x_2}^{x_2+s}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|dr \qquad (5.7)\\ &= \int_{-\infty}^{\infty}\theta\begin{pmatrix}r\\x_2\end{pmatrix}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|dr,\end{aligned}$$

where

$$\theta\begin{pmatrix}r\\x_2\end{pmatrix}=\begin{cases}1, & \text{if } x_2\le r\le x_2+s;\\ 0, & \text{otherwise.}\end{cases}$$

Using (5.7) and Fubini's theorem, for $s > 0$ we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\frac{1}{s}\left|\varphi\begin{pmatrix}x_1\\x_2+s\end{pmatrix}-\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|dx_1dx_2$$

$$\begin{aligned}
&\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\frac{1}{s}\int_{-\infty}^{\infty}\theta\begin{pmatrix}r\\x_2\end{pmatrix}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|drdx_1dx_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|dx_1dr\frac{1}{s}\int_{-\infty}^{\infty}\theta\begin{pmatrix}r\\x_2\end{pmatrix}dx_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|dx_1dr\frac{1}{s}\int_{r-s}^{r}1\,dx_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\left|\partial_2\varphi\begin{pmatrix}x_1\\r\end{pmatrix}\right|dx_1dr.
\end{aligned}$$

Using the above inequality and a change of variable, it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\frac{1}{|s|}\left|\varphi\begin{pmatrix}x_1\\x_2+s\end{pmatrix}-\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|dx_1dx_2\\
&\qquad\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|e^{-2\pi\imath tx_1}-1|}{|t|^{\alpha}}\left|\partial_2\varphi\begin{pmatrix}x_1\\x_2\end{pmatrix}\right|dx_1dx_2, \qquad (5.8)
\end{aligned}$$

for all $s \neq 0$. Using (5.5) and (5.6) with $\beta = 1$, (5.8), and part (i) of Lemma 4.2, it follows that $|\xi_2|\hat{\varphi}$ is strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent $1/2$. This shows that (i) implies (iv).

Suppose that (ii) is satisfied, for some $1/2 < \alpha, \beta \leq 1$. Again, it follows from part (i) of Lemma 4.1 and part (ii) of Lemma 4.2 that $\hat{\varphi}$ is continuous and strongly locally Hölder continuous in the direction e_1 with exponent 1/2. Recall that $|e^{ix} - 1| \leq 2|x|^\alpha$, for all $x \in \mathbb{R}$. Using this, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|e^{-2\pi\imath t x_1} - 1|}{|t|^\alpha} \frac{1}{|s|^\beta} \left| \varphi \begin{pmatrix} x_1 \\ x_2 + s \end{pmatrix} - \varphi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| dx_1 dx_2 \\
&\qquad \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2|2\pi t x_1|^\alpha}{|t|^\alpha} \frac{1}{|s|^\beta} \left| \varphi \begin{pmatrix} x_1 \\ x_2 + s \end{pmatrix} - \varphi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| dx_1 dx_2 \\
&\qquad = 2^{\alpha+1}\pi^\alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|s|^\beta} \left| |x_1|^\alpha \varphi \begin{pmatrix} x_1 \\ x_2 + s \end{pmatrix} - |x_1|^\alpha \varphi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| dx_1 dx_2.
\end{aligned}$$

Combining the above inequality with (5.5) and (5.6), it follows that $|\xi_2|^\beta \hat{\varphi}$ is strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent 1/2. This shows that (ii) implies (iv). □

5.1.2 Main Results

This subsection contains the two main results of the current section—results regarding the nonexistence of certain aB-scaling multifunctions. It turns out that the only aB-MRA properties necessary in the proofs of Theorem 5.1 and Corollary 5.1 are properties (i), (ii), and (v). The first of our main results, Theorem 5.1, involves only these three aB-MRA properties and certain decay and regularity assumptions. Corollary 5.1, the second of our main results, translates Theorem 5.1 into a statement regarding the nonexistence of a large collection of shearlet-like aB-scaling multifunctions which satisfy a minimal amount of decay and regularity.

Theorem 5.1. *Let $a \in GL_2(\mathbb{R})$, let $z > 0$, and let B_0 be an infinite subset of*

$$B = \left\{ b(l) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Suppose that

(i) $a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, *as* $j \to \infty$;

(ii) there exists $\epsilon > 0$ such that $\|kb^{-1}a^j\| \geq \epsilon$, for all $k \in \hat{\mathbb{Z}}^2 \setminus \{0\}$, all $b \in B_0$, and all $j \geq 0$.

Let V be a nonzero closed subspace of $L^2(\mathbb{R}^2)$, let $I \in \mathbb{Z}^+$, and let $\{\varphi_1, \ldots, \varphi_I\} \subset L^2(\mathbb{R}^2)$. Suppose that

(iii) $D_a V \subset V$;

(iv) $\{D_b T_k \varphi_i : b \in B_0, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$ *forms a frame for V.*

Then, the functions $\varphi_1, \ldots, \varphi_I$ cannot satisfy the following:

(v) $|\widehat{\varphi}_i|$ is continuous, for each i;

(vi) there exists $\alpha > 1/2$ such that $|\widehat{\varphi}_i|$ and $|\xi_2|^\alpha |\widehat{\varphi}_i|$ are strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent $1/2$, for each i.

We make the following comments regarding the hypotheses of Theorem 5.1:

(i) Note that (ii) is automatically satisfied if $a \in GL_2(\mathbb{Z})$ and $z \in \mathbb{Z}^+$.

(ii) It is easy to see that both (i) and (ii) are satisfied for any $a \in GL_2(\mathbb{R})$ of the form
$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix},$$
where $|a_1| > 1$ and $|a_3| \geq 1$.

(iii) As will be evident from the proof, the assertion of Theorem 5.1 remains true if (v) and (vi) are only assumed to be satisfied on the ξ_2-axis.

(iv) In general terms, (vi) assumes a certain amount of regularity in the $\hat{e}_1$ direction and a certain amount of decay in the $\hat{e}_2$ direction. There are several other versions of (vi) resembling this general form for which the assertion of Theorem 5.1 remains true. However, the (vi) we have chosen is most convenient for our purposes.

We now prove Theorem 5.1.

Proof of Theorem 5.1. To obtain a contradiction, suppose that the functions $\varphi_1, \ldots, \varphi_I$ satisfy (v) and (vi) in the statement of Theorem 5.1 and assume that (iv) in the same statement is satisfied with frame constants $C \leq D$. It follows from assumption (ii) in the statement of Theorem 5.1 that there exists a bounded measurable subset E of $\hat{\mathbb{R}}^2$ satisfying

(i) $(E + kb^{-1}a^j) \cap E = \emptyset$, for all $k \in \hat{\mathbb{Z}}^2 \setminus \{0\}$, all $b \in B_0$, and all $j \geq 0$;

(ii) $\langle \widehat{\varphi}_i, \chi_E \rangle \neq 0$, for some i.

Define $\theta \in L^2(\mathbb{R}^2)$ by $\hat{\theta} = \chi_E$. For $j \in \mathbb{Z}^+$, write $V_j = D_a^{-j}V$, and note, since the operator D_a is unitary, the collection

$$\{D_a^{-j} D_b T_k \varphi_i : b \in B_0, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

forms a frame for V_j with constants $C \leq D$. Let P denote the orthogonal projection of $L^2(\mathbb{R}^2)$ onto V, and, for each j, let P_j denote the orthogonal projection of $L^2(\mathbb{R}^2)$ onto V_j. Using property (ii) of E, assumption (iii) in the statement of Theorem 5.1, the Plancherel theorem, and a change of variable,

we obtain

$$
\begin{aligned}
0 < \|P\theta\|^2 &\leq \|P_j\theta\|^2 \\
&\leq C^{-1} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} |\langle D_a^{-j} D_b T_k \varphi_i, P_j\theta\rangle|^2 \\
&= C^{-1} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} |\langle D_a^{-j} D_b T_k \varphi_i, \theta\rangle|^2 \\
&= C^{-1} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} |\langle (D_a^{-j} D_b T_k \varphi_i)\widehat{\ }, \hat{\theta}\,\rangle|^2 \qquad (5.9) \\
&= C^{-1} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} \left| \int_E (D_a^{-j} D_b T_k \varphi_i)\widehat{\ }(\xi)\, d\xi \right|^2 \\
&= C^{-1} |\det a|^{-j} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} \left| \int_E \widehat{\varphi}_i(\xi a^{-j} b) e^{-2\pi\imath\xi a^{-j} bk}\, d\xi \right|^2 \\
&= C^{-1} |\det a|^{j} \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} \left| \int_{Ea^{-j}b} \widehat{\varphi}_i(\xi) e^{-2\pi\imath\xi k}\, d\xi \right|^2.
\end{aligned}
$$

Using Lemma 3.1 and property (i) of E, we obtain

$$\begin{aligned}
&|\det a|^j \sum_{i=1}^{I} \sum_{b\in B_0} \sum_{k\in\mathbb{Z}^2} \left| \int_{Ea^{-j}b} \widehat{\varphi}_i(\xi) e^{-2\pi\imath\xi k}\, d\xi \right|^2 \\
&\qquad\qquad = |\det a|^j \sum_{i=1}^{I} \sum_{b\in B_0} \int_{Ea^{-j}b} |\widehat{\varphi}_i(\xi)|^2\, d\xi \\
&\qquad\qquad = |\det a|^j \sum_{i=1}^{I} \sum_{b\in B_0} \int_{Ea^{-j}} |\widehat{\varphi}_i(\xi b)|^2\, d\xi \qquad (5.10) \\
&\qquad\qquad \le |\det a|^j \sum_{i=1}^{I} \sum_{b\in B} \int_{Ea^{-j}} |\widehat{\varphi}_i(\xi b)|^2\, d\xi \\
&\qquad\qquad = |E| \sum_{i=1}^{I} \frac{1}{|Ea^{-j}|} \int_{Ea^{-j}} \sum_{b\in B} |\widehat{\varphi}_i(\xi b)|^2\, d\xi.
\end{aligned}$$

Note that, for each i, φ_i satisfies the hypotheses of Proposition 5.1. Thus, for each i and for all (ξ_1, ξ_2) with $0 < |\xi_1| \le \min\{1, (2z)^{-1}\}$ we have

$$\sum_{b\in B} |\widehat{\varphi}_i((\xi_1, \xi_2)b)|^2 \le M_i(\xi_1), \qquad (5.11)$$

where the function M_i is bounded, measurable, and satisfies $M_i(t) \to 0$, as $t \to 0$. If $(\xi_1, \xi_2) \in Ea^{-j}$, then $(\xi_1, \xi_2) = \eta a^{-j}$, for some $\eta \in E$. Using assumption (i) in the statement of Theorem 5.1 and that E is bounded, we obtain

$$|\xi_1| = \left| \eta a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \le \|\eta\| \left\| a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \le K \left\| a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \to 0, \qquad (5.12)$$

as $j \to \infty$, where the constant K is independent of (ξ_1, ξ_2). By (5.12), the inequalities of (5.11) are valid a.e. in Ea^{-j}, when j is large enough. For these j, using (5.11) and (5.12) yields

$$\begin{aligned}
&\sum_{i=1}^{I} \frac{1}{|Ea^{-j}|} \int_{Ea^{-j}} \sum_{b \in B} |\widehat{\varphi}_i((\xi_1, \xi_2)b)|^2 \, d\xi \\
&\qquad \leq \sum_{i=1}^{I} \frac{1}{|Ea^{-j}|} \int_{Ea^{-j}} M_i(\xi_1) \, d\xi \\
&\qquad \leq \sum_{i=1}^{I} \sup_{(\xi_1, \xi_2) \in Ea^{-j}} M_i(\xi_1) \to 0,
\end{aligned} \tag{5.13}$$

as $j \to \infty$. Combining (5.9), (5.10), and (5.13) obtains the desired contradiction. This completes the proof. $\square$

The below corollary, which follows immediately from Theorem 5.1 and Lemma 5.2, is the second of the two main results of this section.

Corollary 5.1. *Let $z > 0$, let $a \in GL_2(\mathbb{R})$, let $I \in \mathbb{Z}^+$, and let B_0 be an infinite subset of*

$$\left\{ \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Suppose that a and B_0 satisfy properties (i) and (ii) in the statement of Theorem 5.1. Then there does not exist an aB_0-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\} \subset L^2(\mathbb{R}^2)$ such that, for each i, ONE of the following is satisfied:

(i) $\partial_2 \varphi_i(x)$ exists for all x, $\partial_2 \varphi_i \in C(\mathbb{R}^2)$, and $\varphi_i, |x_1|^\alpha \varphi_i, |x_1|^\alpha \partial_2 \varphi_i \in L^1(\mathbb{R}^2)$, for some $1/2 \leq \alpha \leq 1$.

(ii) $\varphi_i \in L^1(\mathbb{R}^2)$ *and* $|x_1|^\alpha \varphi_i$ *is* L^1*-Hölder continuous in the direction* e_2 *with exponent* β*, for some* $1/2 < \alpha, \beta \leq 1$.

(iii) φ_i *is compactly supported and Hölder continuous in the direction* e_2 *with exponent* β*, for some* $1/2 < \beta \leq 1$.

We make the following comments regarding Theorem 5.1 and Corollary 5.1:

(i) Let

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Choose $0 \neq \theta \in C^\infty(\hat{\mathbb{R}}^2)$ that is compactly supported in

$$\{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : \xi_1 \leq \xi_2 \leq 0\} \cup \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 0 \leq \xi_2 \leq \xi_1\}$$

and satisfies $[\theta, \theta](\xi) = 1$, for all ξ. Define φ in the Schwartz class of $\mathbb{R}^2$ by $\hat{\varphi} = \theta$. It follows easily from part (iii) of Lemma 2.1 and part (iv) of Theorem 2.1 that

$$\{D_b T_k \varphi : b \in B, k \in \mathbb{Z}^2\}$$

forms an ON basis for its closed span V. It is interesting to compare this example with the result of Theorem 5.1; in particular, we cannot have $D_a V \subset V$.

(ii) Let a and B be as in (i) above. In section 5 of [3], a Parseval aB-wavelet ψ is constructed that belongs to the Schwartz class of $\mathbb{R}^2$. However, by Corollary 5.1, ψ cannot be associated with an aB-MRA whose aB-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ is such that, for each i, φ_i satisfies ONE of (i), (ii), and (iii) in the statement of Corollary 5.1. In particular, the functions $\varphi_1, \ldots, \varphi_I$ certainly cannot belong to the Schwartz class of $\mathbb{R}^2$.

5.2 Nonexistence Results Regarding Decay and Finite Type Conditions

Let $a \in GL_n(\mathbb{R})$ and let B be an infinite subgroup of $\widetilde{SL_n}(\mathbb{Z})$. Assume that $\{V_j : j \in \mathbb{Z}\}$ is an aB-MRA with ON aB-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$). Let $i' \in \{1, \ldots, I\}$. By properties (i) and (ii) in the aB-MRA definition, we have $D_a \varphi_{i'} \in V_0$. Using aB-MRA property (v) and part (iii) of Lemma 1.1, we may write

$$D_a \varphi_{i'} = \sum_{i=1}^{I} \sum_{b \in B} \sum_{k \in \mathbb{Z}^n} \langle D_a \varphi_{i'}, D_b T_k \varphi_i \rangle D_b T_k \varphi_i, \tag{5.14}$$

where each sum converges unconditionally in V_0. In developing fast algorithmic implementations of the aB-MRA system $\{V_j : j \in \mathbb{Z}\}$ (and its associated aB-multiwavelet), it is very desirable that the sum (5.14) be finite. If each of $\varphi_1, \ldots, \varphi_I$ are compactly supported, then, for each $i \in I$ and $b \in B$, it follows that the collection $\{\langle D_a\varphi_{i'}, D_bT_k\varphi_i\rangle : k \in \mathbb{Z}^n\}$ is finite. However, to ensure that, for each i, the collection

$$\{\langle D_a\varphi_{i'}, D_bT_k\varphi_i\rangle : b \in B, k \in \mathbb{Z}^n\}$$

is finite, we must also assume the existence of a finite subset F of B such that, for each j', $D_a\varphi_{j'}$ belongs to the closed span of

$$\{D_bT_k\varphi_j : b \in F, k \in \mathbb{Z}^n, j = 1, \ldots, I\}. \tag{5.15}$$

In summary, to ensure that (5.14) is finite, we have assumed, in particular, the following two properties of the above aB-MRA system and its aB-scaling multifunction:

(i) $\{\varphi_1, \ldots, \varphi_I\}$ is an ON aB-scaling multifunction; more generally, there exists some "separation" amongst the elements

$$\{D_bT_k\varphi_i : b \in b, k \in \mathbb{Z}^n, i = 1, \ldots, I\};$$

(ii) there exists a finite subset F of B such that, for each j', $D_a\varphi_{j'}$ belongs to the closed span of (5.15).

Properties (i) and (ii) above will form the basis for the more general finite type 1 and finite type 2 conditions (Definitions 5.1 and 5.2). The three main results of this section (Theorems 5.2 and 5.3 and Corollary 5.2) regard the nonexistence of a large collection of shearlet-like aB-scaling multifunctions which satisfy a minimal amount of decay and either the finite type 1 or the finite type 2 condition.

5.2.1 Finite Type Conditions

In this subsection, we introduce the two finite type conditions that will be used in the statements of the main results of this section. Below is the defintion of the first.

Definition 5.1. *Let $a \in GL_n(\mathbb{R})$ and let B be an infinite subgroup of $\widetilde{SL_n}(\mathbb{Z})$. Assume that $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$) is an aB-scaling multifunction. We say that $\{\varphi_1, \ldots, \varphi_I\}$ is of finite type 1 if there exists a finite subset F of B such that the following are satisfied:*

(i) for each i, $D_a\varphi_i$ is in the closed linear span of the collection

$$\{D_b T_k \varphi_{i'} : b \in F, k \in \mathbb{Z}^n, i' = 1, \ldots, I\};$$

(ii) there exists a dual frame to

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i = 1, \ldots, I\}$$

of the form

$$\{D_b T_k \psi_i : b \in B, k \in \mathbb{Z}^n, i = 1, \ldots, I\}$$

such that $\varphi_i \perp D_b T_k \psi_{i'}$*, for all* $b \in B \setminus F$*, all* $k \in \mathbb{Z}^n$*, and all* $i, i' = 1, \ldots, I$*.*

Let a and B be as in Definition 5.1 and let $\{\varphi_1, \ldots, \varphi_I\} \subset L^2(\mathbb{R}^n)$ ($I \in \mathbb{Z}^+$). We make the following comments regarding the above definition:

(i) Suppose that $\{\varphi_1, \ldots, \varphi_I\}$ is an aB-scaling multifunction of finite type 1 and let $\{\psi_1, \ldots, \psi_I\}$ and F be as in Definition 5.1. If each of the functions $\varphi_1, \ldots, \varphi_I, \psi_1, \ldots, \psi_I$ are compactly supported, then $\{\varphi_1, \ldots, \varphi_I\}$ has finite refinement equations. That is, there exists a finite subset Z of $\mathbb{Z}^n$ such that, for each i', there exists a collection

$$\{\alpha_{ibk}^{i'} : i = 1, \ldots, I, b \in F^2, k \in Z\} \subset \mathbb{C}$$

such that

$$D_a \varphi_{i'} = \sum_{i=1}^{I} \sum_{b \in F^2} \sum_{k \in Z} \alpha_{ibk}^{i'} D_b T_k \varphi_i.$$

(ii) If

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^n, i = 1, \ldots, I\}$$

forms a frame for its closed span, then its canonical dual frame

$$\{(D_b T_k \varphi_i)^\sim : b \in B, k \in \mathbb{Z}^n, i = 1, \ldots, I\}$$

automatically satisfies the first part of condition (ii) of Definition 5.1;

that is,

$$(D_bT_k\varphi_i)^\sim = D_bT_k(\varphi_i)^\sim,$$

for all b, k, and i.

(iii) If

$$\{D_bT_k\varphi_i : b \in B, k \in \mathbb{Z}^n, i = 1, \ldots, I\} \tag{5.16}$$

forms a Parseval frame for its closed span, then it is canonically dual to itself. Thus, condition (ii) of Definition 5.1 is satisfied if there exists a finite subset F of B such that $\varphi_i \perp D_bT_k\varphi_{i'}$, for all $b \in B \setminus F$, all $k \in \mathbb{Z}^n$, and all $i, i' = 1, \ldots, I$.

(iv) If (5.16) forms a Riesz basis for its closed span, then condition (ii) in Definition 5.1 is satisfied with $F = \{I\}$.

Below is the definition of the second finite type condition.

Definition 5.2. *Let $a \in GL_2(\mathbb{R})$, let $z \in \mathbb{Z}^+$, and let*

$$B = \left\{ b(l) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Assume that $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$) is an aB-scaling multifunction. We say that $\{\varphi_1, \ldots, \varphi_I\}$ is of finite type 2 if there exists a sequence $L_1 < L_2 < L_3 < \ldots$ of positive integers such that the following are satisfied:

(i) for each i, $D_a\varphi_i$ is in the closed linear span of the collection

$$\{D_{b(l)}T_k\varphi_{i'} : |l| \leq L_1, k \in \mathbb{Z}^2, i' = 1, \ldots, I\};$$

(ii) there exists $0 < C \leq D < \infty$ such that, for each $j \in \mathbb{Z}^+$, the collection

$$\{D_{b(l)}T_k\varphi_i : |l| \leq L_j, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

forms a frame for its closed span with constants $C \leq D$.

Let a and B be as in Definition 5.2 and assume that $\{\varphi_1, \ldots, \varphi_I\}$ is an aB-scaling multifunction satisfying condition (i) in the same definition, for some $L_1 \in \mathbb{Z}^+$. It follows from Proposition 1.5 (and the comments preceding it) that $\{\varphi_1, \ldots, \varphi_I\}$ is of finite filter type 2 if EITHER of the following hold:

(i) The collection

$$\{D_bT_k\varphi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

forms a Riesz basis for its closed span.

(ii) There exists a sequence $L_2 < L_3 < L_4 < \ldots$ of positive integers such that, for each j, the intersection of the closed spans of

$$\{D_{b(l)}T_k\varphi_i : |l| \leq L_j, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

and

$$\{D_{b(l)}T_k\varphi_i : |l| \geq L_j + 1, k \in \mathbb{Z}^2, i = 1, \dots, I\}$$

is the zero subspace.

Definitions 5.1 and 5.2 are very similar. In fact, Corollary 5.2 will imply that the two definitions are trivially equivalent when a and B are as in Lemma 5.3 below and each element of the aB-scaling multifunction $\{\varphi_1, \dots, \varphi_I\}$ satisfies a small amount of fractional polynomial decay. However, when considered within the more general context of collections of the form

$$\{T_k\varphi_i : k \in \mathbb{Z}^n, i \in \mathbb{Z}^+\},$$

it is not hard to see that condition (ii) in Definition 5.1 and condition (ii) in Definition 5.2 are independent.

5.2.2 Preliminary Results

This subsection contains the various lemmas that will be needed in the proofs of Theorems 5.2 and 5.3 and Corollary 5.2.

The above mentioned three results will be formulated in the following generality with respect to a and B: We will consider groups B of the form

$$\left\{ \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\},$$

where $z \in \mathbb{Z}^+$, and we will consider matrices $a \in GL_2(\mathbb{Z})$ that normalize B (i.e., $aBa^{-1} \subset B$) and that satisfy

$$a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

as $j \to \infty$. The following easy lemma characterizes the structure of such a:

Lemma 5.3. *Let $a \in GL_2(\mathbb{Z})$, let $z \in \mathbb{Z}^+$, and let*

$$B = \left\{ b(l) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Consider the following conditions:

(i) a normalizes B;

(ii) $a^{-j} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, as $j \to \infty$.

Then a satisfies (i) and (ii) above if and only if

$$a = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix},$$

for some $p, q, r \in \mathbb{Z}$ satisfying $|p| > 1$, $|q| \geq 1$, and $q \mid p$ (i.e., q divides p).

For the remainder of this subsection, the notation a, z, B, p, q, and r will be as defined in Lemma 5.3 (in particular, we are assuming that (i) and (ii) in the said lemma hold).

Note that $b(l)b(l') = b(l+l')$ and that $ab(l)a^{-1} = b(pq^{-1}l)$. For $M, j \in \mathbb{Z}^+$ and any finite subset E of B, write

$$B(M) = \{b(l) : |l| \leq M\}$$

and

$$E^j = \{a^{j-1}b_{j-1}a^{-(j-1)}a^{j-2}b_{j-2}a^{-(j-2)} \dots ab_1a^{-1}b_0 : b_{j-1}, \dots, b_0 \in E\}.$$

We require the following three lemmas in the proofs of Theorems 5.2 and 5.3:

Lemma 5.4. *For each $M, j \in \mathbb{Z}^+$ we have*

$$B(M)^j \subset \begin{cases} B(M|pq^{-1}|^j), & \text{if } |p| > |q|; \\ B(Mj), & \text{if } |p| = |q|. \end{cases}$$

Proof of Lemma 5.4. Fix $M, j \in \mathbb{Z}^+$ and let $b \in B(M)^j$. We may then write

$$b = a^{j-1}b(l_{j-1})a^{-(j-1)}a^{j-2}b(l_{j-2})a^{-(j-2)} \dots ab(l_1)a^{-1}b(l_0),$$

where $|l_p| \le M$, for each p. We have

$$
\begin{aligned}
b &= a^{j-1}b(l_{j-1})a^{-(j-1)}a^{j-2}b(l_{j-2})a^{-(j-2)}\dots ab(l_1)a^{-1}b(l_0) \\
&= b\big((pq^{-1})^{j-1}l_{j-1}\big)b\big((pq^{-1})^{j-2}l_{j-2}\big)\dots b\big(pq^{-1}l_1\big)b\big(l_0\big) \\
&= b\big((pq^{-1})^{j-1}l_{j-1} + (pq^{-1})^{j-2}l_{j-2} + \dots + pq^{-1}l_1 + l_0\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|(pq^{-1})^{j-1}l_{j-1} + (pq^{-1})^{j-2}l_{j-2} + \dots + pq^{-1}l_1 + l_0\big| \\
&\qquad \le \big|(pq^{-1})^{j-1}l_{j-1}\big| + \big|(pq^{-1})^{j-2}l_{j-2}\big| + \dots + \big|pq^{-1}l_1\big| + \big|l_0\big| \\
&\qquad \le M\big(|pq^{-1}|^{j-1} + |pq^{-1}|^{j-2} + \dots + |pq^{-1}| + 1\big) \\
&\qquad \le \begin{cases} M|pq^{-1}|^{j}, & \text{if } |pq^{-1}| > 1; \\ Mj, & \text{if } |pq^{-1}| = 1. \end{cases}
\end{aligned}
$$

where the last inequality is easily verified. $\square$

Lemma 5.5. *Let E be a finite subset of B and let $\{\varphi_1, \dots, \varphi_I\} \subset L^2(\mathbb{R}^2)$ ($I \in \mathbb{Z}^+$). Suppose that, for each i, we have*

$$
\widehat{\varphi}_i(\xi a) = \sum_{i'=1}^{I} \sum_{b \in E} m^i_{i'b}(\xi)\widehat{\varphi_{i'}}(\xi b), \tag{5.17}
$$

for a.e. ξ, where

$$
\{m^i_{i'b} : i' = 1, \dots, I, b \in E\} \subset L^\infty(\mathbb{T}^2).
$$

Then, for each i and each $j \in \mathbb{Z}^+$, $D_a^j \varphi_i$ is in the closed linear span of

$$\{D_b T_k \varphi_{i'} : b \in E^j, k \in \mathbb{Z}^2, i' = 1, \ldots, I\}.$$

Proof of Lemma 5.5. We will show that, for each i and each $j \in \mathbb{Z}^+$, there exists a collection

$$\{m_{i'b}^{ij} : i' = 1, \ldots, I, b \in E^j\} \subset L^\infty(\mathbb{T}^2)$$

such that

$$\widehat{\varphi}_i(\xi a^j) = \sum_{i'=1}^{I} \sum_{b \in E^j} m_{i'b}^{ij}(\xi) \widehat{\varphi_{i'}}(\xi b), \tag{5.18}$$

for a.e. ξ. In conjunction with part (i) of Theorem 2.1, this will prove the claim.

We proceed by induction on j. The case when $j = 1$ follows by assumption. Fix $j \in \mathbb{Z}^+$ and suppose that, for each i, there exists a collection

$$\{m_{i'b}^{ij} : i' = 1, \ldots, I, b \in E^j\} \subset L^\infty(\mathbb{T}^2)$$

such that (5.18) is satisfied. For fixed i, using (5.18) and (5.17) we obtain

$$\begin{aligned}\widehat{\varphi}_i(\xi a^{j+1}) &= \widehat{\varphi}_i(\xi a a^j)\\ &= \sum_{i'=1}^{I}\sum_{b\in E^j} m^{ij}_{i'b}(\xi a)\widehat{\varphi_{i'}}(\xi ab)\\ &= \sum_{i'=1}^{I}\sum_{b\in E^j} m^{ij}_{i'b}(\xi a)\widehat{\varphi_{i'}}(\xi aba^{-1}a)\\ &= \sum_{i'=1}^{I}\sum_{b\in E^j} m^{ij}_{i'b}(\xi a)\sum_{i''=1}^{I}\sum_{b'\in E} m^{i'}_{i''b'}(\xi aba^{-1})\widehat{\varphi_{i''}}(\xi aba^{-1}b'),\end{aligned} \tag{5.19}$$

for a.e. ξ. Since $b \in E^j$ and $b' \in E$ implies that $aba^{-1}b' \in E^{j+1}$ and since all of the functions $m^{ij}_{i'b}(\xi a)$ and $m^{i'}_{i''b'}(\xi aba^{-1})$ belong to $L^\infty(\mathbb{T}^2)$, it follows that the final quantity in (5.19) may be written as

$$\sum_{i'=1}^{I}\sum_{b\in E^{j+1}} m^{i(j+1)}_{i'b}(\xi)\widehat{\varphi_{i'}}(\xi b),$$

for a suitable collection

$$\{m^{i(j+1)}_{i'b} : i' = 1,\ldots,I, b \in E^{j+1}\} \subset L^\infty(\mathbb{T}^2).$$

This completes the induction and proves the lemma. □

Lemma 5.6. *Let $E \subset \hat{\mathbb{R}}^2$ be bounded and measurable and let $\{M_j\}_{j=1}^\infty$ be a sequence of positive integers. Assume that*

$$\{D_bT_k\varphi_i : b \in B, k \in \mathbb{Z}^2, i = 1,\ldots,I\} \tag{5.20}$$

is a Bessel system in $L^2(\mathbb{R}^2)$ *(*$I \in \mathbb{Z}^+$*).*

(i) Suppose that $|p| > |q|$ *and that, for each* i, $|\widehat{\varphi}_i|$ *is continuous and strongly locally Hölder continuous in the direction* $\hat{e}_1$ *with exponent* α*, where*

$$\alpha = \frac{1}{2}\left(1 - \frac{\log|q|}{\log|p|}\right).$$

Suppose also that there exists a constant K' *such that* $M_j \leq K'|pq^{-1}|^j$*, for all* $j \in \mathbb{Z}^+$*. Then*

$$|\det a|^j \int_{Ea^{-j}} \sum_{i=1}^{I} \sum_{b \in B(M_j)} |\widehat{\varphi}_i(\xi b)|^2 \, d\xi \to 0,$$

as $j \to \infty$*.*

(ii) Suppose that $|p| = |q|$ *and that, for each* i, $|\widehat{\varphi}_i|$ *is continuous and strongly logarithmically continuous in the direction* $\hat{e}_1$ *with exponent* $1/2$*. Suppose also that there exists a constant* K' *such that* $M_j \leq K'j$*, for all* $j \in \mathbb{Z}^+$*. Then*

$$|\det a|^j \int_{Ea^{-j}} \sum_{i=1}^{I} \sum_{b \in B(M_j)} |\widehat{\varphi}_i(\xi b)|^2 \, d\xi \to 0,$$

as $j \to \infty$*.*

Proof of Lemma 5.6. If $j \in \mathbb{Z}^+$ and $\xi \in Ea^{-j}$, then $\xi = \eta a^{-j}$ for some $\eta \in E$ and we have

$$|\xi e_1| = |\eta a^{-j} e_1| \leq \|\eta\| \|a^{-j} e_1\| \leq K|p|^{-j}, \tag{5.21}$$

where $K > 0$ depends only on E.

To prove (i), suppose that $|p| > |q|$ and that, for each i, $|\widehat{\varphi}_i|$ is continuous and strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent α, where

$$\alpha = \frac{1}{2}\left(1 - \frac{\log|q|}{\log|p|}\right).$$

Suppose also that there exists a constant K' such that $M_j \leq K'|pq^{-1}|^j$, for all $j \in \mathbb{Z}^+$. Using part (ii) of Corollary 3.1 and that (5.20) is Bessel, it follows that there exists a bounded increasing function $M : [0,\infty) \longrightarrow [0,\infty)$ such that

$$\sum_{i=1}^{I} |\widehat{\varphi}_i(\xi)|^2 \leq M(|\xi e_1|)|\xi e_1|^{2\alpha}, \tag{5.22}$$

for all ξ with $0 < |\xi e_1| \leq 1$, where $M(t) \to 0$ as $t \to 0$. If $0 < |\xi e_1| \leq 1$ and $j \in \mathbb{Z}^+$, then using (5.22), that B fixes e_1, and that $|B(M)| \leq 3M$ ($M \in \mathbb{Z}^+$), we obtain

$$\begin{aligned}
\sum_{b \in B(M_j)} \sum_{i=1}^{I} |\widehat{\varphi}_i(\xi b)|^2 &\leq \sum_{b \in B(M_j)} M(|\xi b e_1|)|\xi b e_1|^{2\alpha} \\
&= \sum_{b \in B(M_j)} M(|\xi e_1|)|\xi e_1|^{2\alpha} \\
&= |B(M_j)| M(|\xi e_1|)|\xi e_1|^{2\alpha} \qquad (5.23)\\
&\leq 3M_j M(|\xi e_1|)|\xi e_1|^{2\alpha} \\
&\leq 3K'|pq^{-1}|^j M(|\xi e_1|)|\xi e_1|^{2\alpha}.
\end{aligned}$$

By (5.21), (5.23) is valid a.e. in Ea^{-j} for large enough j. For these j, using (5.23), (5.21), and the definition of α we obtain

$$
\begin{aligned}
|\det a|^j \int_{Ea^{-j}} \sum_{i=1}^{I} \sum_{b\in B(M_j)} &|\widehat{\varphi}_i(\xi b)|^2 \, d\xi \\
&\leq |E| \frac{1}{|Ea^{-j}|} \int_{Ea^{-j}} 3K' |pq^{-1}|^j M(|\xi e_1|) |\xi e_1|^{2\alpha} \, d\xi \\
&\leq 3|E|K'|pq^{-1}|^j \sup_{\xi \in Ea^{-j}} M(|\xi e_1|)|\xi e_1|^{2\alpha} \\
&\leq 3|E|K'|pq^{-1}|^j M(K|p|^{-j})(K|p|^{-j})^{2\alpha} \\
&= 3|E|K'K^{2\alpha} M(K|p|^{-j}) \left(\frac{|p|}{|p|^{2\alpha}|q|} \right)^j \\
&= 3|E|K'K^{2\alpha} M(K|p|^{-j}) \to 0,
\end{aligned}
$$

as $j \to \infty$.

To prove (ii), suppose that $|p| = |q|$ and that, for each i, $|\widehat{\varphi}_i|$ is continuous and strongly logarithmically continuous in the direction $\hat{e}_1$ with exponent 1/2. Suppose also that there exists a constant K' such that $M_j \leq K'j$, for all $j \in \mathbb{Z}^+$. We proceed as in the proof of (i). Using part (ii) of Corollary 3.1 and that (5.20) is Bessel, it follows that there exists a bounded increasing function $M : [0, \infty) \longrightarrow [0, \infty)$ such that

$$
\sum_{i=1}^{I} |\widehat{\varphi}_i(\xi)|^2 \leq \frac{M(|\xi e_1|)}{|\log |\xi e_1||}, \tag{5.24}
$$

for all ξ with $0 < |\xi e_1| < 1$, where $M(t) \to 0$ as $t \to 0$. If $0 < |\xi e_1| < 1$ and $j \in \mathbb{Z}^+$, then using (5.24), that B fixes e_1, and that $|B(M)| \leq 3M$ ($M \in \mathbb{Z}^+$),

we obtain

$$
\begin{aligned}
\sum_{b\in B(M_j)} \sum_{i=1}^{I} |\widehat{\varphi}_i(\xi b)|^2 &\le \sum_{b\in B(M_j)} \frac{M(|\xi b e_1|)}{\big|\log|\xi b e_1|\big|} \\
&= \sum_{b\in B(M_j)} \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|} \\
&= |B(M_j)| \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|} \\
&\le 3M_j \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|} \le 3K'j \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|}.
\end{aligned}
\tag{5.25}
$$

If $j \in \mathbb{Z}^+$ is large enough so that $K|p|^{-j} < 1$, then (5.21) implies that (5.25) is valid a.e. in Ea^{-j}. For these j, using (5.25) and (5.21), we obtain

$$
\begin{aligned}
&|\det a|^j \int_{Ea^{-j}} \sum_{i=1}^{I} \sum_{b\in B(M_j)} |\widehat{\varphi}_i(\xi b)|^2 \, d\xi \\
&\qquad \le |E| \frac{1}{|Ea^{-j}|} \int_{Ea^{-j}} 3K'j \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|} \, d\xi \\
&\qquad \le 3|E|K'j \sup_{\xi\in Ea^{-j}} \frac{M(|\xi e_1|)}{\big|\log|\xi e_1|\big|} \\
&\qquad \le 3|E|K'j \frac{M(K|p|^{-j})}{\big|\log(K|p|^{-j})\big|} \\
&\qquad = 3|E|K'M(K|p|^{-j}) \frac{j}{j\log|p| - \log K} \to 0,
\end{aligned}
$$

as $j \to \infty$, where the third inequality and final equality follow from $K|p|^{-j} < 1$. $\square$

5.2.3 Main Results

This subsection contains the three main results of the current section—results regarding the nonexistence of certain aB-scaling multifunctions. Similar to before, the only aB-MRA properties necessary in the proofs of Theorems 5.2 and 5.3 and Corollary 5.2 are properties (i), (ii), and (v). The first two of our main results involve only these three aB-MRA properties, certain decay assumptions, and either the finite type 1 (Theorem 5.2) or the finite type 2 (Theorem 5.3) condition. Corollary 5.1, the third of our main results, translates Theorems 5.2 and 5.3 into a statement regarding the nonexistence of a large collection of shearlet-like aB-scaling multifunctions which satisfy a minimal amount of decay and either the finite type 1 or the finite type 2 condition.

Theorem 5.2. *Let $z \in \mathbb{Z}^+$, let*

$$B = \left\{ b(l) = \begin{pmatrix} 1 & lz \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\},$$

and let

$$a = \begin{pmatrix} p & r \\ 0 & q \end{pmatrix},$$

for some $p, q, r \in \mathbb{Z}$ satisfying $|p| > 1$, $|q| \geq 1$, and $q \mid p$. Let $\{0\} \neq \{\varphi_1, \ldots, \varphi_I\} \subset L^2(\mathbb{R}^2)$ ($I \in \mathbb{Z}^+$). Suppose that there exists a finite subset F of B such that the following conditions are satisfied:

(i) for each i, $D_a\varphi_i$ is in the closed linear span of

$$\{D_bT_k\varphi_{i'} : b \in F, k \in \mathbb{Z}^2, i' = 1, \ldots, I\};$$

(ii) the collection

$$\{D_bT_k\varphi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\} \tag{5.26}$$

forms a frame for its closed linear span;

(iii) there exists a dual frame to (5.26) of the form

$$\{D_bT_k\psi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\} \tag{5.27}$$

such that $\varphi_i \perp D_bT_k\psi_{i'}$, for all $b \in B \setminus F$, all $k \in \mathbb{Z}^2$, and all $i, i' = 1, \ldots, I$.

Then we have the following:

(iv) If $|p| > |q|$, the functions $\varphi_1, \ldots, \varphi_I$ cannot be such that, for each i, $|\widehat{\varphi}_i|$ is continuous and strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent α, where

$$\alpha = \frac{1}{2}\left(1 - \frac{\log|q|}{\log|p|}\right).$$

(v) If $|p| = |q|$, the functions $\varphi_1, \ldots, \varphi_I$ cannot be such that, for each i, $|\widehat{\varphi}_i|$

is continuous and strongly logarithmically continuous in the direction $\hat{e}_1$ with exponent $1/2$.

Theorem 5.3. *Let B and a be as in the statement of Theorem 5.2. Let $\{0\} \neq \{\varphi_1, \ldots, \varphi_I\} \subset L^2(\mathbb{R}^2)$ ($I \in \mathbb{Z}^+$). Suppose that there exists $0 < C \leq D < \infty$ and a sequence $L_1 < L_2 < L_3 < \ldots$ of positive integers such that the following are satisfied:*

(i) for each i, $D_a\varphi_i$ is in the closed span of the collection

$$\{D_{b(l)}T_k\varphi_{i'} : |l| \leq L_1, k \in \mathbb{Z}^2, i' = 1, \ldots, I\}; \tag{5.28}$$

(ii) for each $j \in \mathbb{Z}^+$, the collection

$$\{D_{b(l)}T_k\varphi_i : |l| \leq L_j, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

forms a frame for its closed span with constants $C \leq D$.

Then we have the following:

(iii) If $|p| > |q|$, the functions $\varphi_1, \ldots, \varphi_I$ cannot be such that, for each i, $|\widehat{\varphi}_i|$ is continuous and strongly locally Hölder continuous in the direction $\hat{e}_1$ with exponent α, where

$$\alpha = \frac{1}{2}\left(1 - \frac{\log|q|}{\log|p|}\right).$$

(iv) *If* $|p| = |q|$, *the functions* $\varphi_1, \ldots, \varphi_I$ *cannot be such that, for each* i, $|\widehat{\varphi}_i|$ *is continuous and strongly logarithmically continuous in the direction* $\hat{e}_1$ *with exponent* $1/2$.

We make the following comments regarding the statements of Theorems 5.2 and 5.3:

(i) Using the notation of Theorems 5.2 and 5.3, note the interesting manner in which the regularity of the functions $\varphi_1, \ldots, \varphi_I$ depends on the matrix entries p and q: roughly speaking, the closer $|q|$ gets to $|p|$, the less regularity the functions $\varphi_1, \ldots, \varphi_I$ can attain in the direction $\hat{e}_1$.

(ii) As will be evident from its proof, the statement of Theorem 5.2 can be strengthened in the following fashion: under the assumptions of Theorem 5.2, condition (iv) or (v) (whichever is relevant) cannot be satisfied on the ξ_2-axis. A similar remark can be made regarding the statement of Theorem 5.3.

We now prove Theorems 5.2 and 5.3.

Proof of Theorem 5.2. For $M \in \mathbb{Z}^+$, let $V(M)$ and $W(M)$ denote the closed spans of

$$\{D_{b(l)} T_k \varphi_i : |l| \leq M, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

and

$$\{D_{b(l)} T_k \varphi_i : |l| \geq M, k \in \mathbb{Z}^2, i = 1, \ldots, I\},$$

respectively. Choose $L \in \mathbb{Z}^+$ such that $F \subset B(L)$.

Claim 5.1. *For each $M \in \mathbb{Z}^+$, we have $V(M) \perp W(M+L+1)$.*

Proof of Claim 5.1. Suppose that $|l| \leq M$, that $|l'| \geq M+L+1$, that $k, k' \in \mathbb{Z}^2$, and that $i, i' \in \{1, \ldots, I\}$. Then

$$\begin{aligned}
\langle D_{b(l)} T_k \varphi_i, D_{b(l')} T_{k'} \psi_{i'} \rangle &= \langle \varphi_i, T_k^{-1} D_{b(l)}^{-1} D_{b(l')} T_{k'} \psi_{i'} \rangle \\
&= \langle \varphi_i, D_{b(l'-l)} T_{k'-b(l-l')k} \psi_{i'} \rangle \\
&= 0,
\end{aligned}$$

where the last equality holds since, as a consequence of

$$|l - l'| \geq |l'| - |l| \geq M + L + 1 - N = L + 1,$$

we have $b(l'-l) \notin F$. The claim now follows.

Let $0 < C \leq D < \infty$ denote the frame constants of both of (5.26) and (5.27).

Claim 5.2. *Let $M \in \mathbb{Z}^+$ and let $f \in L^2(\mathbb{R}^2)$. Suppose that $[\hat{f}, \hat{f}] \in L^\infty(\mathbb{T}^2)$ and that $f \in V(M)$. Then*

$$\hat{f}(\xi) = \sum_{i=1}^{I} \sum_{b \in B(M+L)} m_{ib}(\xi) \widehat{\varphi}_i(\xi b),$$

for a.e. ξ, where the collection $\{m_{ib} : 1 \leq i \leq I, b \in B(M+L)\}$ belongs to

$L^\infty(\mathbb{T}^2)$ *and satisfies*

$$\sum_{i=1}^{I} \sum_{b \in B(M+L)} \|m_{ib}\|_2^2 \leq D\|f\|^2.$$

Proof of Claim 5.2. We may write the collection (5.26) and its dual frame (5.27) as

$$\{T_k D_b \varphi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

and

$$\{T_k D_b \psi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\}.$$

Using conditions (ii) and (iii) of this theorem and part (vii) of Lemma 2.1, we may write

$$\hat{f} = \sum_{1 \leq i \leq I, b \in B} [\hat{f}, \widehat{\psi}_i(\cdot b)]\widehat{\varphi}_i(\cdot b), \tag{5.29}$$

with unconditional convergence in $L^2(\mathbb{R}^2)$. Using part (ii) of Theorem 2.1, it follows that

$$\begin{aligned}
\big|[\hat{f}, \widehat{\psi}_i(\cdot b)](\xi)\big| &= \Big| \sum_{k \in \hat{\mathbb{Z}}^2} \hat{f}(\xi + k)\overline{\widehat{\psi}_i\big((\xi + k)b\big)}\Big| \\
&\leq \Big(\sum_{k \in \hat{\mathbb{Z}}^2} |\hat{f}(\xi + k)|^2 \Big)^{1/2} \Big(\sum_{k \in \hat{\mathbb{Z}}^2} |\widehat{\psi}_i\big((\xi + k)b\big)|^2 \Big)^{1/2} \qquad (5.30) \\
&= [\hat{f}, \hat{f}](\xi)^{1/2} [\widehat{\psi}_i(\cdot b), \widehat{\psi}_i(\cdot b)](\xi)^{1/2} \\
&\leq [\hat{f}, \hat{f}](\xi)^{1/2} D^{1/2},
\end{aligned}$$

for a.e. ξ and all b and i. Part (iv) of Lemma 2.1 implies that

$$\sum_{1\leq i\leq I, b\in B} \left\|[\hat{f}, \widehat{\psi}_i(\cdot b)]\right\|_2^2 \leq D\|f\|^2. \tag{5.31}$$

Using Claim 5.1 and part (iii) of Lemma 2.1, it follows that

$$[\hat{f}, \widehat{\psi}_i(\cdot b(l))](\xi) = 0,$$

for a.e. ξ, all $|l| \geq M + L + 1$, and all $i = 1, \ldots, I$. Thus, by (5.29), we have

$$\hat{f}(\xi) = \sum_{i=1}^{I} \sum_{b\in B(M+L)} [\hat{f}, \widehat{\psi}_i(\cdot b)](\xi)\widehat{\varphi}_i(\xi b), \tag{5.32}$$

for a.e. ξ. The claim now follows from (5.30), (5.31), and (5.32).

Using condition (ii) in the statement of this theorem and part (ii) of Theorem 2.1, it follows easily that

$$[(D_a^j\varphi_i)\widehat{\ }, (D_a^j\varphi_i)\widehat{\ }](\xi) \leq |\det a|^j D, \tag{5.33}$$

for a.e. ξ (for each $j \in \mathbb{Z}^+$ and each i). Using the above inequality (with $j = 1$), that $D_a\varphi_i \in V(L)$ (for all i), Claim 5.2, and Lemma 5.5, it follows that $D_a^j\varphi_i$ belongs to the closed span of

$$\{D_bT_k\varphi_{i'} : b \in B(2L)^j, k \in \mathbb{Z}^2, i' = 1, \ldots, I\}$$

(for each $j \in \mathbb{Z}^+$ and each i). Consequently, Lemma 5.4 implies that $D_a^j \varphi_i \in V(L_j)$, where

$$L_j = \begin{cases} 2L|pq^{-1}|^j, & \text{if } |p| > |q|; \\ 2Lj, & \text{if } |p| = |q|. \end{cases}$$

(for each $j \in \mathbb{Z}^+$ and each i). Note that the sequence $\{L_j\}$ defined here bears no relation to the sequence of the same name that appears in the statement of Theorem 5.3. Using (5.33), Claim 5.2, and that $L_j + L \leq 2L_j$, we may write

$$|\det a|^{j/2} \widehat{\varphi}_i(\xi a^j) = (D_a^j \varphi_i)\widehat{\ }(\xi) = \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} m_{i'b}^{ij}(\xi) \widehat{\varphi_{i'}}(\xi b), \tag{5.34}$$

for a.e. ξ, where the collection $\{m_{i'b}^{ij} : 1 \leq i' \leq I, b \in B(2L_j)\}$ belongs to $L^\infty(\mathbb{T}^2)$ and satisfies

$$\sum_{i'=1}^{I} \sum_{b \in B(2L_j)} \|m_{i'b}^{ij}\|_2^2 \leq D \|D_a^j \varphi_i\|^2 = D \|\varphi_i\|^2, \tag{5.35}$$

(for each $j \in \mathbb{Z}^+$ and each i).

Since $\{\varphi_1, \ldots, \varphi_I\} \neq \{0\}$, there is an index i and a measurable bounded subset E of $\hat{\mathbb{R}}^2$ which satisfies $|E \cap (E + k)| = 0$ (for all $k \in \hat{\mathbb{Z}}^2 \setminus \{0\}$) such that

$$\int_E |\widehat{\varphi}_i(\xi)| \, d\xi > 0.$$

Using a change of variable, (5.34), and two applications of the Schwarz inequality, we obtain

$$
\begin{aligned}
0 < \int_E |\widehat{\varphi}_i(\xi)|\, d\xi &= |\det a|^j \int_{Ea^{-j}} |\widehat{\varphi}_i(\xi a^j)|\, d\xi \\
&= |\det a|^{j/2} \int_{Ea^{-j}} \Big| \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} m_{i'b}^{ij}(\xi) \widehat{\varphi_{i'}}(\xi b) \Big|\, d\xi \\
&\le |\det a|^{j/2} \int_{Ea^{-j}} \Big(\sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |m_{i'b}^{ij}(\xi)|^2 \Big)^{1/2} \\
&\qquad\qquad \Big(\sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |\widehat{\varphi_{i'}}(\xi b)|^2 \Big)^{1/2} d\xi \\
&\le |\det a|^{j/2} \Big(\int_{Ea^{-j}} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |m_{i'b}^{ij}(\xi)|^2\, d\xi \Big)^{1/2} \qquad (5.36) \\
&\qquad \Big(\int_{Ea^{-j}} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |\widehat{\varphi_{i'}}(\xi b)|^2\, d\xi \Big)^{1/2} \\
&= \Big(\int_{Ea^{-j}} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |m_{i'b}^{ij}(\xi)|^2\, d\xi \Big)^{1/2} \\
&\qquad \Big(|\det a|^j \int_{Ea^{-j}} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |\widehat{\varphi_{i'}}(\xi b)|^2\, d\xi \Big)^{1/2},
\end{aligned}
$$

for each $j \in \mathbb{Z}^+$. Using (5.35), we obtain

$$
\begin{aligned}
\int_{Ea^{-j}} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |m_{i'b}^{ij}(\xi)|^2\, d\xi &\le \int_{[0,1]^2} \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} |m_{i'b}^{ij}(\xi)|^2\, d\xi \\
&= \sum_{i'=1}^{I} \sum_{b \in B(2L_j)} \|m_{i'b}^{ij}\|_2^2 \qquad (5.37) \\
&\le D \|\varphi_i\|^2,
\end{aligned}
$$

for each $j \in \mathbb{Z}^+$, where the first inequality follows easily from choice of E.

We claim that the proof now follows: if either condition (iv) or condition (v) in the statement of this theorem fails, it would then follow from Lemma 5.6 and (5.37) that the right hand side of (5.36) goes to 0 as $j \to \infty$, a contradiction. $\square$

Proof of Theorem 5.3. For $M \in \mathbb{Z}^+$, let $V(M)$ and $W(M)$ denote the closed spans of

$$\{D_{b(l)}T_k\varphi_i : |l| \leq M, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

and

$$\{D_{b(l)}T_k\varphi_i : |l| \geq M, k \in \mathbb{Z}^2, i = 1, \ldots, I\},$$

respectively.

By condition (ii) in the statement of this theorem (with $j = 1$), the collection (5.28), which we may write as

$$\{T_kD_b\varphi_i : b \in B(L_1), k \in \mathbb{Z}^2, i = 1, \ldots, I\}, \tag{5.38}$$

forms a frame for its closed span with constants $C \leq D$. By comment (i) following Definition 5.1, the canonical dual frame to (5.38) (which has constants $D^{-1} \leq C^{-1}$) may be written as

$$\{T_kD_b\widetilde{\varphi}_i : b \in B(L_1), k \in \mathbb{Z}^2, i = 1, \ldots, I\}.$$

Using condition (i) in the statement of this theorem and part (vii) of Lemma 2.1, we may write

$$
\begin{aligned}
|\det a|^{1/2}\widehat{\varphi}_i(\xi a) &= (D_a\varphi_i)\widehat{\ }(\xi) \\
&= \sum_{i'=1}^{I}\sum_{b\in B(L_1)} [(D_a\varphi_i)\widehat{\ }, \widehat{\widehat{\varphi_{i'}}}(\cdot b)](\xi)\widehat{\varphi_{i'}}(\xi b), \qquad (5.39)
\end{aligned}
$$

for a.e. ξ and for each i. Using the Schwarz inequality and part (ii) of Theorem 2.1, we obtain

$$
\begin{aligned}
\left|[(D_a\varphi_i)\widehat{\ }, \widehat{\widehat{\varphi_{i'}}}(\cdot b)](\xi)\right| &= |\det a|^{1/2}\Big|\sum_{k\in\hat{\mathbb{Z}}^2}\widehat{\varphi}_i((\xi+k)a)\overline{\widehat{\widehat{\varphi_{i'}}}((\xi+k)b)}\Big| \\
&\le |\det a|^{1/2}\Big(\sum_{k\in\hat{\mathbb{Z}}^2}|\widehat{\varphi}_i((\xi+k)a)|^2\Big)^{1/2}\Big(\sum_{k\in\hat{\mathbb{Z}}^2}|\widehat{\widehat{\varphi_{i'}}}((\xi+k)b)|^2\Big)^{1/2} \\
&= |\det a|^{1/2}\Big(\sum_{k\in\hat{\mathbb{Z}}^2}|\widehat{\varphi}_i(\xi a+ka)|^2\Big)^{1/2}[\widehat{\widehat{\varphi_{i'}}}(\cdot b), \widehat{\widehat{\varphi_{i'}}}(\cdot b)](\xi)^{1/2} \\
&\le |\det a|^{1/2}\Big(\sum_{k\in\hat{\mathbb{Z}}^2}|\widehat{\varphi}_i(\xi a+k)|^2\Big)^{1/2}C^{-1/2} \qquad (5.40) \\
&= |\det a|^{1/2}[\widehat{\varphi}_i, \widehat{\varphi}_i](\xi a)^{1/2}C^{-1/2} \\
&\le |\det a|^{1/2}D^{1/2}C^{-1/2},
\end{aligned}
$$

for a.e ξ and for each i. Using (5.39), (5.40), and Lemma 5.5, it follows that $D_a^j\varphi_i$ belongs to the closed span of

$$\{D_bT_k\varphi_{i'} : b\in B(L_1)^j, k\in\mathbb{Z}^2, i'=1,\ldots,I\}$$

(for each $j \in \mathbb{Z}^+$ and each i). Consequently, Lemma 5.4 implies that $D_a^j \varphi_i \in V(M_j)$, where

$$M_j = \begin{cases} L_1 |pq^{-1}|^j, & \text{if } |p| > |q|; \\ L_1 j, & \text{if } |p| = |q|. \end{cases}$$

(for each $j \in \mathbb{Z}^+$ and each i). Choose a sequence $j_1 \le j_2 \le j_3 \le \dots$ of positive integers such that

$$M_{j_s} \le L_s \le M_{j_s+1}, \tag{5.41}$$

for all large enough $s \in \mathbb{Z}^+$.

Since $\{\varphi_1, \dots, \varphi_I\} \neq \{0\}$, there is an index i and a measurable bounded subset E of $\hat{\mathbb{R}}^2$ which satisfies $|E \cap (E+k)| = 0$ (for all $k \in \hat{\mathbb{Z}}^2 \setminus \{0\}$) such that $\langle \widehat{\varphi_i}, \chi_E \rangle \neq 0$. Define $\theta \in L^2(\mathbb{R}^2)$ by $\hat{\theta} = \chi_E$. Using the Plancherel theorem, a change of variable, Lemma 3.1, and choice of E we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}^2} |\langle D_a^{-j} D_b T_k \varphi_{i'}, \theta \rangle|^2 &= \sum_{k \in \mathbb{Z}^2} |\langle (D_a^{-j} D_b T_k \varphi_{i'})\widehat{\ }, \hat{\theta} \rangle|^2 \\ &= \sum_{k \in \mathbb{Z}^2} \left| \int_E (D_a^{-j} D_b T_k \varphi_{i'})\widehat{\ }(\xi)\, d\xi \right|^2 \\ &= \sum_{k \in \mathbb{Z}^2} \left| |\det a|^{-j/2} \int_E \widehat{\varphi_{i'}}(\xi a^{-j} b) e^{-2\pi \imath \xi a^{-j} bk}\, d\xi \right|^2 \\ &= |\det a|^j \sum_{k \in \mathbb{Z}^2} \left| \int_{Ea^{-j}} \widehat{\varphi_{i'}}(\xi b) e^{-2\pi \imath \xi bk}\, d\xi \right|^2 \\ &= |\det a|^j \int_{Ea^{-j}} |\widehat{\varphi_{i'}}(\xi b)|^2\, d\xi, \end{aligned} \tag{5.42}$$

for all b, i', and $j \in \mathbb{Z}^+$.

Let V denote the closed subspace $\{\alpha\varphi_i : \alpha \in \mathbb{C}\}$, and let $P : L^2(\mathbb{R}^2) \longrightarrow V$ be the orthogonal projection of $L^2(\mathbb{R}^2)$ onto V. For $j, M \in \mathbb{Z}^+$, let

$$P(j,M) : L^2(\mathbb{R}^2) \longrightarrow D_a^{-j}V(M)$$

be the orthogonal projection of $L^2(\mathbb{R}^2)$ onto $D_a^{-j}V(M)$. Using (5.41), (5.42), that $V \subset D_a^{-j_s}V(M_{j_s})$, and that

$$\{D_a^{-j_s}D_bT_k\varphi_{i'} : b \in B(L_s), k \in \mathbb{Z}^2, i' = 1, \ldots, I\}$$

forms a frame for $D_a^{-j_s}V(L_s)$ with constants $C \leq D$ (for all $s \in \mathbb{Z}^+$), we obtain

$$\begin{aligned}
0 < \|P\theta\|^2 &\leq \|P(j_s, M_{j_s})\theta\|^2 \\
&\leq \|P(j_s, L_s)\theta\|^2 \\
&\leq C^{-1} \sum_{i'=1}^{I} \sum_{b \in B(L_s)} \sum_{k \in \mathbb{Z}^2} |\langle D_a^{-j_s}D_bT_k\varphi_{i'}, P(j_s, L_s)\theta\rangle|^2 \qquad (5.43) \\
&= C^{-1} \sum_{i'=1}^{I} \sum_{b \in B(L_s)} \sum_{k \in \mathbb{Z}^2} |\langle D_a^{-j_s}D_bT_k\varphi_{i'}, \theta\rangle|^2 \\
&= C^{-1} |\det a|^{j_s} \int_{Ea^{-j_s}} \sum_{i'=1}^{I} \sum_{b \in B(L_s)} |\widehat{\varphi_{i'}}(\xi b)|^2 \, d\xi,
\end{aligned}$$

for all large enough s.

We claim that the proof now follows: First note that condition (ii) in the statement of this theorem easily implies that

$$\{D_b T_k \varphi_i : b \in B, k \in \mathbb{Z}^2, i = 1, \ldots, I\}$$

is a Bessel system. Thus, if either condition (iii) or condition (iv) in the statement of this theorem fails, it would then follow from (5.41) and Lemma 5.6 that the right hand side of (5.43) goes to 0 as $s \to \infty$, a contradiction. $\square$

The below corollary, which follows immediately from Theorems 5.2 and 5.3 and Lemmas 4.2 and 4.4, is the third main result of this section.

Corollary 5.2. *Let B and a be as in the statement of Theorem 5.2.*

(i) If $|p| > |q|$, then there does not exist an aB-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$) of finite filter type 1 or 2 such that, for each i, we have $\varphi_i, |x_1|^\alpha \varphi_i \in L^1(\mathbb{R}^2)$, where

$$\alpha = \frac{1}{2}\left(1 - \frac{\log|q|}{\log|p|}\right).$$

(ii) If $|p| = |q|$, then there does not exist an aB-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$) of finite filter type 1 or 2 such that, for each i, we have

$$\varphi_i, \big(\log(|x_1| + 1)\big)^{1/2} \varphi_i \in L^1(\mathbb{R}^2).$$

In any case, there does not exist an aB-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ ($I \in \mathbb{Z}^+$) of finite filter type 1 or 2 such that, for each i, φ_i is compactly *supported.*

We make the following comment in connection with Corollary 5.2: Let

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

let

$$B = \left\{ b(l) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\},$$

and let $\hat{\varphi} = \chi_I$, where $I = I^+ \cup I^-$, $I^- = -I^+$, and

$$I^+ = \{(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2 : 0 \le \xi_1 \le 1, 0 \le \xi_2 \le \xi_1\}.$$

As observed in Example 3.3, φ is an ON aB scaling function. It is straightforward to verify that

$$\hat{\varphi}(\xi a) = m(\xi)\hat{\varphi}(\xi) + m(\xi)\hat{\varphi}\big(\xi b(-1)\big),$$

for a.e. ξ, where $m \in L^2(\mathbb{T}^2)$ is the $\hat{\mathbb{Z}}^2$-periodic extension to $\hat{\mathbb{R}}^2$ of $\chi_{[0,1/2]\times[0,1]}$. It follows easily that φ is of finite filter type 1 and 2. In accordance with Corollary 5.2, note that $\varphi \notin L^1(\mathbb{R}^2)$ (by the discontinuity of $\hat{\varphi}$).

Chapter 6

Final Comments

In this chapter, we indicate a number of interesting questions that arise from the reproducing system characterizations of Chapter 2 and the scaling multifunction nonexistence results of Chapters 3 and 5.

6.1 Regarding the Reproducing System Characterizations of Chapter 2

Recall that a shift invariant (SI) space is a closed subspace V of $L^2(\mathbb{R}^n)$ that satisfies $T_k V \subset V$, for all $k \in \mathbb{Z}^n$. Such spaces play an important role in many areas of mathematics, particularly in wavelet theory. Clearly, a very natural choice of reproducing system for a SI space V is one of the form

$$\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in I\}, \tag{6.1}$$

where $\{\varphi_i : i \in I\} \subset V$ and I is a countable indexing set.

When $I = \{1\}$, essentially every reproducing property of (6.1)—orthonormality, Riesz basis, frame, minimality, l^2-linear independence, Bessel, Schauder basis ($n = 1$), etc.—is characterized relatively simply in terms of the bracket product $[\widehat{\varphi_1}, \widehat{\varphi_1}]$ (see [5] and [9]). In Chapter 2, we characterize when (6.1) (with I a general countable indexing set) forms a Bessel system, a frame, and a Riesz basis. These characterizations involve operator inequalities of matrices whose entries are the bracket products $[\widehat{\varphi_i}, \widehat{\varphi_j}]$ $(i, j \in I)$. It is very likely that the methods used to prove the results of Chapter 2 can also be used to characterize the other above mentioned reproducing properties (minimality, l^2-linear independence, Schauder basis ($n = 1$), etc.) of (6.1) when I is a general countable indexing set.

6.2 Regarding the Scaling Multifunction Nonexistence Results of Chapters 3 and 5

In this section, we indicate several interesting questions that arise from the nonexistence results of Chapter 3 (regarding aB-scaling multifunctions of Haar-type when B is infinite) and the nonexistence results of Chapter 5 (regarding shearlet-like scaling multifunctions satisfying certain desirable properties).

6.2.1 What Shearlet Scaling Multifunctions Can Exist?

Let

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$B = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} : l \in \mathbb{Z} \right\}.$$

Recall that B is said to be a shear group and that an aB-multiwavelet, an aB-MRA, and an aB-scaling multifunction are said to be a multishearlet, a shearlet MRA, and a shearlet scaling multifunction, respectively.

Corollaries 3.6, 5.1, and 5.2 imply that there does not exist a shearlet scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ $(I \in \mathbb{Z}^+)$ that satisfies ANY of the following properties:

(i) $\{\varphi_1, \ldots, \varphi_I\}$ is of Haar-type;

(ii) φ_i is compactly supported and Hölder continuous in the direction e_2 with exponent β, for some $1/2 < \beta \leq 1$ (for each i);

(iii) φ_i is compactly supported (for each i) and $\{\varphi_1, \ldots, \varphi_I\}$ is of finite type 1 or 2.

In contrast to the above nonexistence results, following Corollary 5.2, an example of a MSF ON shearlet scaling function φ that is of finite filter type

1 and 2 is given. However, although $\varphi \in C^\infty(\mathbb{R}^2)$, it has very slow decay ($\varphi \notin L^1(\mathbb{R}^2)$). φ is essentially the only shearlet scaling multifunction in existence.

Given the relatively large gap between the above existence and nonexistence results, an obvious question is: What is the maximum amount of decay and regularity that the elements of a shearlet scaling multifunction can attain? For example, does there exist a shearlet scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ such that, for each i,

- φ_i belongs to $L^1(\mathbb{R}^2)$?
- φ_i is compactly supported?
- φ_i is compactly supported and exhibits a certain amount of regularity?

Or, on the other hand, can the nonexistence results of Corollaries 5.1 and 5.2 be substantially strengthened?

6.2.2 The Nonexistence of Shearlet-Like Scaling Multifunctions in Higher Dimensions

It is natural to wonder whether Corollaries 5.1 and 5.2 have higher dimensional analogs. In general, a matrix $b \in SL_n(\mathbb{R})$ is said to be shear if $(b - I_n)^2 = 0$, where I_n is the $n \times n$ identity matrix. Given a countable shear group $B \subset SL_n(\mathbb{R})$ (i.e., a matrix group consisting of shear matrices) and a matrix $a \in GL_n(\mathbb{R})$, one can ask how much decay, regularity, and/or other desirable properties an aB-scaling multifunction can exhibit.

The author is currently considering this question for a very large class of countable shear groups $B \subset SL_n(\mathbb{R})$ and matrices $a \in GL_n(\mathbb{R})$. The answer depends very interestingly on the structure of B. For example, consider the following shear groups and matrices:

$$B_1 = \left\{ \begin{pmatrix} 1 & l_1 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : l_1, l_2 \in \mathbb{Z} \right\} \quad B_2 = \left\{ \begin{pmatrix} 1 & l_1 & l_2 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : l_1, l_2, l_3 \in \mathbb{Z} \right\}$$

and

$$a_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The results of Corollaries 5.1 and 5.2 carry over to a_2B_2-scaling multifunctions. For instance, there does not exist an a_2B_2-scaling multifunction $\{\varphi_1, \ldots, \varphi_I\}$ such that φ_i is compactly supported (for each i) and EITHER of the following is satisfied:

(i) $\varphi_i \in C^3(\mathbb{R}^4)$ (for each i);

(ii) $\{\varphi_1, \ldots, \varphi_I\}$ is of finite type 1.

However, the arguments of Corollaries 5.1 and 5.2 fail (for a good reason) when applied to $a_1 B_1$-scaling multifunctions.

An obvious question, therefore, is: How much decay, regularity, and/or other desirable properties can $a_1 B_1$-scaling multifunctions exhibit? In particular, does there exist a Parseval $a_1 B_1$-scaling multifunction that is compactly supported and of finite type 1? Such an $a_1 B_1$-scaling multifunction would most likely be very useful in applications. In any case, an answer to this question would help foster a much greater understanding of aB-scaling multifunctions, when B is a shear group and, more generally, when B is infinite.

6.2.3 The Nonexistence of aB-Scaling Multifunctions for Non-Shear B

aB-scaling multifunctions of Haar-type are clearly the simplest variety of aB-scaling multifunctions. In general, their existence (for a particular a and B) can be taken as evidence in favor of the existence of other aB-scaling multifunctions that, in addition to being compactly supported, also satisfy some degree of regularity. Moreover, if aB-scaling multifunctions of Haar-type can be constructed (for a particular a and B), their method of construction can often provide some insight as to how smoother compactly supported aB-scaling multifunctions may be constructed. Therefore, the nonexistence of aB-scaling multifunctions of Haar type, for all countably infinite $B \subset \widetilde{SL_n}(\mathbb{R})$ and all $a \in GL_n(\mathbb{R})$ (Corollary 3.6), can be seen as strong evidence against the existence of other varieties of aB-scaling multifunctions (in paricular, those satisfying certain amounts of decay, regularity, and/or other desirable properties), at least for certain countably infinite $B \subset \widetilde{SL_n}(\mathbb{R})$. In confirmation of this, Corollaries 5.1 and 5.2 imply the nonexistence of a large class of shearlet-like aB-scaling multifunctions that satisfy a certain amount of decay and either some regularity or one of two finite type conditions.

A natural question to ask, then, is: Do versions of Corollaries 5.1 and 5.2 exist for, say, other subgroups B of $SL_2(\mathbb{Z})$ that are not shear groups? For example, define the matrix

$$b = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Note that b has eigenvalues

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2},$$

where $\lambda_1 > 1 > \lambda_2 > 0$; in particular, b is not shear. Define the infinite group $B = \{b^j : j \in \mathbb{Z}\}$. How much decay, regularity, and/or other desirable properties can aB-scaling multifunctions exhibit (where a is some matrix belonging to, say, $GL_2(\mathbb{Z})$)? In paricular, does there exist a Parseval aB-scaling multifunction that is compactly supported and of finite type 1? Such an aB-scaling multifunction would most likely have many useful applications, and, in any case, an answer to this question would help us to better understand composite scaling multifunctions when the composite group is infinite.

6.2.4 A New aB-MRA Definition When B is Infinite

Classical wavelet systems utilize operators of the form

$$\{D_a^j : j \in \mathbb{Z}\} \quad \text{and} \quad \{T_k : k \in \mathbb{Z}^n\}, \tag{6.2}$$

where $a \in GL_n(\mathbb{R})$. In dimensions two and higher, there are several important applications in which the relative geometric simplicity of classical wavelet systems limits their usefulness. In response to this deficiency, the more geometricly diverse aB-wavelet systems (or composite wavelet systems) were introduced in [3]. Composite wavelet systems, in addition to the operators in (6.2), employ operators of the form $\{D_b : b \in B\}$, where B is a countable subset of $\widetilde{SL_n}(\mathbb{R})$. Taking the composite wavelet definition for granted, the aB-MRA and aB-scaling multifunction defintions follow in a natural and almost obvious manner from the classical MRA and scaling function definitions, particularly when $B \subset \widetilde{SL_n}(\mathbb{Z})$.

When B is finite, aB-wavelet and aB-MRA systems retain many of the fundamental characteristics of classical wavelet systems. Haar-type systems, for instance, continue to exist: In Example 3.2, a compactly supported ON aB-wavelet ψ and associated compactly supported ON aB-scaling function φ of Haar-type are constructed, where a is the quincunx matrix and B is the group of symmetries of the unit square.

When B is infinite, the aB-wavelet definition seems to be well-made, particularly in the case of shearlet systems: As discussed at the beginning of Chapter 5, shearlet systems and their variants have been shown to excel (indeed, to outperform classical wavelet systems) in several important applications. Additionally, very well-behaved shearlet systems exist: In section 5 of [3], a Parseval shearlet ψ is constructed that belongs to the Schwartz class of $\mathbb{R}^2$. However, it is evident from Corollaries 3.6, 5.1, and 5.2 that the aB-MRA definition has serious limitations: aB-scaling multifunctions of Haar-type do not exist, for any countably infinite $B \subset \widetilde{SL_n}(\mathbb{R})$ and any $a \in GL_n(\mathbb{R})$. Moreover, essentially no useful shearlet scaling multifunctions exist.

The above paragraph seems to indicate that, despite its naturality, the aB-MRA and aB-scaling multifunction defintions (when B is infinite) are incorrectly formulated. A natural question, therefore, is:

Do there exist alternative aB-MRA and aB-scaling multifunction defintions (when B is infinite) that, while retaining much of the useful structure of the current definitions, allow the existence of desirable aB-scaling multifunctions—in particular, aB-scaling multifunctions of Haar-type and shearlet scaling multifunctions that satisfy substantial amounts of decay, regularity, and/or finite type conditions?

If yes, such concepts would undoubtedly be very useful in developing fast algorithmic implementations of aB-wavlelet systems and, in particular, shearlet systems. In either case, an answer to the above question would greatly enhance our understanding of aB-wavelet and aB-MRA systems when B is infinite.

Bibliography

[1] John B. Conway, *A Course in Functional Analysis*, Second Edition, Springer-Verlag, New York NY, 1990.

[2] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley and Sons, Inc., New York NY, 1984.

[3] Kanghui Guo, Demetrio Labate, Wang-Q Lim, Guido Weiss, Edward Wilson, *Wavelets with Composite Dilations and Their MRA Properties*, Appl. Comput. Harmon. Anal. 20 (2006) 202-236.

[4] Eugenio Hernández, Demetrio Labate, Guido Weiss, *A Unified Characterization of Reproducing Systems Generated by a Finite Family, II*, J. Geom. Anal. 12(4) (2002) 615–662.

[5] Eugenio Hernández, Hrvoje Šikić, Guido Weiss, Edward Wilson, *On the Properties of the Integer Translates of a Square Integrable Function in* $L^2(\mathbb{R})$ (preprint).

[6] Eugenio Hernández, Guido Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton FL, 1996.

[7] Ilya A. Krishtal, Benjamin D. Robinson, Guido L. Weiss, Edward N. Wilson, *Some Simple Haar-Type Wavelets in Higher Dimensions*, J. Geom. Anal. 17(1) (2007) 87–96.

[8] Demetrio Labate, Guido Weiss, *Continuous and Discrete Reproducing Systems that Arise from Translations: Theory and Applications* (preprint).

[9] Morten Nielson, Hrvoje Šikić, *Schauder Bases of Integer Translates*, Appl. Comput. Harmon. Anal. 23(2) (2007) 259-262.

[10] Walter Rudin, *Functional Analysis*, McGraw-Hill Inc., New York NY, 1973.

[11] Walter Rudin, *Real and Complex Analysis*, Second Edition, McGraw-Hill Inc., New York NY, 1974.

[12] Walter Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, Inc., New York NY, 1976.

[13] Elias M. Stein, Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton NJ, 1971.

[14] Guido Weiss, Edward N. Wilson, *The Mathematical Theory of Wavelets*, in: Twentieth Century Harmonic Analysis–A Celebration (Il Ciocco),

NATO Sci. Ser. II Math. Phys. Chem., Vol. 33, Kluwer Acad. Publ., Doedrecht, 2001, pp. 329-366.

[15] Robert M. Young, *An Introduction to Nonharmonic Fourier Series*, Revised First Edition, Academic Press, San Diego CA, 2001.