5-4-2021

Mathematical Modeling of Stress Strain Behavior of Newborn Mouse Aorta

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Introduction:

The stress-strain behavior of materials is very important in determining their physical and mechanical properties. Whereas the stress-strain behaviors of different metals (isotropic and linear) can be easily determined by Hooke's law, determining that of newborn mouse aorta (anisotropic and nonlinear) needs different approaches. The stress-strain behavior of newborn mouse aorta is highly nonlinear, and previous constitutive equations have been used to model the stress-strain behaviors of only adult mouse aorta. In other words, previous equations do not accurately represent the stress-strain behavior of newborn mouse aorta. This paper proposes a constitutive equation that represents the stress-strain behavior of newborn mice aortae. This strain energy function incorporates exponentials, but has some important differences from the previous models. Regression analysis in MATLAB shows that the proposed strain energy equation can accurately represent the stress-strain behavior of newborn mouse aorta.

History and Background in Modeling:

The aorta is modeled as an incompressible, nonlinear, anisotropic, and homogeneous cylinder that does not undergo shearing. The mathematical relations that govern the deformation are the deformation gradient (**F**), Cauchy strain (**C**), and Grain strain tensors (**E**). Respectively, they are written below:

$$\mathbf{F} = diag\{\lambda_r, \lambda_\theta, \lambda_z\} \qquad \text{(Eqn. 1)}$$

$$\mathbf{C} = diag\{\lambda_r^2, \lambda_\theta^2, \lambda_z^2\} \qquad \text{(Eqn. 2)}$$

$$\mathbf{E} = diag\{1/2(\lambda_r^2 - 1),\ 1/2(\lambda_\theta^2 - 1), 1/2(\lambda_z^2 - 1)\} \qquad \text{(Eqn. 3)}$$

where λ_r,λ_θ,and λ_z represent the stretch ratio in r, θ, and z direction in cylindrical coordinates.

Numerically,

$$\lambda_\theta = r/R \qquad \text{(Eqn. 4)}$$

$$\lambda_r = \partial r/\partial R \qquad \text{(Eqn. 5)}$$

$$\lambda_z = l/L \qquad \text{(Eqn. 6)},$$

where r and l are deformed length and radius; R and L are undeformed length and radius.

Scholars have proposed different phenomenological modeling equations for adult mammal aortae's stress-strain behavior. They model different strain energy functions denoted by W that records strain energy per unit mass. C. J. Chuong and Y. C. Fung proposed an exponential equation that has 6 constants [1]:

$$\rho_o W = \frac{c}{2} e^Q,$$

$$\text{where } Q = b_1 E_\theta^2 + b_2 E_z^2 + b_3 E_r^2 + 2b_4 E_\theta E_z + 2b_5 E_z E_r + 2b_6 E_\theta E_r$$

(Eqn. 7),

and b_{1-6}, c are material constants. Takamizawa and Hayashi postulate a natural log function [2]:

$$W = -C ln(1 - \psi)$$

$$\text{where } \psi = \frac{1}{2} a_{\theta\theta} E_\theta^2 + \frac{1}{2} a_{zz} E_z^2 + a_{\theta z} E_\theta E_z \text{(Eqn. 8)}$$

where $a_{\theta\theta}$, a_{zz},and $a_{\theta z}$are material constants. Vaishnav, Young, and Patel hypothesize polynomial strain energy functions that contain degrees of 3, 7, and 12, respectively [3]:

$$W = A_1 a^2 + B_1 ab + C_1 b^2 \quad \text{(Eqn. 9)}$$

$$W = A_2 a^2 + B_2 ab + C_2 b^2 + D_2 a^3 + E_2 a^2 b + F_2 ab^2 + G_2 b^3 \qquad \text{(Eqn. 10)}$$

$$W = A_3 a^2 + B_3 ab + C_3 b^2 + D_3 a^3 + E_3 a^2 b + F_3 ab^2 + G_3 b^3 + H_3 a^4 + I_3 a^3 b + J_3 a^2 b^2 + K_3 ab^3 + L_3 b^4$$

(Eqn. 11)

where A-L are material constants.

With the proposed strain energy functions, one can derive the theoretical Cauchy stresses in the θ and z directions:

$$\sigma_\theta = \lambda_\theta^2 \frac{\partial W}{\partial E_\theta} - \lambda_r^2 \frac{\partial W}{\partial E_r},\ \sigma_z = \lambda_z^2 \frac{\partial W}{\partial E_z} - \lambda_r^2 \frac{\partial W}{\partial E_r} \quad \text{(Eqn. 12)}$$

when $\lambda_r^2 \frac{\partial W}{\partial E_r}$ serves as a Lagrange multiplier that enforces incompressibility. Alternatively, Vaishnav and Hayashi also suggest that incompressibility can be enforced directly [4]:

$$\sigma_\theta = \lambda_\theta^2 \frac{\partial W}{\partial E_\theta},\ \sigma_z = \lambda_z^2 \frac{\partial W}{\partial E_z} \quad \text{(Eqn. 13).}$$

<u>Experimental Data and Fitting:</u>

Biaxial mechanical tests on newborn mouse aorta provide data that can be used to calculate experimental values for stresses in the θ and z directions. Variables that are obtained during mechanical tests include: undeformed outer and inner diameters R_o and R_i, axial stretch force f, internal pressure P, deformed outer diameter r_o, and axial stretch ratio λ_z. With these variables, one is able to calculate the wall stresses [5], which can then be compared to the theoretical values obtained from the strain energy function:

$$\sigma_\theta = \frac{Pr_i}{r_o - r_i},\ \sigma_z = \frac{f + P\pi r_i^2}{\pi(r_o^2 - r_i^2)} \quad \text{(Eqn. 14).}$$

The strain energy function used for theoretical fitting in this project was inspired by the function for heart valve [6], which is highly nonlinear like newborn mouse aorta:

$$W = B_0(exp(\frac{b_1 E_\theta^2}{2}) + exp(\frac{b_2 E_z^2}{2}) + exp(\frac{b_3 E_\theta E_z}{2}) - 3) \quad \text{(Eqn. 15).}$$

The material constants were found in the fitting process. Specifically, the fmincon function in MATLAB was used to determine the constants. The error function that is applied is:

$$err_1 = \sum_{n=1}^{n=N} (\sigma_{zth-n} - \sigma_{zexp-n})^2 + (\sigma_{\theta th-n} - \sigma_{\theta exp-n})^2 \qquad \text{(Eqn. 16)}$$

where N is the total number of data points. Optimization can be achieved if the gradient of the error function decreases to 0, indicating that it has reached the local minima of the discrepancy between experimental and theoretical data.

In addition to this error function, constraints were imposed on the values of constants as well. The constants were considered to be positive because strains are caused by energy input. Positive constants would denote this relation since they result in a larger value of W as strain increases.

Table 1 indicates the material constants for wild type newborn aorta, and Table 2 indicates the material constants for elastin knockout newborn aorta.

Table 1: Material Constants for Wild Type Newborn Aorta. The column indicates the sample number for each aorta.

	B0 (kPa)	b1	b2	b3	RMSE_t	RMSE_z	r_t	r_z
Sample 1	1.01	16.22	1.02	14.11	9.23	2.76	0.88	0.94
Sample 2	11.62	1.66	1.00	1.40	6.90	5.99	0.98	0.94
Sample 3	17.74	1.54	1.00	1.00	5.14	5.19	0.98	0.95
Sample 4	13.02	1.05	1.00	1.00	10.33	4.82	0.96	0.97
Sample 5	8.48	2.43	1.00	1.77	5.32	4.22	0.97	0.95
Sample 6	19.61	1.54	1.00	1.00	11.19	6.80	0.91	0.90
Sample 7	15.03	1.62	1.00	1.22	4.28	5.81	0.99	0.96
mean	12.36	3.72	1.00	3.07	7.48	5.08	0.95	0.94
stdv	9.91	10.33	0.01	9.11	3.50	2.16	0.08	0.01

Table 2: Material Constants for Elastin Knockout Newborn Aorta. The column indicates the sample number for each aorta.

	B0 (KPa)	b1	b2	b3	RMSE_t	RMSE_z	r_t	r_z
Sample 1	39.32	0.45	0.25	0.17	2.73	3.28	0.99	0.91
Sample 2	24.29	0.37	0.10	0.23	5.30	5.40	0.96	0.84
Sample 3	23.22	1.13	0.59	0.44	2.07	2.03	0.99	0.95
Sample 4	30.49	0.32	0.10	0.14	3.98	3.61	0.97	0.85
Sample 5	50.94	0.11	0.10	0.10	16.51	9.74	0.90	0.83
Sample 6	5.39	1.10	1.00	1.00	4.36	1.75	0.97	0.96
Sample 7	7.68	1.22	1.00	1.00	5.76	2.96	0.96	0.94
mean	25.90	0.67	0.45	0.44	5.81	4.11	0.96	0.90
stdv	16.28	0.46	0.41	0.40	4.89	2.75	0.03	0.06

In Table 2, stress-strain behavior of elastin knockout mice was studied. Unlike wild type mice, elastin knockout mice lack the elastin gene (Eln-/-) and die prematurely because of uncontrolled growth of smooth muscle cells that block their arterial lumen [8].

The above result indicates that the correlation coefficient, r_t and r_z, for both Elastin Knockout and Wild Type aorta are high. They measure the strength of the relationship between the experimental and theoretical value. The value ranges from -1 to 1. The larger the value is, the stronger the relationship exists between the two values. Typically, values between 0.7-1.0 suggest a strong positive relationship.

Figure 1 and Figure 2 visualize the experimental data and theoretical data of Sample 1 of Wild Type Newborn aorta. The dots represent the experimental data and the lines represent theoretical data. The six solid lines represent six protocols in one trial. 3 of the protocols had constant axial stretch ratio (E_z)and underwent luminal inflation, and the other 3 protocols underwent constant luminal pressure (P) and increase in axial stretch ratios. Stresses in the θand z directions are both plotted against E_θ ($E_t$as in the figures).

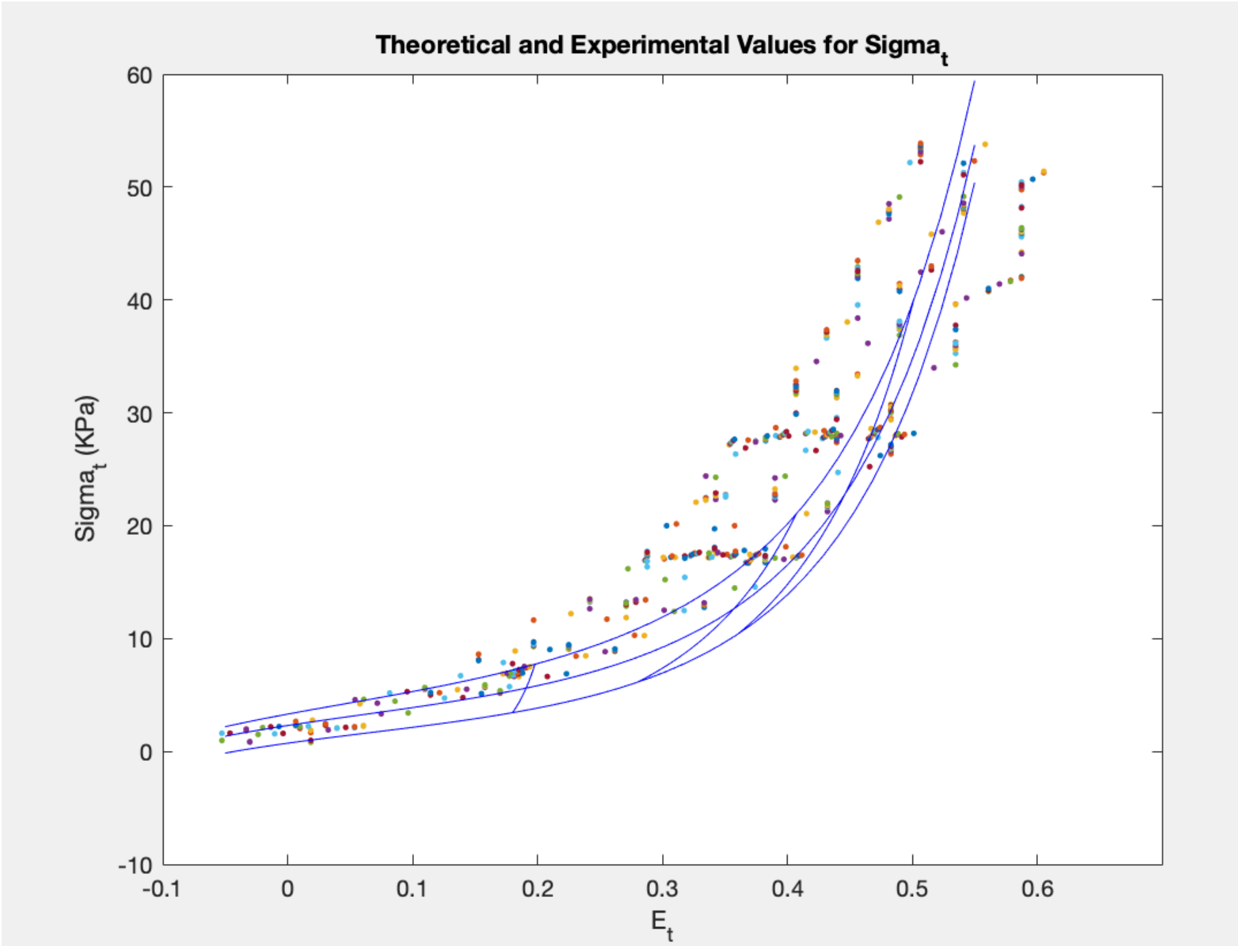

Figure 1: Stress-Strain Relations of σ_θ with respect to E_t.

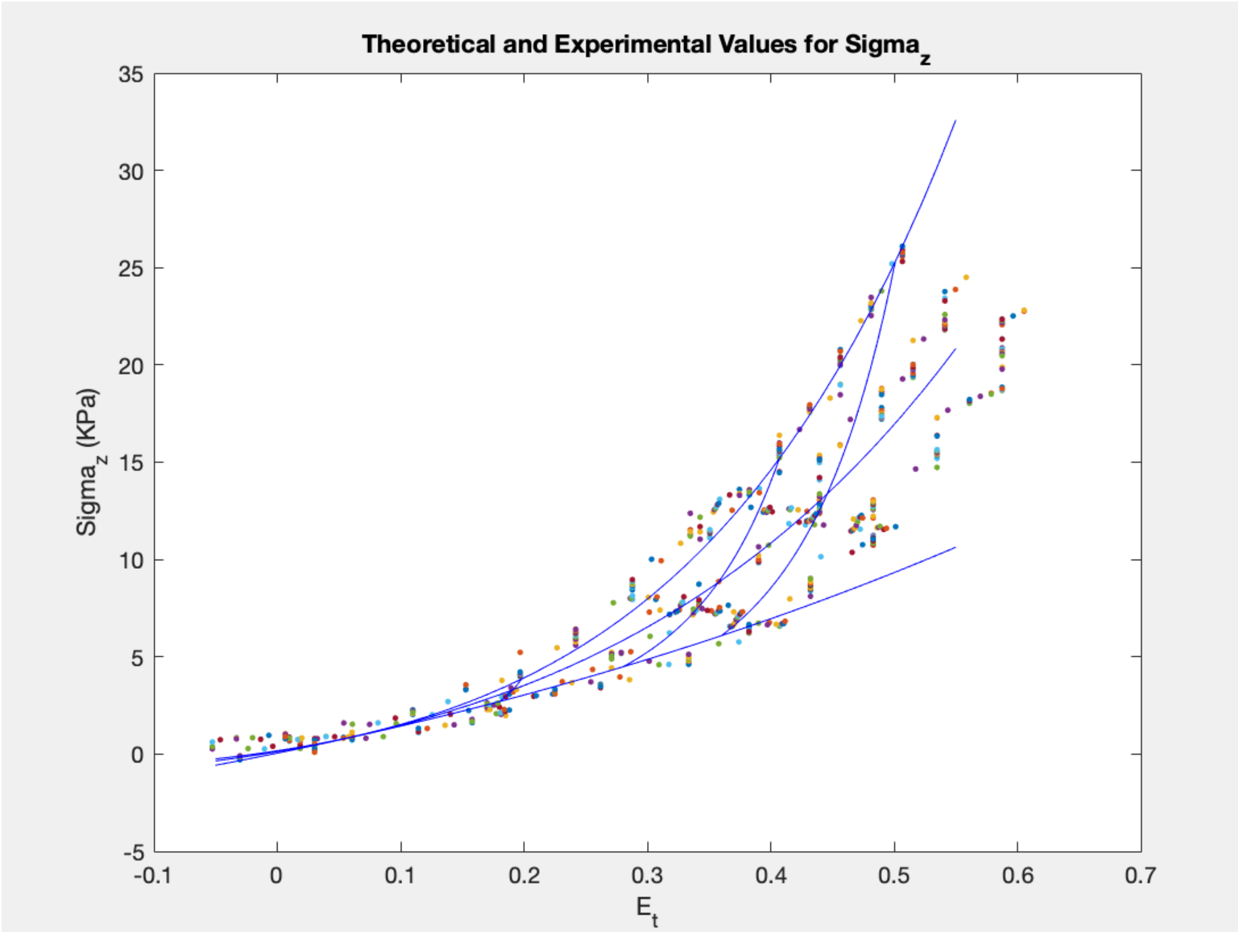

Figure 2: Stress-Strain Relations of σ_z with respect to E_θ

<u>Discussion and Conclusion:</u>

The fitting result accurately reflects the nonlinear stress-strain characteristic of newborn aorta. Nevertheless, because the strain energy function proposed in this paper contains only the deformation in 2 directions, instead of 3, the modeling function can still be improved and tested; likely, it can be written as:

$$W = B_0\left(exp\left(\frac{b_1 E_\theta^{\ 2}}{2}\right) + exp\left(\frac{b_2 E_z^{\ 2}}{2}\right) + exp\left(\frac{b_3 E_r^{\ 2}}{2}\right) + exp\left(\frac{b_4 E_\theta E_z}{2}\right) + exp\left(\frac{b_5 E_r E_z}{2}\right) + exp\left(\frac{b_6 E_\theta E_r}{2}\right) - 3\right)$$

(Eqn. 17)

such that the deformation in r direction is sufficed and that incompressibility can be enforced by a Lagrange multiplier.

Moreover, error functions other than Eqn. 16 can be tested in future fitting process. According to Ferruzzi et al, this error function is especially useful when computing planar stress at high loads [7]. Other error functions that take into account of stresses at low loads and stresses at both high and low loads are written as:

$$err_2 = \sum_{n=1}^{n=N} \left(\frac{P_{th}-P_{exp}}{P_{exp}}\right)_n^2 + \left(\frac{f_{th}-f_{exp}}{f_{exp}}\right)_n^2 ,$$

$$err_3 = \sum_{n=1}^{n=N} \left(\frac{P_{th}-P_{exp}}{\overline{P_{exp}}}\right)_n^2 + \left(\frac{f_{th}-f_{exp}}{\overline{f_{exp}}}\right)_n^2 \qquad \text{(Eqn. 18).}$$

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