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Filling Essential Laminations

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FILLING ESSENTIAL LAMINATIONS

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ABSTRACT OF THE DISSERTATION

Filling Essential Laminations

by

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Thurston and, later, Calegari-Dunfield found superlaminations in certain laminated 3-manifolds, the existence of which implies inclusions into HomeoS^1 of the fundamental groups of those manifolds. The present paper extends the construction of the superlamination, and finds an infinite class of manifolds to which the extension does not yield such an inclusion of groups.

Specifically, Calegari and Dunfield's proof of the existence of such an inclusion into HomeoS^1 depended on their filling lemma, which states that essential laminations with solid torus guts can have leaves added to them to yield essential laminations with solid torus complementary regions. (Roughly, a gut is that part of a complementary region that is not sandwiched between only two leaves.) The present paper finds the leafspace of the resultant lamination, and extends Calegari and Dunfield's operation to more general cases: first to reduce any finite-genus-handlebody complementary

region to its gut, and then to reduce the genus of a complementary region even where doing so modifies the gut itself. In these cases, too, then, there can be an inclusion of the manifold's fundamental group into HomeoS^1.

Cataclysms correspond to non-Hausdorffness in the leafspace of a lamination. A cataclysm is *orderable* if some order on it is invariant under deck transformations. Calegari-Dunfield showed that orderability of cataclysms, which is weaker than Hausdorffness of the leafspace, is sufficient for the existence of an inclusion into HomeoS^1. The present paper finds a criterion for the non-orderability of cataclysms, and a class of examples satisfying the criterion.

1. Basics

1.1 Basic definitions

Throughout, unless otherwise noted, the term *manifold* will refer to a connected, orientable manifold, Hausdorff and second-countable, without boundary. It will be three-dimensional unless otherwise specified. A three-dimensional manifold will be compact and irreducible unless it is specified as being a covering space, or otherwise noted. All covering spaces will be simply connected unless otherwise noted.

A *closed* manifold (of any dimension) is a compact manifold without boundary.

A *surface* is a two-dimensional manifold, possibly with boundary, not necessarily compact.

A *missing-boundary* **X** (or an **X** *missing boundary*), for ***X*** a term indicating membership in a particular class of manifolds with boundary, is obtained from an n-dimensional manifold M in the class of **X**es by removing from its boundary a (not necessarily connected) manifold of dimension $n-2$. Thus, for example, a *missing-boundary disk* is obtained from a closed disk $D^2 \approx I \times I$ by removing some points from its boundary, and we shall later encounter missing-boundary 3-dimensional manifolds also, obtained from particular sorts of manifolds by removal of properly embedded

1-dimensional manifolds from their respective boundaries. (For this purpose, we consider a point a 0-dimensional manifold.)

A 3-manifold M is *(algebraically) atoroidal* if any subgroup of $\pi_1 M$ isomorphic to $\mathbb{Z}\times\mathbb{Z}$ is conjugate in $\pi_1 M$ to a subgroup of $i_*\pi_1\partial M$, where i_* is induced by inclusion. It is *(algebraically) toroidal* otherwise. Thus, a manifold M without boundary is toroidal iff $\mathbb{Z}\times\mathbb{Z} < \pi_1 M$.

A compact manifold M is *Haken* if it is irreducible but contains a properly embedded, two-sided surface incompressible in M; otherwise, it is *non-Haken*. Thus, an irreducible, closed, orientable manifold M is Haken iff it contains a closed orientable surface incompressible in M.

A *handlebody* is a connected, three-dimensional manifold H (with boundary) which contains a set of disjoint properly embedded two-dimensional disks $\{D_1, \ldots, D_n\}$ (or $\{D_1, D_2, \ldots\}$) such that $H \setminus \bigcup_i D_i$ is homeomorphic to a ball with some of its boundary removed ([H, chap. 2]). The D_i are called *meridian disks* and will be assumed to be chosen for each handlebody under discussion.

A handlebody with a finite set of meridian disks — say, n disks — can be constructed (not in a unique way) by attaching n *handles* to a closed ball B; each handle is a ball $D^2 \times I$ attached to B by identifying $D^2 \times \partial I$ with two disjoint disks in B. We will call B the *cube* of the handlebody. We will assume such a decomposition for each handlebody under discussion, and will, moreover, assume that the meridian disks are embedded in the handles, one per handle. A *core curve* of a handle is a loop that is the union, along their boundaries, of the arc $\{\text{point}\} \times I$ in the handle $D^2 \times I$

and an arc in the cube of the handlebody; for a fixed handle, this loop is unique up to isotopy in the handlebody.

We will later examine missing-boundary handlebodies, handlebodies missing loops from their boundaries; the meridian disks, then, may be missing-boundary disks.

A *lamination* of a manifold M is a set of disjoint embedded connected surfaces, called *leaves*, whose union is not necessarily all of M, such that M can be decomposed as a union of 3-balls, called *lamination charts* or *charts*, each of which, B, is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$, such that for each leaf λ, the components of $B \cap \lambda$ are mapped under that homeomorphism to surfaces $\mathbb{R}^2 \times$point, and such that the union of all such components (ranging over all components of all leaves for a fixed B) is mapped to $\mathbb{R}^2\times$ (a closed subset of $\mathbb{R}$) ([GK3, 1.2], [CD, §2]).

We will assume all laminations' leaves are orientable; since we assumed our manifolds orientable, we obtain that every leaf is two-sided in its ambient manifold.

A *codimension-one foliation* of a manifold M is a lamination of M whose leaves comprise all of M ([La, Defn. 1]). We will call this a *foliation*. M is said to be *laminated* if it has a lamination, *foliated* if it has a foliation.

One could define *foliation* first and then say a lamination of M is a foliation of a closed subset of M, and that is a good way of thinking about a lamination ([GK3, 1.2]); we did not define the terms that way merely because we did not wish to define *foliation* for manifolds with boundary.

Note that we are, after [La, §1], defining a lamination as a set of leaves; a lamination is thus not a subset of a manifold. By abuse of notation, we will (as others do; e.g., [CD]) talk about *the complement in M of a lamination*: *lamination* will then be understood to refer to the union of the lamination's leaves, a subspace of the manifold.

Transverse intersection of an arc $\alpha \subset M$ with a leaf λ means that there is some small ball B about each point $p \in \alpha \cap \lambda$ and some homeomorphism from B to $\mathbb{R}^3$ such that $\alpha \cap B$ is mapped to point$\times\mathbb{R}$ and $\lambda \cap B$ is mapped to $\mathbb{R}^2\times$ a subset of $\mathbb{R}$. *Transverse* intersection of a 1-manifold in M with a (foliation or) lamination is transverse intersection at each point of intersection. A simply-connected, Hausdorff 1-manifold that has transverse intersection with a lamination (intersects it transversely) is called a *transversal* of the lamination.

A lamination of M is *transversely orientable* if there is a set of lamination charts $\mathbb{R}^2 \times \mathbb{R}$ whose union is M such that chart has a transverse $\{\text{point}\} \times \mathbb{R}$ that can be oriented so that the orientations of the transversals are coherent where charts overlap. This paper will assume all laminations transversely orientable.

A foliation is *taut* if there is a circle embedded in the manifold that transversely intersects every leaf.

A *complementary region* of a lamination Λ of M is the completion, immersed in M, of a connected component of $M \setminus \Lambda$. Here, the metric being used is the path metric on the complementary region, which is inherited from a metric on M. (Every second-countable manifold is metrizable; if the manifold is smooth, one can use the Riemannian metric, but we make no such assumption.)

The following lemma is well-known (e.g., [GK3, Definition 1.2]), but seemingly not proven anywhere.

Lemma 1 *Any leaf meeting a complementary region will be contained in it.*

Proof Fix a connected component C of $M \setminus \Lambda$ and consider the complementary region that is its completion. Let λ be a leaf meeting that complementary region in a point y, and fix $x \in \lambda$. By definition, there is a chart B_y about y homeomorphic to $\mathbb{R}^3$ under a homeomorphism mapping y to, say, $\mathbf{0}$. Because y is in the completion of C, there is a sequence $\{y_i\}_i$ of points of C convergent in the path metric to y; if necessary, take a subsequence so that these lie (after the homeomorphism) on one side of $\mathbb{R}^2 \times \{0\}$, and say without loss that they are contained in the planes $\mathbb{R}^2 \times \{1/n\}$, and that their first two coordinates are 0.

Because leaves are connected, they are path-connected, so fix a path γ in λ from y to x. There is a chart about each point of that path; since the path is compact, there are finitely many such charts covering it, say $B_y = B_1, B_2, B_3, \ldots, B_{m-1}, B_m$, such that $B_i \cap B_j = \varnothing$ for $|i - j| > 1$ and such that $B_i \cap B_{i+1}$ is connected for each $i < m$. Homeomorphisms should have been chosen so that, on the intersection of two charts, the coordinates match up; for example, the images in B_i and in B_{i+1} of $\mathbb{R}^2 \times \{1\}$ should coincide on the charts' intersection. Then x will be in $\mathbb{R}^2 \times \{0\}$ in B_m. In each such chart, and for each n, consider the path in $\mathbb{R}^2 \times \{1/n\}$ whose points have the same first two coordinates as γ does in that chart. Because of the consistency of coordinates, we obtain a path from each y_i to a point in B_m. The endpoints of these paths accumulate on x and are in C. $\|$

A *boundary leaf* of a lamination is any leaf λ contained in a complementary region. Those boundary leaves contained in a given complementary region are said to be its boundary leaves and to comprise its *boundary*; the rest of the complementary region — namely, the connected component of $M \setminus \Lambda$ — is its *interior*.

A lamination is *essential* if each complementary region is irreducible, the boundary of each complementary region is incompressible and end-incompressible in the complementary region, no leaf of the lamination is a sphere, and no leaf is a torus that bounds a solid torus in the manifold. *End-incompressible* in the complementary region, for the boundary of a complementary region, means that any proper embedding of a half-open disk (a two-dimensional disk with a closed arc deleted from its boundary) into the complementary region is homotopic rel boundary to an embedding into the boundary of the complementary region.

The above definition of *essential* is not the original definition found in [GO, 1.1], but is equivalent to the original definition ([GO, 4.6]) and more easily stated. It is also frequently given (e.g., [C1, Defn. 1.3]).

An essential lamination is *genuine* if it has a complementary region which is not an I-bundle over a surface ([GK1, 1.1], [G7, §0]); we will call such a complementary region *genuine* also. An essential lamination which is not genuine, and not a foliation, can be made a foliation by filling in each complementary region by adding leaves parallel to the complementary region's boundary leaves ([CD, 2.5]).

A genuine lamination is *very full* if each complementary region is a bundle over the circle of an ideal polygon (a polygon with its vertices removed); here, the boundary leaves form the faces of the polygon bundle ([M, §1.1]).

Any complementary region which is not a product surface$\times I$ can be decomposed into guts and interstitial regions. Each *interstitial region* is a product of a connected surface F and I, where $F \times \{0\}$ is closed in one of the boundary leaves of the complementary region and $F \times \{1\}$ is closed in another. F, if compact, is a closed surface. Each *gut* is compact and has no properly embedded surface homeomorphic to a missing-boundary disk with two boundary components. The boundary of the interstitial region is then $(F \times \{0, 1\}) \cup A$ for some union A of annuli that is the intersection of the interstitial region and a gut or guts ([GK3, 1.3]); each such annulus is called an *interstitial annulus*, and the gut's boundary is then the union of several such annuli and several compact parts of boundary leaves of the complementary region. (See also [CB, §4] for similar concepts.) These criteria guarantee a unique (up to isotopy in the complementary region) decomposition into guts and interstitial regions ([GK3, 1.3]).

There is more than one way to decompose a complementary region into guts and interstitial regions; we will, following [GK3], consider the above decomposition canonical.

A complementary region that *is* a surface $\times I$ is then an interstitial region without any gut. Thus, an essential lamination is genuine if and only if it has a gut.

Gabai, in [G2], [G1], and [G4], developed the subject of sutured manifolds. A *sutured manifold* is (M, γ) where M is a manifold (an orientable compact 3-manifold) with boundary and $\gamma = A(\gamma) \cup T(\gamma)$ is a union of pairwise disjoint annuli and tori, all embedded in ∂M. Each annulus in $A(\gamma)$ is assumed to contain a simple closed curve, essential in the annulus, called a *suture*. Finally, $s(\gamma)$ and the complement $\partial M \setminus \text{int}(\gamma)$ in ∂M of the interior of γ must be orientable so that the orientation that each component of $\partial A(\gamma)$ inherits from $\partial M \setminus \text{int}(\gamma)$ makes each suture represent the same fundamental group element as each of its annulus' boundary components.

A sutured manifold (M, γ) is *taut* if M is irreducible and $\partial M \setminus \gamma$ is Thurston-norm–minimizing in H_2. (For more details, including the definition of the Thurston norm, see [G1].)

Every complementary region of an essential lamination in a non-Haken manifold will be a missing-boundary handlebody, possibly with an infinite system of missing-boundary meridian disks ([B, page 320]) (or, equivalently, an infinite number of missing-boundary handles, which we will call *handles*). If a complementary region R has only finitely many handles, one can view it as a sutured manifold, specifically a sutured handlebody, with its boundary leaves corresponding to the components of $\partial R \setminus s(\gamma)$. Even if R has infinitely many handles, it can be useful to consider its gut as a sutured handlebody, with the interstitial annuli serving as the elements of $A(\gamma)$. For the remainder of this paper, our attention is restricted to the case that the gut of the complementary region actually satisfies the definition of sutured handlebody:

that is, the boundary leaves can be oriented so that their orientations match on the sutures.

A lamination all of whose guts are homeomorphic to solid tori is said to have *solid torus guts*. Note that every very full lamination has solid torus guts (but not conversely) ([CD, §3]).

Any lamination Λ of M can be lifted to the universal covering space $\widetilde{M}$ of the manifold, by lifting the inclusion maps of the leaves. What results is a lamination of $\widetilde{M}$, which we will denote $\widetilde{\Lambda}$. The maximal connected preimages of a complementary region of Λ are complementary regions of $\widetilde{\Lambda}$. If Λ is essential, then each leaf of $\widetilde{\Lambda}$ is homeomorphic to a plane ([GK1, 4.6]). For any complementary region R in a manifold M, $\widetilde{R}$ denotes a single component of its lift to the universal cover $\widetilde{M}$; this is, of course, a universal cover of R.

A leaf is *isolated* if its intersection with some chart is isolated from other leaves' intersection with the chart ([GK2, §1]). We assume no leaf is isolated; if one were, we could thicken it to a band (I-bundle) of leaves ([G6, 2.1.1]). That is, we would replace such a leaf by one of its closed neighborhoods, foliated with leaves parallel in the neighborhood to the original leaf ([GO, pg. 47]). (Throughout, a *band* of leaves will refer to the product of a leaf and I.)

The *leafspace* L of a lamination is the quotient space of the manifold by identification of each leaf to a point, except that each complementary region, including all its boundary leaves, is identified to a single point. (Because no leaf is isolated, no leaf meets two complementary regions, so complementary regions are disjoint.)

More interesting, as we shall see, is the leafspace of the cover $\widetilde{\Lambda}$; if this is Hausdorff, Λ is said to be *tight*. For Λ essential, even if the leafspace of the cover is not Hausdorff, it is an order tree. Since we have eliminated isolated leaves, the leafspace is actually an $\mathbb{R}$-order tree.[1] Definitions follow immediately:

An *order tree* is a set T with a collection of totally ordered subsets called *segments*, such that

- each segment σ has at least two points; following [RSS, §5], we denote its initial and final points $i\sigma$ and $f\sigma$ respectively, and write $\sigma = [i\sigma, f\sigma]$;
- any two points $x, y \in T$ can be connected by finitely many segments $\sigma_1, \ldots, \sigma_n$ so that $x = i\sigma_1$, $y = f\sigma_n$, and $f\sigma_k = i\sigma_{k+1} \forall k$;
- each segment, with the reverse order, is also a segment;
- any closed subinterval of a segment, containing at least two points, is a segment;
- if two segments σ, τ satisfy $\sigma \cap \tau = \{x\}$ and $x = f\sigma = i\tau$, then $\sigma \cup \tau$ is a segment; and
- no segment σ has $i\sigma = f\sigma$ (no "loops").

An $\mathbb{R}$-*order tree* is an order tree that can be written as a union of countably many of its segments, with each segment order-isomorphic to a closed interval of $\mathbb{R}$ ([GO], [GK2]).

[1] This was known as early as [GO, §6.II], but was more clearly stated in [GK2, Prop. 1.1] (using [GK1, 4.6]).

Similar to [RSS, §5], we define for each $x \in T$ (for T an order tree) an equivalence relation on $\{\sigma | x = f\sigma\}$ by $\sigma \approx \tau$ iff $\{x\} \neq \sigma \cap \tau$. The *degree* of x is then the number of equivalence classes under this relation; this can be infinite. If x has degree > 2, we call it a *branch point*.

Any Hausdorff $\mathbb{R}$-order tree is the limit of countably many finite simplicial trees ([CD, proof of Theorem 3.1], after [GK2, 3.1]; compare "1-graphs" in [Z]).

The following is lemma well-known, but seemingly not proven anywhere.

Lemma 2 *A branch point of the leafspace T corresponds precisely to a genuine complementary region of $\widetilde{\Lambda}$.*

Proof Any branch point has three segments incident on it which pairwise meet only in it. Since each leaf of $\widetilde{\Lambda}$ is homeomorphic to $\mathbb{R}^2$ ([GK1]), we obtain that the preimage $P \subset \widetilde{M}$ of the branch point satisfies that there exist three planar leaves pairwise nonhomotopic in $\widetilde{M} \setminus P$. If P is a complementary region, then, it has three boundary leaves; if it is not a complementary region, then we obtain by [GK1, 4.6] a foliation by lines of a subset of the plane bounded by three lines, which is impossible.

Conversely, any simply connected 3-manifold bounded by two planes is an I-bundle over the plane, so if a point is not a branch point but is the image of a complementary region then that complementary region is not genuine. $\|$

We will call the image in a leafspace of a complementary region, whether a branch point or not, a *vertex*.[2]

[2]Others, e.g. [CD], use *vertex* to mean any point of an order tree. We use *point* for that purpose. There is no commonly used term for the image in an order tree of a complementary region.

As noted above, every second-countable manifold is metrizable; for a particular laminated manifold M, call its metric d. Following [CD], we consider a leaf λ to be a *limit* of a sequence $\{\lambda_i\}_i$ of leaves, and the sequence to *converge* on λ, if there is convergence on compacta in the Hausdorff metric induced by d. That is, $\lambda_i \to \lambda$ iff for every compact $K \subset M$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $i > N$

$$\sup_{x \in \lambda \cap K} d(x, \lambda_i \cap K), \sup_{x \in \lambda_i \cap K} d(x, \lambda \cap K) < \varepsilon.$$

A non-tight essential lamination has, in its universal cover's leafspace, at least two points not separated from one another by open sets in the topology. These correspond precisely to a cataclysm in the lamination of the universal cover ([CD, 3.4]). The definition of *cataclysm*, though, requires first some other definitions, from [CD]:

Two leaves of a lamination are *comparable* if there is an embedding of $I = [0, 1]$ into the manifold, transverse to the lamination, with endpoints of the image on the two leaves. Otherwise, they're *incomparable*.

A sequence $\{\mu_i\}_i$ of leaves of a lamination is *monotone ordered* if, after passing to the leafspace, all the leaves of the sequence are in one segment of the order tree, and are (as parametrized by i) in order in that segment.

A *cataclysm* in a lamination is a pair $(\{\lambda_k\}, \{\mu_i\}_i)$, where $\{\lambda_k\}$ is a set of pairwise incomparable leaves, called *limit leaves*, and $\{\mu_i\}_i$ is a monotone ordered sequence of leaves, called a *limiting sequence*, such that $\forall k$ μ_i converges to λ_k (on compacta, as described above); we assume further that the set of limit leaves is maximal with respect to these criteria, for the limiting sequence. Two cataclysms are *equivalent* if their sets of limit leaves coincide and their limiting sequences have tails whose union (for example $\{\mu_i\}_{i>M} \cup \{\nu_i\}_{i>N}$) can be totally ordered in a segment of the leafspace. The equivalence class of a cataclysm is determined by its limit leaves ([CD, 3.4]).

A cataclysm in a lamination of $\widetilde{M}$ is *orderable* if there is a (linear) order on its set $\{\widetilde{\lambda_k}\}$ of limit leaves equivariant under the stabilizer in $\pi_1 M$ of $\{\widetilde{\lambda_k}\}$. (Here, we view $\pi_1 M$ as acting on $\widetilde{M}$ as usual.) If all the cataclysms of $\widetilde{M}$ are orderable, M is said to have *orderable cataclysms.*

Additional notational convention: *Countable* will mean finite or countably infinite. $\mathrm{int}(X)$ means the interior of a space X. And the notation $A \setminus B$ will refer to $\{x | x \in A, x \notin B\}$.

1.2 A basic history

Gabai and Oertel introduced (in [GO]) essential laminations, and proved that any essentially laminated manifold has universal cover $\mathbb{R}^3$, so has infinite fundamental group, and is irreducible; and that the lamination's leaves are π_1-injective. They also asked which 3-manifolds can be essentially laminated. This question led to a spate of research, some of the major results of which we outline here.

Manifolds admitting an essential lamination are extremely common: Gabai and others have found that the manifolds resulting from almost all Dehn surgeries on a nontrivial knot in S^3 can be essentially laminated ([F, §1]).

It is interesting to note then which manifolds do not admit such a lamination, besides, of course, those that obviously cannot, such as connected sums (which are necessarily not irreducible). Toward that effort, further implications of being essentially laminated have been found.

An essential (π_1-injective) torus *is* an essential lamination, so that Thurston's geometrization conjecture, which has been proven, implies that we need only examine atoroidal manifolds, and, hence, only Seifert-fibered spaces and hyperbolic manifolds ([S]). And Seifert-fibered spaces have been classified with respect to which ones admit essential laminations ([N]). So hyperbolic manifolds remain ([G7, Remark 1.1], [F, §1]).

Likewise, we need only consider non-Haken manifolds, since an incompressible closed surface is an essential lamination.

One favorite implication of having an essential lamination is that the manifold's fundamental group acts by automorphisms on a one-dimensional (possibly non-Hausdorff) manifold or an order tree. For example, the first examples of a hyperbolic manifold that does not admit a Reebless foliation were found by Roberts, Shareshian, and Stein in [RSS]; this was done by demonstrating that the fundamental group of the foliated manifold would have to act on the foliation's leafspace, which was shown to be impossible. Fenley, in [F], built on this to evince hyperbolic manifolds that do not admit essential laminations. Like the examples of manifolds that *can* be essentially laminated, all these counterexamples were found as Dehn surgeries, though in this case not on S^3.

Others have found further actions on one-dimensional spaces. For example, Thurston, in [T], found an action by homeomorphisms on the circle of the fundamental group of a manifold with a taut foliation. Calegari and Dunfield, in [CD], then showed, further, that if M is atoroidal and has an essential lamination Λ with solid torus guts and orderable cataclysms, then $\pi_1 M$ acts faithfully on the circle by orientation-preserving homeomorphisms. Their method did not specifically look at manifolds as Dehn surgeries. Rather, they used the result of [GK1] that $\widetilde{\Lambda}$ is vertical in $\widetilde{M} \approx \mathbb{R}^3$ to map $\widetilde{\Lambda}$ to a lamination of the disk, and found an action of $\pi_1 M$ on the boundary of that disk.

That their result holds for laminations with solid torus guts, rather than only for very full laminations (which are, recall, those with solid torus complementary regions), is a result of their "filling lemma":

Lemma 3 *Let Λ be a genuine lamination of M with solid torus guts. Then leaves can be added to Λ to yield a very full lamination of M.*

This paper will be concerned with this lemma's generalizations, and consequences. It will also be concerned with orderability of cataclysms, about which very little is known.

Section 2 generalizes filling to the case that the gut of a complementary region has finite genus > 1, and Section 3 finds the leafspace in such a case.

Section 4 then considers the case that, rather than adding leaves to fill an interstitial region (as the filling lemma does), one adds leaves to fill other handles of a complementary region. It finds that the leafspace in this case can have cataclysms, and a criterion under which those cataclysms are not orderable.

The paper concludes with an family of examples that satisfy that criterion:

Theorem 18 *If a* 2*-bridge link* K*'s standard Seifert surface's exterior is a sutured handlebody with suture* K *and is filled along a meridian disk corresponding to a band of* S *with* ≥ 4 *twists, then the result has a non-orderable cataclysm.*

2. A generalization of filling

Lemma 4 *Suppose R is a complementary region of an essential lamination, with gut a solid handlebody with a finite number g of handles. Then leaves can be added to the interior of R to yield a complementary region all of which is homeomorphic to a missing-boundary handlebody with g handles so that each interstitial region is homeomorphic to a ray $\times I \times S^1$.*

This is, of course, a generalization of the filling lemma of [CD], and its proof will be modeled on and similar to theirs. As in [CD]'s lemma, the leaves that exist before the new leaves are added are not touched, so that the lamination that results after adding leaves has the preexisting lamination as a sublamination: All the added leaves are in the interior of R, so that R continues to exist as a subspace of the ambient manifold, but winds up containing more leaves than it started out with.

Proof **Step 1.** If any interstitial region has two boundary annuli, then we wish to add a leaf (actually a band of leaves) so as to separate the interstitial region into two interstitial regions, each with one boundary annulus. (Otherwise, we skip this step.) We do this by considering the boundary circles γ, δ of one of its boundary annuli A; let $\lambda_\gamma, \lambda_\delta$ denote the complementary region's boundary leaves that satisfy $\gamma \subset \lambda_\gamma$ and $\delta \subset \lambda_\delta$. Thicken $\lambda_\gamma, \lambda_\delta$ to disjoint bands $\lambda_\gamma \times I, \lambda_\delta \times I$ and remove all but a countably infinite number of leaves from those bundles, leaving a sequence of leaves parallel to each boundary leaf and limiting only on that boundary leaf (see [G6, 2.1]); call the leaves limiting on λ_γ $(\lambda_\gamma^i)^i$; the others, $(\lambda_\delta^i)^i$. The original (non-product) complementary region now has new boundary leaves λ_γ^1 and λ_δ^1; cut these, and all the other newly added leaves, where they meet A. Glue, along A, the part of λ_γ^1 that lies in the gut to the part of λ_δ^1 that lies in the interstitial region. Stagger the remaining leaves, gluing them up so that none are left with boundary components. This actually adds only one leaf. Thicken it then to a band ([G6, 2.1.1]). We have achieved our desired result.

If the interstitial region had more than two boundary annuli to start with, this process will need to be done more than once. Because R has finitely many handles, we only need to do it finitely many times.

Step $2-\varepsilon$. Each interstitial region is $\Sigma \times I$ where Σ is a surface with one boundary component, a circle we will call γ. Its boundary comprises two copies of Σ and the interstitial annulus $\gamma \times I$.

As a starting point, foliate the interstitial region by copies of Σ parallel to the boundary leaves; we will modify this foliation.

Let us first prove that $\pi_1\Sigma$ is free. By [PM], Σ is homeomorphic to any other orientable surface which shares with Σ the following:

- the number and homeomorphism type of punctures, that is, removed disks (where the homeomorphism type depends on the boundary of the removed disk,

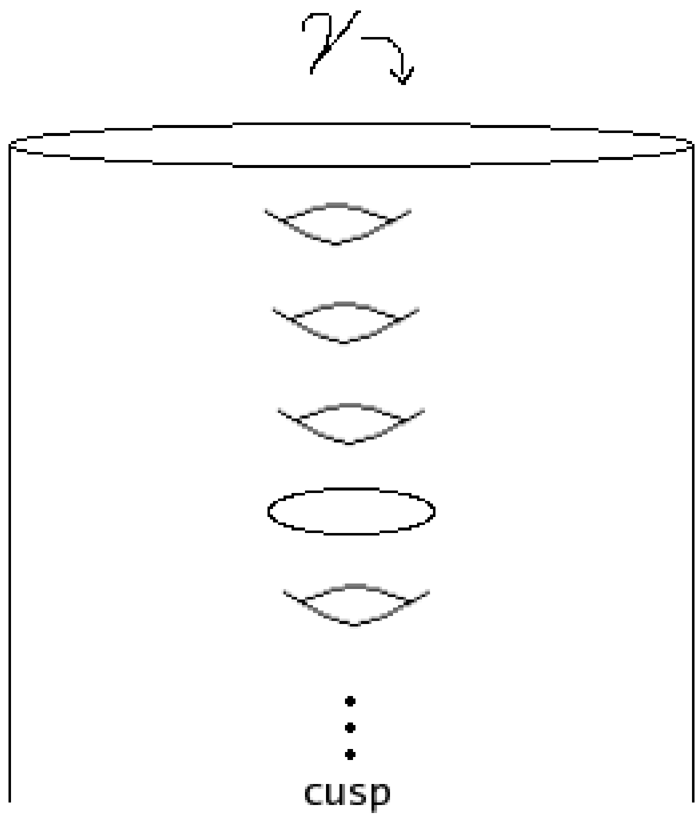

Figure 2.1. A useful view of Σ. Each puncture, such as the one shown, corresponds to the removal of a closed disk.

which is a subset of S^1; for Σ, all removed disks, except one, are closed, as Σ has no boundary component other than γ); and

- the number of annuli attached along pairs of circle boundary components (that is, the genus).

It is thus homeomorphic to such a surface with these subsets arranged nicely as in Figure 2.1, and it is easy to pick out free generators from the picture.

Step 2. Consider the double $D\Sigma$ of Σ. $D\Sigma$ can have a hyperbolic structure put on it ([S, page 421]). $\pi_1 D\Sigma$ then acts faithfully by orientation-preserving homeomorphisms (actually isometries) on $\widetilde{D\Sigma} \approx \mathbb{H}^2$ as a subgroup of $\mathrm{PSL}_2\mathbb{R}$ and thus by orientation-preserving homeomorphisms on the boundary at infinity, S^1, of $\mathbb{H}^2$, so injects into Homeo^+S^1. The injection $\Sigma \hookrightarrow D\Sigma$ induces an injection $\pi_1\Sigma \hookrightarrow \pi_1 D\Sigma$, so that $\pi_1\Sigma$ injects into Homeo^+S^1 also. Since the image of $[\gamma] \in \pi_1 D\Sigma$ in $\mathrm{Homeo}^+\mathbb{H}^2$ is hyperbolic ([CB, page 23]), its image in Homeo^+S^1 has two fixed points, one "stable" and one "unstable"; the same is true for any other nontrivial element of $\pi_1\Sigma$. Denote by ρ the map from $\pi_1\Sigma$ to Homeo^+S^1, and lift ρ to a homomorphism $\widetilde{\rho} : \pi_1\Sigma \to \widetilde{\mathrm{Homeo}}^+S^1$. (That it lifts follows from the fact that $\pi_1\Sigma$ is free, and a homomorphism from a free group to a group G lifts to any group $\widetilde{G}$ that maps epimorphically onto G: just choose preimages in $\widetilde{G}$ of the images in G of the generators of the free group. The lift is not unique, but $\widetilde{\rho}$ is injective.)

$\widetilde{\mathrm{Homeo}}^+S^1$ is the group of orientation-preserving homeomorphisms of $\mathbb{R}$ that are periodic, i.e. that are equivariant with respect to the function $s : \mathbb{R} \to \mathbb{R} : t \mapsto t+1$ ([G, §4]). So it injects into $\mathrm{Homeo}^+\mathbb{R}$, the group of orientation-preserving homeomorphisms of $\mathbb{R}$. The image of $[\gamma] \in \pi_1\Sigma$ (or, again, any other nontrivial element of $\pi_1\Sigma$) in $\mathrm{Homeo}^+\mathbb{R}$ has countably infinitely many fixed points in $\mathbb{R}$, alternatingly stable and unstable, which accumulate nowhere. We can then find a homeomorphism from $\mathbb{R}$ to the interior of I and an isomorphism therefore from $\mathrm{Homeo}^+\mathbb{R}$ to $\mathrm{Homeo}^+\mathrm{int}(I)$ and another to Homeo^+I; in the latter group, all elements fix each endpoint of I, and each nontrivial element of the image of $\pi_1\Sigma$ has countably infinitely many fixed points, which can be taken to accumulate only at the endpoints of I.

$$\pi_1\Sigma \overset{\widetilde{\rho}}{\hookrightarrow} \widetilde{\mathrm{Homeo}^+}S^1 \hookrightarrow \mathrm{Homeo}^+\mathbb{R} \cong \mathrm{Homeo}^+\mathrm{int}(I) \cong \mathrm{Homeo}^+I$$

Denote the map from $\pi_1\Sigma$ to Homeo^+I by σ.

Choose a set G of independent generators of $\pi_1\Sigma$, including $[\gamma]$. (G is countable, possibly infinite.) For each element of $G \setminus [\gamma]$, choose a representative g_i. Choose g_i to be separating iff $[g_i]$ is parallel to a puncture of Σ.

For each $[g_i] \in G$ there is a line ℓ_i properly embedded in Σ that intersects g_i in precisely one point, transversely. Such a line can be properly embedded by assuming Σ is smooth and properly embedding an increasing union of closed subsegments in an increasing union of compact submanifolds of Σ (e.g., [Le, 2.28]). By the same exhaustion, we may — and do — assume that $g_j \cap \ell_i = \varnothing$ for $i \neq j$: we can choose the circles g_i in the order indicated by Figure 2.1 and define a compact subsegment of each corresponding line ℓ_i before choosing the next circle; one can easily then define the circles and lines to satisfy $g_j \cap \ell_i = \varnothing$ for $i \neq j$.

Fixing i, consider $\ell_i \times I \subset \Sigma \times I$, the interstitial region. Replace this rectangle with a cube $\ell_i \times I \times J$ (where g_i, which is oriented, meets $J \approx [0, 1]$ at 0 first; i.e., $(\text{basepoint}, 0 \in J, 1 \in J)$ is in cyclic order on g_i), and foliate it as follows.

(Note that the foliation outside $\ell_i \times I \times J$ is, so far, just copies of Σ parallel to the boundary leaves.) Foliate the cube by I many disks (whose boundaries are in

$I \times \partial J$), so the line $\ell_i \times \{t\} \times \{0\}$ is on the same leaf as the line $\ell_i \times \{(\sigma[g_i])t\} \times \{1\}$. Then glue the cube back into the complementary region.

(The noncompactness of ℓ_i doesn't matter, as the holonomy of the foliation is trivial in that direction.)

For γ itself, do the same: Cut the complementary region along a properly embedded ray$\times I$ with its boundary in $\gamma \times I$, taking care that the ray does not meet any other generating circle. Replace that ray$\times I$ by a ray$\times I \times J$ and foliate the latter per $\sigma[\gamma]$ in the J direction as above.

We are done foliating the interstitial region.

Step $2 + \varepsilon$. Choose one circle leaf of the foliation by circles and lines of $\gamma \times I$.

Claim 5 *The surface S containing this leaf is an annulus, that is, its only essential loop is parallel to γ (or a multiple of γ).*

Remark that one boundary component of S is γ; there are countably many other boundary components where S returns to meet the interstitial annulus. S is thus homeomorphic to a closed annulus with countably many points removed from one boundary circle.

Proof Indeed, otherwise, there is a nontrivial loop in S, and we may choose as our basepoint a point on γ, so there is a nontrivial arc $\delta \subset S$ from a point on γ back to the same point on γ. $[\delta]$ is some word in the generators $[g_i]$, say $[g_{i_1}] \cdots [g_{i_n}]$, and so homotopes to pass through the modified-foliation cubes corresponding to $\ell_{i_1}, \ldots, \ell_{i_n}$. Thus $[g_{i_1}] \cdots [g_{i_n}]$ and $[\gamma]$ share a fixed point in $\mathrm{Homeo}^+ I$, so share a fixed point in $\mathrm{Homeo}^+ S^1$. But then $[g_{i_1}] \cdots [g_{i_n}]$ is a multiple of $[\gamma]$ in $\pi_1 S$ (e.g., [Ra, Exercise 9.6.1]), which contradicts the choice of independent generators. $\|$

References to a *chosen* leaf are to this circle leaf of $\gamma \times I$, or the annulus leaf S containing it.

Extend the foliation of the interstitial region slightly into the gut; do this for every interstitial region meeting that gut, taking care that all chosen circle leaves match up with respect to whether they are "stable" or "unstable" (so that adjacent interstitial regions' leaves will be able to be glued).

Open up each chosen leaf S using the standard operation of [G6, 2.1.2] ([Li, Remark 3.1]), obtaining two leaves S_1, S_2 with complementary region between them. Note that this complementary region will meet what we have been calling our interstitial annulus (though it is no longer interstitial) countably many times, as the chosen leaf met it countably many times.

The foliation of the interstitial region and its extension into the gut are thus divided into two pieces along the opened up chosen leaf, since we are assuming leaves to be two-sided; references to *half* of the foliation are to one of these two foliated subspaces.

Glue adjacent halves of interstitial regions' foliations with one another by extending them along the boundary leaf of the gut. (Recall that the complementary region satisfies the definition of a sutured handlebody, so that sutures cobound surfaces.) Do this so that each chosen leaf, where it meets the interstitial annulus again, is glued to a plane, not an annulus, leaf, so as to avoid adding fundamental group to the chosen leaves. This can be done because there are only finitely many chosen leaves and countably many times each leaf meets the interstitial annulus while remaining in a given interstitial region. (Recall, too, that the leaves of the foliated annuli are countably many and accumulate only on the original boundary leaves, so that they can be paired off.) If any interstitial region is not filled (because it is a ray$\times I \times S^1$), any adjacent interstitial region's foliation will continue into that interstitial region anyway, and extend to its cusp.

The new complementary region is thus bounded in each interstitial region by an annulus, so is the product there of an annulus and a ray.

This concludes the proof of this generalization of the filling lemma. ‖

Filling will mean applying the algorithm in the proof of the filling lemma — Step 1 and/or Step 2, as needed, unless otherwise noted.

Recall that each complementary region is homeomorphic to a missing-boundary handlebody, as described in Section 1.1.

Lemma 6 *Any interstitial region that is not a ray$\times I \times S^1$ contains a missing-boundary disk, essential in the handlebody, that has precisely two boundary components.*

Recall that it is precisely such an interstitial region to which filling applies.

Proof As in the proof of Lemma 4, there is a properly embedded line $\ell \subset \Sigma$. Because Σ is assumed to be nontrivial, it has attached annuli (genus) or missing points, so that ℓ can be taken to be essential. $\ell \times I$ then is a missing-boundary disk essential in the interstitial region. $\|$

A partial converse:

Lemma 7 *If the missing-boundary meridian disk of a handle has precisely two boundary components, then that disk is included into an interstitial region.*

Proof A disk with two boundary components' meeting a gut violates the canonical decomposition of the complementary region into guts and interstitial regions. $\|$

Thus, filling can be performed on a missing-boundary handlebody complementary region that has a missing-boundary meridian disk with precisely two boundary components. Step 1 of filling adds a band of leaves to the interior of the corresponding handle so that the handle is no longer contained in an interstitial region. As that band of leaves has been described in Step 1 of the proof of lemma 4, it can be taken to meet the missing-boundary meridian disk of the handle as in Figure 2.2. Although not all complementary regions contain product missing-boundary disks, and not all are filled by Step 1 of filling, this particular construction is an interesting one that we will generalize in Section 4.

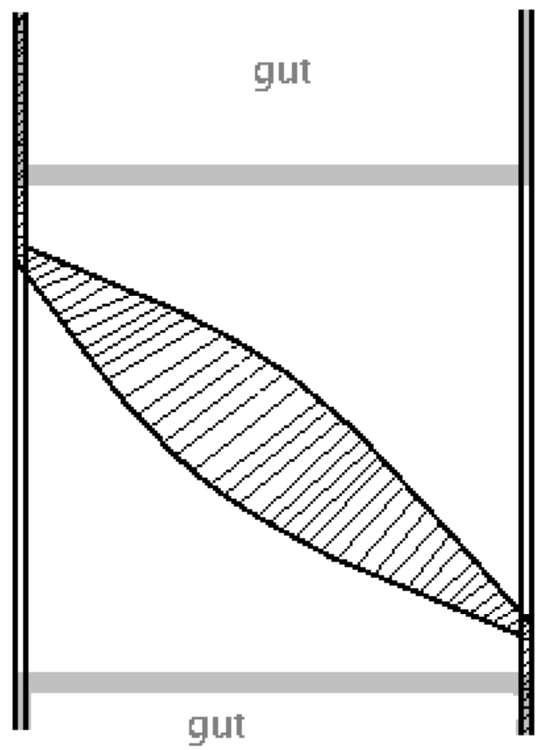

Figure 2.2. A filled handle. In this figure, the interstitial annulus before filling is grey. The slanted missing-boundary disk in the middle is the meridian disk of the handle. Step 1 of filling adds a leaf (actually a band of leaves) to the handle; this is shown in the picture as striped.

3. Leafspace after the generalization of filling

The question then is what filling a lamination of M does to the leafspace of $\widetilde{M}$. We are assuming (without loss[1]) that each of the boundary leaves of $\widetilde{R}$ has a band of leaves on one side (namely, in the complement in $\widetilde{M}$ of $\widetilde{R}$). (Recall the notation: $\widetilde{R}$ denotes a single component of the preimage in $\widetilde{M}$ of the complementary region R; it is a universal cover of R, and, before filling is done, is a complementary region.)

$\widetilde{R}$, before we start filling, maps in the leafspace to a single point (corresponding to the complementary region); this point is in countably many segments that pairwise meet precisely at it (corresponding to $(0,1)$-bundles of leaves that limit on the various boundary leaves of $\widetilde{R}$). Filling adds leaves to the interior of $\widetilde{R}$, making the image in the leafspace of that part of $\widetilde{M}$ that was originally part of $\widetilde{R}$ larger than a point.

Let T be that subspace of the post-filling leafspace $\widetilde{L}$ of $\widetilde{M}$ that is the image in the leafspace of the interior of $\widetilde{R}$. That is, the interior of $\widetilde{R}$ now contains complementary regions, and bands of leaves between complementary regions, and we let T be the leafspace which the image of all those. Note, though, that such original (pre-filling) boundary leaves of $\widetilde{R}$ as are no longer boundary leaves after filling will have images in the leafspace that are not contained in T.

[1]It is possible that a boundary leaf of $\widetilde{R}$ is limited on by leaves that also limit on another leaf, but if this is the case we can thicken our boundary leaf ([G6, 2.1.1]) to move the cataclysm away from the complementary region.

Lemma 8 *T is Hausdorff; moreover, no ray in T limits on distinct points in the closure of T in the leafspace $\widetilde{L}$ of $\widetilde{M}$, so that if Λ was tight before filling it remains so after filling.*

Proof It is sufficient to prove that every sequence $\{\widetilde{\lambda}_i\}_i$ of leaves added to $\widetilde{M}$ has at most one limit leaf in $\widetilde{M}$. Choose such a sequence, and assume without loss that it has at least one limit $\widetilde{\lambda}$.

Choose compact, connected $K \subset \widetilde{M}$, and $\varepsilon > 0$. By definition, recall, there is $N_1 \in \mathbb{N}$ such that for all $i > N_1$

$$\sup_{x \in \widetilde{\lambda} \cap K} d(x, \widetilde{\lambda}_i \cap K), \sup_{x \in \widetilde{\lambda}_i \cap K} d(x, \widetilde{\lambda} \cap K) < \varepsilon$$

for d the metric on $\widetilde{R}$. Suppose that there is a leaf $\widetilde{\mu}$ and some $N_2 \in \mathbb{N}$ such that for all $i > N_2$

$$\sup_{x \in \widetilde{\mu} \cap K} d(x, \widetilde{\lambda}_i \cap K), \sup_{x \in \widetilde{\lambda}_i \cap K} d(x, \widetilde{\mu} \cap K) < \varepsilon.$$

Let $N = \max(N_1, N_2)$ and increase N as necessary so that, for each $i > N$ and for each pre-filling gut or interstitial region $\widetilde{Q}$ of $\widetilde{R}$, $\widetilde{\lambda}_i \cap K \cap \widetilde{Q}$ and $\widetilde{\lambda} \cap K \cap \widetilde{Q}$ are joined by a transversal. This is possible because, in a pre-filling gut, any two added leaves near enough to one another are parallel to the same boundary leaf of the gut and so are joined by a transversal (and in a pre-filling interstitial region all leaves are joined by a transversal). Likewise, increase N as necessary so that, for each $i > N$, $\widetilde{\lambda}_i \cap K \cap \widetilde{Q}$ and $\widetilde{\mu} \cap K \cap \widetilde{Q}$ are joined by a transversal. Then $\widetilde{\lambda} \cap K$ and $\widetilde{\mu} \cap K$ cannot intersect the same gut or interstitial region, since there cannot be two limits of a sequence of leaves joined by a transversal. Suppose then that $\widetilde{\lambda} \cap K$ meets some gut or interstitial region Q_1, and $\widetilde{\mu} \cap K$ meets some Q_2, with $\{\widetilde{\lambda}_i\}_{i>N} \cap K$ meeting both Q_k. Then for each $i > N$ there is some path in $\widetilde{\lambda}_i$ from a point in $Q_1 \cap K$ to a point in $Q_2 \cap K$. (The path does not necessarily lie in K.) This compact path is thus contained in a finite number of lamination charts. Because the $\widetilde{\lambda}_i \cap K \cap Q_1$ limit on $\widetilde{\lambda}$, finitely many charts suffice to cover all the paths, and, by making these slightly larger if necessary, we can assume that $\widetilde{\lambda} \cap K \cap Q_1$ and $\widetilde{\mu} \cap K \cap Q_2$ are also in these charts. (We know finitely many charts suffice to cover all infinitely many paths even outside of K because the charts overlap.) Where the charts, homeomorphic to $\mathbb{R}^3$, overlap, we can assume that the coordinates match, so that λ and μ are seen to coincide. $\|$

Because we have assumed that each boundary leaf of the original complementary region $\widetilde{R}$ is part of a band of leaves exterior to $\widetilde{R}$, a small neighborhood of T in the leafspace $\widetilde{L}$ of $\widetilde{M}$ is also Hausdorff. We will refer to the *degree* of a branch point v of T even if some of the leafspace incident to it is exterior to T.

Lemma 9 *There are countably many vertices in T; these can be partitioned into finitely many sets in each of which all vertices have equal degree.*

If the gut of the complementary region R is a solid torus, then that degree is finite, since the filling lemma yields a finite-sided-solid-torus complementary region in M.

Proof There is one vertex per complementary region in $\widetilde{R}$ after filling. There are only countably many such complementary regions after filling, since there is no more than one per fundamental domain of the covering space $\widetilde{R} \to R$ and $\pi_1 R$ is countable.

All complementary regions are lifts of the same finitely many complementary regions in M, and all lifts of a single complementary region will have the same number of boundary leaves. $\|$

Note that, for each original boundary leaf λ of R, $\pi_1\lambda$ injects into $\pi_1 R$ under the homomorphism induced by $\lambda \hookrightarrow R$. Fix λ, and denote the image of $\pi_1\lambda$ by $H < \pi_1 R$.

Lemma 10 *The number of lifts of λ is the index of H in $\pi_1 R$.*

Note that this will be countable, since $\pi_1 R$ is countable. Thus, the total number of original boundary leaves of $\widetilde{R}$ is countable, since there are only finitely many original boundary leaves of R.

Proof The lifts of λ can be indexed by cosets of H in $\pi_1 R$, since the lifts are one per fundamental domain of the covering space $\widetilde{R} \to R$ except that two lifts coincide if there is a path from one to the other, a lift of a nontrivial path in λ. $\|$

Lemma 11 *A single boundary leaf $\widetilde{\lambda}$ of $\widetilde{R}$ is still a boundary leaf after filling iff its image $\lambda \subset R$ does not meet any filled interstitial region.*

Thus, all lifts of a single original boundary leaf λ of R will still be boundary leaves after filling, or all λ's lifts will not be; we obtain also that some of $\widetilde{R}$'s original boundary leaves are still boundary leaves of complementary regions even after filling iff some original boundary leaf of R does not meet any filled interstitial region.

The criterion that some original boundary leaf not meet a filled interstitial region is equivalent, precisely in the case that precisely one interstitial region is filled, to the criterion that there be more than two original boundary leaves of R. Indeed, precisely two leaves meet the interstitial region, as it is a product.

Proof Suppose first that some original boundary leaf λ of R meets a filled interstitial region. The band of leaves added in the first step of filling meets every neighborhood of each point of λ, so that, by definition, λ no longer meets a complementary region so is not a boundary leaf. Its lifts are then not boundary leaves either.

Now suppose that some original boundary leaf λ of R meets no filled interstitial region. Since the leaves of filling are chosen to limit on other boundary leaves and not on λ, every neighborhood of λ even after filling meets the complementary region.

$\|$

Lemma 12 *Each original boundary leaf of the complementary region that is no longer a boundary leaf is limited on by a properly embedded ray of T.*

Although this is clear from the definition of *boundary leaf*, it is interesting to see how a ray limits on the boundary leaf:

Proof The filling algorithm's Step 2, if performed, guarantees that the annulus A that was, before the step is done, the interstitial annulus, will meet the filled interstitial region multiple times; in this case, since the pre–Step-2 interstitial region has countable π_1, countably many times. Thus, there are countably infinitely many times one meets a complementary region when traveling along the I direction of A from one original boundary leaf to the other. In $\widetilde{M}$, then, one still meets a complementary region countably infinitely many times when traveling along the I direction of each lift of A (which is an $I \times \mathbb{R}$); this time, though, these complementary regions do not coincide (indeed, a path in the complementary region in M returns to A after passing through some nontrivial loop, which in $\widetilde{M}$ becomes a return to *another* lift of A). Thus there are countably infinitely many distinct complementary regions in the band of leaves added by filling. These accumulate, as guaranteed by the choice of homeomorphism made by the filling algorithm, on the original boundary leaves of the complementary region.

Thus we have countably infinitely many complementary regions, linearly ordered, each with a band of leaves between them, and accumulating on the original boundary leaves of the complementary region. This corresponds in the leafspace to some properly embedded line in the tree, with ends original boundary leaves of the complementary region. $\|$

4. In more generality

One can, using the filling lemma (Lemma 3) and its generalization Lemma 4, fill a handle (of a missing-boundary handlebody complementary region) that meets cusps precisely twice (i.e., whose meridian disk, which is a missing-boundary disk, has two boundary components). We will now consider filling a handle that meets cusps more than twice. We will, by analogy, call this process *filling*, and it will be done as filling was: by adding a band of leaves into the handle. The inspiration for doing so is disk decomposition of sutured handlebodies ([G2], [G1]), which can be done to such a handle as well as an interstitial one.

Specifically, choose a (missing-boundary) meridian disk D of the complementary region $R \subset M$, viewed as a missing-boundary handlebody, and for each boundary leaf λ of the complementary region such that λ meets D, thicken λ to a band $\lambda \times I$ in the original complementary region, then remove all but countably infinitely many leaves from that band; ensure that the countably many that are left limit only on the original boundary leaf (see [G6, 2.1]). Call these new leaves $\lambda^1, \lambda^2, \ldots$, so that $\lambda^i \to \lambda$ as i increases. Each of these new leaves λ^n meets D in a disjoint union of finitely many lines.

Denote by $\partial_i D$ the boundary components of D and, for each i, denote the leaves λ^n added parallel to $\partial_i D$ by λ_i^n, noting that indices are not necessarily unique (that is, we may have $\lambda_i^n = \lambda_j^n$ for $i \neq j$. However, $\lambda_i^n = \lambda_j^m$ implies $n = m$). Denote the line in $D \cap \lambda_i^n$ parallel in D to $\partial_i D$ by ℓ_i^n. Noting that there are an even number of boundary components $\partial_i D$ of D, call them alternatingly *positive* and *negative*, so that each positive (respectively negative) boundary component is adjacent in the cyclic order around D to two negative (positive) components. Then let D' be a missing-boundary disk with each of its boundary components $\partial_i D'$ parallel in D to the corresponding $\partial_i D$, but such that D' is contained in the connected component of $D \setminus \bigcup_i \ell_i^1$ that contains no boundary component of D. D is two-sided in and separates the handle h of which it is a meridian; for each connected component of $h \setminus D$ consider the subspace of h which is the union of that component with D itself; call the two subspaces so formed the *positive* and *negative* parts of h, h_+, h_-. ($h_+ \cap h_- = D$, then.) Likewise, for any subspace $K \subset M$ that meets h, call $K \cap h_+$ the *positive* and $K \cap h_-$ the *negative* components of $K \cap h$.

Cut each λ_i^n along the line ℓ_i^n where the leaf meets D. For a positive (respectively negative) boundary component $\partial_i D$, glue the positive (negative) component of λ_i^1 in D to $\partial_i D'$ along the line$\times I$ in D between ℓ_i^1 and $\partial_i D'$. Then glue across a line$\times I$ in D the negative (positive) component of λ_i^n with the positive (negative) component of λ_i^{n+1} for each $n \geq 1$. For each boundary leaf λ meeting D, we obtain a single added leaf parallel to and limiting on λ; thicken it then to a band of leaves ([G6, 2.1.1]).

Note that, unlike in the case of the filling lemma itself and its generalization to higher-genus handlebodies, described in previous sections, this does not necessarily yield an essential lamination. (The resulting complementary region may have a meridian disk *not* missing boundary and thus be compressible.[1]) If it does yield an essential lamination, then it will cause a change in the decomposition of the complementary region into guts and interstitial regions, and a new interstitial region may not be a ray$\times I \times S^1$: it may have some nontrivial loops besides the core of its boundary, the interstitial annulus A, and multiples of that core. In that case, the second step of filling can be performed, getting rid of such nontrivial fundamental group. This Step 2 is done precisely as in Section 2: first foliate the interstitial region by leaves parallel to its boundary leaves $\Sigma \times S^0$, then, noting that $\pi_1\Sigma$ is free and $\gamma = A \cap \Sigma$ is hyperbolic, lift Σ's action on S^1 to an action on I. Modify the foliated interstitial region, along lines chosen to meet the generators' representatives, by the images of those generators in $\mathrm{Homeo}^+ I$, and then open up along a chosen leaf of this foliation. Extend the foliation into the gut of the complementary region; because we started with a sutured handlebody and Step 1 of filling meets the criteria in [G1, 3.1], [G2, 3.8], we still have a sutured handlebody, so that adjacent interstitial regions' foliations can be made to match up as described in Section 2. We obtain an interstitial region whose fundamental group is generated by γ.

Lemma 13 *That second step of filling keeps an already essential lamination essential.*

Proof The complementary region is still a missing-boundary handlebody, so is still irreducible; no handles were added to allow new compressing disks, so it still has incompressible boundary. As to end-incompressibility: Any properly embedded half-

[1]If the original complementary region was disk-decomposable as a sutured handlebody, in the sense of [G2], [G1], then the disk along which filling is done can be chosen so that that will not occur.

open disk in the complementary region that does not homotope rel boundary into a leaf must, because the interstitial region is trivial, meet the interstitial region in a ray$\times I$; the rest of the disk existed before also the second step of filling was done, as the gut does not change in the second step of filling (except that its boundary leaves are replaced by others parallel to them). Thus the disk existed before the second step of filling, contradicting that we started with an essential lamination. $\|$

Lemma 14 *Assume that filling is done as above, to one handle only, and that the result is an essential lamination. Consider a loop that, before filling, was the to-be-filled handle's core curve (i.e., represented that handle in $\pi_1 R$), and whose intersection with the lamination is precisely where that core curve meets the leaves filling the handle. That loop homotopes to meet transversely all of the leaves added during filling, so that no subarc of the loop cobounds, with an arc in a leaf of the lamination, a disk in the post-filling complementary region.*

This lemma applies also to the Section-2 case that a handle meeting two cusps was filled along its missing-boundary meridian disk. The proof is the same.

Proof The complementary region after filling is bounded by the leaves S_1, S_2 formed by opening up the chosen leaf S. The loop when it meets the lamination meets S_1, say, first, as S_1 is a boundary leaf, and then passes through the leaves comprising half the modified foliation added in Step 2 of filling. Since there is, as described in the proof of the filling lemma (Lemma 4), a transversal α joining all those leaves, our loop can be taken transverse to them also. Then our loop meets the band of leaves added in Step 1 of filling, as the Step-2 leaves limit on it. That band is just a $[0, 1]$ band of leaves, so α continues as a transversal into the band; our loop then can also. Likewise, the transversal of the other half of the modified foliation of Step 2, which limits on the other side of the Step-1 leaves, continues α, so our loop continues transversely out into the complementary region through S_2. (See Figure 4.1.) $\|$

As we have done, let T be that subspace of the leafspace of filled lamination that is the image in the leafspace of the subspace of $\widetilde{M}$ that was, before filling, the interior of $\widetilde{R}$.

As above, since we assume each original boundary leaf of $\widetilde{R}$ is part of a band of leaves exterior to $\widetilde{R}$, we use the terms *degree* even for a vertex of T that has leafspace outside T incident to it.

Remark that there are zero or countably infinitely many vertices in T, and all vertices in T have equal degree; the proof is the same as that of Lemma 9. (Zero vertices will occur precisely in the case that Step 1 of filling yields a product complementary region so that Step 2 simply fills in that complementary region with foliation.)

Lemma 15 *Each original boundary leaf of $\widetilde{R}$ that is no longer a boundary leaf of any complementary region is limited on by a properly embedded ray in T.*

Proof As discussed in the remark immediately following the statement of Lemma 12, this is clear, and the proof is the same as that lemma's proof. $\|$

Lemma 16 *If the original complementary region had ≥ 2 handles with precisely one of them meeting > 2 cusps, that handle is filled by Step 1 of filling only, and the result*

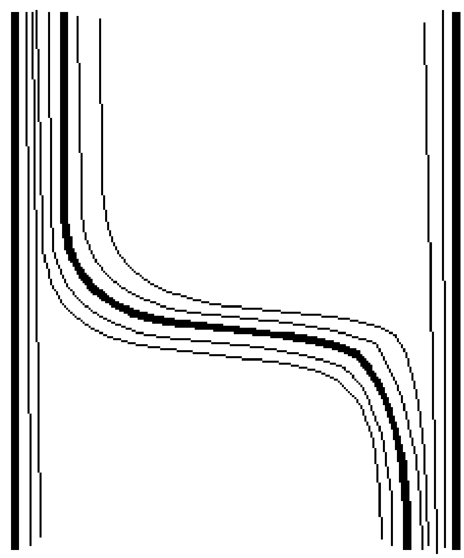

Figure 4.1. A filled handle. In this figure the thick lines represent the band of leaves added in Step 1 of filling; it wraps around the interior of the original (missing-boundary) handlebody and returns countably many times, but we only show it three times. The thin lines represent the leaves added during Step 2. These are infinitely many (in countably many bands), but four are drawn. Note that the leaves that extend downward out of the figure to the bottom left wrap around the interior of the complementary region and return as the "innermost" leaves depicted on the top left; thus, all of the leaves depicted here meet a vertical transversal drawn down the middle of the figure, as described in Lemma 14. (The original boundary leaves of the complementary region are not shown. Also, the schematic shows only two boundary components of the meridian disk, but there are more; in fact, the lower left leaves, when they wrap around, may do so elsewhere than at the same boundary component of the meridian disk.)

is an essential lamination, then the degree of each vertex in its leafspace T is 2, and none of the original boundary leaves of $\widetilde{R}$ are boundary leaves of any complementary region.

In this case, we are filling the solid-torus gut of the complementary region to yield a non-genuine complementary region. (Step 2 of filling will then foliate the rest of that complementary region. The corresponding leafspace will then still be a line, but with no vertices along it corresponding to complementary regions, since the complementary region in M will have been filled in by leaves.)

Proof After filling, there are precisely two boundary components to each (missing-boundary) meridian disk of R, so that each handle of R is interstitial and the complementary region is not genuine but is, rather, $F \times I$ for some surface F. Thus, its universal cover is $\widetilde{F} \times I$, which also has but two boundary components. The degree of each vertex in T is two, as required; moreover, the complementary regions in $\widetilde{M}$ have the boundary leaves just mentioned, so they do not have the original boundary leaves of $\widetilde{R}$ as boundary leaves. $\|$

A specific example of Lemma 16 is filling the "four-cusp" handle of the solid–genus-2–missing–boundary–handlebody complementary region R pictured in Figure 4.2. We will show in Theorem 18 that the resulting lamination in this case is essential.

Theorem 17 *Consider a complementary region R filled as described in this section (page 33), with D denoting the filled handle's (missing-boundary) meridian disk.*

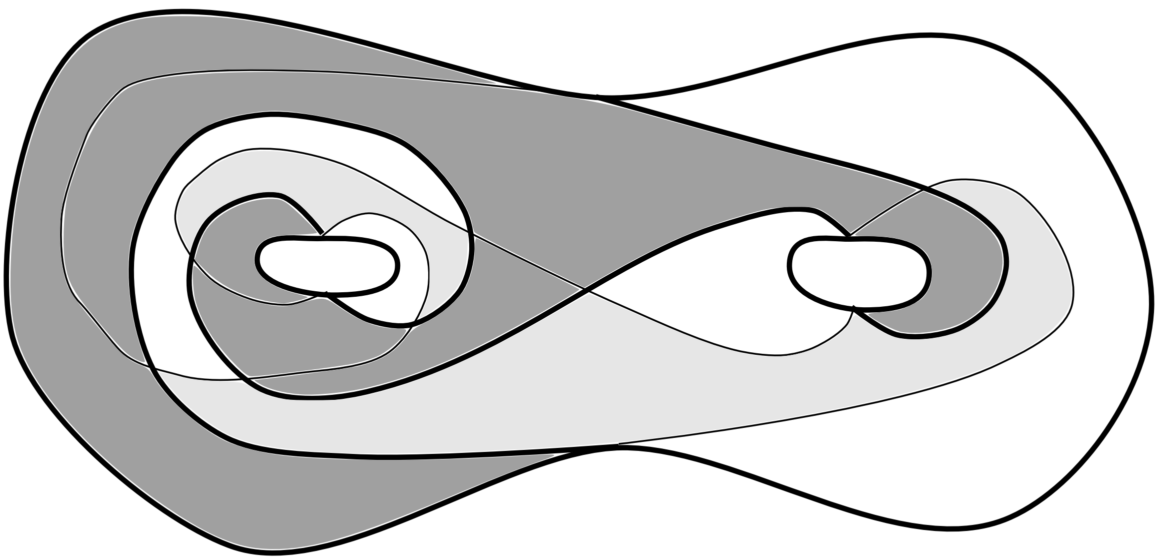

Figure 4.2. A missing-boundary handlebody complementary region with two boundary leaves, one of which is shaded. (Here, we use the sutured-handlebody view described in Section 1.1.)

Suppose there is a band (that is, an annulus or a Möbius band) A immersed in R by $f : A \to R$ such that $f\partial A \subset \partial R$, such that fA does not in R homotope rel $f\partial A$ into ∂R, and such that, for some component of $f\partial A$, D cannot be homotoped to meet that component in fewer than two components of ∂D. Denote the boundary components of D meeting that component of $f\partial A$ by $\partial_i D$, $1 \leq i \leq n$, and suppose moreover that there is a subset of those n components, say $\{\partial_1 D, \ldots, \partial_k D\}$, with $2 \leq k \leq n$, such that $\forall i \neq j$ there is no arc joining $\partial_i D$ and $\partial_j D$ trivial in $\pi_1(R/D)$ (where R/D is R with D identified to a point).

Then the filling yields a non-orderable cataclysm, with lifts of the leaves containing those $\partial_i D$ $(i \leq k)$ as the limit leaves.

Proof Consider a lift $\widetilde{D}_0 \subset \widetilde{R}$ of D. The lifts of fA that $\widetilde{D}_0$'s boundary components, lifts of $\partial_i D$ for $i \leq k$, meet are pairwise distinct. Indeed, for $i \neq j$, the arc of $f\partial A$ joining $\partial_i D$ and $\partial_j D$ is nontrivial in R/D, as there is no trivial arc joining them. Because these lines — call them η_m — are the boundary components of a lift of a band, they will intersect the same fundamental domains as one another.

Where the η_m meet another lift $\widetilde{D}_\ell$ of D, let $\widetilde{\mu}_\ell$ denote the lift to $\widetilde{D}_\ell$ of the leaf added to R, i.e. the leaf added to the fundamental domain (of $\widetilde{R} \to R$) containing $\widetilde{D}_\ell$. Consider a path in $\widetilde{\mu}_\ell$ ending in $\widetilde{D}_\ell$, meeting $\widetilde{D}_0$'s fundamental domain, and parallel in lamination charts to η_m. It is contained in a leaf among those limiting on $\partial_i \widetilde{D}_0$, the lift to $\widetilde{D}_0$ of $\partial_i D$, so we obtain a monotone sequence.

Thus, there is a cataclysm. (The $\partial_i \widetilde{D}_0$ cannot coincide, since any path from $\partial_i D$ to $\partial_j D$, $i \neq j$, lifts to a path between fundamental domains of $\widetilde{R} \to R$.)

Fix an order on the cataclysm's limit leaves and, by inheritence, on the boundary components of $\widetilde{D}_0$ they contain, since, as just noted, different boundary components are in distinct leaves: say $\partial_1 \widetilde{D}_0 < \cdots < \partial_k \widetilde{D}_0$, contained respectively in $\widetilde{\lambda}_1 < \cdots < \widetilde{\lambda}_k$. Because the image in R of each η_m contains multiple components of $D \cap f\partial A$, there is an integer ℓ such that the deck transformation mapping $\widetilde{D}_0$ to $\widetilde{D}_\ell$ permutes the indices of λ_i nontrivially, and thus does not keep the order on the cataclysm's limit leaves invariant. $\|$

An example of a complementary region that meets the criteria in the preceding theorem is the missing-boundary handlebody in Figure 4.2. Note that that handlebody is the complement in S^3 of the Seifert surface of the knot formed by plumbing a 4-half-twist band and a 2-half-twist band, with the knot as suture. Let us now explain and generalize this a bit.

Any 2-bridge (also called rational) link K is the boundary of a surface S formed by taking g bands $A_1, \ldots, A_g$, where, for each p, A_p has $t_p \neq 0$ half-twists, and plumbing A_p to A_{p+1}, $p < g$, along a plumbing disk. (If g and the t_p are all even, K is a knot.) That is, a disk in the one band is identified with a disk in the other so that the boundary of the identified disk comprises four closed arcs, disjoint except at endpoints, alternatingly from the boundaries of the two bands. We may — and do — assume that the half-twists in A_p are in one component of the complement in A_p of the plumbing disks. Moreover, for simplicity of exposition, we assume that the bands are stacked vertically (with A_1 on top) before plumbing, so that the two sides of each plumbing disk are well-defined as *top* and *bottom*. ([C], [HT].)

The complement in S^3 of that surface is a handlebody, which can be viewed as having suture $\partial S = K$. (Technically, one needs to consider $S^3 \setminus \text{int}(S \times I)$. The space is nonetheless usually called the complement of S.) Consider such a handlebody as a complementary region of an essential lamination of some manifold; we assume each complementary region satisfies the definition of a sutured handlebody, so that necessarily each of the bands A_p has an even number of half-twists and each meridian disk of the handlebody, taken as the longitudinal disk of an A_p, meets sutures in an even number of components.

Theorem 18 *Suppose the complement of the surface S corresponding to the 2-bridge link K, formed by plumbing $g > 1$ bands each with an even number of half-twists, is the complementary region of an essential lamination, and is filled along a meridian disk D corresponding (as described above) to a band A_p with $t_p \geq 4$ half-twists. Then the filled lamination is essential and has an non-orderable cataclysm.*

Proof To prove the existence of a non-orderable cataclysm, it is sufficient to show that the conditions of Theorem 17 are satisfied. That is, there is a band A immersed in R by $f : A \to R$ such that $f\partial A \subset \partial R$, such that fA does not homotope rel $f\partial A$ into ∂R, and such that $fA \cap D \neq \varnothing$, and the boundary components of D meeting a boundary one component of fA, $\partial_1 D, \ldots, \partial_r D$, satisfy that $r > 2$ and $\forall i \neq j$ there is no arc joining $\partial_i D$ and $\partial_j D$ trivial in $\pi_1(R/D)$.

Let B_1 and B_2 be A_p's plumbing disks. (There will be only one, though, if $p = 1$ or $p = g$.)

Choose a boundary leaf meeting D as follows. When thickening S to $S \times I$, each B_i becomes $B_i \times I$ with the components of $B_i \times \partial I$ corresponding to the top and bottom of B_i. For the plumbing disk B_i joining A_p to A_{p-1} (respectively A_{p+1}) we prefer the component of $B_i \times \partial I$ corresponding to the bottom (top) of B_i; choose the leaf λ meeting D that meets that component of $B_i \times \partial I$. This will be the same leaf for both B_i's because one component of $A_p \setminus \bigsqcup_i B_i$ does not twist.

Consider R as a sutured handlebody. λ, where it meets $(A_p \setminus \bigsqcup_i \mathrm{int} B_i) \times I$ is $I^2 \sqcup I^2$ where, for each component I^2, $I \times \partial I$ is the sutures and $(\partial I) \times I \subset (\bigsqcup_i B_i) \times I$. Choose the arc $I \times \{\frac{1}{2}\}$ in each component, connect these arcs across the $B_i \times I$ to form a simple closed curve, and call the whole loop thus formed γ. (Note that γ is isotopic in A_p to the core curve of the annulus A_p.)

$\gamma \cap D$ can be homotoped to comprise precisely $t_p/2$ points, all of which lie in ∂D; we assume that this has been done. Consider these points in cyclic order about ∂D, and choose two of them, x, y, adjacent in that order. Let J be an arc properly embedded in D with $\partial J = \{x, y\}$.

A_p is an annulus $S^1 \times I$; homotope J (with endpoints not fixed but within γ) in the S^1 direction. In doing so, homotope it so that at every stage of the homotopy $J \subset D$ or $J \cap D = \varnothing$. Because there are only finitely many points in $\gamma \cap D$, eventually this homotopy will come back around to our original arc J; we obtain a band A swept out by the moving J.

The lack of trivial (in R) arc in λ from one $\partial_i D$ to another is due to the twists in A_p.

Essentiality of the filled lamination is shown as follows, which proof is based on [G3]: The complementary region is irreducible, as it remains a handlebody; it is end-incompressible because it is a sutured manifold and thus satisfies that no one leaf appears on both sides of a suture. Compressibility will result from showing that handles corresponding to other bands, A_q, $q \neq p$, have their meridian disks still cusped so that there is no non-cusped meridian disk to compress along.

Consider the band A_{p-1} (or A_{p+1}, but for ease of notation we will use A_{p-1}). It has, before plumbing, two boundary components, since it has an even number of half-twists. Those two components meet the plumbing disk B_i (where the band is plumbed to A_p) in four points; let x, y be the points that are in the component of $A_{p-1} \setminus B_i$ in which A_{p-1} twists. (If A_{p-1} has only one plumbing disk, let x, y be on one side of it.) There are three seeming possibilities: Filling can modify the sutures — the boundary of A_p — so that there is now a length of suture joining x and y, transverse to the meridian disk of A_{p-1}; filling can modify the sutures so that x (and hence y) is joined by a length of suture to A_{p+1}; or filling can modify the sutures so that x (and hence y) is joined to a point on the other component of $A_{p-1} \setminus B_i$. If the first of these possibilities occurs, then it is possible to untwist the sutures of A_{p-1}, so that its meridian disk is not cusped; otherwise, it is not, so we wish to show that that x is not joined by a length of suture to y.

Consider the points z_i where sutures meet D; these are in cyclic order about D, and the disk decomposition effected by filling D's handle connects (by a length of suture) each z_i to a point z_j adjacent to it in that cyclic order. Since ∂A_p has two components (before plumbing) corresponding to the sutures, each z_i is connected by filling to a z_j on the opposing boundary component of A_p. Since x and y are on the same component of ∂A_p, filling does not join them by a length of suture. $\|$

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REFERENCES

[B] M. Brittenham. Essential laminations in non-Haken 3-manifolds. Topology Appl. 53:3 (1993), 317–24. doi:10.1016/0166-8641(93)90124-V. MR 1254875 (94k:57027).

[C] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. Computational problems in abstract algebra (Proceedings of a conference held at Oxford, 1967). Pergamon Press, 1970. 329–358. MR 0258014 (41 #2661).

[C1] D. Calegari. Problems in foliations and laminations of 3-manifolds. "Version 0.78", September 8, 2002. math.GT/0209081v1.

[C2] D. Calegari. Foliations and the geometry of 3-manifolds. Oxford mathematical monographs. Oxford, 2007. MR 2327361 (2008k:57048).

[CB] A. J. Casson, S. A. Bleiler. Automorphisms of surfaces after Nielsen and Thurston. LMS student texts 9. Cambridge, 1998. MR 0964685 (89k:57025).

[CD] D. Calegari, N. M. Dunfield. Laminations and groups of homeomorphisms of the circle. Invent. Math. 152 (2003), 149–204. doi:10.1007/s00222-002-0271-6. MR 1965363 (2005a:57013).

[F] S. Fenley. Laminar free hyperbolic 3-manifolds. Comment. Math. Helv. 82:2 (2007), 247–321. math.GT/021048v1. MR 2319930 (2008g:57020).

[G] É. Ghys. Groups acting on the circle. Enseign. Math. (2) 47 (2001), 329–407. MR 1876932 (2003a:37032).

[G1] D. Gabai. Foliations and the topology of 3-manifolds. J. Diff. Geom. 18 (1983), 445–503. MR 0723813 (86a:57009).

[G2] D. Gabai. Foliations and genera of links. Topology 23:4 (1984), 381–394. MR 0780731 (86h:57006).

[G3] D. Gabai. Genera of the alternating links. Duke Math. J. 53:3 (1986), 677–681. MR 0860665 (87m:57004).

[G4] D. Gabai. Foliations and the topology of 3-manifolds. II. J. Diff. Geom. 26 (1987), 461–478. MR 0910017 (89a:57014a).

[G5] D. Gabai. Foliations and the topology of 3-manifolds. III. J. Diff. Geom. 26 (1987), 479–536. MR 0910018 (89a:57014b).

[G6] D. Gabai. Taut foliations of 3-manifolds and suspensions of S^1. Ann. Inst. Fourier (Grenoble) 42:1–2 (1992), 193–208. MR 1162560 (93d:57028).

[G7] D. Gabai. Problems in foliations and laminations. Geometric topology (Proceedings of the 1993 Georgia International Topology Conference). AMS/IP studies in advanced mathematics 2. American Mathematical Society and International Press, 1997. Part 2, 1–33. MR 1470750.

[GK1] D. Gabai, W. H. Kazez. Homotopy, isotopy and genuine laminations of 3-manifolds. Geometric topology (Proceedings of the 1993 Georgia International Topology Conference). AMS/IP studies in advanced mathematics 2. American Mathematical Society and International Press, 1997. Part 1, 123–138. MR 1470725 (98k:57026).

[GK2] D. Gabai, W. H. Kazez. Order trees and laminations of the plane. Math. Res. Lett. 4 (1997), 603–616. MR 1470429 (98k:57024).

[GK3] D. Gabai, W. H. Kazez. Group negative curvature for 3-manifolds with genuine laminations. Geometry & Topology 2 (1998), 65–77. MR 1619168 (99e:57023).

[GO] D. Gabai, U. Oertel. Essential laminations in 3-manifolds. Ann. Math. (2) 130:1 (1989), 41–73. MR 1005607 (90h:57012).

[H] J. Hempel. 3-manifolds. Annals of mathematics studies 86. Princeton, 1976. MR 0415619 (54 #3702).

[HT] A. Hatcher, W. Thurston. Incompressible surfaces in 2-bridge knot complements. Invent. Math. 79:2 (1985), 225–246. MR 0778125 (86g:57003).

[La] H. B. Lawson. Foliations. Bull. Amer. Math. Soc. 80:3 (1974), 369–418. MR 0343289 (49 #8031).

[Le] J. M. Lee. Introduction to smooth manifolds. GTM 218. Springer, 2003. MR 1930091 (2003k:58001).

[Li] T. Li. Laminar branched surfaces in 3-manifolds. Geometry & Topology 6 (2002), 153–194. MR 1914567 (2003h:57019).

[M] L. Mosher. Laminations and flows transverse to finite depth foliations, Part I: Branched surfaces and dynamics. Preprint, December 16, 1996, version.

[N] R. Naimi. Foliations transverse to fibers of Seifert manifolds. Comment. Math. Helv. 69:1 (1994), 155–162. MR 1259611 (94m:57060).

[PM] A. O. Prishlyak, K. I. Mischenko. Classification of noncompact surfaces with boundary. Methods Funct. Anal. Topology 13:1 (2007), 62–66. MR 2308580 (2008f:57026).

[Ra] J. G. Ratcliffe. Foundations of hyperbolic manifolds. GTM 149. Second edition. Springer, 2006. MR 2249478 (2007d:57029).

[RSS] R. Roberts, J. Shareshian, M. Stein. Infinitely many hyperbolic 3-manifolds which contain no Reebless foliation. J. Amer. Math. Soc. 16:3 (2003), 639–679. MR 1969207 (2004e:57023).

[S] P. Scott. The geometries of 3-manifolds. Bull. London Math. Soc. 15:5 (1983), 401–487. MR 0705527 (84m:57009).

[T] W. Thurston. Three-manifolds, foliations and circles, II. Preprint.

[Z] A. H. Zemanian. Tranfinite graphs and electrical networks. Trans. Amer. Math. Soc. 334:1 (1992), 1–36. MR 1066452 (93i:94026).