Witten Index and spectral shift function

Abstract

Let $D$ be a selfadjoint unbounded operator on a Hilbert space and let \{B(t)\} be a one parameter norm continuous family of self-adjoint bounded operators that converges in norm to asymptotes $B_{\pm}$. Then setting $A(t) = D + B(t)$ one can consider the operator $D_A = d/dt + A(t)$ on the Hilbert space $L^2(\mathbb{R}, H)$. We present a connection between the theory of spectral shift function for the pair of the asymptotes $(A_+, A_-)$ and index theory for the operator $D_A$.

Under the condition that the operator $B_+ + t$ is a $p$-relative trace-class perturbation of $A_-$ and some additional smoothness assumption we prove a heat kernel formula for all $t > 0$,

$$\text{tr}\left(e^{-tD_A D_A^*} - e^{-tD_A^* D_A}\right) = -\left(\frac{1}{\pi}\right)^{1/2} \int_0^1 \text{tr}\left(e^{-tA_s^2(A_+ - A_-)}\right) ds,$$

where $A_s, s \in [0, 1]$ is a straight path joining $A_-$ and $A_+$.

Using this heat kernel formula we obtain the description of the Witten index of the operator $D_A$ in terms of the spectral shift function for the pair $(A_+, A_-)$.

**Theorem.** If $0$ is a right and a left Lebesgue point of the spectral shift function $\xi(\cdot; A_+, A_-)$ for the pair $(A_+, A_-)$ (denoted by $\xi_{L}(0_+; A_+, A_-)$ and $\xi_{L}(0_-; A_+, A_-)$, respectively), then the Witten index $W_s(D_A)$ of the operator $D_A$ exists and equals

$$W_s(D_A) = \frac{1}{2}(\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)).$$

We note that our assumptions include the cases studied earlier. In particular, we impose no assumption on the spectra of $A_{\pm}$ and we can treat differential operators in any dimension.

As a corollary of this theorem we have the following result.

**Corollary.** Assume that the asymptotes $A_{\pm}$ are boundedly invertible. Then the operator $D_A$ is Fredholm and for the Fredholm index $\text{index}(D_A)$ of the operator $D_A$ we have

$$\text{index}(D_A) = \xi(0; A_+, A_-) = \text{sf}\{A(t)\}_{t=-\infty}^{+\infty},$$

where $\text{sf}\{A(t)\}_{t=-\infty}^{+\infty}$ denotes the spectral flow along the path $\{A(t)\}_{t=-\infty}^{+\infty}$.

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