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Zhangjian HU

Xiaofen Lv

Brett D. Wick

Washington University in St. Louis, bwick@wustl.edu

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LOCALIZATION AND COMPACTNESS OF OPERATORS ON FOCK SPACES

ZHANGJIAN HU¹, XIAOFEN LV¹, AND BRETT D. WICK²

ABSTRACT. For $0 < p \leq \infty$, let F_φ^p be the Fock space induced by a weight function φ satisfying $dd^c\varphi \simeq \omega_0$. In this paper, given $p \in (0, 1]$ we introduce the concept of weakly localized operators on F_φ^p , we characterize the compact operators in the algebra generated by weakly localized operators. As an application, for $0 < p < \infty$ we prove that an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F_φ^p if and only if its Berezin transform satisfies certain vanishing property at ∞ . In the classical Fock space, we extend the Axler-Zheng condition on linear operators T , which ensures T is compact on F_α^p for all possible $0 < p < \infty$.

1. INTRODUCTION

Let $H(\mathbb{C}^n)$ be the collection of all entire functions on \mathbb{C}^n , and let $\omega_0 = dd^c|z|^2$ be the Euclidean Kähler form on \mathbb{C}^n , where $d^c = \frac{\sqrt{-1}}{4}(\bar{\partial} - \partial)$. Set $B(z, r)$ to be the Euclidean ball in \mathbb{C}^n with center z and radius r , and $B(z, r)^c = \mathbb{C}^n \setminus B(z, r)$. Throughout the paper, we assume that $\varphi \in C^2(\mathbb{C}^n)$ is real-valued and there are two positive numbers M_1, M_2 such that

$$(1.1) \quad M_1\omega_0 \leq dd^c\varphi \leq M_2\omega_0$$

in the sense of currents. The expression (1.1) will be denoted as $dd^c\varphi \simeq \omega_0$. Given $0 < p < \infty$ and a positive Borel measure μ on \mathbb{C}^n , let $L_\varphi^p(\mu)$ be the space defined by

$$L_\varphi^p(\mu) = \{f \text{ is } \mu\text{-measurable on } \mathbb{C}^n : f(\cdot)e^{-\varphi(\cdot)} \in L^p(\mathbb{C}^n, d\mu)\}.$$

When $d\mu = dV$, the Lebesgue measure on \mathbb{C}^n , we write L_φ^p for $L_\varphi^p(\mu)$ and set

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} |f(z)e^{-\varphi(z)}|^p dV(z) \right)^{\frac{1}{p}}.$$

For $0 < p < \infty$ the Fock space F_φ^p is defined as $F_\varphi^p = L_\varphi^p \cap H(\mathbb{C}^n)$, and

$$F_\varphi^\infty = \left\{ f \in H(\mathbb{C}^n) : \|f\|_{\infty,\varphi} = \sup_{z \in \mathbb{C}^n} |f(z)|e^{-\varphi(z)} < \infty \right\}.$$

F_φ^p is a Banach space with norm $\|\cdot\|_{p,\varphi}$ when $1 \leq p \leq \infty$ and F_φ^p is a Fréchet space with distance $\rho(f, g) = \|f - g\|_{p,\varphi}^p$ if $0 < p < 1$. The typical model of φ is $\varphi(z) = \frac{\alpha}{2}|z|^2$ with $\alpha > 0$, which induces the classical Fock space. For this particular special weight φ , F_φ^p and $\|\cdot\|_{p,\varphi}$ will be written as F_α^p and $\|\cdot\|_{p,\alpha}$, respectively. The space F_α^p has been studied by

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many authors, see [2, 5, 7, 18–21] and the references therein. Another special case is with $\varphi(z) = \frac{\alpha}{2}|z|^2 - \frac{m}{2}\ln(A + |z|^2)$ with suitable $A > 0$, and then F_φ^p is the Fock-Sobolev space $F_\alpha^{p,m}$ studied in [3, 4].

It is well-known that F_φ^2 is a Hilbert space with inner product

$$\langle f, g \rangle_{F_\varphi^2} = \int_{\mathbb{C}^n} f(z)\overline{g(z)}e^{-2\varphi(z)}dV(z).$$

Given $z, w \in \mathbb{C}^n$, the reproducing kernel of F_φ^2 will be denoted by $K_z(w) = K(w, z)$. We write $k_z = \frac{K_z}{\|K_z\|_{2,\varphi}}$ to denote the normalized reproducing kernel. Given some bounded linear operator T on F_φ^p , the Berezin transform of T is well defined as

$$\tilde{T}(z) = \langle Tk_z, k_z \rangle_{F_\varphi^2},$$

since $Tk_z \in F_\varphi^p \subset F_\varphi^\infty$ and $k_z \in F_\varphi^1$. Set P to be the projection from L_φ^2 to F_φ^2 , that is

$$Pf(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)}dV(w) \quad \text{for } f \in L_\varphi^2.$$

For a complex Borel measure μ on \mathbb{C}^n and $f \in F_\varphi^p$, we define the Toeplitz operator T_μ to be

$$T_\mu f(z) = \int_{\mathbb{C}^n} f(w)K(z, w)e^{-2\varphi(w)}d\mu(w).$$

If $d\mu = gdV$, for short, we will use T_g to stand for the induced Toeplitz operator and will use that $\tilde{g} = \tilde{T}_g$.

In the case of Fock spaces F_α^2 , fixed g bounded on \mathbb{C}^n , $|\langle T_g k_z, k_w \rangle|$ as a function of (z, w) decays very fast off the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$, see [20, Proposition 4.1]. From this point of view, Xia and Zheng in [20] introduced the notion of “sufficiently localized” operators on F_α^2 which include the algebra generated by Toeplitz operators with bounded symbols, and they proved that, if T is in the C^* -algebra generated by the class of sufficiently localized operators, T is compact on F_α^2 if and only if its Berezin transform tends to zero when z goes to infinity. In [10], Isralowitz extended [20] to the generalized Fock space F_φ^2 with $dd^c\varphi \simeq \omega_0$. Isralowitz, Mitkovski and the third author extended Xia and Zheng’s idea further in [11] to what they called “weakly localized” operators on F_φ^p with $1 < p < \infty$. They showed that, if T is in the C^* -algebra generated by the class of weakly localized operators, T is compact on F_φ^p if and only if its Berezin transform shares certain vanishing property near infinity. We would like to emphasize that the prior results in the area, for example [1, 2, 5, 8–10, 14, 15, 17, 19–22], depend strongly on two points. The first is the use of Weyl unitary operators induced by holomorphic self mappings of the domain; and the second is the restriction on the range of the exponent p , for example $p = 2$ or $1 < p < \infty$, so that Banach space techniques are applicable. But on F_φ^p with $0 < p < 1$ and $dd^c\varphi \simeq \omega_0$ these two points are not available.

The main purpose of this work is, on F_φ^p with $0 < p < 1$ and $dd^c\varphi \simeq \omega_0$, to study the so called “weakly localized” operators WL_p^φ and to characterize those compact operators $T \in WL_p^\varphi$. The paper is divided into four sections. In Section 2, we introduce the concept of weakly localized operators WL_p^φ for $0 < p \leq 1$, we will characterize the compact operators in WL_p^φ , and furthermore give a quantity equivalent to the essential norm of an operator in WL_p^φ . Section 3 is devoted to the compactness of Toeplitz operators induced by

BMO symbols acting on F_φ^p for all $0 < p < \infty$, our theorem shows an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F_φ^p if and only if its Berezin transform satisfies a certain vanishing property at ∞ (more precisely, $\lim_{z \rightarrow \infty} \tilde{T}(z) = 0$ when $\varphi(z) = \frac{\alpha}{2}|z|^2$). In Section 5, we extend Axler-Zheng's condition on linear operators T , which insures T are bounded (or compact) on F_α^p for all possible $0 < p < \infty$. In the final section, we provide some remarks and point to some open problems.

In what follows, C will denote a positive constant whose value may change from one occasion to another but does not depend on the functions or operators in consideration. For two positive quantities A and B , the expression $A \simeq B$ means there is some $C > 0$ such that $\frac{1}{C}B \leq A \leq CB$.

2. THE OPERATOR CLASS WL_p^φ WITH $0 < p \leq 1$

As a generalization of the “strongly localized” operators of Xia and Zheng in [20], Isralowitz, Mitkovski and the third author introduced “weakly localized” operators on F_φ^p with $1 < p < \infty$, see [11]. In this section, we first give the definition of weakly localized operators on F_φ^p when $0 < p \leq 1$. We use \mathcal{D} to stand for the linear span of all normalized reproducing kernel functions $k_z(\cdot)$. It is obvious that \mathcal{D} is dense in F_φ^p . As in [11], we will assume that the domain of every linear operator T appearing in this paper contains \mathcal{D} , and that the function $z \mapsto TK_z$ is conjugate holomorphic. We also assume the range of T is in F_φ^∞ . Then $\langle Tk_z, k_w \rangle_{F_\varphi^2}$ can make sense.

Definition 2.1. *Let $0 < p < \infty$, set $s = \min\{1, p\}$. A linear operator T from \mathcal{D} to F_φ^∞ is called weakly localized for F_φ^p if*

$$(2.1) \quad \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^s dV(w) < \infty, \quad \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} \left| \langle k_z, Tk_w \rangle_{F_\varphi^2} \right|^s dV(w) < \infty;$$

and

$$(2.2) \quad \lim_{r \rightarrow \infty} \sup_{z \in \mathbb{C}^n} \int_{B(z,r)^c} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^s dV(w) = 0,$$

$$(2.3) \quad \lim_{r \rightarrow \infty} \sup_{z \in \mathbb{C}^n} \int_{B(z,r)^c} \left| \langle k_z, Tk_w \rangle_{F_\varphi^2} \right|^s dV(w) = 0.$$

The algebra generated by weakly localized operators for F_φ^p will be denoted by WL_p^φ . For $\varphi(z) = \frac{\alpha}{2}|z|^2$, we write $WL_p^\varphi = WL_p^\alpha$ for convenience.

When $1 \leq p < \infty$ $WL_p^\varphi = WL_1^\varphi$ by definition, and then Definition 2.1 was first introduced in [11]. Let \mathcal{T}_p^φ denote the Toeplitz algebra on F_φ^p generated by L^∞ symbols, and let $\mathcal{K}(F_\varphi^p)$ be the set of all compact operators on F_φ^p . We use $\|T\|_{e, F_\varphi^p}$ to stand for the essential norm of a given operator T on F_φ^p

$$\|T\|_{e, F_\varphi^p} = \inf \left\{ \|T - A\|_{F_\varphi^p \rightarrow F_\varphi^p} : A \in \mathcal{K}(F_\varphi^p) \right\}.$$

The purpose of this section is to characterize compact operators in WL_p^φ , $0 < p \leq 1$. To carry out our analysis, we need some preliminary facts.

Lemma 2.2 ([16]). *Given φ as in the introduction, the Bergman kernel $K(\cdot, \cdot)$ for F_φ^2 satisfies the following estimates:*

(1) *There exists C and $\theta > 0$ such that*

$$|K(z, w)|e^{-\varphi(z)}e^{-\varphi(w)} \leq Ce^{-\theta|z-w|} \quad \text{for } z, w \in \mathbb{C}^n.$$

(2) *There exists some $r > 0$ such that*

$$|K(z, w)|e^{-\varphi(z)}e^{-\varphi(w)} \simeq 1 \quad \text{whenever } w \in B(z, r) \text{ and } z \in \mathbb{C}^n.$$

(3) *For $0 < p \leq \infty$ fixed,*

$$\|K(\cdot, z)\|_{p, \varphi} \simeq e^{\varphi(z)} \simeq \sqrt{K(z, z)}, \quad z \in \mathbb{C}^n.$$

Lemma 2.3 ([8]). *Suppose $0 < p < \infty$ and $r > 0$. Then there exists C such that for $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, we have*

$$|f(z)e^{-\varphi(z)}|^p \leq C \int_{B(z, r)} |f(w)e^{-\varphi(w)}|^p dV(w)$$

and

$$\int_{\mathbb{C}^n} |f(z)e^{-\varphi(z)}|^p d\mu(z) \leq C \int_{\mathbb{C}^n} |f(z)e^{-\varphi(z)}|^p \hat{\mu}_r(z) dV(z)$$

where μ is some given positive Borel measure and $\hat{\mu}_r(\cdot) = \frac{\mu(B(\cdot, r))}{V(B(\cdot, r))}$.

Let $d(\cdot, \cdot)$ be the Euclidean distance on \mathbb{C}^n . Given some domain $\Omega \subseteq \mathbb{C}^n$, write $\Omega^+ = \{z \in \mathbb{C}^n : d(z, \Omega) < 1\}$, and Ω^+ is again a domain. Set $\mathcal{L} = \{a + bi : a, b \in \frac{1}{4}\mathbb{Z}^n\}$, \mathcal{L} is countable so that we may write $\mathcal{L} = \{z_1, z_2, \dots, z_j, \dots\}$. It is obvious that \mathcal{L} forms a 1/4-lattice in \mathbb{C}^n (see [21] for the definition). For $E \subset \mathbb{C}^n$, let χ_E be the characteristic function of E . We have some absolute constant $N > 0$ such that

$$(2.4) \quad \sum_{z_j \in \mathcal{L}} \chi_{B(z_j, \frac{1}{2})}(w) \leq N \quad \text{for } w \in \mathbb{C}^n.$$

Lemma 2.4. *For $0 < p \leq 1$ there is some constant C (depending only on p and n) such that for any domain $\Omega \subset \mathbb{C}^n$ and $f \in H(\mathbb{C}^n)$,*

$$\left(\int_{\Omega} |f(w)e^{-\varphi(w)}| dV(w) \right)^p \leq C \int_{\Omega^+} |f(w)e^{-\varphi(w)}|^p dV(w).$$

Proof. It is trivial to see that $(u + v)^p \leq u^p + v^p$ for positive u, v and $0 < p \leq 1$. Applying Lemma 2.3 and (2.4), for $f \in H(\mathbb{C}^n)$ we have

$$\begin{aligned}
 \left(\int_{\Omega} |f(w)e^{-\varphi(w)}| dV(w) \right)^p &\leq \left(\sum_{z_j \in \mathcal{L}} \int_{\Omega \cap B(z_j, 1/4)} |f(w)e^{-\varphi(w)}| dV(w) \right)^p \\
 &\leq C \sum_{z_j \in \mathcal{L}, d(z_j, \Omega) < 1/4} \max_{|w-z_j| \leq 1/4} |f(w)e^{-\varphi(w)}|^p \\
 &\leq C \sum_{z_j \in \mathcal{L}, d(z_j, \Omega) < 1/4} \int_{|w-z_j| < 1/2} |f(w)e^{-\varphi(w)}|^p dV(w) \\
 &\leq C \int_{\Omega^+} \sum_{z_j \in \mathcal{L}, d(z_j, \Omega) < 1/4} \chi_{B(z_j, 1/2)}(w) |f(w)e^{-\varphi(w)}|^p dV(w) \\
 &\leq C \int_{\Omega^+} |f(w)e^{-\varphi(w)}|^p dV(w).
 \end{aligned}$$

It is easy to check that the constants C above depend only on p and n . \square

With the assumption that $w \mapsto TK_w$ is conjugate holomorphic, we know $\langle TK_w, K_z \rangle$ is conjugate holomorphic with w . And also, $\langle TK_z, K_w \rangle_{F_\varphi^2}$ is holomorphic with w . For $0 < p < 1$, apply Lemma 2.4 to get

$$\begin{aligned}
 \int_{\Omega} |\langle Tk_z, k_w \rangle_{F_\varphi^2}| dV(w) &= \int_{\Omega} |\langle Tk_z, K_w \rangle_{F_\varphi^2} e^{-\varphi(w)}| dV(w) \\
 &\leq C \left(\int_{\Omega^+} |\langle Tk_z, K_w \rangle_{F_\varphi^2} e^{-\varphi(w)}|^p dV(w) \right)^{\frac{1}{p}} \\
 &= C \left(\int_{\Omega^+} |\langle Tk_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \right)^{\frac{1}{p}}.
 \end{aligned}$$

And, similarly

$$\int_{\Omega} |\langle k_z, Tk_w \rangle_{F_\varphi^2}| dV(w) \leq C \left(\int_{\Omega^+} |\langle k_z, Tk_w \rangle_{F_\varphi^2}|^p dV(w) \right)^{\frac{1}{p}}.$$

These two inequalities tell us $WL_p^\varphi \subset WL_1^\varphi$ with $0 < p \leq 1$. With the relation $\langle TK_z, K_w \rangle_{F_\varphi^2} = \langle K_z, T^*K_w \rangle_{F_\varphi^2}$, we know T^* is well defined on \mathcal{D} . In [11] it is pointed out that WL_1^φ is contained in the set of all bounded operators on F_φ^p for all $1 \leq p < \infty$. When $0 < p < 1$, we have the two following lemmas.

Lemma 2.5. *For $0 < p \leq 1$, if $T \in WL_p^\varphi$ then T is bounded on F_φ^p .*

Proof. Set

$$G(T) = \max \left\{ \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w \rangle_{F_\varphi^2}|^p dV(w), \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |\langle k_z, Tk_w \rangle_{F_\varphi^2}|^p dV(w) \right\}.$$

Then, $G(T) < \infty$. For $f \in \mathcal{D}$, we have

$$(2.5) \quad Tf(z) = \langle Tf, K_z \rangle_{F_\varphi^2} = \langle f, T^*K_z \rangle_{F_\varphi^2} = \int_{\mathbb{C}^n} f(w) \langle K_w, T^*K_z \rangle_{F_\varphi^2} e^{-2\varphi(w)} dV(w).$$

Applying (2.5), Lemma 2.2 (estimate (3)) and Lemma 2.4 with $\Omega = \mathbb{C}^n$ to have

$$\begin{aligned} |Tf(z)e^{-\varphi(z)}|^p &\leq C \left(\int_{\mathbb{C}^n} |f(w)\langle K_w, T^*k_z \rangle_{F_\varphi^2}| e^{-2\varphi(w)} dV(w) \right)^p \\ &\leq C \left(\int_{\mathbb{C}^n} |f(w)\langle Tk_w, k_z \rangle_{F_\varphi^2}| e^{-\varphi(w)} dV(w) \right)^p \\ &\leq C \int_{\mathbb{C}^n} |f(w)\langle Tk_w, k_z \rangle_{F_\varphi^2} e^{-\varphi(w)}|^p dV(w). \end{aligned}$$

Now, integrate both sides over \mathbb{C}^n , and apply Fubini's Theorem to obtain

$$\|Tf\|_{p,\varphi}^p \leq C \int_{\mathbb{C}^n} |f(w)e^{-\varphi(w)}|^p dV(w) \int_{\mathbb{C}^n} |\langle Tk_w, k_z \rangle_{F_\varphi^2}|^p dV(z) = CG(T)\|f\|_{p,\varphi}^p.$$

When $0 < p < 1$, although F_φ^p is only a Fréchet space, with $P|_{F_\varphi^p} = \text{Id}$ we know that \mathcal{D} is dense in F_φ^p . Therefore, T is bounded on F_φ^p . \square

Isralowitz, Mitkovski and the third author demonstrated in [11] that WL_1^φ is a *-algebra. Lemma 2.6 tells us WL_p^φ is closed under the F_φ^p operator norm while $0 < p \leq 1$.

Lemma 2.6. *For $0 < p \leq 1$, WL_p^φ is closed under the operator norm on F_φ^p .*

Proof. We only need to prove $\overline{\text{WL}}_p^\varphi = \text{WL}_p^\varphi$. For $T \in \overline{\text{WL}}_p^\varphi$ we show

$$\limsup_{r \rightarrow \infty} \int_{z \in \mathbb{C}^n} \int_{B(z,r)^c} |\langle Tk_z, k_w \rangle_{F_\varphi^2}|^p dV(w) = 0.$$

In fact, for any $\varepsilon > 0$ we have some $A_\varepsilon \in \text{WL}_p^\varphi$ such that $\|T - A_\varepsilon\|_{F_\varphi^p \rightarrow F_\varphi^p} < \varepsilon$. For this A_ε we have some r such that

$$\sup_{z \in \mathbb{C}^n} \int_{B(z,r)^c} |\langle A_\varepsilon k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) < \varepsilon.$$

This implies

$$\begin{aligned} &\int_{B(z,r)^c} |\langle Tk_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \\ &\leq \int_{B(z,r)^c} |\langle (T - A_\varepsilon)k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) + \int_{B(z,r)^c} |\langle A_\varepsilon k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \\ &\leq \int_{\mathbb{C}^n} |\langle (T - A_\varepsilon)k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) + \int_{B(z,r)^c} |\langle A_\varepsilon k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \\ &= \|(T - A_\varepsilon)k_z\|_{p,\varphi}^p + \int_{B(z,r)^c} |\langle A_\varepsilon k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \\ &\leq \|T - A_\varepsilon\|_{F_\varphi^p \rightarrow F_\varphi^p}^p \|k_z\|_{p,\varphi}^p + \int_{B(z,r)^c} |\langle A_\varepsilon k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) \\ &< C\varepsilon, \end{aligned}$$

where the constant C does not depend on ε . \square

To characterize the compactness of those $T \in \text{WL}_p^\varphi$ in the case $0 < p \leq 1$, we will borrow ideas from [17] and will be approximating a given operator $T \in \text{WL}_p^\varphi$ by infinite sums of well

localized pieces. To get this approximation we need the following covering lemma from [11]. See also [2, Lemma 3.1].

Lemma 2.7. *There exists a positive integer N such that for each $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}_{j=1}^\infty$ of \mathbb{C}^n by disjoint Borel sets satisfying:*

- (1) *every point of \mathbb{C}^n belongs to at most N of the sets $G_j = \{z : d(z, F_j) \leq r\}$;*
- (2) *diam $F_j \leq 2r$ for every j .*

Notice that, if $r > 1$, we have some absolute constant $N > 0$ such that

$$(2.6) \quad \sum_{j=1}^{\infty} \chi_{F_j^+}(w) \leq \sum_{j=1}^{\infty} \chi_{G_j}(w) \leq \sum_{j=1}^{\infty} \chi_{G_j^+}(w) \leq N \quad \forall w \in \mathbb{C}^n.$$

This covering \mathcal{F}_r can also be chosen in a simple way. For example, let $\{a_j\}$ be an enumeration of the lattice $\frac{2r}{\sqrt{n}}\mathbb{Z}^{2n}$. And take F_j to be the cube with centers a_j , side-length $\frac{2r}{\sqrt{n}}$ and half of the boundary so that $\cup_{j=1}^\infty F_j = \mathbb{C}^n$, $F_j \cap F_k = \emptyset$ if $j \neq k$.

Proposition 2.8. *Let $0 < p \leq 1$ and $T \in WL_p^\varphi$. Then for every $\varepsilon > 0$, there is some $r > 0$ sufficiently large such that, for the covering $\{F_j\}_{j=1}^\infty$ and $\{G_j\}_{j=1}^\infty$ (associated to r) from Lemma 2.7, we have*

$$(2.7) \quad \left\| T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right) \right\|_{F_\varphi^p \rightarrow F_\varphi^p} < \varepsilon.$$

Proof. Let $T \in WL_p^\varphi$ be given. For $\varepsilon > 0$, we have some $r > 0$ sufficiently large (we may assume $r > 10$) such that

$$\int_{B(z, r-1)^c} |\langle T k_z, k_w \rangle_{F_\varphi^2}|^p dV(w) < \varepsilon \quad \text{and} \quad \int_{B(z, r-1)^c} |\langle k_z, T k_w \rangle_{F_\varphi^2}|^p dV(w) < \varepsilon.$$

Take $\{F_j\}_{j=1}^\infty$ and $\{G_j\}_{j=1}^\infty$ to be as in Lemma 2.7 with r . For $w \in F_j^+$ and $u \in G_j^c$ we have $|u - w| > r - 1$, then $u \in B(w, r - 1)^c$. That is $G_j^c \subset B(w, r - 1)^c$ whenever $w \in F_j^+$. Hence, for $w \in F_j^+$,

$$\begin{aligned} \left| \left(T P M_{\chi_{G_j^c}} f \right) (w) \right| &= \left| \langle P M_{\chi_{G_j^c}} f, T^* K_w \rangle_{F_\varphi^2} \right| \\ &= \left| \langle M_{\chi_{G_j^c}} f, T^* K_w \rangle_{F_\varphi^2} \right| \\ &\leq \int_{G_j^c} |f(u)| \left| \langle K_u, T^* K_w \rangle_{F_\varphi^2} \right| e^{-2\varphi(u)} dV(u) \\ &\leq \int_{B(w, r-1)^c} |f(u)| \left| \langle K_u, T^* K_w \rangle_{F_\varphi^2} \right| e^{-2\varphi(u)} dV(u). \end{aligned}$$

Set $S = TP - \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. Then

$$\begin{aligned}
& |PSf(z)|^p \\
& \leq \left(\int_{\mathbb{C}^n} |Sf(w)K(z, w)e^{-2\varphi(w)}| dV(w) \right)^p \\
& = \left(\int_{\mathbb{C}^n} \left| \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} f(w) \right| |K(z, w)| e^{-2\varphi(w)} dV(w) \right)^p \\
& = \left(\sum_{j=1}^{\infty} \int_{\mathbb{C}^n} \left| M_{\chi_{F_j}} TPM_{\chi_{G_j}} f(w) \right| |K(z, w)| e^{-2\varphi(w)} dV(w) \right)^p \\
& \leq \sum_{j=1}^{\infty} \left(\int_{F_j} \left| TPM_{\chi_{G_j}} f(w) \right| |K(z, w)| e^{-2\varphi(w)} dV(w) \right)^p.
\end{aligned}$$

Notice that $|K(z, w)| = |K(w, z)|$, applying Lemma 2.4 twice to above, we get

$$\begin{aligned}
& |PSf(z)|^p \\
& \leq C \sum_{j=1}^{\infty} \int_{F_j^+} \left| TPM_{\chi_{G_j}} f(w) \right|^p |K(w, z)|^p e^{-2p\varphi(w)} dV(w) \\
& \leq C \sum_{j=1}^{\infty} \int_{F_j^+} |K(w, z)|^p e^{-p\varphi(w)} \left(\int_{B(w, r-1)^c} |f(u)e^{-\varphi(u)} \langle Tk_u, k_w \rangle_{F_\varphi^2}| dV(u) \right)^p dV(w) \\
& \leq C \sum_{j=1}^{\infty} \int_{F_j^+} |K(w, z)|^p e^{-p\varphi(w)} \left(\int_{B(w, r-2)^c} |f(u)e^{-\varphi(u)} \langle Tk_u, k_w \rangle_{F_\varphi^2}|^p dV(u) \right) dV(w).
\end{aligned}$$

By Fubini's Theorem and (2.6), we get $\|PSf\|_{p, \varphi}^p$ is no more than

$$\begin{aligned}
& C \sum_{j=1}^{\infty} \int_{\mathbb{C}^n} |f(u)e^{-\varphi(u)}|^p \int_{F_j^+} \chi_{B(u, r-1)^c}(w) \left| \langle Tk_u, k_w \rangle_{F_\varphi^2} \right|^p e^{-p\varphi(w)} \\
& \quad \times \int_{\mathbb{C}^n} |K(w, z)|^p e^{-p\varphi(z)} dV(z) dV(w) dV(u) \\
& \leq CN \int_{\mathbb{C}^n} |f(u)e^{-\varphi(u)}|^p \left(\int_{B(u, r-1)^c} \left| \langle Tk_u, k_w \rangle_{F_\varphi^2} \right|^p dV(w) \right) dV(u) \\
& \leq C\varepsilon \|f\|_{p, \varphi}^p.
\end{aligned}$$

The constants C above are independent of ε . Notice that $PTP = T$ on F_φ^p , so $PS = T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right)$ is well defined on F_φ^p and the estimate (2.7) is proved under the restriction that $T \in \text{WL}_p^\varphi$. \square

Lemma 2.9. *Given $0 < p \leq 1$, there is some constant C such that for all bounded linear operator T on F_φ^p and $\{F_j\}_{j=1}^\infty, \{G_j\}_{j=1}^\infty$ associated to $r > 1$ as in Lemma 2.7 and each*

positive integer m , we have

$$(2.8) \quad \limsup_{m \rightarrow \infty} \|PT_m\|_{F_\varphi^p \rightarrow F_\varphi^p} \leq C \limsup_{m \rightarrow \infty} \sup_{w \in \cup_{j>m} G_j^+} \|Tk_w\|_{p,\varphi},$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} T P M_{\chi_{G_j}}$.

Proof. First, we are going to show

$$(2.9) \quad \sup_{f \in F_\varphi^p \setminus \{0\}} \left\| TP \left(\frac{\chi_{G_j} f}{\|\chi_{G_j^+} f\|_{p,\varphi}} \right) \right\|_{p,\varphi} \leq C \sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}.$$

In fact, given $f \in F_\varphi^p$ not identically zero, set

$$g_j = P \left(\frac{\chi_{G_j} f}{\|\chi_{G_j^+} f\|_{p,\varphi}} \right).$$

Then

$$g_j(z) = \int_{G_j} \frac{f(w) \overline{K(z,w)} e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w).$$

It is trivial to see that $g_j \in F_\varphi^p$ because of the compactness of $\overline{G_j}$. Since T is bounded on F_φ^p , then

$$|T(g_j)(z)| \leq \int_{G_j} \frac{|f(w)| |TK_w(z)| e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w)$$

Note that TK_w is conjugate holomorphic respecting to w . From Lemma 2.4 we have

$$\begin{aligned} \|T(g_j)\|_{p,\varphi}^p &\leq \int_{\mathbb{C}^n} \left(\int_{G_j} \frac{|f(w)| |\overline{TK_w(z)}| e^{-2\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w) \right)^p e^{-p\varphi(z)} dV(z) \\ &\leq C \int_{\mathbb{C}^n} \left(\int_{G_j^+} \frac{|f(w)|^p |\overline{TK_w(z)}|^p e^{-2p\varphi(w)}}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} dV(w) \right) e^{-p\varphi(z)} dV(z) \\ &\leq C \int_{G_j^+} \frac{|f(w) e^{-\varphi(w)}|^p}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} \left(\int_{\mathbb{C}^n} |TK_w(z) e^{-\varphi(z)}|^p dV(z) \right) dV(w) \\ &\leq C \sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}^p \int_{G_j^+} \frac{|f(w) e^{-\varphi(w)}|^p dV(w)}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} \\ &= C \sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}^p. \end{aligned}$$

This gives (2.9). To prove (2.8), we have from Lemma 2.4 that

$$\begin{aligned}
& \left| P \left(\chi_{F_j}(\cdot) \int_{G_j} \frac{\langle f, k_w \rangle_{F_\varphi^2}(Tk_w)(\cdot)}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w) \right) (z) \right|^p \\
& \leq \left| \int_{F_j} K(z, u) e^{-2\varphi(u)} \int_{G_j} \frac{\langle f, k_w \rangle_{F_\varphi^2}(Tk_w)(u)}{\|\chi_{G_j^+} f\|_{p,\varphi}} dV(w) dV(u) \right|^p \\
& \leq C \int_{F_j^+} |K(z, u)|^p e^{-2p\varphi(u)} \left| \int_{G_j} \frac{\langle f, k_w \rangle_{F_\varphi^2}(Tk_w)(u)}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} dV(w) \right|^p dV(u) \\
& \leq C \int_{F_j^+} |K(z, u)|^p e^{-2p\varphi(u)} \left(\int_{G_j^+} \frac{|\langle f, k_w \rangle_{F_\varphi^2}|^p |(Tk_w)(u)|^p}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} dV(w) \right) dV(u).
\end{aligned}$$

Hence, integrating both sides and interchanging the order of integrations we obtain

$$\begin{aligned}
& \left\| P \left(M_{\chi_{F_j}} Tg_j \right) \right\|_{p,\varphi}^p \\
& \leq C \int_{G_j^+} \frac{|\langle f, k_w \rangle_{F_\varphi^2}|^p}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} \int_{F_j^+} |(Tk_w)(u)|^p e^{-2p\varphi(u)} \int_{\mathbb{C}^n} |K(z, u)|^p e^{-p\varphi(z)} dV(z) dV(u) dV(w) \\
& \leq C \int_{G_j^+} \frac{|\langle f, k_w \rangle_{F_\varphi^2}|^p}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} \left(\int_{F_j^+} |(Tk_w)(u)|^p e^{-p\varphi(u)} dV(u) \right) dV(w).
\end{aligned}$$

This gives

$$\left\| P \left(M_{\chi_{F_j}} Tg_j \right) \right\|_{p,\varphi}^p \leq C \left(\sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}^p \right) \int_{G_j^+} \frac{|\langle f, k_w \rangle_{F_\varphi^2}|^p}{\|\chi_{G_j^+} f\|_{p,\varphi}^p} dV(w) = C \sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}^p.$$

Therefore, (2.6) yields

$$\begin{aligned}
\|PT_m f\|_{p,\varphi}^p & \leq \sum_{j>m} \|PM_{\chi_{F_j}} TPM_{\chi_{G_j}} f\|_{p,\varphi}^p \\
& = \sum_{j>m} \|P \left(M_{\chi_{F_j}} Tg_j \right)\|_{p,\varphi}^p \|\chi_{G_j^+} f\|_{p,\varphi}^p \\
& \leq C \sum_{j>m} \sup_{w \in G_j^+} \|Tk_w\|_{p,\varphi}^p \|\chi_{G_j^+} f\|_{p,\varphi}^p \\
& \leq CN \left(\sup_{w \in \cup_{j>m} G_j^+} \|Tk_w\|_{p,\varphi}^p \right) \|f\|_{p,\varphi}^p.
\end{aligned}$$

From this, (2.8) follows. \square

In the case of $1 \leq p < \infty$, the projection P is bounded from L_φ^p to F_φ^p , and so is PM_{χ_E} when $E \subset \mathbb{C}^n$ is measurable. But P is not bounded on L_φ^p if $0 < p < 1$. The following lemma, Lemma 2.10, says PM_{χ_E} is still bounded on F_φ^p .

Lemma 2.10. *Suppose $0 < p \leq 1$. There exists some constant C such that for any domain E in \mathbb{C}^n we have $\|PM_{\chi_E}\|_{F_\varphi^p \rightarrow F_\varphi^p} \leq C$.*

Proof. Suppose $E \in \mathbb{C}^n$ is a domain. For $f \in F_\varphi^p$, we have $|f(w)K(z, w)| = |f(w)K(w, z)|$. Lemma 2.4 and Lemma 2.2, estimate (3) gives

$$\begin{aligned} \|PM_{\chi_E} f\|_{p, \varphi}^p &= \int_{\mathbb{C}^n} \left| \int_E f(w)K(z, w)e^{-2\varphi(w)} dV(w) \right|^p e^{-p\varphi(z)} dV(z) \\ &\leq C \int_{\mathbb{C}^n} \left(\int_{E^+} |f(w)K(w, z)e^{-2\varphi(w)}|^p dV(w) \right) e^{-p\varphi(z)} dV(z) \\ &= C \int_{E^+} |f(w)|^p e^{-2p\varphi(w)} \left(\int_{\mathbb{C}^n} |K(w, z)e^{-\varphi(z)}|^p dV(z) \right) dV(w). \\ &\leq C \|f\|_{p, \varphi}^p. \end{aligned}$$

□

Lemma 2.11. *Suppose $0 < p \leq 1$ and $T \in \mathcal{K}(F_\varphi^p)$. Then*

$$\lim_{R \rightarrow \infty} \|PM_{\chi_{B(0, R)}} T - T\|_{F_\varphi^p \rightarrow F_\varphi^p} = 0.$$

Proof. Notice that, $PT = T$ on F_φ^p . For $f \in F_\varphi^p$ with $\|f\|_{p, \varphi} \leq 1$, we get

$$\begin{aligned} \left\| \left(PM_{\chi_{B(0, R)}} T - T \right) f \right\|_{p, \varphi}^p &= \left\| \left(PM_{\chi_{B(0, R)}} T - PT \right) f \right\|_{p, \varphi}^p \\ &= \int_{\mathbb{C}^n} \left| \int_{|w| \geq R} Tf(w)K(z, w)e^{-2\varphi(w)} dV(w) \right|^p e^{-p\varphi(z)} dV(z). \end{aligned}$$

Then by Lemma 2.4,

$$\begin{aligned} \left\| \left(PM_{\chi_{B(0, R)}} T - T \right) f \right\|_{p, \varphi}^p &\leq C \int_{\mathbb{C}^n} \left(\int_{|w| \geq R-1} |Tf(w)K(w, z)e^{-2\varphi(w)}|^p dV(w) \right) e^{-p\varphi(z)} dV(z) \\ &= \int_{|w| \geq R-1} |Tf(w)e^{-2\varphi(w)}|^p \left(\int_{\mathbb{C}^n} |K(w, z)|^p e^{-p\varphi(z)} dV(z) \right) dV(w) \\ &\leq C \int_{|w| \geq R-1} |Tf(w)e^{-\varphi(w)}|^p dV(w). \end{aligned}$$

Since $T \in \mathcal{K}(F_\varphi^p)$, $\{Tf : f \in F_\varphi^p \text{ with } \|f\|_{p, \varphi} \leq 1\} \subset F_\varphi^p$ is relatively compact. By [8, Lemma 3.2], for each $\varepsilon > 0$ there is some $R > 0$ such that

$$\sup_{f \in F_\varphi^p, \|f\|_{p, \varphi} \leq 1} \int_{|w| > R-1} |Tf(w)e^{-\varphi(w)}|^p dV(w) < \varepsilon^p.$$

Therefore,

$$\|PM_{\chi_{B(0, R)}} T - T\|_{F_\varphi^p \rightarrow F_\varphi^p} = \sup_{f \in F_\varphi^p, \|f\|_{p, \varphi} \leq 1} \left\| \left(PM_{\chi_{B(0, R)}} T - T \right) f \right\|_{p, \varphi} < C\varepsilon,$$

where C is independent of ε . □

Lemma 2.12. *Suppose $0 < p \leq 1$. Then for T bounded on F_φ^p we have*

$$\|T\|_{e, F_\varphi^p} \simeq \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0, R)^c}} T\|_{F_\varphi^p \rightarrow F_\varphi^p}.$$

Proof. For any $R > 0$, $PM_{\chi_{B(0,R)}}$ is a Toeplitz operator induced by $\chi_{B(0,R)}$, Lemma 2.9 from [8] tells us it is compact on F_φ^p . Given T bounded on F_φ^p , $PM_{\chi_{B(0,R)}}T$ is compact. Thus,

$$\|T\|_{e, F_\varphi^p} \leq \|T - PM_{\chi_{B(0,R)}}T\|_{F_\varphi^p \rightarrow F_\varphi^p}.$$

This yields

$$\|T\|_{e, F_\varphi^p} \leq \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}T\|_{F_\varphi^p \rightarrow F_\varphi^p}.$$

On the other hand, for any $A \in \mathcal{K}(F_\varphi^p)$, Lemma 2.11 shows

$$\limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}A\|_{F_\varphi^p \rightarrow F_\varphi^p} = 0.$$

From Lemma 2.10, we know

$$\begin{aligned} \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}T\|_{F_\varphi^p \rightarrow F_\varphi^p} &= \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}(T - A)\|_{F_\varphi^p \rightarrow F_\varphi^p} \\ &\leq \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}\|_{F_\varphi^p \rightarrow F_\varphi^p} \|T - A\|_{F_\varphi^p \rightarrow F_\varphi^p} \\ &\leq C \|T - A\|_{F_\varphi^p \rightarrow F_\varphi^p}. \end{aligned}$$

Hence,

$$\limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}^c}T\|_{F_\varphi^p \rightarrow F_\varphi^p} \leq C \|T\|_{e, F_\varphi^p}.$$

□

Now we are in the position to characterize those compact operators in WL_p^φ with $0 < p \leq 1$, which extends the main results in [10, 11, 20] to the small exponential case.

Theorem 2.13. *Let $0 < p \leq 1$ and $T \in WL_p^\varphi$. The following statements are equivalent:*

- (A) $T \in \mathcal{K}(F_\varphi^p)$;
- (B) $\lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0$ for any $r > 0$;
- (C) $\lim_{z \rightarrow \infty} \sup_{w \in \mathbb{C}^n} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0$;
- (D) $\lim_{z \rightarrow \infty} \|Tk_z\|_{p, \varphi} = 0$.

Proof. It is trivial that (C) \Rightarrow (B). We will show the implication (B) \Rightarrow (D) under the hypothesis $T \in WL_p^\varphi$. In fact, for any $\varepsilon > 0$, by (2.2) we have some $r > 0$ such that

$$\sup_{z \in \mathbb{C}^n} \int_{B(z,r)^c} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^p dV(w) < \varepsilon.$$

Combining the above inequality with (B), we get

$$\begin{aligned}
 \|Tk_z\|_{p,\varphi}^p &= \int_{\mathbb{C}^n} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^p dV(w) \\
 &= \left(\int_{B(z,r)^c} + \int_{B(z,r)} \right) \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^p dV(w) \\
 &\leq \varepsilon + A(B(z,r)) \sup_{w \in B(z,r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right|^p \\
 &\leq \varepsilon + Cr^{2n} \left(\sup_{w \in B(z,r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| \right)^p \\
 &< 2\varepsilon
 \end{aligned}$$

whenever $|z|$ is sufficiently large. Therefore, (B) implies (D).

Suppose T satisfies (D). By Lemma 2.3 we know

$$(2.10) \quad \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = |Tk_z(w)e^{-\varphi(w)}| \leq C \left(\int_{B(w,1)} |Tk_z(u)e^{-\varphi(u)}|^p dV(u) \right)^{\frac{1}{p}} \leq C \|Tk_z\|_{p,\varphi}.$$

Then,

$$\sup_{w \in \mathbb{C}^n} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| \leq C \|Tk_z\|_{p,\varphi}$$

which gives the implication (D) \Rightarrow (C).

To prove (D) \Rightarrow (A), given $\varepsilon > 0$ we pick some $r > 10$ with sets $\{F_j\}_j$ and $\{G_j\}_j$ as in Proposition 2.8 so that

$$\left\| T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right) \right\|_{F_\varphi^p \rightarrow F_\varphi^p} < \varepsilon.$$

For each positive integer m , set $T_m = \sum_{j>m} M_{\chi_{F_j}} T P M_{\chi_{G_j}}$. Since $P \left(\sum_{j=1}^m M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right)$ is compact on F_φ^p , we get

$$(2.11) \quad \|T\|_{e, F_\varphi^p}^p \leq \left\| T - P \left(\sum_{j=1}^m M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right) \right\|_{F_\varphi^p \rightarrow F_\varphi^p}^p < \varepsilon^p + \|PT_m\|_{F_\varphi^p \rightarrow F_\varphi^p}^p.$$

Suppose T satisfies (D), then there exists $t > 0$ such that $\|Tk_z\|_{p,\varphi} < \varepsilon$ for $|z| \geq t$. Notice that, $\cup_{j>m} G_j^+ \subset B(0,t)^c$ whenever m is large enough. So, (2.8) in Lemma 2.9 and (2.11) imply $\|T\|_{e, F_\varphi^p} = 0$ which gives the compactness of T .

To finish our proof, we only need to prove the implication (A) \Rightarrow (B). Given $T \in \mathcal{K}(F_\varphi^p)$, Lemma 2.11 tells us

$$(2.12) \quad \lim_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)}} T - T\|_{F_\varphi^p \rightarrow F_\varphi^p} = 0.$$

First, we claim that

$$(2.13) \quad \lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} \left| \langle PM_{\chi_{B(0,R)}} Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0.$$

In fact, Lemma 2.10 shows $PM_{\chi_{B(0,R)}}Tk_z \in F_\varphi^p \subset F_\varphi^2$, we obtain

$$\begin{aligned}
\left| \langle PM_{\chi_{B(0,R)}}Tk_z, k_w \rangle_{F_\varphi^2} \right| &= \left| \langle M_{\chi_{B(0,R)}}Tk_z, k_w \rangle_{F_\varphi^2} \right| \\
&\leq \int_{B(0,R)} |Tk_z(u)\overline{k_w(u)}| e^{-2\varphi(u)} dV(u) \\
&\leq \|Tk_z\|_{\infty,\varphi} \int_{B(0,R)} |k_w(u)| e^{-\varphi(u)} dV(u) \\
&\leq C \|Tk_z\|_{p,\varphi} \sup_{u \in B(0,R)} |k_w(u)| e^{-\varphi(u)} \\
&\leq C \|T\|_{F_\varphi^p \rightarrow F_\varphi^p} \|k_z\|_{p,\varphi} e^{-\theta|w|} \\
&\leq C e^{-\theta|w|},
\end{aligned}$$

where the constants C are independent of z and w . Hence, (2.13) is true. Using (3) in Lemma 2.2 and (2.12) to get that

$$\begin{aligned}
\left| \left\langle \left(T - PM_{\chi_{B(0,R)}}T \right) k_z, k_w \right\rangle_{F_\varphi^2} \right| &\leq \left\| \left(T - PM_{\chi_{B(0,R)}}T \right) k_z \right\|_{\infty,\varphi} \|k_w\|_{1,\varphi} \\
&\leq C \left\| \left(T - PM_{\chi_{B(0,R)}}T \right) k_z \right\|_{p,\varphi} \\
&\leq C \|T - PM_{\chi_{B(0,R)}}T\|_{F_\varphi^p \rightarrow F_\varphi^p} \|k_z\|_{p,\varphi} \\
&\leq C \|T - PM_{\chi_{B(0,R)}}T\|_{F_\varphi^p \rightarrow F_\varphi^p} \rightarrow 0
\end{aligned}$$

as $R \rightarrow \infty$. Combining the above with (2.13), we obtain

$$\sup_{w \in B(z,r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| \leq \sup_{w \in B(z,r)} \left| \langle PM_{\chi_{B(0,R)}}Tk_z, k_w \rangle_{F_\varphi^2} \right| + \sup_{w \in B(z,r)} \left| \left\langle \left(T - PM_{\chi_{B(0,R)}}T \right) k_z, k_w \right\rangle_{F_\varphi^2} \right|$$

goes to 0 as $z \rightarrow \infty$. \square

If $1 < p < \infty$, $k_z \rightarrow 0$ weakly on F_φ^p , which implies $\lim_{z \rightarrow \infty} \|T(k_z)\|_{p,\varphi} = 0$ for $T \in \mathcal{K}(F_\varphi^p)$. Theorem 1.2 in [11] tells us that the equivalence from (A) to (D) remains true for $T \in \text{WL}_p^\varphi$ if $1 < p < \infty$. For our later applications, we exhibit the following result.

Theorem 2.14. *Let $0 < p < \infty$ and $T \in \text{WL}_p^\varphi$. The following statements are equivalent:*

- (A) $T \in \mathcal{K}(F_\varphi^p)$;
- (B) $\lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0$ for any $r > 0$;
- (C) $\lim_{z \rightarrow \infty} \sup_{w \in \mathbb{C}^n} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0$;
- (D) $\lim_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi} = 0$.

From Theorem 2.13, it is natural to ask whether the essential norm of $T \in \text{WL}_p^\varphi$ can be dominated by its behavior on normalized reproducing kernel k_z ? This problem has attracted much interest, see [10, Section 6] for example. Our Theorem 2.15 says the answer is affirmative when $0 < p \leq 1$.

Theorem 2.15. *Suppose $0 < p \leq 1$. Then for $T \in \text{WL}_p^\varphi$ we have*

$$(2.14) \quad \|T\|_{e,F_\varphi^p} \simeq \limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi}.$$

Proof. Suppose $T \in \text{WL}_p^\varphi$. From Lemma 2.5 we know T is bounded on F_φ^p which implies $\limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi} < \infty$. By Theorem 2.13, $\|T\|_{e,F_\varphi^p} = 0$ if $\limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi} = 0$. So, we may assume $\limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi} = \varepsilon_1 > 0$. From Proposition 2.8, we have two sequences of sets $\{F_j\}_j$ and $\{G_j\}_j$ so that

$$\left\| T - P \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right) \right\|_{F_\varphi^p \rightarrow F_\varphi^p} < \varepsilon_1.$$

Then, for $m = 1, 2, \dots$, from (2.8) and (2.11) we have

$$\|T\|_{e,F_\varphi^p} \leq \varepsilon_1 + \|PT_m\|_{F_\varphi^p \rightarrow F_\varphi^p} \leq \varepsilon_1 + C \sup_{z \in \cup_{j>m} G_j^+} \|Tk_z\|_{p,\varphi}.$$

Since $0 < p \leq 1$, Lemma 2.9 tells us that the constants C above do not depend on the precise choice of $\{F_j\}_j$ and $\{G_j\}_j$, and hence do not depend on T . Let $m \rightarrow \infty$, we have the desired estimate

$$\|T\|_{e,F_\varphi^p}^p \leq \varepsilon_1^p + C \limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi}^p = C \limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi}^p.$$

On the other hand, fixed $R > 0$, notice that $PM_{\chi_{B(0,R)}}$ is a Toeplitz operator induced by $\chi_{B(0,R)}$, which is a bounded function. So $PM_{\chi_{B(0,R)}} \in \text{WL}_p^\varphi$ and $PM_{\chi_{B(0,R)}} T \in \text{WL}_p^\varphi$, because WL_p^φ is a algebra. Since $PM_{\chi_{B(0,R)}}$ is compact and T is bounded on F_φ^p (see Lemma 2.5), we get that $PM_{\chi_{B(0,R)}} T$ is compact on F_φ^p . Theorem 2.13 tells us

$$(2.15) \quad \lim_{z \rightarrow \infty} \left\| PM_{\chi_{B(0,R)}} Tk_z \right\|_{p,\varphi} = 0.$$

Therefore, Lemma 2.12, (2.15) and the fact that $PT = T$ yield

$$\begin{aligned} \|T\|_{e,F_\varphi^p}^p &\simeq \limsup_{R \rightarrow \infty} \|PM_{\chi_{B(0,R)^c}} T\|_{F_\varphi^p \rightarrow F_\varphi^p}^p \\ &\geq \limsup_{R \rightarrow \infty} \limsup_{z \rightarrow \infty} \left\| PM_{\chi_{B(0,R)^c}} Tk_z \right\|_{p,\varphi}^p \\ &\geq \limsup_{R \rightarrow \infty} \limsup_{z \rightarrow \infty} \left(\|Tk_z\|_{p,\varphi}^p - \left\| PM_{\chi_{B(0,R)}} Tk_z \right\|_{p,\varphi}^p \right) \\ &\geq \limsup_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi}^p. \end{aligned}$$

□

3. TOEPLITZ OPERATORS WITH BMO SYMBOLS

In this section, we are going to discuss the characterizations on Toeplitz operators with BMO symbols. First, we will characterize the boundedness (and the compactness) of Toeplitz operators T_f on F_φ^p with BMO symbols f . Furthermore, we will characterize those compact operators on F_φ^p which are in the algebra generated by bounded Toeplitz operators with BMO symbols. For this purpose, we need some more auxiliary function spaces.

Fixed $r > 0$, recall that $B(\cdot, r) = \{w \in \mathbb{C}^n : |w - \cdot| < r\}$. Given a locally Lebesgue integrable function f on \mathbb{C}^n (written as $f \in L_{loc}^1(\mathbb{C}^n)$), write

$$\omega_r(f)(\cdot) = \sup \{|f(w) - f(\cdot)| : w \in B(\cdot, r)\}$$

and

$$MO_r(f)(\cdot) = \frac{1}{V(B(\cdot, r))} \int_{B(\cdot, r)} |f - \widehat{f}_r(\cdot)| dV$$

where

$$\widehat{f}_r(\cdot) = \frac{1}{V(B(\cdot, r))} \int_{B(\cdot, r)} f dV.$$

For f on \mathbb{C}^n with $f(\cdot)|k_z(\cdot)|^2 \in L^1_\varphi$ for all $z \in \mathbb{C}^n$, the Berezin transform of f is defined as

$$\widetilde{f}(z) = \int_{\mathbb{C}^n} f(w) |k_z(w)|^2 e^{-2\varphi(w)} dV(w).$$

Let BO_r be the collection of all continuous functions f on \mathbb{C}^n such that $\omega_r(f)$ is bounded. We use BA_r and BMO_r to denote respectively the set of all $f \in L^1_{loc}(\mathbb{C}^n)$ such that $|\widehat{f}|_r$ and $MO_r(f)$ are bounded on \mathbb{C}^n . The space BMO is the family of all measurable function f on \mathbb{C}^n satisfying $f(\cdot)|k_z(\cdot)|^2 \in L^1_\varphi$ for $z \in \mathbb{C}^n$ and

$$\|f\|_{BMO} = \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |f(w) - \widetilde{f}(z)| |k_z(w)|^2 e^{-2\varphi(w)} dV(w) < \infty.$$

By Lemma 3.33 in [21], we obtain that the spaces BO_r and BA_r are independent of r , they will be denoted as BO and BA below. The next lemma says BMO_r is independent of r as well.

Lemma 3.1. *Suppose $f \in L^1_{loc}(\mathbb{C}^n)$. The following three statements are equivalent:*

- (A) $f \in BMO_r$ for some (or any) $r > 0$;
- (B) $f \in BMO$;
- (C) $f = f_1 + f_2$, where $f_1 \in BA$ and $f_2 \in BO$.

Proof. For $n = 1$ and $\varphi(z) = \frac{\alpha}{2}|z|^2$, this is Theorem 3.34 from [21]. For general n and φ satisfying $dd^c\varphi \simeq \omega_0$, the proof can be carried out as that of [21] with a little modification. The details will be omitted here. \square

For $f \in BMO$, say $f = f_1 + f_2$ with $f_1 \in BA$ and $f_2 \in BO$, similar to [7, Lemma 4.1], we know the Toeplitz operator T_{f_1} is well defined on F^p_φ . From [21], $|f_2(z)| \leq a|z| + b$ with constants $a, b > 0$, T_{f_2} is also well defined on F^p_φ . Thus, T_f is well defined on F^p_φ , where $0 < p < \infty$. Moreover,

$$(3.1) \quad \langle T_f k_z, k_w \rangle = \int_{\mathbb{C}^n} k_z(u) \overline{k_w(u)} f(u) e^{-2\varphi(u)} dV(u).$$

Coburn, Isralowitz and Li [5] proved that T_f ($f \in BMO$) is compact on the classical Fock space $F^2_{1/2}$ if and only if the Berezin transform \widetilde{f} vanished at the infinity. The first two authors extended this result to the setting of $F^p_{1/2}$ with $0 < p < \infty$ in [8]. Under the assumption that S is a linear combination of operators of form $T_{f_1} \cdots T_{f_m}$ with each function f_j satisfying $|\widetilde{f}_j|$ bounded, Isralowitz proved that S is compact on $F^2_{1/2}$ if and only if \widetilde{S} vanishes at the infinity, see [9] for details. In all these references, the Weyl unitary operators acting on F^2_α by $W_z f(\cdot) = k_z f(\cdot - z)$ and the involutive unitary operators $U_z f(\cdot) = k_z f(z - \cdot)$ play as a very crucial role. Unfortunately, there are not these kinds of unitary operators on our generalized Fock space F^p_φ .

We will use \mathcal{B}_p^φ to denote the collection of all linear combination of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$, where each function $f_j \in \text{BMO}$ and \tilde{f}_j is bounded on \mathbb{C}^n .

Theorem 3.2. *Let $0 < p < \infty$.*

(A) *If $f \in \text{BMO}$, then T_f is bounded on F_φ^p if and only if \tilde{f} is bounded on \mathbb{C}^n ; T_f is compact on F_φ^p if and only if*

$$(3.2) \quad \lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} \left| \langle T_f k_z, k_w \rangle_{F_\varphi^2} \right| = 0 \quad \forall r > 0.$$

(B) *If $S \in \mathcal{B}_p^\varphi$, then S is compact on F_φ^p if and only if*

$$(3.3) \quad \lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} \left| \langle S k_z, k_w \rangle_{F_\varphi^2} \right| = 0 \quad \forall r > 0.$$

Proof. We claim $T_f \in \text{WL}_p^\varphi$ if $f \in \text{BMO}$ and \tilde{f} remains bounded. In fact, similar to [5, Lemma 1] it is trivial to verify

$$(3.4) \quad \sup_{z \in \mathbb{C}^n} |\tilde{f}|(z) \leq \|f\|_{\text{BMO}} + \sup_{z \in \mathbb{C}^n} |\tilde{f}(z)| < \infty.$$

By [8, Theorem 3.5], $|f|dV$ is a Fock-Carleson measure. Hence,

$$\begin{aligned} \left| \langle T_f k_z, k_w \rangle_{F_\varphi^2} \right| &\leq \int_{\mathbb{C}^n} |k_z(u)k_w(u)| e^{-2\varphi(u)} |f(u)| dV(u) \\ &\leq C \sup_{u \in \mathbb{C}^n} |\widehat{f}|_r(u) \int_{\mathbb{C}^n} |k_z(u)k_w(u)| e^{-2\varphi(u)} dV(u) \\ &\leq C \sup_{u \in \mathbb{C}^n} |\tilde{f}|(u) \int_{\mathbb{C}^n} e^{-\theta|z-u|-\theta|w-u|} dV(u) \\ &\leq C e^{-\frac{\theta}{2}|z-w|}. \end{aligned}$$

This implies $T_f \in \text{WL}_p^\varphi$ for any $p \in (0, \infty)$ (and also, T_f is strongly localized in the sense of Xia and Zheng, see [20]).

(A). Suppose $f \in \text{BMO}$. If \tilde{f} is bounded on \mathbb{C}^n , then $T_f \in \text{WL}_p^\varphi$ which implies T_f is bounded on F_φ^p for any $p \in (0, \infty)$. Conversely, the condition that T_f is bounded implies \tilde{f} is bounded, which can be proved in a standard way with $\tilde{f}(z) = \langle T_f k_z, k_z \rangle_{F_\varphi^2}$.

Now we deal with the compactness of T_f . If (3.2) holds, then $\tilde{f}(z) = \langle T_f k_z, k_z \rangle_{F_\varphi^2}$ is bounded, hence $T_f \in \text{WL}_p^\varphi$. Therefore, by (3.2) and Theorem 2.13, T_f is compact on F_φ^p for all $0 < p < \infty$. Conversely, if T_f is compact on F_φ^p for some $0 < p < \infty$. If $1 < p < \infty$, we have $\lim_{z \rightarrow \infty} \|T_f k_z\|_{p,\varphi} = 0$ because k_z tends to zero weakly, from which (3.2) follows for any $r > 0$. If $0 < p \leq 1$, we know \tilde{f} to be bounded. Then, $T_f \in \text{WL}_p^\varphi$. Now the estimate (3.2) comes from Theorem 2.13.

(B) Since each $T_{f_j} \in \text{WL}_p^\varphi$, we have $\mathcal{B}_p^\varphi \subset \text{WL}_p^\varphi$ for $0 < p < \infty$. Now the conclusion follows from Theorem 2.13. \square

As shown by Isralowitz in [10, Proposition 1.5], on the classical Fock space F_α^p the estimate (3.3) is equivalent to $\lim_{z \rightarrow \infty} \tilde{S}(z) = 0$. Therefore, Theorem 3.2 extends [5, 8].

As in [9], set BT to be the collection of all measurable functions f on \mathbb{C}^n with $|\widetilde{f}|$ bounded. As shown in the proof of Theorem 4.2, $T_f \in \text{WL}_p^\varphi$ if $f \in \text{BT}$. We have Corollary 3.3 at once.

Corollary 3.3. *Let $0 < p < \infty$, and let S be in the family all linear combination of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$, where each function $f_j \in \text{BT}$. Then S is compact on F_φ^p if and only if one of the following three statements holds:*

- (A) $\lim_{z \rightarrow \infty} \sup_{w \in B(z,r)} |\langle Tk_z, k_w \rangle| = 0$ for any $r > 0$;
- (B) $\lim_{z \rightarrow \infty} \sup_{w \in \mathbb{C}^n} |\langle Tk_z, k_w \rangle| = 0$;
- (C) $\lim_{z \rightarrow \infty} \|Tk_z\|_{p,\varphi} = 0$.

While $\varphi(z) = \frac{1}{4}|z|^2$, $p = 2$ and S is a linear combination of operators of the form $T_{f_1}T_{f_2}\cdots T_{f_m}$ with each $f_j \in \text{BT}$, Corollary 3.3 gives the main result of [9].

4. OPERATORS SATISFYING AXLER AND ZHENG'S CONDITION

In this section, we will restrict ourselves to the classical Fock space F_α^p , that is $\varphi(z) = \frac{\alpha}{2}|z|^2$ with $\alpha > 0$. We are going to characterize the boundedness and compactness of linear operators with the Axler-Zheng condition on F_α^p .

Let ϕ_z be the holomorphic self-map of \mathbb{C}^n , $\phi_z(\cdot) = z - \cdot$. U_z is the operator on F_α^p defined by $U_z f = (f \circ \phi_z)k_z$. Given some linear operator S on F_α^p , define

$$S_z = U_z S U_z^*.$$

In the context of Bergman space $A^2(\mathbb{D})$ on the unit disc \mathbb{D} , with $\phi_z(w) = \frac{w-z}{1-\bar{z}w}$ and $U_z f = (f \circ \phi_z)\phi'(z)$, Axler-Zheng introduced the condition

$$\sup_{z \in \mathbb{D}} \|S_z 1\|_{A^p} < \infty \text{ with some } p > 2$$

in [1]. The work in [5, 6, 13, 15, 22] also explored the condition $\|S_z 1\|_{A^p} \leq C$. In the Fock space setting, Wang, Cao, and Zhu carried out related research in [19] to obtain that, if there exist some $p > 2$ such that

$$\sup_{z \in \mathbb{C}^n} \|S_z 1\|_{p, \frac{2\alpha}{p}} < \infty \quad (\text{or } \|S_z 1\|_{p, \frac{2\alpha}{p}} \rightarrow 0 \text{ as } z \rightarrow \infty),$$

the operator S is bounded (or compact) on F_α^2 .

Theorem 4.1. *Suppose S is a linear operator defined on \mathcal{D} . If there are some $0 < \sigma < p < \infty$ such that*

$$(4.1) \quad M = \sup_{z \in \mathbb{C}^n} \int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) < \infty,$$

then

$$|\langle S k_z, k_w \rangle_{F_\alpha^s}| \leq C M^{\frac{1}{p}} e^{-\frac{\alpha(p-\sigma)}{2p}|z-w|^2},$$

so S is bounded on F_α^s for all $0 < s < \infty$. Furthermore, if both (4.1) and

$$(4.2) \quad \lim_{z \rightarrow \infty} \int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) = 0$$

hold, then S is compact on F_α^s for $0 < s < \infty$.

Proof. Since $K(\cdot, \cdot) = e^{\alpha(\cdot, \cdot)}$, it is easy to verify $k_z(z - u)k_z(u) = 1$ and

$$K(z - u, z - u) = K(z, z)K(u, u)|K(u, z)|^{-2}.$$

By the equality $S_z 1(u) = k_z(u)(Sk_z)(z - u)$ (see [21]) and Lemma 2.3 we have

$$\begin{aligned} \int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) &= \int_{\mathbb{C}^n} |k_z(u)(Sk_z)(z - u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) \\ &= \int_{\mathbb{C}^n} |k_z(z - u)(Sk_z)(u)|^p e^{-\frac{\alpha\sigma}{2}|u-z|^2} dV(u) \\ &= \int_{\mathbb{C}^n} |(Sk_z)(u)|^p |k_z(u)|^{-p} e^{-\frac{\alpha\sigma}{2}|u-z|^2} dV(u) \\ &\geq \int_{B(w, 1)} |(Sk_z)(u)|^p |k_z(u)|^{-(p-\sigma)} e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) \\ &\geq C |(Sk_z)(w)|^p |k_z(w)|^{-(p-\sigma)} e^{-\frac{\alpha\sigma}{2}|w|^2} \\ &= C |\langle Sk_z, k_w \rangle_{F_\alpha^2}|^p e^{\frac{\alpha(p-\sigma)}{2}|z-w|^2}. \end{aligned}$$

From the above inequalities and (4.1), we get

$$|\langle Sk_z, k_w \rangle_{F_\alpha^2}| \leq CM^{\frac{1}{p}} e^{-\frac{\alpha(p-\sigma)}{2p}|z-w|^2}.$$

Since $p - \sigma > 0$, S is weakly localized for F_α^s , so S is bounded on F_α^s for all $0 < s < \infty$.

Furthermore, if both (4.1) and (4.2) are valid, from the proof above we have $S \in \text{WL}_s^\alpha$. And also, for $p \in (0, \infty)$ there is some constant C_r such that

$$\sup_{w \in B(z, r)} |\langle Sk_z, k_z \rangle_{F_\alpha^2}| \leq C_r \left(\int_{\mathbb{C}^n} |S_z 1(u)|^p e^{-\frac{\alpha\sigma}{2}|u|^2} dV(u) \right)^{\frac{1}{p}} \rightarrow 0$$

as $z \rightarrow \infty$. By Theorem 2.13 S is compact on F_α^s for all $0 < s < \infty$. \square

Remark. If $p > \sigma = 2$ and $s = 2$, then Theorem 4.1 reduces to Theorems A and B in [19].

5. FURTHER REMARKS

An important theme in analysis on function spaces is to characterize when a given operator is compact. In the setting of the Bergman space $A^p(\mathbb{B}_n)$ on the unit ball \mathbb{B}_n , $1 < p < \infty$, in 2007 Suárez proved, see [17], that a bounded operator S is compact if and only if S is in the Toeplitz algebra and the Berezin transform of S vanishes on the boundary. Later on, Mitkovski, Suárez and the third author [14] extended [17] to the weighted Bergman space $A_\alpha^p(\mathbb{B}_n)$. On the classical Fock space F_α^p for $1 < p < \infty$, in [2] Bauer and Isralowitz showed that Suárez's characterization on compact operators is valid. For general φ with $dd^c\varphi \simeq \omega_0$ and $1 < p < \infty$, most recently in [10] Isralowitz obtained $\mathcal{K}(F_\varphi^p) = \mathcal{T}_\varphi^p(C_c^\infty(\mathbb{C}^n))$, which implies the results in [2].

For Toeplitz operators T_μ with positive Borel measures μ as symbols, the boundedness (or compactness) on F_φ^p with $0 < p \leq 1$ can be characterized with the same condition as that on F_φ^q with $q > 1$. Unfortunately, some differences appear when we talk about the structure of $\mathcal{K}(F_\varphi^p)$. For example, we find $\mathcal{K}(F_\varphi^p) \setminus \mathcal{T}_\varphi^p \neq \emptyset$ if $0 < p \leq 1$. To see this, from [21, Lemma

4.39] (or [12]) we take a separated sequence $\{z_j\}_{j=1}^\infty$ which is an interpolating sequence for F_α^∞ . Hence, we have some $f \in F_\alpha^\infty$ such that

$$(5.1) \quad f(z_k)e^{-\frac{\alpha}{2}|z_k|^2} = 1, \quad \forall k \in \mathbb{N}.$$

Although [21] is only concerned with one variable interpolation, take $\{z_j\} \subset \mathbb{C}$ and $f \in H(\mathbb{C})$ satisfying the interpolation above, extend f to \mathbb{C}^n with the equation $f(z, z') = f(z)$ for $(z, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$, we will have f satisfying (5.1) in \mathbb{C}^n . Furthermore, for $0 < p \leq 1$ take $g \in F_\alpha^p$ so that $g(0) \neq 0$. Define the operator T on F_α^p as

$$(5.2) \quad T(\cdot) = \langle \cdot, f \rangle_{F_\alpha^2} g.$$

T is bounded and of rank 1, so T is compact on F_α^p . Also,

$$|\langle Tk_{z_k}, k_w \rangle_{F_\alpha^2}| = \left| \overline{f(z_k)} e^{-\frac{\alpha}{2}|z_k|^2} g(w) e^{-\frac{\alpha}{2}|w|^2} \right|.$$

Because $\{z_j\}_{j=1}^\infty$ is separated, we have $\lim_{j \rightarrow \infty} z_j = \infty$. For each $r > 0$, as k is large enough we have from Lemma 2.3 that

$$\begin{aligned} \int_{B(z_k, r)^c} |\langle Tk_{z_k}, k_w \rangle_{F_\alpha^2}|^p dV(w) &= \int_{B(z_k, r)^c} \left| g(w) e^{-\frac{\alpha}{2}|w|^2} \right|^p dV(w) \\ &\geq \int_{B(0, 1)} \left| g(w) e^{-\frac{\alpha}{2}|w|^2} \right|^p dV(w) \\ &\geq C |g(0)|^p. \end{aligned}$$

$T \notin \text{WL}_p^\alpha$ by Definition 2.1. Hence, $\mathcal{K}(F_\alpha^p) \setminus \mathcal{T}_\alpha^p \neq \emptyset$ for $0 < p \leq 1$. This tells us the characterization of compact operators T on F_α^p with $0 < p \leq 1$ is quite different from that with $1 < p < \infty$.

For $0 < p \leq 1$, $\{k_z : z \in \mathbb{C}^n\}$ does not converge weakly to zero in F_α^p as z goes to ∞ . In fact, take $f \in F_\alpha^\infty$ satisfying (5.1), since the dual space of F_α^p is F_α^∞ under the pairing $\langle g, f \rangle_{F_\alpha^2}$ (see [21]), we know that $F_f = \langle \cdot, f \rangle_{F_\alpha^2}$ is a bounded linear functional on F_α^p . However,

$$F_f(k_{z_k}) = \langle k_{z_k}, f \rangle_{F_\alpha^2} = 1$$

for all k .

The operator T defined as (5.2) also shows $\lim_{z \rightarrow \infty} \|Tk_z\|_{p, \alpha} \neq 0$, because $Tk_{z_j} = g$ for $j = 1, 2, \dots$, which says Tk_z need not converge to 0 in F_α^p even if T is compact while $0 < p \leq 1$. So, the hypothesis $T \in \text{WL}_p^\alpha$ both in Theorem 2.13 and 2.14 can not be removed. But for the Berezin transform, we have the following Proposition 5.1.

Proposition 5.1. *Suppose $0 < p \leq 1$ and $T \in \mathcal{K}(F_\alpha^p)$. Then $\tilde{T}(z) \rightarrow 0$ as $z \rightarrow \infty$.*

Proof. For $R > 0$ fixed, $PM_{\chi_{B(0,R)}}Tk_z \in F_\varphi^p \subset F_\varphi^2$. Lemma 2.2 estimate (1) and Lemma 2.2 estimate (3) give

$$\begin{aligned}
 \left| \langle PM_{\chi_{B(0,R)}}Tk_z, k_z \rangle_{F_\varphi^2} \right| &= \left| \langle M_{\chi_{B(0,R)}}Tk_z, k_z \rangle_{F_\varphi^2} \right| \leq \int_{B(0,R)} \left| Tk_z(u) \overline{k_z(u)} \right| e^{-2\varphi(u)} dV(u) \\
 &\leq \|Tk_z\|_{\infty, \varphi} \int_{B(0,R)} |k_z(u)| e^{-\varphi(u)} dV(u) \\
 &\leq C \|Tk_z\|_{p, \varphi} \sup_{|u| \leq R} |k_z(u)| e^{-\varphi(u)} \\
 &\leq \|T\|_{F_\varphi^p \rightarrow F_\varphi^p} \|k_z\|_{p, \varphi} e^{-\theta|z|} \\
 &\leq C e^{-\theta|z|} \rightarrow 0
 \end{aligned}$$

as $z \rightarrow \infty$. Since $T \in \mathcal{K}(F_\varphi^p)$, Lemma 2.11 tells us

$$\begin{aligned}
 \left| \left\langle \left(T - PM_{\chi_{B(0,R)}}T \right) k_z, k_z \right\rangle_{F_\varphi^2} \right| &\leq \left\| \left(T - PM_{\chi_{B(0,R)}}T \right) k_z \right\|_{\infty, \varphi} \|k_z\|_{1, \varphi} \\
 &\leq C \left\| \left(T - PM_{\chi_{B(0,R)}}T \right) k_z \right\|_{p, \varphi} \\
 &\leq C \|T - PM_{\chi_{B(0,R)}}T\|_{F_\varphi^p \rightarrow F_\varphi^p} \|k_z\|_{p, \varphi}, \\
 &\leq C \|T - PM_{\chi_{B(0,R)}}T\|_{F_\varphi^p \rightarrow F_\varphi^p} \rightarrow 0
 \end{aligned}$$

as $R \rightarrow \infty$. Therefore, taking $z \rightarrow \infty$,

$$\left| \tilde{T}(z) \right| = \left| \langle Tk_z, k_z \rangle_{F_\varphi^2} \right| \leq \left| \langle PM_{\chi_{B(0,R)}}Tk_z, k_z \rangle_{F_\varphi^2} \right| + \left| \left\langle \left(T - PM_{\chi_{B(0,R)}}T \right) k_z, k_z \right\rangle_{F_\varphi^2} \right| \rightarrow 0.$$

□

Summarizing the discussion above we put forward the following problem.

Problem 5.2. For $0 < p \leq 1$, what are the necessary and sufficient conditions to characterize the membership in $\mathcal{K}(F_\varphi^p)$?

Under the restriction $0 < p \leq 1$, we dominate the essential norm of $T \in \text{WL}_p^\varphi$ by its behavior on k_z , see Theorem 2.14. Our second problem is whether the estimate (2.14) still holds for $1 < p < \infty$?

Problem 5.3. Suppose $1 < p < \infty$. Does

$$\|T_f\|_{e, F_\varphi^p} \simeq \limsup_{z \rightarrow \infty} \|T_f k_z\|_{p, \varphi}$$

hold for bounded f on \mathbb{C}^n ?

In the previous section, with $f \in \text{BMO}$ we have obtained the compactness of Toeplitz operators T_f on F_φ^p . However, to consider the compactness of finite product $T_{f_1} T_{f_2} \cdots T_{f_m}$ of Toeplitz operators with BMO symbols we have assumed each symbol f_j has a bounded Berezin transform. Is this hypothesis necessary in the statement (B) of Theorem 3.2?

Problem 5.4. Suppose $0 < p < \infty$, and T is in the set of all linear combination of the form $T_{f_1} T_{f_2} \cdots T_{f_m}$, where each function $f_j \in \text{BMO}$. Can we conclude that T is compact on F_φ^p if and only if

$$\lim_{z \rightarrow \infty} \sup_{w \in B(z, r)} \left| \langle Tk_z, k_w \rangle_{F_\varphi^2} \right| = 0$$

holds for each $r > 0$?

We also point to the general question of how the story is similar, or different, in the case of the Bergman space $A^p(\mathbb{D})$ when $0 < p < 1$.

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Z. HU, DEPARTMENT OF MATHEMATICS, HUZHOU UNIVERSITY, HUZHOU, ZHEJIANG, 313000, CHINA
E-mail address: huzj@zjhu.edu.cn

X. LV, DEPARTMENT OF MATHEMATICS, HUZHOU UNIVERSITY, HUZHOU, ZHEJIANG, 313000, CHINA
E-mail address: lvxf@zjhu.edu.cn

B. D. WICK, DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY - ST. LOUIS, ONE BROOKINGS DRIVE, ST. LOUIS, MO USA 63110.
E-mail address: wick@math.wustl.edu