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A non-commutative Julia inequality

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Abstract

We prove a Julia inequality for bounded non-commutative functions on polynomial polyhedra. We use this to deduce a Julia inequality for holomorphic functions on classical domains in $\mathbb{C}^d$. We look at differentiability at a boundary point for functions that have a certain regularity there.

1 Introduction

The classical Julia inequality asserts that if a holomorphic function ϕ maps the unit disk $\mathbb{D}$ to itself, and if at some boundary point $\tau \in \partial\mathbb{D}$ one has

$$\liminf_{z \to \tau} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \;=\; \alpha \;< \infty, \tag{1.1}$$

then there exists $\omega \in \partial\mathbb{D}$ such that

$$\frac{|\varphi(z) - \omega|^2}{1 - |\varphi(z)|^2} \;\leq\; \alpha \, \frac{|z - \tau|^2}{1 - |z|^2}. \tag{1.2}$$

The inequality was proved, with an extra regularity hypothesis on φ, by G. Julia in [22], and in the form stated by C. Carathéodory in [16]. D. Sarason found a proof using model theory [33, Chap VI].

Generalizations of Julia's inequality have been found for functions on the ball by M. Hervé [20], W. Rudin [32, Sec. 8.5] and M. Jury [23], and on the

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polydisk by K. Wlodarczyk [37], F. Jafari [21] and M. Abate [1]. In the case of the bidisk, a detailed analysis of points for which the analogue of the Julia quotient (1.1) remains bounded (these are called B-points) has been carried out in [7, 8, 10, 15].

It is the purpose of this note to extend Julia's inequality, and the study of B-points, to non-commutative functions (which we shall define in Subsection 1.1 below) that are bounded on polynomial polyhedra. Our methods rely on the model-theoretic ideas of [8].

Our results are of interest even in the commutative case, because they provide a unified approach to proving boundary versions of the Schwarz-Pick lemma in the Schur-Agler class of various domains, such as the polydisk, or the multipliers of the Drury-Arveson space. The methods also show that at B-points where the function is not analytic, it does have directional derivatives in all directions pointing into the set, and the derivative is a holomorphic (but not necessarily linear) function of the direction. This is explained in Section 6 below.

1.1 Non-commutative Functions

Non-commutative function theory, which originated in the work of J.L. Taylor [34, 35], has recently started to flourish [6, 11, 12, 14, 17–19, 25–28, 30, 31]. The foundations are developed in the book [24].

The idea is to study functions of non-commuting variables that are generalized non-commuting polynomials in an analogous fashion to thinking of a holomorphic function as a generalization of a polynomial. Our domains are domains of d-tuples of matrices, but they don't reside in just one dimension. We let $\mathbb{M}_n$ denote the n-by-n complex matrices, and let

$$\mathbb{M}^{[d]} \;=\; \cup_{n=1}^{\infty} \mathbb{M}_n^d.$$

We shall call a function ϕ defined on a subset of $\mathbb{M}^{[d]}$ *graded* if, whenever $x \in \mathbb{M}_n^d$, then $\phi(x) \in \mathbb{M}_n$. If $x \in \mathbb{M}_n^d$ and $y \in \mathbb{M}_m^d$, we shall let $x \oplus y$ denote the d-tuple in $\mathbb{M}_{n+m}$ obtained by direct summing each component. If $x \in \mathbb{M}_n^d$ and s is an invertible matrix in $\mathbb{M}_n$, then $s^{-1}xs$ denotes the d-tuple $(s^{-1}x^1s, \ldots, s^{-1}x^ds)$.

Definition 1.3. *An nc-function ϕ on a set $\Omega \subseteq \mathbb{M}^{[d]}$ is a graded function* *that respects direct sums and joint similiarities, i.e.*

$$\begin{aligned} \phi(x \oplus y) &= \phi(x) \oplus \phi(y) \\ \phi(s^{-1}xs) := \phi(s^{-1}x^1 s, \ldots, s^{-1}x^d s) &= s^{-1}\phi(x)s, \end{aligned}$$

where the equations are only required to hold when the arguments on both sides are in Ω*, and in the second one if* $x \in \mathbb{M}_n^d$*, then* s *is invertible in* $\mathbb{M}_n$*.*

(We use superscripts for components, since we shall have many sequences indexed by subscripts.) Notice that every non-commutative polynomial is an nc-function on $\mathbb{M}^{[d]}$. A particularly nice class of domains on which to study nc-functions are polynomial polyhedra. These are defined in terms of a matrix δ, each of whose entries is a non-commutative polynomial in d variables. Then $\mathcal{G}_\delta$ is defined as

$$\mathcal{G}_\delta \ := \ \cup_{n=1}^\infty \{x \in \mathbb{M}_n^d \ : \ \|\delta(x)\| < 1\}, \tag{1.4}$$

where $\delta(x)$, a matrix of matrices, is given the operator norm. A primary example is the d-dimensional noncommutative polydisk, which is the set

$$\{x \in \mathbb{M}^{[d]} : \|x\| := \max_{1 \le r \le d} \|x^r\| < 1\}. \tag{1.5}$$

For this set, we can take δ to be the diagonal d-by-d matrix with the coordinate functions on the diagonal. Another well-studied example (see *e.g.* [29]) is the non-commutative ball, which we shall take to be the column contractions[1],

$$\{x \in \mathbb{M}^{[d]} \ : \ \sum_{r=1}^{d} x^{r*} x^r \le I\}, \tag{1.6}$$

which is obtained by letting

$$\delta(x) \ = \ \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^d \end{pmatrix}.$$

[1]It is more common to consider the row contractions, but we choose column contractions so that what we call the distinguished boundary will be non-empty. It is easy to pass between these two sets, since the column contractions are just the adjoints of the row contractions.

1.2 Principal Results

Let $\mathcal{G}_\delta$ be defined by (1.4). We shall let $\mathcal{B}_\delta$ denote the topological boundary of $\mathcal{G}_\delta$. If $x \in \mathcal{B}_\delta$, then $\|\delta(x)\| = 1$, but the converse need not hold — *e.g.* with $d = 1$, take $\delta(x) = \begin{pmatrix} x-1 & 0 \\ 0 & x+1 \end{pmatrix}$. Then $\mathcal{G}_\delta$ is empty, but $\|\delta(0)\| = 1$.

The distinguished boundary of $\mathcal{G}_\delta$, which we shall denote $\mathcal{I}_\delta$, is

$$\mathcal{I}_\delta \; := \; \{x \in \mathcal{B}_\delta \; : \; \delta(x) \text{ is an isometry}\}. \tag{1.7}$$

The reader should keep in mind the example of the non-commutative polydisk (1.5), in which case $\mathcal{I}_\delta$ is $\mathcal{U}^{[d]}$, the set of d-tuples of unitary matrices in $\mathbb{M}^{[d]}$, and $\mathcal{B}_\delta$ is the set of contractive d-tuples such that at least one has norm equal to one. For the column-ball (1.6), the distinguished boundary will agree with $\mathcal{B}_\delta$ when $n = 1$, but will be smaller when $n > 1$.

Definition 1.8. *The Schur class of $\mathcal{G}_\delta$, denoted $\mathcal{S}(\mathcal{G}_\delta)$, is the set of nc functions ϕ on $\mathcal{G}_\delta$ such that $\|\phi(x)\| < 1 \; \forall \; x \in \mathcal{G}_\delta$.*

A B-point for ϕ is a point in the boundary where a certain regularity occurs.

Definition 1.9. *Let $\phi \in \mathcal{S}(\mathcal{G}_\delta)$, and let $T \in \mathcal{B}_\delta$. Then T is a B-point for ϕ if*

$$\lim_{j\to\infty} \frac{\|I - \phi(Z_j)^*\phi(Z_j)\|}{1 - \|\delta(Z_j)\|^2} \; < \; \infty \tag{1.10}$$

for some sequence Z_j converging to T.

Here is our non-commutative Julia inequality, which says that at a B-point, one has a boundary version of the Schwarz-Pick inequality, akin to (1.2).

Theorem 3.6 *Suppose $\phi \in \mathcal{S}(\mathcal{G}_\delta)$ and $T \in \mathcal{B}_\delta$. If*

$$\liminf_{Z\to T} \frac{\|I - \phi(Z)^*\phi(Z)\|}{1 - \|\delta(Z)\|^2} = \alpha,$$

then there exists $W \in \mathcal{U}_n$ such that for all Z in $\mathcal{G}_\delta$

$$\frac{\|\phi(Z) - W\|^2}{\|I - \phi(Z)^*\phi(Z)\|} \; \leq \; \alpha \left(\frac{\|I - \delta(T)^*\delta(Z)\|^2}{1 - \|\delta(Z)\|^2} \right).$$

We note that the noncommutative structure then gives that if T is a B-point, then so is U^*TU for any unitary matrix U. Furthermore, given two B-points S and T, then $S \oplus T$ must also be a B-point.

Our second main result, Theorem 4.17, gives a characterization of when a point in $\mathcal{I}_\delta$ is a B-point, in terms of a realization of a δ nc-model for ϕ. We shall defer an exact statement until Section 4.

Our third result, Theorem 5.3, holds under the assumption that there are a lot of inward directions at T. For now, we shall just give a special case.

Theorem 1.11. *Suppose $\mathcal{G}_\delta$ is either* (1.5) *or* (1.6)*. Suppose $T \in \mathcal{I}_\delta$ is a B-point of ϕ. Then*

$$\eta(H) \;=\; \lim_{t\downarrow 0} \frac{\phi(T+tH)-\phi(T)}{t}$$

exists for all H satisfying $T+tH \in \mathcal{G}_\delta$ for t small and positive. Moreover η is a holomorphic function of H, which is homogeneous of degree 1.

There is compatibility between the directional derivative at a B-point T, given by the function η in Theorem 1.11, and the directional derivative at a direct sum of m copies of T, which we will denote by $T^{(m)}$. In particular, if a scalar point is a B-point, then one obtains B-points at all levels.

Corollary 1.12. *Suppose $\mathcal{G}_\delta$ is either* (1.5) *or* (1.6)*. Suppose $T \in \mathcal{I}_\delta \cap \mathbb{M}_1^d$ is a B-point of ϕ. Then*

$$\eta(H) \;=\; \lim_{t\downarrow 0} \frac{\phi(T^{(m)}+tH)-\phi(T^{(m)})}{t}$$

exists for all m, and for all H satisfying $T^{(m)}+tH \in \mathcal{G}_\delta$ for t small and positive. Moreover η is a free function of H, which is homogeneous of degree 1.

Corollary 1.12 follows immediately from the definition of a free function, and the fact that T is a d-tuple of scalars. We leave the details to the interested reader, but emphasize that the noncommutative Julia-Carathéodory really captures a regularity that holds not only within each level $G_\delta \cap \mathbb{M}_n^d$, but also between levels.

In Example 7.1 we consider the function

$$\phi(Z^1,Z^2) \;=\; \frac{1}{2}(Z^1+Z^2)+\frac{1}{2}(Z^1-Z^2)(2-Z^1-Z^2)^{-1}(Z^1-Z^2)$$

which we show is in the Schur class of (1.4).

2 Background material

2.1 The one variable Julia-Carathéodory theorem

The Julia-Carathéodory Theorem, due to G. Julia [22] in 1920 and C. Carathéodory [16] in 1929, is the following.

Theorem 2.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, and $\tau \in \mathbb{T}$. The following are equivalent:*

(A) $\liminf_{z\to\tau} \frac{1-|\varphi(z)|^2}{1-|z|^2} < \infty$.

(B) The quotient $\frac{1-|\varphi(z)|^2}{1-|z|^2}$ has a non-tangential limit as z tends to τ.

(C) The function φ has both a non-tangential limit $\omega \in \mathbb{T}$ at τ and also an angular derivative $\eta \in \mathbb{C}$, that is the difference quotient

$$\frac{\varphi(z) - \omega}{z - \tau}$$

has a non-tangential limit η at τ.

(D) There exist ω in $\mathbb{T}$ and η in $\mathbb{C}$ so that at τ, $\varphi(z)$ tends to ω non-tangentially and $\phi'(\zeta)$ tends to η non-tangentially.

Furthermore, if (1.1) *holds, then* (1.2) *does.*

On the bidisk, the analogue of *(B)* does not imply *(C)*; but it is proved in [8] that *(C)* implies *(D)*. Moreover, it is shown that even when ϕ does not have a holomorphic differential pointing into the bidisk, the one sided derivative exists and is holomorphic in the direction.

2.2 Background on free holomorphic functions

A free holomorphic function is an nc function that is locally bounded with respect to the topology generated by all the sets $\mathcal{G}_\delta$, as δ ranges over all matrices with entries that are free polynomials. These functions are studied in [3], and two principal results are obtained. One is that a bounded function on $\mathcal{G}_\delta$ is nc if and only if it is the pointwise limit of a sequence of non-commutative polynomials. The other is that every bounded nc-function on $\mathcal{G}_\delta$ has an nc δ-model. Alternative proofs of both these results have been found in [13] and [2]. Before explaining what this is, we need to slightly expand definition 1.3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ denote the bounded linear operators from $\mathcal{E}_1$ to $\mathcal{E}_2$. Following [28], we shall *write tensor* *products vertically* to enhance readability and condense realization formulas, so $\overset{A}{\underset{B}{\otimes}}$ represents the same object as $A \otimes B$. We shall assume that the domain Ω of any nc function is closed w.r.t. direct sums. Also, we shall for notational convenience assume that δ is a square J-by-J matrix — we can always add rows or columns of zeroes to ensure this. We shall let Ω_n denote $\Omega \cap \mathbb{M}_n^d$, and $\mathcal{I}_n$ denote the invertible matrices in $\mathbb{M}_n$.

Definition 2.2. *An $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$-valued nc function F on a set $\Omega \subseteq \mathbb{M}^{[d]}$ is a function satisfying*

$$\begin{aligned} F(x) &\in \mathcal{L}\Big(\overset{\mathcal{E}_1}{\underset{\mathbb{C}^n}{\otimes}} , \overset{\mathcal{E}_2}{\underset{\mathbb{C}^n}{\otimes}} \Big) \quad \forall\, n,\ \forall x \in \Omega_n \\ F(x \oplus y) &= F(x) \oplus F(y) \quad \forall\, m, n\ \forall\, x \in \Omega_n,\ y \in \Omega_m \\ F(s^{-1} x s) &= \overset{I_{\mathcal{E}_2}}{\underset{s^{-1}}{\otimes}}\ F(x)\ \overset{I_{\mathcal{E}_1}}{\underset{s}{\otimes}} \quad \forall\, n,\ \forall\, s \in \mathcal{I}_n \text{ s.t. } x, s^{-1} x s \in \Omega_n. \end{aligned}$$

Definition 2.3. *Let δ be a J-by-J matrix of free polynomials. Let ϕ be an nc-function on $\mathcal{G}_\delta$. A δ nc-model for ϕ is an $\mathcal{L}(\mathbb{C}, \overset{\mathcal{E}}{\underset{\mathbb{C}^J}{\otimes}})$-valued nc function u on Ω that satisfies*

$$I_{\mathbb{C}^n} - \phi(y)^* \phi(x) \;=\; u(y)^* \left(\overset{I_{\mathcal{E}}}{\underset{I_{\mathbb{C}^J \otimes \mathbb{C}^n} - \delta(y)^* \delta(x)}{\otimes}} \right) u(x). \tag{2.4}$$

Theorem 2.5. [3] *An nc function ϕ defined on $\mathcal{G}_\delta$ is bounded by 1 in norm if and only if it has a δ nc-model.*

To help the reader, let us rewrite Theorem 2.5 in the case that $\mathcal{G}_\delta$ is the non-commutative d-polydisk, given by 1.5, and both $\mathcal{E}_1$ and $\mathcal{E}_2$ are one dimensional. Then $J = d$, and Theorem 2.5 becomes:

Theorem 2.6. *Let Ω be the non-commutative d-polydisk,* (1.5). *The graded function ϕ is in the Schur class of Ω if and only if there are d $\mathcal{L}(\mathbb{C}, \mathcal{E})$-valued* *nc functions* $u^1, \dots u^d$ *so that, for all* n*, for all* $x, y \in \Omega \cap \mathbb{M}_n^d$*, we have*

$$
\begin{aligned}
I_{\mathbb{C}^n} - \phi(y)^*\phi(x) \ &= \\
&\begin{pmatrix} u^1(y) \\ u^2(y) \\ \vdots \\ u^d(y) \end{pmatrix}^* \begin{pmatrix} I - \substack{I_{\mathcal{E}} \\ \otimes \\ (y^1)^*x^1} & 0 & \dots & 0 \\ 0 & I - \substack{I_{\mathcal{E}} \\ \otimes \\ (y^2)^*x^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \substack{I_{\mathcal{E}} \\ \otimes \\ I-(y^d)^*x^d} \end{pmatrix} \begin{pmatrix} u^1(x) \\ u^2(x) \\ \vdots \\ u^d(x) \end{pmatrix} \\
&= \sum_{j=1}^d u^j(y)^* \left(I_{\mathcal{E}\otimes\mathbb{C}^n} - \substack{I_{\mathcal{E}} \\ \otimes \\ (y^j)*x^j} \right) u^j(x).
\end{aligned}
$$

3 Julia's Inequality and Consequences

We shall assume for the remainder of the paper that δ is a J-by-J matrix of non-commutative polynomials, that $\phi \in \mathcal{S}(\mathcal{G}_\delta)$, and u is a δ nc-model for ϕ, with values in $\mathcal{L}(\mathbb{C}, \substack{\mathcal{E} \\ \otimes \\ \mathbb{C}^J})$. We shall further assume that $T \in \mathcal{B}_\delta$ is in $\mathbb{M}_n^d$. We shall let $\mathcal{U}_n$ denote the n-by-n unitaries.

If ϕ is in $\mathcal{S}(\mathcal{G}_\delta)$, it follows from Theorem 2.5 that

$$\|I - \phi(x)^*\phi(x)\| \ \geq \ \|u(x)\|^2(1 - \|\delta(x)\|^2).$$

So if (1.10) holds, then $\{\|u(Z_j)\|\}$ is bounded, and the following definition is non-vacuous.

Definition 3.1. *Let* T *be a B-point for* ϕ*. Let* $Y_T = Y_T(u)$ *denote the set of all weak-limits of* $u(Z_j)$*, where* Z_j *is a sequence in* $\mathcal{G}_\delta$ *that converges to* T *and has*

$$\frac{\|I - \phi(Z_j)^*\phi(Z_j)\|}{1 - \|\delta(Z_j)\|^2} \tag{3.2}$$

bounded. We shall call Y_T *the cluster set of the model* u*.*

Proposition 3.3. *Let* T *be a B-point for* ϕ*. Then there exists* $W \in \mathcal{U}_n$ *such that for all* $v \in Y_T$*, for all* Z *in* $\mathcal{G}_\delta$*, we have*

$$I - W^*\phi(Z) \ = \ v^* \left(\substack{I \\ \otimes \\ I-\delta(T)^*\delta(Z)} \right) u(Z). \tag{3.4}$$

Moreover, if

$$\lim_{j\to\infty} \frac{\|I - \phi(Z_j)^*\phi(Z_j)\|}{1 - \|\delta(Z_j)\|^2} \ \leq \alpha \tag{3.5}$$

holds for some sequence, then there exists $v \in Y_T$ with $\|v\|^2 \leq \alpha$.

Proof: Suppose (1.10) holds and $u(Z_j)$ tends weakly to v. Then $\|I - \phi(Z_j)^*\phi(Z_j)\| \to 0$, so by passing to a subsequence we can assume that $\phi(Z_j)$ tends to some unitary W. Taking the limit in

$$I - \phi(Z_j)^*\phi(Z) \ = \ u(Z_j)^* \left(\begin{smallmatrix} I \\ \otimes \\ I - \delta(Z_j)^*\delta(Z) \end{smallmatrix} \right) u(Z).$$

we get (3.4). To see that W is unique, if another sequence Z_j' tending to T with $u(Z_j')$ weakly convergent had $\phi(Z_j') \to W'$, then letting $Z = Z_j'$ in (3.4) we get

$$I - W^*\phi(Z_j') \ = \ v^* \left(\begin{smallmatrix} I \\ \otimes \\ I - \delta(T)^*\delta(Z_j') \end{smallmatrix} \right) u(Z_j').$$

Since $\phi(Z_j)'$ and $\delta(Z_j')$ act on finite dimensional spaces, we can pass to a subsequence so that they converge in norm, and $u(Z_j')$ converges weakly. In the limit, we get

$$I - W^*W' \ = \ 0,$$

so $W = W'$.

For the latter part, note

$$\begin{aligned} (1 - \|\delta(Z_j)\|^2)u(Z_j)^*u(Z_j) \quad &\leq \quad u(Z_j)^* \left(\begin{smallmatrix} I \\ \otimes \\ I - \delta(Z_j)^*\delta(Z_j) \end{smallmatrix} \right) u(Z_j) \\ &= \quad I - \phi(Z_j)^*\phi(Z_j), \end{aligned}$$

so

$$\|u(Z_j)\|^2 \ \leq \ \frac{\|1 - \phi(Z_j)^*\phi(Z_j)\|}{1 - \|\delta(Z_j)\|^2}.$$

Taking v to be any weak cluster point of $u(Z_j)$, we get $\|v\|^2 \leq \alpha$. $\Box$

If T is a B-point for ϕ, we shall let $\phi(T)$ denote the matrix W that satisifies (3.4).

Here is the nc Julia inequality.

Theorem 3.6. *Suppose $T \in \mathcal{B}_\delta$, and*

$$\liminf_{\mathcal{G}_\delta \ni Z \to T} \frac{\|I - \phi(Z)^*\phi(Z)\|}{1 - \|\delta(Z)\|^2} = \alpha.$$

Then there exists $W \in \mathcal{U}_n$ such that for all Z in $\mathcal{G}_\delta$

$$\frac{\|\phi(Z) - W\|^2}{\|I - \phi(Z)^*\phi(Z)\|} \;\leq\; \alpha \left(\frac{\|I - \delta(T)^*\delta(Z)\|^2}{1 - \|\delta(Z)\|^2} \right). \tag{3.7}$$

Proof: By Proposition 3.3, we can choose v in Y_T with $\|v\|^2 \leq \alpha$. From (3.4) we have

$$I - W^*\phi(Z) \;=\; v^* \left(\begin{array}{c} I \\ \otimes \\ I - \delta(T)^*\delta(Z) \end{array} \right) u(Z),$$

so

$$\begin{aligned} \|\phi(Z) - W\|^2 \quad &= \quad \|v^* \left(\begin{array}{c} I \\ \otimes \\ I - \delta(T)^*\delta(Z) \end{array} \right) u(Z)\|^2 \\ &\leq \quad \alpha \, \|u(Z)\|^2 \|I - \delta(T)^*\delta(Z)\|^2. \end{aligned} \tag{3.8}$$

Now

$$\begin{aligned} \|I - \phi(Z)^*\phi(Z)\| \quad &= \quad \|u(Z)^* \left(\begin{array}{c} I \\ \otimes \\ I - \delta(Z)^*\delta(Z) \end{array} \right) u(Z)\| \\ &\geq \quad \|u(Z)\|^2 (1 - \|\delta(Z)\|^2). \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\|\phi(Z) - W\|^2 \;\leq\; \alpha \left(\frac{\|I - \delta(T)^*\delta(Z)\|^2}{1 - \|\delta(Z)\|^2} \right) \|I - \phi(Z)^*\phi(Z)\|,$$

which yields (3.7). ☐

If $T \in \mathcal{I}_\delta$, a non-tangential approach region is a region of the form

$$\{Z : \|\delta(Z) - \delta(T)\| \leq c(1 - \|\delta(Z)\|^2)\}. \tag{3.10}$$

A corollary of Julia's lemma is that a function's behavior is controlled non-tangentially at a B-point on the distinguished boundary.

Proposition 3.11. *If $T \in \mathcal{I}_\delta$ is a B-point for ϕ, then*

$$\frac{\|I - \phi(Z)^*\phi(Z)\|}{1 - \|\delta(Z)\|^2}$$

is bounded on all sets that approach T non-tangentially.

PROOF: Suppose v and W are such that (3.4) holds:

$$I - W^* \phi(Z) \ = \ v^* \left(\begin{smallmatrix} I \\ \otimes \\ I - \delta(T)^* \delta(Z) \end{smallmatrix} \right) u(Z).$$

Fix c, and let S be the non-tangential approach region

$$S \ = \ \{ Z : \| \delta(T) - \delta(Z) \| \leq c(1 - \|\delta(Z)\|^2) \}.$$

By (3.7), we have for Z in S that

$$\|\phi(Z) - W\|^2 \ \leq \ \alpha \left(\frac{\|I - \delta(T)^* \delta(Z)\|^2}{1 - \|\delta(Z)\|^2} \right) \|I - \phi(Z)^* \phi(Z)\|. \qquad (3.12)$$

Now

$$\begin{aligned} \|I - \phi(Z)^* \phi(Z)\| \ &= \ \|W^* W - \phi(Z)^* \phi(Z)\| \\ &= \ \|W^* (W - \phi(Z)) - (\phi(Z)^* - W^*) \phi(Z)\| \\ &\leq \ \|W - \phi(Z)\| (\|W\| + \|\phi(Z)\|) \\ &\leq \ 2 \|W - \phi(Z)\|. \qquad (3.13) \end{aligned}$$

Squaring and using (3.12), we get

$$\|I - \phi(Z)^* \phi(Z)\|^2 \ \leq \ 4\alpha \left(\frac{\|I - \delta(T)^* \delta(Z)\|^2}{1 - \|\delta(Z)\|^2} \right) \|I - \phi(Z)^* \phi(Z)\|. \quad (3.14)$$

Since $T \in \mathcal{I}_\delta$, we have $\delta(T)$ is an isometry, so

$$\|\delta(T) - \delta(Z)\| \ = \ \|I - \delta(T)^* \delta(Z)\|.$$

Therefore if $Z \in S$, the expression in parantheses on the right-hand side of (3.14) is bounded by $c^2(1 - \|\delta(Z)\|^2)$, so we get

$$\|I - \phi(Z)^* \phi(Z)\| \ \leq \ 4\alpha c^2 (1 - \|\delta(Z)\|^2),$$

as required. $\square$

4 Models and B-points

In this section we shall study how being a B-point is related to properties of the δ nc model. In the case of the bidisk, these results are in [8].

Proposition 4.1. *Let $T \in \mathcal{I}_\delta$, and suppose Z_j in $\mathcal{G}_\delta$ converges to T non-tangentially in the region* (3.10). *The following are equivalent:*
(i) $\|I - \phi(Z_j)^*\phi(Z_j)\| \leq M\|I - \delta(Z_j)^*\delta(Z_j)\|$.
(ii) $\|I - \phi(Z_j)^*\phi(Z_j)\| \leq M'(1 - \|\delta(Z_j)\|^2)$.
(iii) $\|u(Z_j)\|^2 \leq M''$ *for some u satisfying* (2.4).
(iv) $\|u(Z_j)\|^2 \leq M''$ *for every u satisfying* (2.4).

Proof:
(i) $\Rightarrow$ *(iv)*

$$\begin{aligned}(1 - \|\delta(Z_j)\|^2)u(Z_j)^*u(Z_j) &\leq u(Z_j)^* \left(\underset{I-\delta(Z_j)^*\delta(Z_j)}{\overset{I}{\otimes}} \right) u(Z_j) \\ &= I - \phi(Z_j)^*\phi(Z_j) \\ &\leq M\|I - \delta(Z_j)^*\delta(Z_j)\| \; I.\end{aligned}$$

Therefore

$$(1 - \|\delta(Z_j)\|^2)\|u(Z_j)\|^2 \leq M\|I - \delta(Z_j)^*\delta(Z_j)\|.$$

But

$$\begin{aligned}\|I - \delta(Z_j)^*\delta(Z_j)\| &= \|\delta(T)^*\delta(T) - \delta(Z_j)^*\delta(Z_j)\| \\ &= \|\delta(T)^*(\delta(T) - \delta(Z_j)) + (\delta(T)^* - \delta(Z_j)^*)\delta(Z_j)\| \\ &\leq (1 + \|\delta(Z_j)\|)(\|\delta(T) - \delta(Z_j)\|) \\ &\leq c(1 + \|\delta(Z_j)\|)(1 - \|\delta(Z_j)\|^2).\end{aligned}$$

Therefore we get

$$\begin{aligned}\|u(Z_j)\|^2 &\leq Mc(1 + \|\delta(Z_j)\|) \\ &\leq 2Mc. \qquad \lhd\end{aligned}$$

(iii) $\Rightarrow$ *(i)* Since

$$I - \phi(Z_j)^*\phi(Z_j) = u(Z_j)^* \left(\underset{I-\delta(Z_j)^*\delta(Z_j)}{\overset{I}{\otimes}} \right) u(Z_j),$$

taking norms we get

$$\|I - \phi(Z_j)^*\phi(Z_j)\| \leq M''\|I - \delta(Z_j)^*\delta(Z_j)\|. \qquad \lhd$$

$(i) \Leftrightarrow (ii)$ In the region (3.10), we have

$$\begin{aligned} 1 - \|\delta(Z)\|^2 &\leq \|I - \delta(Z)^* \delta(Z)\| \\ &\leq 2c(1 - \|\delta(Z)\|^2). \end{aligned}$$

□

We can now give a different characterization of B-points that are on the distinguished boundary.

Corollary 4.2. *Let $T \in \mathcal{I}_\delta$, and let u be a δ nc-model for ϕ. Suppose:*
(NT) There exists some sequence in $\mathcal{G}_\delta$ that approaches T non-tangentially.
Then The following are equivalent:
(i) T is a B-point of ϕ.
(ii) $u(Z_j)$ is bounded on some sequence Z_j that approaches T non-tangentially.
(iii) $u(Z)$ is bounded on every set that approaches T non-tangentially.
(iv) $\frac{\|I - \phi(Z)^*\phi(Z)\|}{1 - \|\delta(Z)\|^2}$ is bounded on every set that approaches T non-tangentially.

Proof: (i) $\Rightarrow$ (iv) by Proposition 3.11, and (iii) $\Rightarrow$ (ii) is trivial.
(iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) both follow from Proposition 4.1, and the observation that the proof shows that all the constants M, M', M'' are comparable once the aperture of the non-tangential approach region is fixed. □

Remark: Condition (NT) is very mild. It will hold if $\Gamma(T)$ (see Def. 4.6 below) is non-empty.

If u is a δ nc model for ϕ, then by [3, Cor. 8.2], there is an isometry (which is called a *realization of the model*)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C} \oplus \begin{matrix} \mathcal{E} \\ \otimes \\ \mathbb{C}^J \end{matrix} \to \mathbb{C} \oplus \begin{matrix} \mathcal{E} \\ \otimes \\ \mathbb{C}^J \end{matrix} \tag{4.3}$$

so that for $x \in \mathcal{G}_\delta \cap \mathbb{M}_n^d$,

$$\left[\begin{matrix} I_{\mathcal{E}} \\ \otimes \\ I_{\mathbb{C}^J} \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} - \begin{matrix} D \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} \begin{matrix} I_{\mathcal{E}} \\ \otimes \\ \delta(x) \end{matrix} \right] u(x) = \begin{matrix} C \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} , \tag{4.4}$$

and

$$\begin{aligned} \phi(x) &= \begin{matrix} A \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} + \begin{matrix} B \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} \begin{matrix} I_{\mathcal{E}} \\ \otimes \\ \delta(x) \end{matrix} u(x) \\ &= \begin{matrix} A \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} + \begin{matrix} B \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} \begin{matrix} I_{\mathcal{E}} \\ \otimes \\ \delta(x) \end{matrix} \left[I - \begin{matrix} D \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} \begin{matrix} I_{\mathcal{E}} \\ \otimes \\ \delta(x) \end{matrix} \right]^{-1} \begin{matrix} C \\ \otimes \\ I_{\mathbb{C}^n} \end{matrix} . \end{aligned} \tag{4.5}$$

For $T \in \mathcal{I}_\delta$, the inward directions for T are those H such that $T + tH$ is inside $\mathcal{G}_\delta$ for t small and positive. Formally, if $T \in \mathbb{M}_n^d$, and $H \in \mathbb{M}_n^d$, let

$$\nabla\delta(T)[H] \;=\; \lim_{t\to 0} \frac{1}{t}[\delta(T+tH) - \delta(T)]$$

denote the derivative of δ at T in the direction H. If A is a self-adjoint matrix, we write $A < 0$ to mean A is negative definite.

Definition 4.6. *Let $T \in \mathbb{M}_n^d$ be in $\mathcal{I}_\delta$. The inward set of T is the set*

$$\Gamma(T) \;=\; \{H \in \mathbb{M}_n^d \;:\; \|H\| \leq 1, \text{ and } \mathrm{Re}\,[\delta(T)^*\nabla\delta(T)[H]] < 0\}.$$

The transverse inward set of T is the subset of $\Gamma(T)$ defined by

$$\Delta(T) \;=\; \{H \in \mathbb{M}_n^d \;:\; \|H\| \leq 1, \text{ and } \delta(T)^*\nabla\delta(T)[H] < 0\}.$$

We have the following elementary result.

Lemma 4.7. *Let $H \in \Gamma(T)$. Then there exists $\varepsilon > 0$ such that*

$$T + tH \in \mathcal{G}_\delta \;\forall\; 0 < t < \varepsilon.$$

Moreover, $T + tH$ approaches T non-tangentially as $t \downarrow 0$.

Proof: Let $V = \delta(T)$, an isometry since $T \in \mathcal{I}_\delta$. Then

$$\mathrm{Re}\,[V^*\nabla\delta(T)[H]] \;\leq\; -\beta I,$$

so

$$\begin{aligned} I - \delta(T+tH)^*\delta(T+tH) \quad &= \quad -2\mathrm{Re}\,[tV^*\nabla\delta(T)[H]] + O(t^2) \\ &\geq \quad 2\beta t I + O(t^2). \end{aligned} \tag{4.8}$$

This yields the first assertion, and the second follows from this and the mean value theorem, which implies that

$$\|\delta(T+tH) - \delta(T)\| \;=\; O(t).$$

□

For the rest of this section, we shall make the following assumption:

(A1) The set $\Delta(T)$ is non-empty, so there exists $K \in \mathbb{M}_n^d$ and $\beta > 0$ so that

$$\delta(T)^*\nabla\delta(T)[K] \;\leq\; -\beta I. \tag{4.9}$$

Lemma 4.10. *Let $T \in \mathcal{I}_\delta$, and u be a δ nc model for ϕ. Suppose that T is a B-point for ϕ, and that $K \in \Delta(T)$ satisfies (4.9). Let $Z_j = T + t_j K$, where $0 < t_j < 1$ and $t_j \to 0$. If $u(Z_j)$ converges weakly to v, then $u(Z_j)$ converges in norm to v.*

Proof: We have

$$\begin{aligned} I - W^*\phi(Z) &= v^* \Big(\overset{I}{\underset{I-\delta(T)^*\delta(Z)}{\otimes}} \Big) u(Z) && (4.11) \\ I - \phi(Z)^*\phi(Z) &= u(Z)^* \Big(\overset{I}{\underset{I-\delta(Z)^*\delta(Z)}{\otimes}} \Big) u(Z). && (4.12) \end{aligned}$$

Let $V = \delta(T)$ and $X = \nabla\delta(T)[K]$. Let $Z = T + tK$, so $\delta(Z) = V + tX + O(t^2)$, and recall that $V^*V = I$ since $T \in \mathcal{I}_\delta$. So the lower parts of the terms in parentheses on the right-hand sides of (4.11) and (4.12) are, respectively, $-tV^*X + O(t^2)$ and $-2tV^*X + O(t^2)$. From (4.12) we get

$$(u(Z)^* - v^*) \ \overset{I}{\underset{-2tV^*X}{\otimes}} \ u(Z) \ = \ I - \phi(Z)^*\phi(Z) - v^* \ \overset{I}{\underset{-2tV^*X}{\otimes}} \ u(Z) \ + O(t^2).$$

Take the real part, and subtract and add twice the real part of (4.11) to get

$$\begin{aligned} \mathrm{Re}\left[(u(Z)^* - v^*) \ \overset{I}{\underset{-2tV^*X}{\otimes}} \ u(Z) \right] &= I - \phi(Z)^*\phi(Z) - 2\mathrm{Re}\,(I - W^*\phi(Z)) + O(t^2) \\ &= -(I - \phi(Z)^*W)(I - W^*\phi(Z)) + O(t^2). \end{aligned}$$

Therefore

$$\|\mathrm{Re}\left[(u(Z)^* - v^*) \ \overset{I}{\underset{V^*X}{\otimes}} \ u(Z) \right] \| \ \leq \ \frac{\|\phi(Z) - W\|^2}{2t} + O(t). \qquad (4.13)$$

As $Z_j \to T$ within a non-tangential approach region, by Theorem 3.6 and Proposition 4.1 $(iii) \Rightarrow (i)$, we get some constant M so that

$$\|\phi(Z_j) - W\|^2 \ \leq \ M\|\delta(T) - \delta(Z_j)\|^2 \ = \ M\|X\|^2 t^2 + O(t^3).$$

Therefore the right-hand side of (4.13) is $O(t)$, and we conclude that, since $V^*X \leq -\beta I$,

$$\begin{aligned} \lim_{j\to\infty} \|u(Z_j) - v\|^2 &\leq \frac{1}{\beta} \lim_{j\to\infty} (u(Z_j)^* - v^*) \ \overset{I}{\underset{V^*X}{\otimes}} \ (u(Z_j) - v) \\ &= \frac{1}{\beta} \lim_{j\to\infty} \mathrm{Re}\left[(u(Z_j)^* - v^*) \ \overset{I}{\underset{V^*X}{\otimes}} \ u(Z_j) \right] \\ &= 0, \end{aligned}$$

so

$$\lim_{j\to\infty} \|v - u(Z_j)\|^2 \;=\; 0,$$

as desired. $\Box$

Lemma 4.14. *Under the assumptions of Lemma 4.10, there exists a unique u_T such that*

$$u_T \perp \ker \left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right] \tag{4.15}$$

and

$$\left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right] u_T \;=\; \overset{C}{\underset{I}{\otimes}} \,. \tag{4.16}$$

Proof: By Corollary 4.2, as t decreases to 0, the vectors $u(T+tK)$ stay bounded; so there is some sequence t_j so that $u(T+t_jK)$ converges weakly, Choose a sequence r_j increasing to 1 so that $u(r_jT)$ converges weakly, and hence, by Lemma 4.10, also in norm, to a vector v. Writing $Z_j = T + t_jK$,

$$\left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(Z_j)}{\otimes}} \right] u(Z_j) \;=\; \overset{C}{\underset{I}{\otimes}} \,,$$

so taking the limit we get

$$\left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right] v \;=\; \overset{C}{\underset{I}{\otimes}} \,.$$

Since $\overset{C}{\underset{I}{\otimes}}$ is in the range of $\left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right]$, there exists a a unique vector u_T satisfying (4.16) and (4.15). $\Box$

We can now give a characterization of B-points, for homogeneous δ's, in terms of realizations.

Theorem 4.17. *Let $\phi \in \mathcal{S}(\mathcal{G}_\delta)$, let u be a δ nc model for ϕ, and let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a realization as in (4.3 – 4.5). Let $T \in \mathcal{I}_\delta$, and assume that (A1) holds. Then T is a B-point for ϕ if and only if*

$$\overset{C}{\underset{I}{\otimes}} \;\in\; \mathrm{Ran} \left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right].$$

Proof: If T is a B-point, then the inclusion follows from Lemma 4.14. Conversely, suppose

$$\left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right] v \;=\; \overset{C}{\underset{I}{\otimes}}$$

for some vector v. By (4.4), for any Z in $\mathcal{G}_\delta$ we have

$$
\begin{aligned}
u(Z) &= \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(Z)}{\overset{I}{\otimes}}\right]^{-1} \underset{I}{\overset{C}{\otimes}} \\
&= \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(Z)}{\overset{I}{\otimes}}\right]^{-1} \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(T)}{\overset{I}{\otimes}}\right] v \\
&= v - \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(Z)}{\overset{I}{\otimes}}\right]^{-1} \left[\underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(T)-\delta(Z)}{\overset{I}{\otimes}}\right] v.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|u(Z)\| &\leq \|v\| \left(1 + \frac{\|\delta(Z) - \delta(T)\|}{1 - \|\delta(Z)\|}\right). \\
&\leq \|v\| \left(1 + \frac{2\|\delta(Z) - \delta(T)\|}{1 - \|\delta(Z)\|^2}\right). \qquad (4.18)
\end{aligned}
$$

Now let Z approach T non-tangentially (such as along $T+tK$), and the right-hand side of (4.18) stays bounded; therefore T is a B-point by Corollary 4.2. $\square$

If δ is homogeneous, then the norm of the vector u_T introduced in Lemma 4.14 is $\lim_{r\uparrow 1} \|u(rT)\|$. We shall only prove it when δ is homogeneous of order 1, though the argument can be modified for any positive homogeneity.

Proposition 4.19. *Assume that $\delta(rZ) = r\delta(Z)$. Let $T \in \mathcal{I}_\delta$ be a B-point for ϕ. Then u_T satisfies*

$$
\begin{aligned}
\|u_T\|^2 &= \lim_{r\uparrow 1} \|u(rT)\|^2 \qquad (4.20) \\
&= \liminf_{\mathcal{G}_\delta \ni Z \to T} \frac{\|I - \phi(Z)^*\phi(Z)\|}{1 - \|\delta(Z)\|^2} \qquad (4.21)
\end{aligned}
$$

Proof: Let r_j be a sequence increasing to 1 so that $u(r_jT)$ converge weakly, and hence by Lemma 4.10 in norm, to some vector v. By continuity,

$$
\left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(T)}{\overset{I}{\otimes}}\right] v = \underset{I}{\overset{C}{\otimes}},
$$

so

$$
\begin{aligned}
u(r_jT) &= \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(r_jT)}{\overset{I}{\otimes}}\right]^{-1} \underset{I}{\overset{C}{\otimes}} \\
&= \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{r_j\delta(T)}{\overset{I}{\otimes}}\right]^{-1} \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(T)}{\overset{I}{\otimes}}\right] v \\
&= \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{\delta(T)}{\overset{I}{\otimes}}\right] \left[I - \underset{I}{\overset{D}{\otimes}} \quad \underset{r_j\delta(T)}{\overset{I}{\otimes}}\right]^{-1} v.
\end{aligned}
$$

As $\overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}}$ is a contraction, we have that

$$\ker \left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right] \perp \operatorname{ran} \left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right].$$

So each $u(r_j T)$ is perpendicular to $\ker \left[I - \overset{D}{\underset{I}{\otimes}} \quad \overset{I}{\underset{\delta(T)}{\otimes}} \right]$ and hence v is also. Therefore v is the vector u_T from Lemma 4.14, and (4.20) holds.

To prove (4.21), we need to show that

$$\liminf_{\mathcal{G}_\delta \ni Z \to T} \frac{\|I - \phi(Z)^* \phi(Z)\|}{1 - \|\delta(Z)\|^2} = \lim_{r \uparrow 1} \frac{\|I - \phi(rT)^* \phi(rT)\|}{1 - \|\delta(rT)\|^2} \tag{4.22}$$

Let α denote the left-hand side of (4.22). By Theorem 3.6, we have

$$\begin{aligned} \frac{\|\phi(rT) - W\|^2}{\|I - \phi(rT)^* \phi(rT)\|} &\leq \alpha \left(\frac{\|I - \delta(T)^* \delta(rT)\|^2}{1 - \|\delta(rT)\|^2} \right) \\ &= \alpha \frac{1 - r}{1 + r}. \end{aligned}$$

As in (3.13), we have $\|I - \phi(rT)^* \phi(rT)\| \leq 2 \|W - \phi(rT)\|$. So we have

$$\begin{aligned} \liminf_{r \uparrow 1} \frac{\|I - \phi(rT)^* \phi(rT)\|^2}{(1 - \|\delta(rT)\|^2)^2} &\leq \liminf_{r \uparrow 1} \frac{4 \|W - \phi(rT)\|^2}{(1 - r^2)^2} \\ &\leq \liminf_{r \uparrow 1} \frac{4}{(1 - r^2)^2} \alpha \frac{1 - r}{1 + r} \|I - \phi(rT)^* \phi(rT)\| \end{aligned}$$

Dividing by

$$\frac{\|I - \phi(rT)^* \phi(rT)\|}{1 - r^2}$$

we get

$$\begin{aligned} \liminf_{r \uparrow 1} \frac{\|I - \phi(rT)^* \phi(rT)\|}{1 - \|\delta(rT)\|^2} &\leq \liminf_{r \uparrow 1} \frac{4\alpha}{(1 + r)^2} \\ &= \alpha. \end{aligned}$$

So we have proved that (4.22) holds. $\square$

5 Derivatives at B-points

Let T be a B-point of ϕ in $\mathcal{I}_\delta$, and $W = \phi(T)$. We will keep these fixed for the remainder of the section. Let us make the following assumption:

(A) The complex span of $\Delta(T)$ is all of $\mathbb{M}_n^d$.

This condition ensures that we have a full set of transverse directions pointing into G_δ.. Assumption (A) is equivalent to the following two conditions holding:

(A1) The set $\Delta(T)$ is non-empty.

(A2) The complex span of

$$\Sigma(T) \ := \ \{H \in \mathbb{M}_n^d \ : \ \delta(T)^* \nabla \delta(T)[H] \ \text{ is self-adjoint}\}$$

is all of $\mathbb{M}_n^d$.

Let $H \in \Gamma(T)$; we want to show that

$$\lim_{t \downarrow 0} \frac{1}{t}[\phi(T + tH) - W]$$

exists and is holomorphic in H.

First, let us sharpen Lemma 4.7.

Lemma 5.1. *Let $\beta > 0$. Then there exists $\varepsilon > 0$ such that, if*

$$\mathrm{Re}\ [\delta(T)^* \nabla \delta(T)[H]] \ \leq \ -\beta I \tag{5.2}$$

then

$$T + tH \in \mathcal{G}_\delta \ \forall\ 0 < t < \varepsilon.$$

Proof: This follows from (4.8), and the observation that the error term can be bounded by some absolute constant (which depends on δ and its derivatives in a neighborhood of T) times t^2. □

Let

$$U \ = \ \{(z, H) \in \mathbb{C} \times \mathbb{M}_n^d : \ H \in \Gamma(T),\ zH \in \Gamma(T),\ T + tzH \in \mathcal{G}_\delta \ \forall\, 0 < t \leq 1\}.$$

Consider the set of functions

$$\{\xi_t(z, H) = \frac{1}{t}[\phi(T + tzH) - W] \ : \ 0 < t < 1\}.$$

These functions are all defined on U, and are locally bounded by Corollary 4.2, and are holomorphic in both z and H. So they form a normal family by Montel's theorem. Let $S = (t_n)$ be a sequence decreasing to 0 such that

$$\lim_{n\to\infty} \xi_{t_n}(z, H)$$

exists; call this limit $\eta_S(z, H)$. We wish to show that η_S does not, in fact, depend on the choice of sequence S.

Let K be in $\Delta(T)$. Multiplying K by a small positive number if necessary, we can assume that

$$(z, K) \in U \ \forall\ z \in \{\mathrm{Re}\,(z) > 0 \text{ and } |z| < 2\}.$$

Let v be a unit vector in $\mathbb{C}^n$, and define the function

$$f(z) \ = \ \langle \phi(T + zK)v, \phi(T)v \rangle.$$

Then $f : \mathbb{D}(1, 1) \to \mathbb{D}$, where $\mathbb{D}(1, 1)$ is the disk centered at 1 of radius 1. Moreover, 0 is a B-point for f, because, for $t \in (0, 1)$,

$$1 - |f(t)| \ \leq \ 2\|I - \phi(T + tK)^*\phi(T + tK)\|$$

and, letting M denote $\|\nabla\delta(T)\|$,

$$\|I - \delta(T + tK)^*\delta(T + tK)\| \ \leq \ 2Mt + O(t^2),$$

so

$$\lim_{t\downarrow 0} \frac{1 - |f(t)|}{t} \ \leq \ 4M \ \lim_{t\downarrow 0} \frac{\|I - \phi(T + tK)^*\phi(T + tK)\|}{1 - \|\delta(T + tK)\|^2},$$

and the right-hand side is bounded since T is a B-point of ϕ.

So we can apply the one variable Julia-Carathéodory Theorem 2.1 to conclude that

$$\lim_{t\downarrow 0} \frac{f(t) - f(0)}{t}$$

exists — indeed the limit exists as 0 is approached non-tangentially from within $\mathbb{D}(1, 1)$. Since this holds for every unit vector v, by polarization we can conclude that

$$\lim_{t\downarrow 0} \frac{1}{t}[\phi(T + tK) - W]$$

exists, so every function η_S agrees on points of the form (t, K), and, by holomorphicity, on points in U of the form (z, K), whenever $K \in \Delta(T)$.

Now, fix some element $K \in \Delta(T)$ such that $(2, K) \in U$. Then, for some $\varepsilon > 0$, if H is in $\Sigma(T)$ and $\|H\| < \varepsilon$, then $K + H$ is in $\Delta(T)$. So all the η_S agree on points in U of the form $(t, K + H)$, with $t > 0$ and H in $\Sigma(T)$. By assumption (A2), since η_S is holomorphic in H, we get that in fact η_S is independent of the choice of S.

Thus we have proved:

Theorem 5.3. *Suppose $T \in \mathcal{I}_\delta$ is a B-point of ϕ, and assumption (A) holds. Then*

$$\eta(H) \;=\; \lim_{t \downarrow 0} \frac{\phi(T + tH) - \phi(T)}{t} \tag{5.4}$$

exists for all $H \in \Gamma(T)$. Moreover η is a holomorphic function of H, homogeneous of degree 1.

6 Deducing the scalar case from the nc theorem

Knowing Theorem 3.6, how could we deduce the classical Julia inequality? We would need to know that any holomorphic function $\psi : \mathbb{D} \to \mathbb{D}$ could be extended to a function in the Schur class of $\{x \in \mathbb{M}^{[1]} : \|x\| < 1\}$. This indeed holds, by von Neumann's inequality [36].

More generally, suppose Ω is a domain in $\mathbb{C}^d$, and $\psi : \Omega \to \mathbb{D}$ is holomorphic. If we wish to deduce a Julia inequality using the results of the previous section, first we need to find a matrix γ of polynomials in d commuting variables so that $\Omega = \{z \in \mathbb{C}^d : \|\gamma(z)\| < 1\}$. This of course may not be possible, though it is for the polydisk, or any polynomial polyhedron, and for the ball.

We can define G_γ to be the subset of $\mathcal{G}_\gamma$ consisting of *commuting* d-tuples of matrices $x = (x^1, \ldots, x^d)$ for which $\|\gamma(x)\| < 1$. The original function ψ, which is holomorphic and bounded on $G_\gamma \cap \mathbb{M}_1^d$, can be extended to all of G_γ, either by approximating ψ by polynomials, or using the Taylor functional calculus (see [4, 9] for a discussion). Let us define $H^\infty(G_\gamma)$ to be those holomorphic functions ψ so that

$$\|\psi\|_{H^\infty(G_\gamma)} \;:=\; \sup\{\|\psi(x)\| : x \in G_\gamma\}$$

is finite. (If Ω were the ball, we would get the multiplier algebra of the Drury-Arveson space). By [5], any function in $H^\infty(G_\gamma)$ can be extended to a bounded function on $\mathcal{G}_\gamma$ of the same norm. So we can deduce the following corollary of Theorem 3.6.

Corollary 6.1. *Let $\Omega = G_\gamma \cap \mathbb{M}_1^d$ be a domain in $\mathbb{C}^d$, and let ψ be a holomorphic function on Ω. Assume that $\|\psi\|_{H^\infty(G_\gamma)} = 1$. Suppose $\tau \in \partial\Omega$ satisfies*

$$\liminf_{z \to \tau} \frac{1 - |\psi(z)|^2}{1 - \|\gamma(z)\|^2} \;=\; \alpha.$$

Then there exists a complex number ω of modulus 1 so that

$$\frac{|\psi(z) - \omega|^2}{1 - |\psi(z)|^2} \;\leq\; \alpha \left(\frac{\|I - \gamma(\tau)^* \gamma(z)\|^2}{1 - \|\gamma(z)\|^2} \right).$$

In the case of the ball, Corollary 6.1 is proved as Theorem 8.5.3 in [32], though with the weaker assumption that ψ is bounded by 1 in the sup-norm, not in the multiplier norm of the Drury-Arveson space. In [37], K. Wlodarczyk obtained a version for the unit ball of any J^*-algebra, which includes the polydisk.

Theorem 5.3 also can be applied to the scalar case. Assumptions (A1) and (A2) can be checked in many concrete cases, such as the ball or the polydisk.

Corollary 6.2. *Assume Ω, ψ and τ are as in Corollary 6.1, and that (A) holds at τ. Then ψ has a directional derivative in all inward directions at τ, and moreover this directional derivative is a holomorphic function of the direction.*

Of course, if ψ were regular at τ, the directional derivative would be a linear function of the direction.

7 Examples

Example 7.1. Suppose

$$\delta(Z) \;=\; \begin{pmatrix} Z^1 & & \\ & \ddots & \\ & & Z^d \end{pmatrix}$$

Then $T \in \mathcal{I}_\delta$ if and only if each T^r is an isometry. We have

$$\Delta(T) \;=\; \{\|K\| \leq 1 \text{ and } T^{r*}K^r < 0,\ 1 \leq r \leq d\},$$

and this is non-empty (*e.g.* take $K = -T$). The set $\Sigma(T)$ is the set of H such that each H^r is T^r times a self-adjoint, so (A2) is also satisfied.

Let $d = 2$, and consider the scalar rational inner function

$$f(z,w) \;=\; \frac{z + w - 2wz}{2 - z - w}. \tag{7.2}$$

This has a B-point at $(1,1)$. By Andô's theorem, f is of norm one on G_δ, so by [5], we can extend f to a function of norm one on $\mathcal{G}_\delta$. It is not immediately obvious how to do so.

Claim: The function

$$\phi(Z^1,Z^2) \;=\; \frac{1}{2}(Z^1+Z^2) + \frac{1}{2}(Z^1-Z^2)(2-Z^1-Z^2)^{-1}(Z^1-Z^2) \tag{7.3}$$

is in $\mathcal{S}(\mathcal{G}_\delta)$ and agrees with f on commuting variables.

Let us temporarily accept the claim. For each n, the point (I_n, I_n) is a B-point, because

$$\begin{aligned} \lim_{r\uparrow 1} \frac{\|I - \phi(rI)^*\phi(rI)\|}{1-r^2} \quad &= \quad \lim_{r\uparrow 1} \frac{1-r^2}{1-r^2} \\ &= \quad 1. \end{aligned}$$

Theorem 3.6 then says that for ϕ as in (7.3), and Z a pair of contractions,

$$\frac{\|\phi(Z) - I\|^2}{\|I - \phi(Z)^*\phi(Z)\|} \;\leq\; \frac{\max_{r=1,2}\|I - Z^r\|^2}{1 - \max_{r=1,2}\|Z^r\|^2}. \tag{7.4}$$

If $Z^1 = Z^2$, we get equality in (7.4).

If we calculate $\eta(H)$ as in (5.4), we get that for all H with $\mathrm{Re}\,(H^1)$ and $\mathrm{Re}\,(H^2)$ negative definite, the derivative of ϕ at (I_n, I_n) in the direction H is

$$\eta(H) \;=\; \frac{1}{2}(H^1+H^2) - \frac{1}{2}(H^1-H^2)(H^1+H^2)^{-1}(H^1-H^2),$$

which is clearly holomorphic and homogeneous of degree 1.

Proof of claim: We shall write down a realization for f, and extend it to non-commutative variables.

Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \left(\begin{array}{c|cc} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \hline \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{array}\right). \tag{7.5}$$

Let

$$\delta(z,w) \;=\; \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix},$$

and let

$$u(z,w) \;=\; (I - D\delta(z,w))^{-1} C.$$

Since (7.5) is a unitary matrix, the function $A + B\delta(z,w)u(z,w)$ is a rational inner function on $\mathbb{D}^2$, which by inspection agrees with f in (7.2). Now, we keep the same unitary, and using (4.3) - (4.5) (where $\mathcal{E}$ is just $\mathbb{C}$, and $J = 2$), we get a formula for ϕ, which, after some algebra, becomes

$$\phi(Z) \;=\; \frac{1}{2}\begin{pmatrix} Z^1 & Z^2 \end{pmatrix} \begin{pmatrix} I - \frac{1}{2}Z^1 & \frac{1}{2}Z^2 \\ \frac{1}{2}Z^1 & I - \frac{1}{2}Z^2 \end{pmatrix}^{-1} \begin{pmatrix} I \\ I \end{pmatrix}. \tag{7.6}$$

Inverting the matrix on the right-hand side of (7.6) using the Boltz-Banachiewicz formula does not lead to a nice formula; but if one expands the inverse in a Neumann series, and observes that

$$\begin{aligned} \begin{pmatrix} Z^1 & -Z^2 \\ -Z^1 & Z^2 \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix} &\;=\; (Z^1 - Z^2)\begin{pmatrix} I \\ -I \end{pmatrix} \\ \begin{pmatrix} Z^1 & -Z^2 \\ -Z^1 & Z^2 \end{pmatrix} \begin{pmatrix} A \\ -A \end{pmatrix} &\;=\; (Z^1 + Z^2)\begin{pmatrix} A \\ -A \end{pmatrix}, \end{aligned}$$

then for $k \geq 1$, we get

$$\frac{1}{2^k}\begin{pmatrix} Z^1 & -Z^2 \\ -Z^1 & Z^2 \end{pmatrix}^k \begin{pmatrix} I \\ I \end{pmatrix} \;=\; \frac{1}{2^k}(Z^1 + Z^2)^{k-1}(Z^1 - Z^2)\begin{pmatrix} I \\ -I \end{pmatrix}.$$

Then (7.6) becomes

$$\phi(Z) \;=\; \frac{1}{2}(Z^1 + Z^2) + \frac{1}{4}(Z^1 - Z^2)\left[\sum_{j=0}^{\infty} \frac{1}{2^j}(Z^1 + Z^2)^j\right](Z^1 - Z^2)$$

and summing the Neumann series we get (7.3), as claimed. $\triangleleft$

Remark: The function

$$\psi(Z) \;=\; (Z^1 + Z^2 - Z^1 Z^2 - Z^2 Z^1)(2I - Z^1 - Z^2)^{-1}$$

is another nc extension of f, but it is not in $\mathcal{S}(\mathcal{G}_\delta)$. Indeed, evaluating it on the pair of unitaries

$$Z = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

we get

$$\psi(Z) \;=\; \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix},$$

which has norm $1 + \sqrt{2}$. So by continuity

$$\|\psi\|_{\mathcal{G}_\delta} \;\geq\; \sup_{0<r<1} \|\psi(rZ)\| \;>\; 1.$$

Example 7.7. Let Ω be the classical Cartan domain of symmetric J-by-J contractive matrices in $d = \frac{J(J+1)}{2}$ dimensions. There is an obvious embedding δ that takes d numbers and writes them as a J-by-J symmetric matrix, and we can extend this map to matrices, giving $\mathcal{G}_\delta$ and the commutative version G_δ. A point is in the distinguished boundary of $\mathcal{G}_\delta$ when $\delta(T)$ is a symmetric isometry. (A1) holds at every distinguished boundary point, and so does (A2), since $\Sigma(T)$ is the set of H such that $\delta(H)$ can be written as the sum of $d(T)$ times a self-adjoint matrix and $(I - \delta(T)^*\delta(T))$ times anything. So if ϕ is in the Schur class of $\mathcal{G}_\delta$ or G_δ, we can apply both Theorem 3.6 and Theorem 5.3.

References

[1] M. Abate, *The Julia-Wolff-Carathéodory theorem in polydisks*, J. Anal. Math. **74** (1998), 275–306. ↑2

[2] J. Agler and J.E. McCarthy, *Global holomorphic functions in several non-commuting variables II.* Canadian Math. Bull., To appear. ↑6

[3] ———, *Global holomorphic functions in several non-commuting variables*, Canad. J. Math. **67** (2015), no. 2, 241–285. ↑6, 7, 13

[4] ———, *Operator theory and the Oka extension theorem*, Hiroshima Math. J. **45** (2015), no. 1, 9–34. ↑21

[5] ———, *Pick interpolation for free holomorphic functions*, Amer. J. Math. **137** (2015), no. 6, 1685–1701. ↑21, 23

[6] ———, *Aspects of non-commutative function theory*, Concrete Operators **3** (2016), 15–24. ↑2

[7] J. Agler, J.E. McCarthy, and N.J. Young, *Facial behavior of analytic functions on the bidisk* **43** (2011), 478–494. ↑2

[8] ———, *A Carathéodory theorem for the bidisk via Hilbert space methods*, Math. Ann. **352** (2012), no. 3, 581–624. ↑2, 6, 11

[9] ———, *On the representation of holomorphic functions on polyhedra*, Michigan Math. J. **62** (2013), no. 4, 675–689. ↑21

[10] J. Agler, R. Tully-Doyle, and N. J. Young, *Boundary behavior of analytic functions of two variables via generalized models*, Indag. Math. (N.S.) **23** (2012), no. 4, 995–1027. ↑2

[11] D. Alpay and D. S. Kalyuzhnyi-Verbovetzkii, *Matrix-J-unitary non-commutative rational formal power series*, The state space method generalizations and applications, 2006, pp. 49–113. ↑2

[12] Sriram Balasubramanian, *Toeplitz corona and the Douglas property for free functions*, J. Math. Anal. Appl. **428** (2015), no. 1, 1–11. ↑2

[13] J.A. Ball, G. Marx, and V. Vinnikov, *Interpolation and transfer function realization for the non-commutative Schur-Agler class.* To appear. arXiv:1602.00762. ↑6

[14] Joseph A. Ball, Gilbert Groenewald, and Tanit Malakorn, *Conservative structured noncommutative multidimensional linear systems*, The state space method generalizations and applications, 2006, pp. 179–223. ↑2

[15] Kelly Bickel and Greg Knese, *Inner functions on the bidisk and associated Hilbert spaces*, J. Funct. Anal. **265** (2013), no. 11, 2753–2790. ↑2

[16] C. Carathéodory, *Über die Winkelderivierten von beschränkten analytischen Funktionen*, Sitzunber. Preuss. Akad. Wiss. (1929), 39–52. ↑1, 6

[17] Jakob Cimpric, J. William Helton, Scott McCullough, and Christopher Nelson, *A noncommutative real nullstellensatz corresponds to a noncommutative real ideal: algorithms*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 5, 1060–1086. ↑2

[18] J. William Helton, Igor Klep, and Scott McCullough, *Proper analytic free maps*, J. Funct. Anal. **260** (2011), no. 5, 1476–1490. ↑2

[19] J. William Helton and Scott McCullough, *Every convex free basic semi-algebraic set has an LMI representation*, Ann. of Math. (2) **176** (2012), no. 2, 979–1013. ↑2

[20] Michel Hervé, *Quelques propriétés des applications analytiques d'une boule à m dimensions dan elle-même*, J. Math. Pures Appl. (9) **42** (1963), 117–147. ↑1

[21] F. Jafari, *Angular derivatives in polydisks*, Indian J. Math. **35** (1993), 197–212. ↑2

[22] G. Julia, *Extension nouvelle d'un lemme de Schwarz*, Acta Math. **42** (1920), 349–355. ↑1, 6

[23] M.T. Jury, *An improved Julia-Caratheodory theorem for Schur-Agler mappings of the unit ball.* arXiv:0707.3423. ↑1

[24] Dmitry S. Kaliuzhnyi-Verbovetskyi and Victor Vinnikov, *Foundations of free noncommutative function theory*, AMS, Providence, 2014. ↑2

[25] John E. McCarthy and R. Timoney, *Non-commutative automorphisms of bounded non-commutative domains*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), no. 5, 1037–1045. ↑2

[26] Paul S. Muhly and Baruch Solel, *Tensorial function theory: from Berezin transforms to Taylor's Taylor series and back*, Integral Equations Operator Theory **76** (2013), no. 4, 463–508. ↑2

[27] J. E. Pascoe, *The inverse function theorem and the Jacobian conjecture for free analysis*, Math. Z. **278** (2014), no. 3-4, 987–994. ↑2

[28] J. E. Pascoe and R. Tully-Doyle, *Free Pick functions: representations, asymptotic behavior and matrix monotonicity in several noncommuting variables*, J. Funct. Anal. **273** (2017), no. 1, 283–328. ↑2, 6

[29] G. Popescu, *Von Neumann inequality for* $(B(\mathcal{H})^n)_1$, Math. Scand. **68** (1991), 292–304. ↑3

[30] Gelu Popescu, *Free holomorphic functions on the unit ball of* $B(\mathcal{H})^n$, J. Funct. Anal. **241** (2006), no. 1, 268–333. ↑2

[31] ———, *Free biholomorphic classification of noncommutative domains*, Int. Math. Res. Not. IMRN **4** (2011), 784–850. ↑2

[32] W. Rudin, *Zeros of holomorphic functions in balls*, Indag. Math. **38** (1976), 57–65. ↑1, 22

[33] D. Sarason, *Sub-Hardy Hilbert spaces in the unit disk*, University of Arkansas Lecture Notes, Wiley, New York, 1994. ↑1

[34] J.L. Taylor, *A general framework for a multi-operator functional calculus*, Advances in Math. **9** (1972), 183–252. ↑2

[35] ———, *Functions of several non-commuting variables*, Bull. Amer. Math. Soc. **79** (1973), 1–34. ↑2

[36] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. **4** (1951), 258–281. ↑21

[37] K. Wlodarczyk, *Julia's lemma and Wolff's theorem for* $J*$*-algebras*, Proc. Amer. Math. Soc. **99** (1987), 472–476. ↑2, 22