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Weak Factorizations of the Hardy space $H^1(R^n)$ in terms of Multilinear Riesz Transforms

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Abstract

This paper provides a constructive proof of the weak factorizations of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms. As a direct application, we obtain a new proof of the characterization of $\mathrm{BMO}(\mathbb{R}^n)$ (the dual of $H^1(\mathbb{R}^n)$) via commutators of the multilinear Riesz transforms.

Keywords: Hardy space, BMO space, multilinear Riesz transform, weak factorization.

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1 Introduction and Statement of Main Results

The real-variable Hardy space theory on n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 1$) plays an important role in harmonic analysis and has been systematically developed. An important result about the Hardy space is the weak factorization obtained by Coifman, Rochberg and Weiss [2]. This factorization proves that all $H^1(\mathbb{R}^n)$ can be written in terms of bilinear forms associated to the Riesz transforms, with the basic building blocks being:

$$\Pi_j(f,g) = fR_jg + gR_jf,$$

with R_j the jth Riesz transform $R_jf(x) = \displaystyle\int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x-y|^{n+1}}\,dy$. This result follows as a corollary of the characterization of the function space $\mathrm{BMO}(\mathbb{R}^n)$ in terms of the boundedness of the commutators $[b, R_j](f) = bR_jf - R_j(bf)$.

The main goals of this paper are to provide a constructive proof of the weak factorizations of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms. As a direct corollary, we obtain a full characterization of $\mathrm{BMO}(\mathbb{R}^n)$ (the dual of $H^1(\mathbb{R}^n)$) via commutators of the multilinear Riesz transforms. Our strategy and approach will be to modify the direct constructive proof of Uchiyama in [10] for the weak factorization of the Hardy spaces.

We now recall the multilinear Calderón–Zygmund operators (see for example the statements in [5]). Let $K(y_0, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $\{y_0 = y_1 = \cdots = y_m\}$. K is said to be an m-linear Calderón–Zygmund kernel if there exist positive constants A and ϵ such that

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A}{\left(\sum_{k,l=0} |y_k - y_l|\right)^{mn}} \tag{1.1}$$

and

$$|K(y_0, y_1, \ldots, y_j, \ldots, y_m) - K(y_0, y_1, \ldots, y_j', \ldots, y_m)| \le \frac{A|y_j - y_j'|^\epsilon}{\left(\sum_{k,l=0} |y_k - y_l|\right)^{mn+\epsilon}} \tag{1.2}$$

for all $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0\le k\le m} |y_j - y_k|$.

Suppose T is an m-linear operator defined on $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ associated with the m-linear Calderón–Zygmund kernel K, i.e.,

$$T(f_1, \ldots, f_m)(x) := \int_{\mathbb{R}^{mn}} K(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j)\, dy_1 \cdots dy_m, \tag{1.3}$$

for all $x \notin \cap_{j=1}^m \mathrm{supp}(f_j)$, where $f_1, \ldots, f_m$ are m functions on $\mathbb{R}^n$ with $\cap_{j=1}^m \mathrm{supp}(f_j) \neq \emptyset$. If

$$T : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

for some $1 < p_1, \ldots, p_m$ and p with $p^{-1} = \sum_{j=1}^m p_j^{-1}$, then we say T is an m-linear Calderón–Zygmund operator. According to [5, Theorem 3], T can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all for $1 < p_1, \ldots, p_m$ and p with $p^{-1} = \sum_{j=1}^m p_j^{-1}$.

We also define that T is mn-homogeneous if T satisfies

$$|T(\chi_{B_0}, \ldots, \chi_{B_m})(x)| \ge \frac{C}{M^{mn}}$$

for $m + 1$ balls $B_0 = B_0(x_0, r), \ldots, B_m = B_m(x_m, r)$ satisfying $|y_0 - y_l| \approx Mr$ for $l = 1, 2, \ldots, m$ and for all $x \in B_0$, where $r > 0$ and $M > 10$ a positive number.

Another stronger version of mn-homogeneous is as follows.

$$K(x_0, \ldots, x_m) \ge \frac{C}{M^{mn}}$$

or

$$K(x_0, \ldots, x_m) \le -\frac{C}{M^{mn}}$$

for $m + 1$ pairwisely disjoint balls $B_0 = B_0(x_0, r), \ldots, B_m = B_m(x_m, r)$ satisfying $|y_0 - y_l| \approx Mr$ and for all $x_l \in B_l$ for $l = 1, 2, \ldots, m$, where $r > 0$ and $M > 10$ is a positive number. It is easy to see that this stronger version implies the version above.

In analogy with the linear case, we define the lth possible multilinear commutators of the mth multilinear Calderón–Zygmund operator T as follows.

Definition 1.1. *Suppose T is an m-linear Calderón–Zygmund operator as defined above. For $l = 1, 2, \dots, m$, we set*

$$[b,T]_l(f_1,\dots,f_m)(x) := T(f_1,\dots,bf_l,\dots,f_m)(x) - bT(f_1,\dots,f_m)(x). \tag{1.4}$$

This is simply measuring the commutation properties in each linear coordinate separately.

Dual to the multilinear commutator, in both language and via a formal computation, we define the multilinear "multiplication" operators Π_l:

Definition 1.2. *Suppose T is an m-linear Calderón–Zygmund operator as defined above. For $l = 1, 2, \dots, m$,*

$$\Pi_l(g,h_1,\dots,h_m)(x) := h_l T_l^*(h_1,\dots,h_{l-1},g,h_{l+1},\dots,h_m)(x) - gT(h_1,\dots,h_m)(x), \tag{1.5}$$

where T_l^* is the lth partial adjoint of T, defined as

$$T_l^*(h_1,\dots,h_m)(x) := \int_{\mathbb{R}^{mn}} K(y_l,y_1,\dots,y_{l-1},x,y_{l+1},\dots,y_m)\prod_{j=1}^m h_j(y_j)\,dy_1\cdots dy_m. \tag{1.6}$$

Our main result is then the following factorization result for $H^1(\mathbb{R}^n)$ in terms of the multilinear operators Π_l. Again, this is in direct analogy with the rest in the linear case obtained by Coifman, Rochberg, and Weiss in [2].

Theorem 1.3. *Suppose $1 \le l \le m$, and $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$ with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

And suppose that T is an m-linear Calderón–Zygmund operator, which is mn-homogeneous in the lth component. Then for every $f \in H^1(\mathbb{R}^n)$, there exists sequences $\{\lambda_s^k\} \in \ell^1$ and functions $g_s^k \in L^{p'}(\mathbb{R}^n)$, $h_{s,1}^k \in L^{p_1}(\mathbb{R}^n)$,..., $h_{s,m}^k \in L^{p_m}(\mathbb{R}^n)$ such that

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k\, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) \tag{1.7}$$

in the sense of $H^1(\mathbb{R}^n)$. Moreover, we have that:

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \inf\left\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k|\, \|g_s^k\|_{L^{p'}(\mathbb{R}^n)} \|h_{s,1}^k\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_{s,m}^k\|_{L^{p_m}(\mathbb{R}^n)} \right\},$$

where the infimum above is taken over all possible representations of f that satisfy (1.7).

We then obtain the following new characterization of $\mathrm{BMO}(\mathbb{R}^n)$ in terms of the commutators with the multilinear Riesz transforms; again in analogy with the main results in [2].

Theorem 1.4. *Let $1 \le l \le m$. Suppose that T is an m-linear Calderón–Zygmund operator. If b is in* $\mathrm{BMO}(\mathbb{R}^n)$*, then the commutator $[b,T]_l(f_1,\ldots,f_m)(x)$ is a bounded map from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1<p_1,\ldots,p_m<\infty$ and $1\le p<\infty$, with*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}$$

and with the operator norm

$$\|[b,T]_l : L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)\| \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Conversely, for $b\in\cup_{q>1}L^q_{loc}(\mathbb{R}^n)$, if T is mn-homogeneous in the lth component, and $[b,T]_l$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $1<p_1,\ldots,p_m<\infty$ and $1\le p<\infty$, with

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p},$$

then b is in $\mathrm{BMO}(\mathbb{R}^n)$ *and $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\le C\|[b,T]_l : L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)\|$.*

As a specific example of such operator T which is an m-linear Calderón–Zygmund operator and is mn-homogeneous, we now recall the multilinear Riesz transforms, see [5, Page 162] for example.

Definition 1.5. *Suppose $f_1,\ldots,f_m$ are m functions on $\mathbb{R}^n$. For $j=1,2,\ldots,m$,*

$$\vec{R}_j(f_1,\ldots,f_m)(x) := \int_{\mathbb{R}^{mn}} \vec{K}_j(x,y_1,\ldots,y_m)\prod_{s=1}^m f_s(y_s)\,dy_1\cdots dy_m, \tag{1.8}$$

where the kernel $\vec{K}_j(x,y_1,\ldots,y_m)$ is defined as

$$\vec{K}_j(x,y_1,\ldots,y_m) := \frac{x-y_j}{|(x-y_1,\ldots,x-y_m)|^{mn+1}}. \tag{1.9}$$

To be more specific,

$$\vec{R}_j = (R_j^{(1)},\ldots,R_j^{(n)}),$$

where for each $i=1,2,\ldots,n$, $R_j^{(i)}$ is the multilinear operator with the kernel

$$K_j^{(i)}(x,y_1,\ldots,y_m) := \frac{x^i-y_j^i}{|(x-y_1,\ldots,x-y_m)|^{mn+1}}.$$

Here $x=(x^1,\ldots,x^m)$ and $y_j=(y_j^1,\ldots,y_j^m)$. According to [5, Corollary 2], $\vec{R}_j$ is an m-linear Calderón–Zygmund operator for $j=1,2,\ldots,m$. Moreover, we have that

$$|\vec{R}_j(\chi_{B_0},\ldots,\chi_{B_m})(x)| = \left|\int_{B_1}\cdots\int_{B_m}\frac{x-y_j}{|(x-y_1,\ldots,x-y_m)|^{mn+1}}dy_1\cdots dy_m\right| \ge \frac{C}{M^{mn}}$$

for $m+1$ pairwisely disjoint balls $B_0=B_0(x_0,r),\ldots,B_m=B_m(x_m,r)$ satisfying $|y_0-y_l|\approx Mr$ for $l=1,2,\ldots,m$, $x\in B_0$, $r>0$, and $M>10$ a positive number.

Thus, $\vec{R}_j$ is mn-homogeneous.

Remark 1.6. *As in Corollary 2 in [5, Page 162], they listed a specific multilinear Calderón–Zygmund kernel of the form*

$$K(x_1,\dots,x_m)=\frac{\Omega\Big(\frac{(x_1,\dots,x_m)}{|(x_1,\dots,x_m)|}\Big)}{|(x_1,\dots,x_m)|^{mn}},$$

where Ω is an integrable function with mean value zero on the sphere $\mathbb{S}^{mn-1}$ which is Lipschitz of order $\epsilon>0$. We point out that it is possible to choose kernels of this type that satisfy the mn-homogeneous condition as we stated above. The Riesz transforms $\vec{R}_j$ are special examples of this form.

Remark 1.7. *We remark that Theorem 1.4 was obtained by Chaffee in [1]. His proof uses a technique applied by Janson [6], which is different than that used here. One advantage of the approach taken in this paper is that it provides for a constructive algorithm to produce the weak factorization of $H^1(\mathbb{R}^n)$. As mentioned in [1] it would be interesting to show the equivalence between $\mathrm{BMO}(\mathbb{R}^n)$ and the commutators when $p<1$. Both the methods used there and in this paper hinge upon duality, which won't be a viable strategy when $p<1$.*

2 Weak Factorization of the Hardy space $H^1(\mathbb{R}^n)$

In this section we turn to proving Theorem 1.3. We collect some facts that will be useful in proving the main result.

We first provide the following estimate of the multilinear operator Π_l, which is defined in Definition 1.2.

Proposition 2.1. *Suppose $1\le l\le m$. Let $1<p_1,\dots,p_m<\infty$ and $1\le p<\infty$ with*

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

There exists a positive constant C such that for any $g\in L^{p'}(\mathbb{R}^n)$ and $h_i\in L^{p_i}(\mathbb{R}^n)$, $i=1,\dots,m$,

$$\|\Pi_l(g,h_1,\dots,h_m)\|_{H^1(\mathbb{R}^n)}\le C\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_m\|_{L^{p_m}(\mathbb{R}^n)}.$$

Proof. Note that for $p_1,\dots,p_m\in(1,\infty)$, $p\in[1,\infty)$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, and for any $g\in L^{p'}(\mathbb{R}^n)$ and $h_i\in L^{p_i}(\mathbb{R}^n)$, $i=1,\dots,m$, we have $\Pi_l(g,h_1,\dots,h_m)(x)\in L^1(\mathbb{R}^n)$ by Hölder duality. Moreover, we have

$$\int_{\mathbb{R}^n}\Pi_l(g,h_1,\dots,h_m)(x)\,dx=0.$$

Hence, for $b\in\mathrm{BMO}(\mathbb{R}^n)$, we have

$$\left|\int_{\mathbb{R}^n}b(x)\Pi_l(g,h_1,\dots,h_m)(x)dx\right|=\left|\int_{\mathbb{R}^n}g(x)[b,T]_l(h_1,\dots,h_m)(x)dx\right|$$

$$\leq C\|h_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_m\|_{L^{p_m}(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Here in the last equality we use [7, Theore 3.18] which provides an estimate for the multilinear commutator in terms of BMO. Therefore, $\Pi_l(g, h_1, \ldots, h_m)$ is in $H^1(\mathbb{R}^n)$, with

$$\|\Pi_l(g, h_1, \ldots, h_m)\|_{H^1(\mathbb{R}^n)} \leq C\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_m\|_{L^{p_m}(\mathbb{R}^n)}.$$

The proof of Proposition 2.1 is completed. □

Next, we recall a technical lemma about certain $H^1(\mathbb{R}^n)$ functions.

Lemma 2.2. *Suppose f is a function defined on $\mathbb{R}^n$ satisfying: $\int_{\mathbb{R}^n} f(x)\,dx = 0$, and $|f(x)| \leq \chi_{B(x_0,1)}(x) + \chi_{B(y_0,1)}(x)$, where $|x_0 - y_0| := M > 10$. Then we have*

$$\|f\|_{H^1(\mathbb{R}^n)} \leq C_n \log M. \tag{2.1}$$

We can obtain this lemma using the maximal function characterization of $H^1(\mathbb{R}^n)$, as well as the atomic decomposition characterization of $H^1(\mathbb{R}^n)$. For details of the proof, we refer to similar versions of this lemma in [3] and [8].

Suppose $1 \leq l \leq m$. Ideally, given an $H^1(\mathbb{R}^n)$-atom a, we would like to find functions $g \in L^{p'}(\mathbb{R}^n)$, $h_1 \in L^{p_1}(\mathbb{R}^n), \ldots, h_m \in L^{p_m}(\mathbb{R}^n)$ such that $\Pi_l(g, h_1, \ldots, h_m) = a$ pointwise. While this can not be accomplished in general, the Theorem below shows that it is "almost" true.

Theorem 2.3. *Suppose $1 \leq l \leq m$. Suppose that T is an m-linear Calderón–Zygmund operator, which is mn-homogeneous in the lth component. For every $H^1(\mathbb{R}^n)$-atom $a(x)$ and for all $\varepsilon > 0$ and for all $1 < p_1, \ldots, p_m < \infty$ and $1 \leq p < \infty$, with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

there exists $g \in L^{p'}(\mathbb{R}^n)$, $h_1 \in L^{p_1}(\mathbb{R}^n)$,..., $h_m \in L^{p_m}(\mathbb{R}^n)$ and a large positive number M (depending only on ε) such that:

$$\|a - \Pi_l(g, h_1, \ldots, h_m)\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and that $\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_m\|_{L^{p_m}(\mathbb{R}^n)} \leq CM^{mn}$, where C is an absolute positive constant.

Proof. Let $a(x)$ be an $H^1(\mathbb{R}^n)$-atom, supported in $B(x_0, r)$, satisfying that

$$\int_{\mathbb{R}^n} a(x)dx = 0 \quad \text{and} \quad \|a\|_{L^\infty(\mathbb{R}^n)} \leq r^{-n}.$$

Fix $1 \leq l \leq m$ and fix $\varepsilon > 0$. Choose M sufficiently large so that

$$\frac{\log M}{M^\epsilon} < \varepsilon,$$

where the constant ϵ appeared in the power of M is from the regularity condition (1.2) of the multilinear Calderón–Zygmund kernel K. Now select $y_l \in \mathbb{R}^n$ so that $y_{l,i} - x_{0,i} = \frac{Mr}{\sqrt{n}}$, where $x_{0,i}$ (reps. $y_{l,i}$) is the ith coordinate of x_0 (reps. y_l) for $i = 1, 2, \dots, n$. Note that for this y_l, we have $|x_0 - y_l| = Mr$. Similar to the relation of x_0 and y_l, we choose y_1 such that y_0 and y_1 satisfies the same relationship as x_0 and y_l do. Then by induction we choose $y_2, \dots, y_{l-1}, y_{l+1}, \dots, y_m$.

We then set

$$\begin{aligned} g(x) &:= \chi_{B(y_l,r)}(x), \\ h_j(x) &:= \chi_{B(y_j,r)}(x), \quad j \neq l, \\ h_l(x) &:= \frac{a(x)}{T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)}. \end{aligned}$$

Since T is mn homogeneous, and so is T_l^*, for the specific choice of the functions $h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m$ as above, we have that there exists a positive constant C such that

$$|T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)| \geq CM^{-mn} \qquad \text{for } \ 1 \leq l \leq m. \tag{2.2}$$

From the definitions of the functions g and h_j, we obtain that supp $g = B(y_0, r)$ and supp $h_i = B(x_0, r)$. Moreover,

$$\|g\|_{L^{p'}(\mathbb{R}^n)} \approx r^{\frac{n}{p'}} \quad \text{and} \quad \|h_i\|_{L^{p_i}(\mathbb{R}^n)} \approx r^{\frac{n}{p_i}}$$

for $i = 1, \dots, l-1, l+1, \dots, m$. Also we have

$$\|h_l\|_{L^{p_l}(\mathbb{R}^n)} = \frac{1}{|T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)|} \|a\|_{L^{p_l}(\mathbb{R}^n)} \leq CM^{mn} r^{-n} r^{\frac{n}{p_l}},$$

where the last inequality follows from (2.2). Hence we obtain that

$$\begin{aligned} \|g\|_{L^{p'}(\mathbb{R}^n)} \|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)} &\leq CM^{mn} r^{-n} r^{n(\frac{1}{p'} + \frac{1}{p_1} + \cdots + \frac{1}{p_m})} \\ &\leq CM^{mn}. \end{aligned}$$

Next, we have

$$\begin{aligned} &a(x) - \Pi_l(g, h_1, \dots, h_m)(x) \\ &= a(x) - \Big(h_l T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_{l,m})(x) - gT(h_1, \dots, h_m)(x) \Big) \\ &= a(x) \frac{T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0) - T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x)}{T_l^*(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)} \\ &\qquad + g(x) T(h_1, \dots, h_m)(x) \\ &=: W_1(x) + W_2(x). \end{aligned}$$

By definition, it is obvious that $W_1(x)$ is supported on $B(x_0, r)$ and $W_2(x)$ is supported on $B(y_0, r)$. We first estimate W_1. For $x \in B(x_0, r)$, we have

$$
\begin{aligned}
&|W_1(x)|\\
&= |a(x)| \frac{|T_l^*(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0) - T_l^*(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x)|}{|T_l^*(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|}\\
&\leq C \frac{\|a\|_{L^\infty(\mathbb{R}^n)}}{M^{-mn}} \int_{\prod_{j=1}^m B(y_j, r)} |K(z_l, z_1, \ldots, z_{l-1}, x_0, z_{l+1}, \ldots, z_m)\\
&\qquad\qquad\qquad - K(z_l, z_1, \ldots, z_{l-1}, x, z_{l+1}, \ldots, z_m)| \, dz_1 \cdots dz_m\\
&\leq C M^{mn} r^{-n} \int_{\prod_{j=1}^m B(y_j, r)} \frac{|x_0 - x|^\epsilon}{\big(\sum_{i=1,\ i\neq l}^m |z_l - z_i| + |z_l - x_0|\big)^{mn+\epsilon}} \, dz_1 \cdots dz_m\\
&\leq C M^{mn} r^{-n} r^{mn} \frac{r^\epsilon}{(Mr)^{mn+\epsilon}}\\
&\leq C \frac{1}{M^\epsilon r^n},
\end{aligned}
$$

where in the second inequality we use the regularity condition (1.2) of the multilinear kernel K. Hence we obtain that

$$|W_1(x)| \leq C \frac{1}{M^\epsilon r^n} \chi_{B(x_0, r)}(x).$$

Next we estimate $W_2(x)$. From the definition of $g(x)$ and $h_l(x)$, we have

$$
\begin{aligned}
&|W_2(x)|\\
&= \chi_{B(y_l, r)}(x) |T(h_1, \ldots, h_m)(x)|\\
&= \chi_{B(y_l, r)}(x) \frac{1}{|T_l^*(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|}\\
&\qquad \Big| \int_{\prod_{j=1, j\neq l}^m B(y_j, r) \times B(x_0, r)} \big(K(y_1, \ldots, y_{l-1}, x_0, y_{l+1}, \ldots, y_m)\\
&\qquad\qquad - K(y_1, \ldots, y_{l-1}, x, y_{l+1}, \ldots, y_m)\big) a(y_l) \, dy_1 \cdots dy_m \Big|\\
&\leq C \chi_{B(y_l, r)}(x) M^{mn} \int_{\prod_{j=1, j\neq l}^m B(y_j, r) \times B(x_0, r)} \|a\|_{L^\infty(\mathbb{R}^n)} \frac{|x_0 - x|^\epsilon}{\big(\sum_{s=1}^m |x_0 - z_s|\big)^{mn+\epsilon}} \, dz_1 \cdots dz_m\\
&\leq C \chi_{B(y_l, r)}(x) M^{mn} r^{-n} \frac{r^\epsilon \cdot r^{mn}}{(Mr)^{mn+\epsilon}}\\
&= \frac{C}{M^\epsilon r^n},
\end{aligned}
$$

where in the second equality we use the cancelllation property of the atom $a(y_l)$ Hence we have

$$|W_2(x)| \leq \frac{C}{M^\epsilon r^n} \chi_{B(y_l, r)}(x).$$

Combining the estimates of W_1 and W_2, we obtain that

$$\Big|a(x) - \Pi_l(g, h_1, \ldots, h_m)(x)\Big)\Big| \leq \frac{C}{M^{\epsilon} r^n}(\chi_{B(x_0,r)}(x) + \chi_{B(y_l,r)}(x)). \tag{2.3}$$

Next we point out that

$$\int_{\mathbb{R}^n} \Big[a(x) - \Pi_l(g, h_1, \ldots, h_m)(x)\Big)\Big] dx = 0 \tag{2.4}$$

since the atom $a(x)$ has cancellation and the second integral equals 0 just by the definitions of Π_l.

Then the size estimate (2.3) and the cancellation (2.4), together with Lemma 2.2, imply that

$$\Big\|a(x) - \Pi_l(g, h_1, \ldots, h_m)(x)\Big\|_{H^1(\mathbb{R}^n)} \leq C \frac{\log M}{M^{\epsilon}} < C\varepsilon.$$

This proves the result. □

With this approximation result, we can now prove the main Theorem 1.3.

Proof of Theorem 1.3. By Proposition 2.1, we have that

$$\|\Pi_l(g, h_1, \ldots, h_m)\|_{H^1(\mathbb{R}^n)} \leq C\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)}.$$

It is immediate that for any representation of f as in (1.7), i.e.,

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k),$$

We have that $\|f\|_{H^1(\mathbb{R}^n)}$ is bounded by

$$C \inf \left\{ \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} |\lambda_s^k| \|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} : f \text{ satisfies } (1.7) \right\}.$$

We turn to show that the other inequality holds and that it is possible to obtain such a decomposition for any $f \in H^1(\mathbb{R}^n)$. Utilizing the atomic decomposition, for any $f \in H^1(\mathbb{R}^n)$ we can find a sequence $\{\lambda_s^1\} \in \ell^1$ and sequence of $H^1(\mathbb{R}^n)$-atoms $\{a_s^1\}$ so that $f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1$ and $\sum_{s=1}^{\infty} |\lambda_s^1| \leq C \|f\|_{H^1(\mathbb{R}^n)}$.

We explicitly track the implied absolute constant C appearing from the atomic decomposition since it will play a role in the convergence of the algorithm. Fix $\varepsilon > 0$ so that $\varepsilon C < 1$. We apply Theorem 2.3 to each atom a_s^1. So there exists $g_s^1 \in L^{p'}(\mathbb{R}^n)$, $h_{s,1}^1 \in L^{p_1}(\mathbb{R}^n)$,..., $h_{s,m}^1 \in L^{p_m}(\mathbb{R}^n)$ with

$$\|a_s^1 - \Pi_{j,l}(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1)\|_{H^1(\mathbb{R}^n)} < \varepsilon, \quad \forall s$$

and $\|g_s^1\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_m\|_{L^{p_m}(\mathbb{R}^n)} \leq C(\varepsilon)$, where $C(\varepsilon) = CM^{nm}$ is a constant depending on ε which we can track from Theorem 2.3. Now note that we have

$$\begin{aligned} f &= \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 = \sum_{s=1}^{\infty} \lambda_s^1 \,\Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1) + \sum_{s=1}^{\infty} \lambda_s^1 \left(a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1)\right) \\ &=: M_1 + E_1. \end{aligned}$$

Observe that we have

$$\|E_1\|_{H^1(\mathbb{R}^n)} \leq \sum_{s=1}^{\infty} \left|\lambda_s^1\right| \left\|a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1)\right\|_{H^1(\mathbb{R}^n)} \leq \varepsilon \sum_{s=1}^{\infty} \left|\lambda_s^1\right| \leq \varepsilon C \|f\|_{H^1(\mathbb{R}^n)}.$$

We now iterate the construction on the function E_1. Since $E_1 \in H^1(\mathbb{R}^n)$, we can apply the atomic decomposition in $H^1(\mathbb{R}^n)$ to find a sequence $\{\lambda_s^2\} \in \ell^1$ and a sequence of $H^1(\mathbb{R}^n)$-atoms $\{a_s^2\}$ so that $E_1 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2$ and

$$\sum_{s=1}^{\infty} \left|\lambda_s^2\right| \leq C \|E_1\|_{H^1(\mathbb{R}^n)} \leq \varepsilon C^2 \|f\|_{H^1(\mathbb{R}^n)}.$$

Again, we will apply Theorem 2.3 to each atom a_s^2. So there exists $g_s^2 \in L^{p'}(\mathbb{R}^n)$, $h_{s,1}^2 \in L^{p_1}(\mathbb{R}^n)$,..., $h_{s,m}^2 \in L^{p_m}(\mathbb{R}^n)$ with

$$\left\|a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)\right\|_{H^1(\mathbb{R}^n)} < \varepsilon, \quad \forall s.$$

We then have that:

$$\begin{aligned} E_1 &= \sum_{s=1}^{\infty} \lambda_s^2 a_s^2 = \sum_{s=1}^{\infty} \lambda_s^2 \,\Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2) + \sum_{s=1}^{\infty} \lambda_s^2 \left(a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)\right) \\ &:= M_2 + E_2. \end{aligned}$$

But, as before, observe that

$$\begin{aligned} \|E_2\|_{H^1(\mathbb{R}^n)} &\leq \sum_{s=1}^{\infty} \left|\lambda_s^2\right| \left\|a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)\right\|_{H^1(\mathbb{R}^n)} \leq \varepsilon \sum_{s=1}^{\infty} \left|\lambda_s^2\right| \\ &\leq (\varepsilon C)^2 \|f\|_{H^1(\mathbb{R}^n)}. \end{aligned}$$

And, this implies for f that we have:

$$\begin{aligned} f &= \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 = \sum_{s=1}^{\infty} \lambda_s^1 \,\Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1) + \sum_{s=1}^{\infty} \lambda_s^1 \left(a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1)\right) \\ &= M_1 + E_1 = M_1 + M_2 + E_2 \\ &= \sum_{k=1}^{2} \sum_{s=1}^{\infty} \lambda_s^k \,\Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_2. \end{aligned}$$

Repeating this construction for each $1 \leq k \leq K$ produces functions $g_s^k \in L^{p'}(\mathbb{R}^n)$, $h_{s,1}^k \in L^{p_1}(\mathbb{R}^n), \ldots, h_{s,m}^k \in L^{p_m}(\mathbb{R}^n)$ with $\left\|g_s^k\right\|_{L^{p'}(\mathbb{R}^n)} \left\|h_{s,1}^k\right\|_{L^{p_1}(\mathbb{R}^n)} \cdots \left\|h_{s,m}^k\right\|_{L^{p_m}(\mathbb{R}^n)} \leq C(\varepsilon)$ for all s, sequences $\{\lambda_s^k\} \in \ell^1$ with $\left\|\{\lambda_s^k\}\right\|_{\ell^1} \leq \varepsilon^{k-1} C^k \|f\|_{H^1(\mathbb{R}^n)}$, and a function $E_K \in H^1(\mathbb{R}^n)$ with $\|E_K\|_{H^1(\mathbb{R}^n)} \leq (\varepsilon C)^K \|f\|_{H^1(\mathbb{R}^n)}$ so that

$$f = \sum_{k=1}^{K} \sum_{s=1}^{\infty} \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_K.$$

Passing $K \to \infty$ gives the desired decomposition of

$$f = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k).$$

We also have that:

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \left|\lambda_s^k\right| \leq \sum_{k=1}^{\infty} \varepsilon^{-1} (\varepsilon C)^k \|f\|_{H^1(\mathbb{R}^n)} = \frac{C}{1 - \varepsilon C} \|f\|_{H^1(\mathbb{R}^n)}.$$

□

Finally, we dispense with the proof of Theorem 1.4.

Proof of Theorem 1.4. The upper bound in this theorem is contained in [7, Theorem 3.18]. For the lower bound, suppose that $f \in H^1(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$, where $L_c^\infty(\mathbb{R}^n)$ is the subspace of $L^\infty(\mathbb{R}^n)$ consisting of functions with compact support in $\mathbb{R}^n$. Then using the weak factorization in Theorem 1.3 we have that for every $b \in \cup_{q>1} L_{loc}^q(\mathbb{R}^n)$,

$$\begin{aligned}
\langle b, f \rangle_{L^2(\mathbb{R}^n)} &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \langle b, \Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) \rangle_{L^2(\mathbb{R}^n)} \\
&= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k \langle g_s^k, [b, T]_l(h_{s,1}^k, \ldots, h_{s,m}^k) \rangle_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\left|\langle b, f \rangle_{L^2(\mathbb{R}^n)}\right| &\leq \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \left|\lambda_s^k\right| \left\|[b, T]_l(h_{s,1}^k, \ldots, h_{s,m}^k)\right\|_{L^p(\mathbb{R}^n)} \left\|g_s^k\right\|_{L^{p'}(\mathbb{R}^n)} \\
&\leq \|[b, T]_l : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\| \\
&\qquad \times \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \left|\lambda_s^k\right| \left\|g_s^k\right\|_{L^{p'}(\mathbb{R}^n)} \prod_{j=1}^{m} \left\|h_{s,j}^k\right\|_{L^{p_j}(\mathbb{R}^n)} \\
&\leq C \|[b, T]_l : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\| \, \|f\|_{H^1(\mathbb{R}^n)}.
\end{aligned}$$

By the duality between $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ we have that:

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \approx \sup_{\|f\|_{H^1(\mathbb{R}^n)} \leq 1} \left|\langle b, f \rangle_{L^2(\mathbb{R}^n)}\right| \leq C \|[b, T]_l : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\|.$$

□

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