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The Corona Problem for Kernel Multiplier Algebras

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THE CORONA PROBLEM FOR KERNEL MULTIPLIER ALGEBRAS

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Abstract. We prove an alternate Toeplitz corona theorem for the algebras of pointwise kernel multipliers of Besov-Sobolev spaces on the unit ball in $\mathbb{C}^n$, and for the algebra of bounded analytic functions on certain strictly pseudoconvex domains and polydiscs in higher dimensions as well. This alternate Toeplitz corona theorem extends to more general Hilbert function spaces where it does not require the complete Pick property. Instead, the kernel functions $k_x(y)$ of certain Hilbert function spaces $\mathcal{H}$ are assumed to be invertible multipliers on $\mathcal{H}$, and then we continue a research thread begun by Agler and McCarthy in 1999, and continued by Amar in 2003, and most recently by Trent and Wick in 2009. In dimension $n = 1$ we prove the corona theorem for the kernel multiplier algebras of Besov-Sobolev Banach spaces in the unit disk, extending the result for Hilbert spaces $H^\infty \cap Q_p$ by A. Nicolau and J. Xiao.

Contents

1. Introduction

In 1962 L. Carleson [Car] proved the corona theorem for the algebra of bounded analytic functions on the unit disk. The proof used a beautiful 'corona construction' together with properties of Blaschke products. While there is a large literature on corona theorems for domains in one complex dimension (see e.g. [Nik]), progress in higher dimensions prior to 2011 had been limited to the H^p corona theorem on various bounded domains in $\mathbb{C}^n$, and weaker results restricting N to 2 generators. In fact, apart from the simple cases in

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which the maximal ideal space of the algebra can be identified with a compact subset of $\mathbb{C}^n$, no complete corona theorem was proved in higher dimensions until the 2011 results of S. Costea and the authors in which the corona theorem was established for the multiplier algebras $M_{H_n^2}$ and $M_{\mathcal{D}_n}$ of the Drury-Arveson Hardy space H_n^2 and the Dirichlet space $\mathcal{D}_n$ on the ball in n dimensions. These latter results used the abstract Toeplitz corona theorem for Hilbert function spaces with a complete Pick kernel, see Ball, Trent and Vinnikov [BaTrVi] and Ambrosie and Timotin [AmTi]. The unresolved corona question for the algebra of bounded analytic functions on the ball in higher dimensions has remained a tantalizing problem for over half a century now (see e.g. [CoSaWi2] and [DoKrSaTrWi] for a more detailed history of this problem to date). We note in particular that Varopoulos [Var] gave an example of Carleson measure data in dimension $n=2$ for which there is *no* bounded solution to the $\overline{\partial}$ equation. This poses a significant obstacle to using the $\overline{\partial}$ equation for the multiplier problem, and suggests a more operator theoretic approach akin to the Toeplitz corona theorem in order to solve the corona problem in higher dimensions.

In this paper we prove in particular an alternate Toeplitz corona theorem for all of the algebras of kernel multipliers of Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$ on the ball. These spaces include $H^\infty(\mathbb{B}_n)$, and the alternate Toeplitz corona theorem also extends to the algebra $H^\infty(\Omega)$ of bounded analytic functions on Ω, where Ω is either the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, a sufficiently small C^∞ perturbation of the unit ball $\mathbb{B}_n$, or a bounded strictly pseudoconvex homogeneous complete circular domain in $\mathbb{C}^n$. Moreover, this alternate Toeplitz corona theorem extends to the kernel multiplier space of certain Hilbert function spaces *without* assuming the complete Pick property. This essentially shows that whenever a Hilbert space has one of these special kernels, then the Corona Property for its kernel multiplier algebra reduces to what we call the Convex Poisson Property, a property which can be addressed by methods involving solution of the $\overline{\partial}$ problem.

To illustrate in a very special case, we show that for Ω as above, the algebra $H^\infty(\Omega)$ has the *Corona Property* if and only if the Bergman space $A^2(\Omega)$ has the *Convex Poisson Property* if and only if the Hardy space $H^2(\Omega)$ has the *Convex Poisson Property* (in order to define the Hardy space we need to assume that $\partial\Omega$ is C^2). The Corona Property for $H^\infty(\Omega)$ is this: given $\varphi_1,\dots,\varphi_N \in H^\infty(\Omega)$ satisfying

$$1 \geq \max\left\{|\varphi_1(z)|^2,\dots,|\varphi_N(z)|^2\right\} \geq c^2 > 0, \qquad z\in\Omega,$$

there are a positive constant C and $f_1,\dots,f_N \in H^\infty(\Omega)$ satisfying

$$\begin{aligned}\max\left\{|f_1(z)|^2,\dots,|f_N(z)|^2\right\} &\leq C^2, \qquad z\in\Omega,\\ \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z) &= 1, \qquad z\in\Omega.\end{aligned}$$

For points $\mathbf{a}=(a_1,\dots,a_M)\in\Omega^M$ and $\boldsymbol{\theta}=\theta_0,\dots,\theta_M\in[0,1]^{M+1}$, and for $h\in\mathcal{H}=A^2(\Omega)$ or $H^2(\Omega)$, where Ω is as above, set $\widetilde{k_{a_0}}\equiv 1$ by convention. Then we have

$$\|h\|^2_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} \equiv \sum_{m=0}^M \theta_m \int_\Omega |h|^2 \left|\widetilde{k_{a_m}}\right|^2 dv < \infty.$$

The Convex Poisson Property for the Hilbert space $\mathcal{H}$ (either Bergman or Hardy) is then this: given $\varphi_1,\dots,\varphi_N\in H^\infty(\Omega)$ satisfying

$$1 \geq \max\left\{|\varphi_1(z)|^2,\dots,|\varphi_N(z)|^2\right\} \geq c^2 > 0, \qquad z\in\Omega,$$

there is a positive constant C such that for all points $\mathbf{a}=(a_1,\dots,a_M)\in\Omega^M$ and all $\boldsymbol{\theta}=\theta_0,\dots,\theta_M\in[0,1]^{M+1}$ with $\sum_{m=0}^M\theta_m=1$, there are $f_1,\dots,f_N\in\mathcal{H}$ satisfying

$$\begin{aligned}\sum_{\ell=1}^N \|f_\ell\|^2_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} &\leq C^2,\\ \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z) &= 1, \qquad z\in\Omega.\end{aligned}$$

More generally we prove an *alternate* Toeplitz corona theorem for the kernel multiplier space $K_{\mathcal{H}}$ of a Hilbert function space $\mathcal{H}$ whose reproducing kernel k need not be a complete Pick kernel, but must have the property (among others) that the kernel functions k_a are invertible multipliers on $\mathcal{H}$. Here the Banach space $K_{\mathcal{H}}$ of kernel multipliers consists of those functions $\varphi\in\mathcal{H}$ such that $\sup_{a\in\Omega}\frac{\|\varphi k_a\|_{\mathcal{H}}}{\|k_a\|_{\mathcal{H}}}<\infty$. The roots of this abstract result can be traced back to a research thread begun by Agler and McCarthy [AgMc] in the bidisc using Ando's theorem to reduce the corona problem to estimates on weighted Hardy spaces (see also [AgMc2, Chapters 11 and 13] for a nice survey of related prior work), and continued by work of Amar [Amar] who introduced the use of the von Neumann minimax theorem to circumvent Ando's theorem and go beyond the bidisc, and more recently by work of Trent and the second author [TrWi] who further reduced matters to checking weights whose densities are the modulus squared of nonvanishing H^∞ functions whose boundary values have bounded reciprocals. In this paper we extend and refine this approach to more general Hilbert function spaces and reduce matters to solving the Bezout equation $\varphi \cdot f = 1$ with solutions f in an *analogue* of a weighted L^2 space - e.g. the method applies to Besov-Sobolev spaces on the ball whose norms *cannot* be given as an L^2 norm with respect to some measure.

In dimension $n = 1$ we can prove new Corona theorems for the kernel multiplier algebras of the Besov-Sobolev spaces B_p^σ of the unit disk (See Section 7 for definitions of these spaces). In the special case of $B_2^\sigma(\mathbb{D})$, the kernel multiplier algebra coincides with the Hilbert spaces $H^\infty \cap Q_{2\sigma}$, see Nicolau and Xiao [NiXi], Essén and Xiao [EsXi], Aulaskari, Stegenga and Xiao [AuStXi], or Xiao [Xia, Xia3] for the definitions of these spaces.

1.1. **Organization of the paper.** In Section 2 of this paper we introduce background material on a Hilbert function space on a set Ω, as well as our new definition of the Banach space $K_{\mathcal{H}}$ of kernel multipliers on a Hilbert function space $\mathcal{H}$. The space $K_{\mathcal{H}}$ is often an algebra and plays the pivotal role in our alternate Toeplitz corona theorem, given as Theorem 29 below.

In Section 3 we use von Neumann's minimax theorem, following the thread begun by Amar [Amar] and continued by Trent and Wick [TrWi], to characterize those vectors of corona data $\varphi \in \oplus^N L^\infty$ for which solutions $f \in \oplus^N K_{\mathcal{H}}$ exist to Bezout's equation $\varphi \cdot f = 1$ in Ω.

In Section 4 we use this characterization to obtain an alternate Toeplitz corona theorem for the spaces $K_{\mathcal{H}}$ that uses a Convex Poisson Property instead of a Baby Corona Property as in the Toeplitz corona theorem for $M_{\mathcal{H}}$ in [BaTrVi] and [AmTi]. Instead of requiring a complete Pick kernel we require the following four properties:

(1) the reproducing kernels k_a for $\mathcal{H}$ are invertible multipliers of $\mathcal{H}$, and the map $a \to k_a$ is lower semicontinuous from Ω to $M_{\mathcal{H}}$,
(2) the kernel multiplier space $K_{\mathcal{H}}$ is an algebra,
(3) the constant function 1 is in $\mathcal{H}$, and
(4) the unit ball of $\mathcal{H}$ enjoys a Montel property.

This alternate Toeplitz corona theorem thus reduces the Corona Property for $K_{\mathcal{H}}$ to the Convex Poisson Property for $\mathcal{H}$. As an application, we give examples of domains Ω in $\mathbb{C}^n$ for which the Corona Property for $H^\infty(\Omega)$ is equivalent to the Convex Poisson Property with $\mathcal{H}$ taken to be either the Bergman or Hardy space on Ω. Section 5 is devoted to these higher dimensional examples, which of course include the ball and polydisc. However, we are unable to obtain any new corona theorems in higher dimensions.

Then in Section 6, we discuss the Invertible Multiplier Property, which when it holds, gives corona theorems in many situations as a corollary of our alternate Toeplitz corona theorem. However, the Invertible Multiplier Property is known to hold only for the Szegö kernel $\frac{1}{1-\overline{a}z}$ in dimension $n = 1$, where it can be used to prove a corona theorem in the disk, the annulus, and any other planar domain for which the reproducing kernel is essentially the Szegö kernel $\frac{1}{1-\overline{a}z}$. In particular, we show the Invertible Multiplier Property fails for the Szegö kernel on both the ball and polydisc in higher dimensions.

Finally, in Section 7:

(1) we prove that the space $K_{\mathcal{H}}$ is an algebra when $\mathcal{H}$ is a Besov-Sobolev Hilbert space $B_2^\sigma(\mathbb{B}_n)$ in the ball, $\sigma > 0$, and conclude that $K_{\mathcal{H}}$ has the Corona Property if and only if the Convex Poisson Property holds for $B_2^\sigma(\mathbb{B}_n)$,
(2) we provide a new proof of A. Nicolau and J. Xiao's result that the Corona Property holds for the algebras $K_{\mathcal{H}}$ when $\mathcal{H} = B_2^\sigma(\mathbb{D})$ is a Besov-Sobolev space in the disk with $\sigma > 0$, see [NiXi]; and we further demonstrate that the Corona Property holds for the algebra K_p^σ when $0 < \sigma < \frac{1}{p}$ and $1 < p < \infty$, where this space is defined using the standard reproducing kernels and duality pairings for the spaces of holomorphic functions $B_p^\sigma(\mathbb{D})$,
(3) and we show the existence of Hilbert function spaces $\mathcal{H}$ of solutions to an elliptic PDE, to which our alternate Toeplitz corona theorem applies, and which are *not* spaces of holomorphic functions.

2. Preliminaries

We begin with a quick review of reproducing kernel Hilbert spaces, also known as Hilbert function spaces.

2.1. **Hilbert function spaces.** A Hilbert space $\mathcal{H}$ is said to be a *Hilbert function space* (also called a reproducing kernel Hilbert space) on a set Ω if the elements of $\mathcal{H}$ are complex-valued functions f on Ω with the usual vector space structure, such that each point evaluation on $\mathcal{H}$ is a nonzero continuous linear functional, i.e. for every $x \in \Omega$ there is a positive constant C_x such that

$$|f(x)| \leq C_x \|f\|_{\mathcal{H}}, \qquad \forall f \in \mathcal{H}, \tag{2.1}$$

and there is some f with $f(x) \neq 0$. Since point evaluation at $x \in \Omega$ is a continuous linear functional, there is a unique element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}} \text{ for all } x \in \Omega.$$

The element k_x is called the reproducing kernel at x, and satisfies

$$k_y(x) = \langle k_y, k_x \rangle_{\mathcal{H}}, \qquad x, y \in \Omega.$$

In particular we have

$$\|k_x\|_{\mathcal{H}}^2 = \langle k_x, k_x \rangle_{\mathcal{H}} = k_x(x),$$

and so the normalized reproducing kernel is given by $\widetilde{k_x} \equiv \frac{k_x}{\sqrt{k_x(x)}}$.

The function $k(y,x) \equiv \langle k_x, k_y \rangle_{\mathcal{H}} = k_x(y)$ is self-adjoint ($k(x,y) = \overline{k(y,x)}$), and for every finite subset $\{x_i\}_{i=1}^N$ of Ω, the matrix $[k(x_i,x_j)]_{1 \leq i,j \leq N}$ is positive semidefinite:

$$\sum_{i,j=1}^N \xi_i \overline{\xi_j} k(x_j, x_i) = \sum_{i,j=1}^N \xi_i \overline{\xi_j} \langle k_{x_i}, k_{x_j} \rangle_{\mathcal{H}} = \left\langle \sum_{i=1}^N \xi_i k_{x_i}, \sum_{j=1}^N \xi_j k_{x_j} \right\rangle_{\mathcal{H}} = \left\| \sum_{i=1}^N \xi_i k_{x_i} \right\|_{\mathcal{H}}^2 \geq 0.$$

Altogether we have shown that k is a *kernel function* in the following sense.

Definition 1. *A function $k : \Omega \times \Omega \to \mathbb{C}$ is a* kernel function *on Ω if k is self-adjoint and positive on the diagonal, and if for every finite subset $x = \{x_i\}_{i=1}^N \in \Omega^N$ of Ω, the matrix $[k(x_i,x_j)]_{1 \leq i,j \leq N}$ is positive semidefinite, written $[k(x_i,x_j)]_{1 \leq i,j \leq N} \succcurlyeq 0$, i.e.*

$$\sum_{i,j=1}^N \xi_i \overline{\xi_j} k(x_i, x_j) \geq 0, \qquad \xi \in \mathbb{C}^N, x \in \Omega^N, N \geq 1. \tag{2.2}$$

We write $k \succcurlyeq 0$ if k is a kernel function.

E. H. Moore discovered the following bijection between Hilbert function spaces and kernel functions. Given a kernel function k on $\Omega \times \Omega$, define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_k}$ on finite linear combinations $\sum_{i=1}^N \xi_i k_{x_i}$ of the functions $k_{x_i}(\zeta) = k(\zeta, x_i)$, $\zeta \in \Omega$, by

$$\left\langle \sum_{i=1}^N \xi_i k_{x_i}, \sum_{j=1}^N \eta_j k_{x_j} \right\rangle_{\mathcal{H}_k} = \sum_{i,j=1}^N \xi_i \overline{\eta_j} k(x_j, x_i). \tag{2.3}$$

If the forms in (2.2) are positive definite, then finite collections of the kernel functions k_{x_i} are linearly independent, and the inner product in (2.3) is well-defined. See [AgMc2, page 19] for a proof that (2.3) is well-defined in general.

Definition 2. *Given a kernel function $k : \Omega \times \Omega \to \mathbb{C}$ on a set Ω, define the associated Hilbert function space $\mathcal{H}_k$ to be the completion of the functions $\sum_{i=1}^N \xi_i k_{x_i}$ under the norm corresponding to the inner product (2.3).*

Proposition 3. *The Hilbert space $\mathcal{H}_k$ has kernel k. If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert function spaces on Ω that have the same kernel function k, then there is an isometry from $\mathcal{H}$ onto $\mathcal{H}'$ that preserves the kernel functions k_x, $x \in \Omega$.*

We will need the notion of rescaling a kernel as given in [AgMc2, page 25]. Given a nonvanishing complex-valued function $\rho : \Omega \to \mathbb{C} \setminus \{0\}$ and a kernel function $k(y,x) = k_x(y)$, define the ρ-rescaled kernel k^ρ by

$$k^\rho (y,x) = \rho(y)\, k(y,x)\, \overline{\rho(x)}, \qquad x,y \in \Omega.$$

It is easy to see that k^ρ is self-adjoint and positive semidefinite, and hence is a kernel function. We refer to the associated Hilbert function space $\mathcal{H}_{k^\rho}$ as the ρ-rescaling of the Hilbert function space $\mathcal{H}_k$. A crucial choice of rescaling for us below is the *point* rescaling with $\rho = \frac{1}{k_a}$ that results in $k_a^\rho \equiv 1$. We note that the δ used in [AgMc2] is our $\overline{\rho}$.

2.1.1. *Multipliers.* Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a Hilbert function space on a set Ω. Let $L^\infty = L^\infty(\Omega)$ denote the space of bounded functions on Ω normed by the supremum norm

$$\|h\|_\infty \equiv \sup_{x\in\Omega} |h(x)|.$$

The supremum norm is relevant here as point evaluations are continuous in $\mathcal{H}$, and so $\|h\|_\infty$ is a supremum of moduli of continuous linear functionals. We define the space

$$\mathcal{H}^\infty = \mathcal{H}^\infty(\Omega) \equiv \{h \in \mathcal{H} : \|h\|_\infty < \infty\} = \mathcal{H} \cap L^\infty(\Omega)$$

to consist of the bounded functions in $\mathcal{H}$, and we norm this space by

$$\|h\|_{\mathcal{H}^\infty(\Omega)} \equiv \max\{\|h\|_{\mathcal{H}}, \|h\|_\infty\},$$

so that $\mathcal{H}^\infty$ is a Banach space.

A function φ is said to be a (pointwise) multiplier of $\mathcal{H}$ if $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. The collection of all multipliers of $\mathcal{H}$ is known to be a Banach algebra which we denote by $M_{\mathcal{H}}$. Indeed, (see e.g. [AgMc2]) if φ is a multiplier of $\mathcal{H}$, and if we denote the linear operator of multiplication by

$$\mathcal{M}_\varphi f \equiv \varphi f,$$

then by the closed graph theorem

$$\|\varphi\|_{M_{\mathcal{H}}} \equiv \|\mathcal{M}_\varphi\|_{\mathcal{H}\to\mathcal{H}} \equiv \sup_{f\in\mathcal{H}:\ f\neq 0} \frac{\|\varphi f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} < \infty.$$

If in addition $1 \in \mathcal{H}$ we have $\varphi \in \mathcal{H}$ and

$$\|\varphi\|_{\mathcal{H}} = \|\mathcal{M}_\varphi 1\|_{\mathcal{H}} \le \|\mathcal{M}_\varphi\|_{\mathcal{H}\to\mathcal{H}} \|1\|_{\mathcal{H}} .$$

Finally and most importantly, φ is bounded in Ω by $\|\mathcal{M}_\varphi\|_{\mathcal{H}\to\mathcal{H}}$, i.e.

$$\|\varphi\|_\infty \le \|\mathcal{M}_\varphi\|_{\mathcal{H}\to\mathcal{H}}. \tag{2.4}$$

Indeed, for all $x \in \Omega$ we have

$$|\varphi(x)|\, \|k_x\|_{\mathcal{H}}^2 = |\varphi(x)|\, k_x(x) = |\langle \varphi k_x, k_x\rangle_{\mathcal{H}}| \le \|\varphi\|_{M_{\mathcal{H}}} \|k_x\|_{\mathcal{H}}^2 .$$

Moreover we have $\mathcal{M}_\varphi^* k_x = \overline{\varphi(x)} k_x$ for all $x \in \Omega$ since

$$\left\langle f, \overline{\varphi(x)} k_x \right\rangle_{\mathcal{H}} = \varphi(x) \langle f, k_x\rangle_{\mathcal{H}} = \varphi(x) f(x) = \mathcal{M}_\varphi f(x) = \langle \mathcal{M}_\varphi f, k_x\rangle_{\mathcal{H}} = \left\langle f, \mathcal{M}_\varphi^* k_x\right\rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}$. Thus we have shown that $M_{\mathcal{H}}$ embeds in $\mathcal{H}^\infty$ with

$$\|\varphi\|_{\mathcal{H}^\infty} \le \max\{1, \|1\|_{\mathcal{H}}\}\ \|\varphi\|_{M_{\mathcal{H}}} .$$

2.1.2. *Kernel multipliers.* There is a Banach space $K_{\mathcal{H}}$ intermediate between $M_{\mathcal{H}}$ and $\mathcal{H}^{\infty}(\Omega)$ that plays a major role in this paper, namely the Banach space $K_{\mathcal{H}}$ of *kernel* multipliers consisting of all functions φ on Ω for which

$$\|\varphi\|_{K_{\mathcal{H}}} \equiv \max\left\{\left\|\varphi\widetilde{1}\right\|_{\mathcal{H}}, \sup_{a\in\Omega}\left\|\varphi\widetilde{k_a}\right\|_{\mathcal{H}}\right\} < \infty,$$

where $\widetilde{1}$ is the constant function $\frac{1}{\|1\|_{\mathcal{H}}}$ normalized to have $\mathcal{H}$-norm 1. Let $\varphi \in K_{\mathcal{H}}$. Clearly, from $\widetilde{k_a} = \frac{k_a}{\sqrt{k_a(a)}}$ and the reproducing property of k_a, we have

$$|\varphi(a)| = \frac{1}{k_a(a)}\left|\langle\varphi k_a, k_a\rangle_{\mathcal{H}}\right| = \left|\left\langle\varphi\widetilde{k_a}, \widetilde{k_a}\right\rangle_{\mathcal{H}}\right| \le \left\|\varphi\widetilde{k_a}\right\|_{\mathcal{H}}\left\|\widetilde{k_a}\right\|_{\mathcal{H}} \le \|\varphi\|_{K_{\mathcal{H}}} ,$$

and so $K_{\mathcal{H}}$ embeds in $\mathcal{H}^{\infty}$ with

$$\|\varphi\|_{\mathcal{H}^{\infty}} \le \max\{1, \|1\|_{\mathcal{H}}\} \ \|\varphi\|_{K_{\mathcal{H}}} .$$

Moreover, $M_{\mathcal{H}}$ embeds in $K_{\mathcal{H}}$ with $\|\varphi\|_{K_{\mathcal{H}}} \le \|\varphi\|_{M_{\mathcal{H}}}$ since $\left\|\varphi\widetilde{k_a}\right\|_{\mathcal{H}} \le \|\varphi\|_{M_{\mathcal{H}}}\left\|\widetilde{k_a}\right\|_{\mathcal{H}} = \|\varphi\|_{M_{\mathcal{H}}}$ if $\varphi \in M_{\mathcal{H}}$. Thus we have the embeddings

$$M_{\mathcal{H}} \hookrightarrow K_{\mathcal{H}} \hookrightarrow \mathcal{H}^{\infty} \hookrightarrow \mathcal{H} ,$$

that show that the multiplier algebra $M_{\mathcal{H}}$ is contained in the kernel multiplier space $K_{\mathcal{H}}$ which is contained in the space $\mathcal{H}^{\infty}$. Finally, we note that $M_{\mathcal{H}}$ multiplies the spaces $K_{\mathcal{H}}$ and $\mathcal{H}^{\infty}$ as well as $\mathcal{H}$, i.e. that $M_{\mathcal{H}}$ is contained in both $M_{K_{\mathcal{H}}}$ and $M_{\mathcal{H}^{\infty}}$. Indeed, if $\varphi \in M_{\mathcal{H}}$ and $f \in K_{\mathcal{H}}$, then

$$\|\varphi f\|_{K_{\mathcal{H}}} = \max\left\{\left\|\varphi f\widetilde{1}\right\|_{\mathcal{H}}, \sup_{a\in\Omega}\left\|\varphi f\widetilde{k_a}\right\|_{\mathcal{H}}\right\} \le \|\varphi\|_{M_{\mathcal{H}}}\max\left\{\left\|f\widetilde{1}\right\|_{\mathcal{H}}, \sup_{a\in\Omega}\left\|f\widetilde{k_a}\right\|_{\mathcal{H}}\right\} = \|\varphi\|_{M_{\mathcal{H}}}\|f\|_{K_{\mathcal{H}}} ,$$

and

$$\|\varphi f\|_{\mathcal{H}^{\infty}} = \max\{\|\varphi f\|_{\mathcal{H}}, \|\varphi f\|_{\infty}\} \le \|\varphi\|_{M_{\mathcal{H}}}\max\{\|f\|_{\mathcal{H}}, \|f\|_{\infty}\} = \|\varphi\|_{M_{\mathcal{H}}}\|f\|_{\mathcal{H}^{\infty}} .$$

Of particular importance in this paper is the case when $K_{\mathcal{H}}$ is an algebra. This occurs for example in the case $\mathcal{H}^{\infty} = M_{\mathcal{H}}$, as happens when $\mathcal{H}$ is the classical Hardy or Bergman space on a bounded domain with C^2 boundary in $\mathbb{C}^n$. We also note that $K_{\mathcal{H}}$ may be an algebra even if $\mathcal{H}^{\infty} \ne M_{\mathcal{H}}$. For example, $K_{\mathcal{H}}$ is an algebra when $\mathcal{H}$ is any of the Besov-Sobolev spaces $B_2^{\sigma}(\mathbb{B}_n)$, $\sigma > 0$ and $n \ge 1$, of analytic functions on the ball $\mathbb{B}_n$. See Subsection 7.1 below for this.

We recall at this point that the Corona Property has been proved for $M_{\mathcal{H}}$ when $\mathcal{H} = B_2^{\sigma}(\mathbb{B}_n)$ for $0 \le \sigma \le \frac{1}{2}$ and $n \ge 1$; the case $n = 1$ is in [Car], [Tol], [ArBlPa] and [Xia2], and the case $n > 1$ is in [CoSaWi]. In addition the Corona Property has been proved by Nicolau [Nic] for the algebra $\mathcal{H}^{\infty}$ when $\mathcal{H} = B_2^0(\mathbb{D})$ is the classical Dirichlet space on the disk. In Subsection 7.2 we use the Peter Jones solution [Jo], [Jo2] to the $\overline{\partial}$-equation in the unit disk $\mathbb{D}$, together with an adaptation of the argument of Arcozzi, Blasi and Pau [ArBlPa], to prove the Corona Property for the algebras $K_{B_2^{\sigma}(\mathbb{D})}$. No other corona theorems for the algebras $M_{\mathcal{H}}$, $K_{\mathcal{H}}$ or $\mathcal{H}^{\infty}$ are currently known when $\mathcal{H} = B_2^{\sigma}(\mathbb{B}_n)$. However, we will reduce the Corona Property for the algebra of kernel multipliers $K_{\mathcal{H}}$ associated to $\mathcal{H} = B_2^{\sigma}(\mathbb{B}_n)$, to a simpler property we call the Convex Poisson Property for $\mathcal{H}$. See Theorem 29 where this is shown to hold for more general Hilbert function spaces $\mathcal{H}$ and their associated kernel multiplier algebra $K_{\mathcal{H}}$.

2.1.3. *Shifted spaces and multiplier stability.*

Definition 4. *If $\mathcal{H}$ is a Hilbert function space on a set Ω with* nonvanishing *kernel function k, then for each $a \in \Omega$, we define the a-*shifted *Hilbert space $\mathcal{H}^a$ to be $\mathcal{H}_{k^{\delta}}$ where $\delta = \frac{1}{k_a}$, the $\frac{1}{k_a}$-rescaling of $\mathcal{H}$, and where $\widetilde{k_a} = \frac{1}{\sqrt{k_a(a)}}k_a$ is the normalized reproducing kernel for $\mathcal{H}$.*

Lemma 5. *Let $\mathcal{H}$ be a Hilbert function space on a set Ω with nonvanishing kernel function k. Then the space $\mathcal{H}^a$ consists of those complex-valued functions f on Ω such that $\widetilde{k_a}f \in \mathcal{H}$, and the inner product in $\mathcal{H}^a$ is given by*

$$\langle f, g\rangle_{\mathcal{H}^a} = \left\langle\widetilde{k_a}f, \widetilde{k_a}g\right\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}.$$

In particular $\|f\|_{\mathcal{H}^a} = \left\|\widetilde{k_a}f\right\|_{\mathcal{H}}$.

Proof. Define $\mathcal{G}^a$ to be the linear space of functions on Ω having the form $\frac{1}{\widetilde{k_a}}h$ with $h \in \mathcal{H}$, and define an inner product on $\mathcal{G}^a$ by $\langle f, g\rangle_{\mathcal{G}^a} \equiv \left\langle \widetilde{k_a}f, \widetilde{k_a}g\right\rangle_{\mathcal{H}}$ for $f, g \in \mathcal{G}^a$. We have

$$k^\delta(\zeta,\eta) = \frac{k(\zeta,\eta)}{\widetilde{k_a}(\zeta)\overline{\widetilde{k_a}(\eta)}} \text{ and so } k^\delta_\eta = \frac{1}{\overline{\widetilde{k_a}(\eta)}}\frac{k_\eta}{\widetilde{k_a}} \in \mathcal{G}^a.$$

It is easy to see that $\mathcal{G}^a$ is complete in the norm derived from this inner product, hence is a Hilbert space, and we now show that point evaluations are continuous on $\mathcal{G}^a$. Indeed, if $f = \frac{1}{\widetilde{k_a}}h$ with $h \in \mathcal{H}$ then

$$\begin{aligned} f(\eta) &= \frac{1}{\widetilde{k_a}(\eta)}h(\eta) = \frac{1}{\widetilde{k_a}(\eta)}\langle h, k_\eta\rangle_{\mathcal{H}} = \frac{1}{\widetilde{k_a}(\eta)}\left\langle \widetilde{k_a}f, k_\eta\right\rangle_{\mathcal{H}} \\ &= \frac{1}{\widetilde{k_a}(\eta)}\left\langle \widetilde{k_a}f, \overline{\widetilde{k_a}(\eta)}\widetilde{k_a}k^\delta_\eta\right\rangle_{\mathcal{H}} = \left\langle \widetilde{k_a}f, \widetilde{k_a}k^\delta_\eta\right\rangle_{\mathcal{H}} = \left\langle f, k^\delta_\eta\right\rangle_{\mathcal{G}^a} \end{aligned}$$

and so

$$|f(\eta)| \leq \|f\|_{\mathcal{G}^a}\left\|k^\delta_\eta\right\|_{\mathcal{G}^a} = \|f\|_{\mathcal{G}^a}\sqrt{\left\langle \widetilde{k_a}k^\delta_\eta, \widetilde{k_a}k^\delta_\eta\right\rangle_{\mathcal{H}}} = \|f\|_{\mathcal{G}^a}\sqrt{\left\langle \frac{k_\eta}{\overline{\widetilde{k_a}(\eta)}}, \frac{k_\eta}{\overline{\widetilde{k_a}(\eta)}}\right\rangle_{\mathcal{H}}} = \|f\|_{\mathcal{G}^a}\frac{\|k_\eta\|_{\mathcal{H}}}{\left|\widetilde{k_a}(\eta)\right|}.$$

Thus $\mathcal{G}^a$ is a Hilbert function space on Ω, and the above calculation shows that the reproducing kernel for $\mathcal{G}^a$ is k^δ. By Proposition 3 $\mathcal{G}^a = \mathcal{H}^a$, and this completes the proof of the lemma.

We can give an alternate proof by computing that if $f = \sum_{i=1}^J x_i k^\delta_{\eta_i}(\zeta) \in \mathcal{H}^a$, then

$$\|f\|^2_{\mathcal{H}^a} = \left\langle \sum_{i=1}^J x_i k^\delta_{\eta_i}, \sum_{j=1}^J x_j k^\delta_{\eta_j}\right\rangle_{\mathcal{H}^a} = \sum_{i,j=1}^J x_i\overline{x_j}\left\langle k^\delta_{\eta_i}, k^\delta_{\eta_j}\right\rangle_{\mathcal{H}^a} = \sum_{i,j=1}^J x_i\overline{x_j}k^\delta(\eta_j,\eta_i) = \sum_{i,j=1}^J x_i\overline{x_j}\frac{k(\eta_j,\eta_i)}{\overline{\widetilde{k_a}(\eta_i)}\widetilde{k_a}(\eta_j)}$$

and

$$\begin{aligned} \left\langle \widetilde{k_a}f, \widetilde{k_a}f\right\rangle_{\mathcal{H}} &= \left\langle \sum_{i=1}^J x_i\widetilde{k_a}k^\delta_{\eta_i}, \sum_{j=1}^J x_j\widetilde{k_a}k^\delta_{\eta_j}\right\rangle_{\mathcal{H}} = \left\langle \sum_{i=1}^J x_i\widetilde{k_a}\frac{k_{\eta_i}}{\overline{\widetilde{k_a}(\eta_i)}\widetilde{k_a}}, \sum_{j=1}^J x_j\widetilde{k_a}\frac{k_{\eta_j}}{\overline{\widetilde{k_a}(\eta_j)}\widetilde{k_a}}\right\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^J x_i\overline{x_j}\left\langle \frac{k_{\eta_i}}{\overline{\widetilde{k_a}(\eta_i)}}, \frac{k_{\eta_j}}{\overline{\widetilde{k_a}(\eta_j)}}\right\rangle_{\mathcal{H}} = \sum_{i,j=1}^J x_i\overline{x_j}\frac{k_{\eta_i}(\eta_j)}{\overline{\widetilde{k_a}(\eta_i)}\widetilde{k_a}(\eta_j)}, \end{aligned}$$

are equal. Now use that functions of the form $f = \sum_{i=1}^J x_i k^\delta_{\eta_i}(\zeta)$ are dense in $\mathcal{H}^a$ by definition. $\square$

Despite the difference in norms of the shifted spaces $\mathcal{H}^a$, the multiplier algebras coincide and have identical norms. This is proved in [AgMc2, p.25] where it is shown that rescaling a kernel leaves the multiplier algebra and the multiplier norms unchanged. We give the simple proof in our setting here.

Lemma 6. *Let $\mathcal{H}$ be a Hilbert function space on a set Ω with nonvanishing kernel function. Then $M_{\mathcal{H}^a} = M_{\mathcal{H}}$ with equality of norms for all $a \in \Omega$.*

Proof. Fix $a \in \Omega$. Suppose first that $\varphi \in M_{\mathcal{H}}$. We claim that $\varphi \in M_{\mathcal{H}^a}$ with $\|\varphi\|_{M_{\mathcal{H}^a}} \leq \|\varphi\|_{M_{\mathcal{H}}}$. Indeed, if $f \in \mathcal{H}^a$, then $f = \frac{1}{\widetilde{k_a}}g$ where $g \in \mathcal{H}$ with $\|g\|_{\mathcal{H}} = \|f\|_{\mathcal{H}^a}$, and we have

$$\varphi f = \varphi\frac{1}{\widetilde{k_a}}g = \frac{1}{\widetilde{k_a}}\varphi g = \frac{1}{\widetilde{k_a}}G\ ,$$

where $G \equiv \varphi g \in \mathcal{H}$ with $\|G\|_{\mathcal{H}} \leq \|\varphi\|_{M_{\mathcal{H}}}\|g\|_{\mathcal{H}}$, and hence

$$\|\varphi f\|_{\mathcal{H}^a} = \left\|\widetilde{k_a}\varphi f\right\|_{\mathcal{H}} = \|G\|_{\mathcal{H}} \leq \|\varphi\|_{M_{\mathcal{H}}}\|g\|_{\mathcal{H}} = \|\varphi\|_{M_{\mathcal{H}}}\|f\|_{\mathcal{H}^a}\ .$$

This proves the claimed inequality: $\|\varphi\|_{M_{\mathcal{H}^a}} \leq \|\varphi\|_{M_{\mathcal{H}}}$.

Conversely, suppose that $\varphi \in M_{\mathcal{H}^a}$. We claim that $\varphi \in M_{\mathcal{H}}$ with $\|\varphi\|_{M_{\mathcal{H}}} \leq \|\varphi\|_{M_{\mathcal{H}^a}}$. Indeed, if $g \in \mathcal{H}$, then $g = \widetilde{k_a}f$ where $f \in \mathcal{H}^a$ with $\|f\|_{\mathcal{H}^a} = \|g\|_{\mathcal{H}}$, and we have

$$\varphi g = \varphi\widetilde{k_a}f = \widetilde{k_a}\varphi f = \widetilde{k_a}F$$

where $F \equiv \varphi f \in \mathcal{H}^a$ with $\|F\|_{\mathcal{H}^a} \leq \|\varphi\|_{M_{\mathcal{H}^a}} \|f\|_{\mathcal{H}^a}$, and hence

$$\|\varphi g\|_{\mathcal{H}} = \left\|\widetilde{k_a}\varphi f\right\|_{\mathcal{H}} = \|F\|_{\mathcal{H}^a} \leq \|\varphi\|_{M_{\mathcal{H}^a}} \|f\|_{\mathcal{H}^a} = \|\varphi\|_{M_{\mathcal{H}^a}} \|g\|_{\mathcal{H}} .$$

Hence we have: $\|\varphi\|_{M_{\mathcal{H}}} \leq \|\varphi\|_{M_{\mathcal{H}^a}}$. These two inequalities show that $M_{\mathcal{H}^a} = M_{\mathcal{H}}$ with equality of norms. □

At this point we introduce the first main assumption needed for our alternate Toeplitz corona theorem.

Definition 7. *We say that the Hilbert function space $\mathcal{H}$ is* multiplier stable *if*

(1) *the reproducing kernel functions k_x are nonvanishing and are invertible multipliers on $\mathcal{H}$, i.e. $k_x \in M_{\mathcal{H}}$ and $\frac{1}{k_x} \in M_{\mathcal{H}}$, for all $x \in \Omega$, and*
(2) *the map $x \to k_x$ from Ω to $M_{\mathcal{H}}$ is lower semicontinuous.*

Note that we make no assumptions regarding the size of the norms of the multipliers k_x and $\frac{1}{k_x}$ in this definition. We will see below that all the Besov-Sobolev spaces on the ball are multiplier stable, as well as the Bergman and Hardy spaces on strictly pseudoconvex domains with C^2 boundary. A crucial consequence of the multiplier stable assumption is the $\mathcal{H}$-Poisson reproducing formula below.

Lemma 8. *Suppose $\mathcal{H}$ is a Hilbert function space on a set Ω with nonvanishing kernel and containing the constant functions. Suppose furthermore that $k_x \in M_{\mathcal{H}}$ for all $x \in \Omega$. Then for each $a \in \Omega$ we have the $\mathcal{H}$-Poisson reproducing formula*

$$f(a) = \langle f, 1\rangle_{\mathcal{H}^a}, \qquad f \in \mathcal{H}(\Omega), \ a \in \Omega. \tag{2.5}$$

Proof. Since $k_a \in M_{\mathcal{H}}$ by hypothesis, the function $F_a(w) \equiv \widetilde{k_a}(w) f(w)$ is in $\mathcal{H}$ for $f \in \mathcal{H}$, and so using Lemma 5,

$$\sqrt{k_a(a)} f(a) = \widetilde{k_a}(a) f(a) = F_a(a) = \langle F_a, k_a\rangle_{\mathcal{H}} = \sqrt{k_a(a)} \left\langle \widetilde{k_a} f, \widetilde{k_a}\right\rangle_{\mathcal{H}} = \sqrt{k_a(a)} \langle f, 1\rangle_{\mathcal{H}^a},$$

which gives (2.5). □

The spaces $\mathcal{H}^a$ are in fact all equal to $\mathcal{H}$ as sets, with different but comparable norms (the constants of comparability need not be bounded in a).

Lemma 9. *Suppose $\mathcal{H}$ is multiplier stable. For $a \in \Omega$ we have comparability of the norms for $\mathcal{H}$ and $\mathcal{H}^a$:*

$$\frac{1}{\left\|\frac{1}{\widetilde{k_a}}\right\|_{M_{\mathcal{H}}}} \|h\|_{\mathcal{H}} \leq \|h\|_{\mathcal{H}^a} \leq \left\|\widetilde{k_a}\right\|_{M_{\mathcal{H}}} \|h\|_{\mathcal{H}} , \qquad h \in \mathcal{H} \cup \mathcal{H}^a.$$

Proof. Since $k_a, \frac{1}{k_a} \in M_{\mathcal{H}}$ we see that

$$\begin{aligned} \|f\|_{\mathcal{H}^a} &= \left\|\widetilde{k_a} f\right\|_{\mathcal{H}} \leq \left\|\widetilde{k_a}\right\|_{M_{\mathcal{H}}} \|f\|_{\mathcal{H}} , \\ \|g\|_{\mathcal{H}} &= \left\|\frac{1}{\widetilde{k_a}} \widetilde{k_a} g\right\|_{\mathcal{H}} \leq \left\|\frac{1}{\widetilde{k_a}}\right\|_{M_{\mathcal{H}}} \left\|\widetilde{k_a} g\right\|_{\mathcal{H}} = \left\|\frac{1}{\widetilde{k_a}}\right\|_{M_{\mathcal{H}}} \|g\|_{\mathcal{H}^a} , \end{aligned}$$

and these two inequalities prove the lemma. □

2.2. **Interpolation by rescalings.**

Definition 10. *Given a multiplier stable Hilbert function space $\mathcal{H}$ on Ω with kernel k, and $(\mathbf{a}, \boldsymbol{\theta}) \in \Omega^M \times [0,1]^{M+1}$, define the Hilbert function space $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ to be $\mathcal{H}$ with inner product given by*

$$\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} \equiv \theta_0 \langle f, g\rangle_{\mathcal{H}} + \sum_{m=1}^{M} \theta_m \langle f, g\rangle_{\mathcal{H}^{a_m}} , \qquad f, g \in \mathcal{H}.$$

We recall from Lemma 9 that all of the spaces $\mathcal{H}^{a_m}$ are comparable, hence the inner product $\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}}$ is defined for $f, g \in \mathcal{H}$, and all of the spaces $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ are comparable with $\mathcal{H}$. We will often use the convention $k_\eta^{a_0} = k_\eta$ in order to simplify the sum above to $\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \sum_{m=0}^{M} \theta_m \langle f, g\rangle_{\mathcal{H}^{a_m}}$. We will refer to the spaces $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ as the *convex shifted* spaces associated with $\mathcal{H}$. They are normed by $\|f\|_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \sqrt{\langle f, f\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}}}$.

Let

$$\Sigma_M \equiv \left\{ \boldsymbol{\theta} = \{\theta_m\}_{m=0}^M \subset [0,1]^{M+1} : \sum_{j=0}^M \theta_j = 1 \right\}$$

denote the unit $(M+1)$-dimensional simplex.

Assumption: We now make the standing assumption, in force for the remainder of the paper, that $\mathcal{H}$ contains the constant functions on Ω, and upon multiplying by a positive constant, we may assume that

$$\|1\|_{\mathcal{H}} = 1.$$

Corollary 11. *The norm of the constant function* 1 *in the space* $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ *is* 1 *for all* $\mathbf{a} \in \Omega^M$ *and* $\boldsymbol{\theta} \in \Sigma_M$*.*

Proof. We have

$$\|1\|_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}}^2 = \langle 1,1\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \sum_{m=0}^M \theta_m \langle 1,1\rangle_{\mathcal{H}^{a_m}} = \sum_{m=0}^M \theta_m \left\langle \widetilde{k_{a_m}}, \widetilde{k_{a_m}} \right\rangle_{\mathcal{H}} = \sum_{m=0}^M \theta_m = 1.$$

□

2.2.1. *Vector-valued norms.* Let $N \geq 1$ be fixed. Given Hilbert spaces $\mathcal{H}_\ell$ for $1 \leq \ell \leq N$, define a complete inner product on the direct sum $\oplus_{\ell=1}^N \mathcal{H}_\ell$ by

$$\langle f, g\rangle_{\oplus_{\ell=1}^N \mathcal{H}_\ell} \equiv \sum_{\ell=1}^N \langle f_\ell, g_\ell\rangle_{\mathcal{H}_\ell} , \qquad f = (f_\ell)_{\ell=1}^N , g = (g_\ell)_{\ell=1}^N \in \oplus_{\ell=1}^N \mathcal{H}_\ell .$$

When all the spaces $\mathcal{H}_\ell$ are equal to the same space $\mathcal{H}$, we write simply $\oplus^N \mathcal{H}$ in place of $\oplus_{\ell=1}^N \mathcal{H}_\ell$.

Given Banach spaces B_ℓ for $1 \leq \ell \leq N$, define a complete norm on the direct sum $\oplus_{\ell=1}^N B_\ell$ by

$$\|\varphi\|_{\oplus_{\ell=1}^N B_\ell} \equiv \max_{1\leq \ell \leq N} \|\varphi_\ell\|_{B_\ell} , \qquad \varphi = (\varphi_\ell)_{\ell=1}^N \in \oplus_{\ell=1}^N B_\ell ,$$

and again, when all the spaces B_ℓ are equal to the same space B, we write simply $\oplus^N B$ in place of $\oplus_{\ell=1}^N B_\ell$. In the case when the B_ℓ are also Hilbert spaces, this definition differs from the previous one, but the intended definition should always be clear from the context. We will be mainly concerned with the cases $B = M_{\mathcal{H}}$ and $B = K_{\mathcal{H}}$.

When $B = M_{\mathcal{H}}$ is the multiplier algebra of a Hilbert function space $\mathcal{H}$ on a set Ω, there are two additional natural norms to consider, namely the row and column norms, to which we now turn. Let $\mathcal{H}$ be a Hilbert function space on a set Ω. For $\varphi \in \oplus^N M_{\mathcal{H}}$ define

$$\begin{aligned} \mathcal{M}_\varphi &: \oplus^N \mathcal{H} \to \mathcal{H} \text{ by } \mathcal{M}_\varphi f = \sum_{\alpha=1}^N \varphi_\alpha f_\alpha, \\ \mathbb{M}_\varphi &: \mathcal{H} \to \oplus^N \mathcal{H} \text{ by } \mathbb{M}_\varphi h = (\varphi_\alpha h)_{\alpha=1}^N . \end{aligned}$$

Then we have $\mathcal{M}_\varphi^* h = \left(\mathcal{M}_{\varphi_\alpha}^* f\right)_{\alpha=1}^N$, and in particular,

$$\mathcal{M}_\varphi^* \widetilde{k}_z = \left(\mathcal{M}_{\varphi_\alpha}^* \widetilde{k}_z\right)_{\alpha=1}^N = \left(\overline{\varphi_\ell(z)} \widetilde{k}_z\right)_{\alpha=1}^N . \tag{2.6}$$

We also define the row and column norms by

$$\|\mathcal{M}_\varphi\|_{\text{op}} \equiv \sup_{f\neq 0} \frac{\|\mathcal{M}_\varphi f\|_{\mathcal{H}}}{\|f\|_{\oplus^N \mathcal{H}}} \text{ and } \|\mathbb{M}_\varphi\|_{\text{op}} \equiv \sup_{h\neq 0} \frac{\|\mathbb{M}_\varphi h\|_{\oplus^N \mathcal{H}}}{\|h\|_{\mathcal{H}}}.$$

The two key inequalities involving these norms are:

$$\begin{aligned} \|\varphi\|_{L^\infty(\ell_N^2)} &\equiv \sup_{z\in\Omega} \left(\sum_{\ell=1}^N |\varphi_\ell(z)|^2 \right)^{\frac{1}{2}} \leq \|\mathcal{M}_\varphi\|_{\text{op}} , \\ \|\varphi\|_{\oplus^N \mathcal{H}} &\leq \|\mathbb{M}_\varphi\|_{\text{op}} \|1\|_{\mathcal{H}} . \end{aligned}$$

The first inequality follows from (2.6),

$$\begin{aligned}\sum_{\ell=1}^{N}\left|\varphi_{\ell}(z)\right|^{2} &= \sum_{\ell=1}^{N}\left|\varphi_{\ell}(z)\right|^{2}\left\|\widetilde{k_{z}}\right\|_{\mathcal{H}}^{2}=\sum_{\ell=1}^{N}\left\|\overline{\varphi_{\ell}(z)}\widetilde{k_{z}}\right\|_{\mathcal{H}}^{2}=\left\|\mathcal{M}_{\varphi}^{*}\widetilde{k_{z}}\right\|_{\oplus^{N}\mathcal{H}}^{2}\\ &\leq \left\|\mathcal{M}_{\varphi}^{*}\right\|_{\mathrm{op}}^{2}\left\|\widetilde{k_{z}}\right\|_{\mathcal{H}}^{2}=\left\|\mathcal{M}_{\varphi}\right\|_{\mathrm{op}}^{2}\left\|\widetilde{k_{z}}\right\|_{\mathcal{H}}^{2}=\left\|\mathcal{M}_{\varphi}\right\|_{\mathrm{op}}^{2},\end{aligned}$$

and the second inequality follows from

$$\begin{aligned}\|\varphi\|_{\oplus^{N}\mathcal{H}}^{2} &= \sum_{\ell=1}^{N}\|\varphi_{\ell}\|_{\mathcal{H}}^{2}=\sum_{\ell=1}^{N}\|\varphi_{\ell}1\|_{\mathcal{H}}^{2}=\sum_{\ell=1}^{N}\left\|\mathcal{M}_{\varphi_{\ell}}1\right\|_{\mathcal{H}}^{2}=\|\mathbb{M}_{\varphi}1\|_{\oplus^{N}\mathcal{H}}^{2}\\ &\leq \|\mathbb{M}_{\varphi}\|_{\mathrm{op}}^{2}\|1\|_{\mathcal{H}}^{2}.\end{aligned}$$

Then from our assumption that the norm of 1 in the space $\mathcal{H}$ is 1, we have both of the inequalities $\|\varphi\|_{L^{\infty}(\ell^{2})}\leq\|\mathcal{M}_{\varphi}\|_{\mathrm{op}}$ and $\|\varphi\|_{\oplus^{N}\mathcal{H}}\leq\|\mathbb{M}_{\varphi}\|_{\mathrm{op}}$. We now have three norms on the Banach space $\oplus^{N}M_{\mathcal{H}}$.

Definition 12. *Given $\varphi\in\oplus^{N}M_{\mathcal{H}}$, define the three norms*

$$\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{row}}\equiv\|\mathcal{M}_{\varphi}\|_{\mathrm{op}}\ ,\quad \|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{column}}\equiv\|\mathbb{M}_{\varphi}\|_{\mathrm{op}}\ ,\quad \|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\max}\equiv\max_{1\leq\ell\leq N}\|\varphi_{\ell}\|_{M_{\mathcal{H}}}\ .$$

These norms are comparable since

$$\begin{aligned}\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\max} &\leq \min\left\{\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{row}},\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{column}}\right\}\\ &\leq \max\left\{\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{row}},\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{column}}\right\}\leq\sqrt{N}\,\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\max}\ .\end{aligned}$$

When $B=K_{\mathcal{H}}$ is the Banach space of kernel multipliers on $\mathcal{H}$, and in the presence of the standing assumption $\|1\|_{\mathcal{H}}=1$, we will use the following natural norm on the direct sum $\oplus_{\ell=1}^{N}K_{\mathcal{H}}$.

Definition 13. *Let $\mathcal{H}$ be a Hilbert function space on a set Ω. For $N\geq1$ and $\varphi\in\oplus_{\ell=1}^{N}K_{\mathcal{H}}$ define the following norm on $\oplus^{N}K_{\mathcal{H}}$:*

$$\|\varphi\|_{\oplus^{N}K_{\mathcal{H}}}\equiv\max\left\{\|\varphi\|_{\oplus^{N}\mathcal{H}},\sup_{a\in\Omega}\left\|\varphi\widetilde{k_{a}}\right\|_{\oplus^{N}\mathcal{H}}\right\}.$$

In the event that $K_{\mathcal{H}}=M_{\mathcal{H}}$ isometrically, then the norm just introduced on $\oplus^{N}K_{\mathcal{H}}$ is comparable to the three introduced on $\oplus^{N}M_{\mathcal{H}}$ above:

$$\|\varphi\|_{\oplus^{N}K_{\mathcal{H}}}\leq\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\mathrm{column}}\leq\sqrt{N}\,\|\varphi\|_{\oplus^{N}M_{\mathcal{H}}}^{\max}\leq\sqrt{N}\,\|\varphi\|_{\oplus^{N}K_{\mathcal{H}}}\ . \tag{2.7}$$

2.3. **A characterization of rescaling.** We end this section on preliminaries with a characterization of when two kernels are rescalings of each other. For this we begin with a quick review of relevant properties of positive matrices. Recall that an $n\times n$ self-adjoint matrix A of complex numbers is said to be *positive*, denoted $A\succcurlyeq0$, if all of its eigenvalues are nonnegative; and said to be *strictly positive*, denoted $A\succ0$, if all of its eigenvalues are positive. Clearly sums of positive matrices are positive. Moreover, given any self-adjoint A, there is a unitary matrix U such that $UAU^{*}=\mathrm{Diag}(\lambda_{1},\ldots,\lambda_{n})=\begin{bmatrix}\lambda_{1}&0&\cdots&0\\0&\lambda_{2}&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&\cdots&0&\lambda_{n}\end{bmatrix}$, with $\lambda_{j}\in\mathbb{R}$. A *dyad* is a rank one matrix of the form $v\otimes v^{*}=\begin{bmatrix}|v_{1}|^{2}&v_{1}\overline{v_{2}}&\cdots&v_{1}\overline{v_{n}}\\v_{2}\overline{v_{1}}&|v_{2}|^{2}&\ddots&\vdots\\\vdots&\ddots&\ddots&v_{n-1}\overline{v_{n}}\\v_{n}\overline{v_{1}}&\cdots&v_{n}\overline{v_{n-1}}&|v_{n}|^{2}\end{bmatrix}$. Every dyad is positive, and conversely, every positive matrix A is a sum $\sum_{i=1}^{I}\alpha_{i}v_{i}\otimes v_{i}^{*}$ of dyads $v_{i}\otimes v_{i}^{*}$ with nonnegative coefficients α_{i}. However, such decompositions are not in general unique. For example, any positive sum of dyads is a positive matrix A, and the spectral theorem for A gives in general a different decomposition into dyads with pairwise orthogonal vectors (the eigenvectors of the matrix A). One important consequence is that the Schur product of two positive matrices is positive. Indeed, if $A = \sum_{i=1}^{I} \alpha_i v_i \otimes v_i^*$ and $B = \sum_{j=1}^{J} \beta_j w_j \otimes w_j^*$, then $A \circ B = \sum_{i=1}^{I} \sum_{j=1}^{J} \alpha_i \beta_j \left(v_i \otimes v_i^*\right) \circ \left(w_j \otimes w_j^*\right)$, and it is easily verified that $\left(v_i \otimes v_i^*\right) \circ \left(w_j \otimes w_j^*\right) = \left(v_i \circ w_i\right) \otimes \left(v_j \circ w_j\right)^*$.

Now we turn to the problem of deciding when two self-adjoint nonvanishing functions $K(x,y)$ and $k(x,y)$ on a product set $\Omega \times \Omega$ are rescalings of each other. The surprisingly simple answer depends only on the 2×2 and 3×3 principal submatrices of the infinite matrices $\left[K(x,y)\right]_{(x,y)\in\Omega\times\Omega}$ and $\left[k(x,y)\right]_{(x,y)\in\Omega\times\Omega}$.

Proposition 14. *Suppose that K and k are two self-adjoint nonvanishing functions on a product set $\Omega \times \Omega$. Then K and k are rescalings of each other, i.e. there is a nonvanishing function ψ on Ω such that $K = \left(\psi \otimes \psi^*\right) \circ k$ if and only if the following two conditions hold:*

(1) $\frac{|K(x,y)|^2}{K(x,x)K(y,y)} = \frac{|k(x,y)|^2}{k(x,x)k(y,y)}$ *for all* $x, y \in \Omega$,

(2) $\arg \frac{K(x,z)}{k(x,z)} = \arg \frac{K(x,y)}{k(x,y)} + \arg \frac{K(y,z)}{k(y,z)} \pmod{2\pi}$ *for all* $x, y, z \in \Omega$.

Proof. If $K = \left(\psi \otimes \psi^*\right) \circ k$, then $K(x,y) = \psi(x)\, k(x,y)\, \overline{\psi(y)}$ and we have

$$\frac{|K(x,y)|^2}{K(x,x)\,K(y,y)} = \frac{\left|\psi(x)\, k(x,y)\, \overline{\psi(y)}\right|^2}{\psi(x)\, k(x,x)\, \overline{\psi(x)} \psi(y)\, k(y,y)\, \overline{\psi(y)}} = \frac{|k(x,y)|^2}{k(x,x)\, k(y,y)},$$

and

$$\begin{aligned} \arg \frac{K(x,z)}{k(x,z)} &= \arg \psi(x)\, \overline{\psi(z)} = \arg \psi(x)\, |\psi(y)|^2\, \overline{\psi(z)} \\ &= \arg \psi(x)\, \overline{\psi(y)} + \arg \psi(y)\, \overline{\psi(z)} \\ &= \arg \frac{K(x,y)}{k(x,y)} + \arg \frac{K(y,z)}{k(y,z)}. \end{aligned}$$

Conversely assume that both conditions (1) and (2) hold. Then given any finite set of points $\{x_j\}_{j=1}^{J}$ in Ω, define a matrix

$$U \equiv [u_{ij}]_{i,j=1}^{J}; \qquad u_{ij} = \frac{\frac{K(x_i,x_j)}{\sqrt{K(x_i,x_i)}\sqrt{K(x_j,x_j)}}}{\frac{k(x_i,x_j)}{\sqrt{k(x_i,x_i)}\sqrt{k(x_j,x_j)}}}.$$

and note that U is self-adjoint since K and k are, and that U is unimodular by condition (1). If we set $u_{ij} = e^{i\theta_{ij}}$, then condition (2) says that

$$\begin{aligned} \theta_{i\ell} &= \arg \frac{K(x_i,x_\ell)}{k(x_i,x_\ell)} = \arg \frac{K_{i\ell}}{k_{i\ell}} = \arg \frac{K_{ij}}{k_{ij}} + \arg \frac{K_{j\ell}}{k_{j\ell}} \\ &= \arg \frac{K(x_i,x_j)}{k(x_i,x_j)} + \arg \frac{K(x_j,x_\ell)}{k(x_j,x_\ell)} = \theta_{ij} + \theta_{j\ell} \ . \end{aligned} \tag{2.8}$$

Now define $\theta_1 = 0$ and $\theta_j = \theta_{1j}$ for $1 \le j \le N$, so that $\theta_j = -\theta_{j1}$ since U is self-adjoint. Then $\theta_{1i} - \theta_{1j} = -\theta_{i1} - \theta_{1j} = -\theta_{ij}$ by (2.8), and so with $\psi(x_j) \equiv e^{-i\theta_j} \frac{\sqrt{K(x_j,x_j)}}{\sqrt{k(x_j,x_j)}}$, we obtain

$$\begin{aligned} \psi(x_i)\, k(x_i,x_j)\, \overline{\psi(x_j)} &= e^{-i(\theta_{1i}-\theta_{1j})} \left\{ \frac{\sqrt{K(x_i,x_i)}}{\sqrt{k(x_i,x_i)}} k(x_i,x_j)\, \frac{\sqrt{K(x_j,x_j)}}{\sqrt{k(x_j,x_j)}} \right\} \\ &= e^{i\theta_{ij}} \left\{ \frac{1}{u_{ij}} K(x_i,x_j) \right\} = K(x_i,x_j)\,. \end{aligned}$$

Note that the vector $\{\psi(x_i)\}_{i=1}^{J}$ is determined uniquely up to a unimodular constant.

Finally, we apply any appropriate form of transfinite induction. From what we have done above, we get a *consistent* definition of ψ on an increasing maximal chain of subsets of Ω since two dyads $\psi_1 \otimes \psi_1^*$ and $\psi_2 \otimes \psi_2^*$ are equal if and only if there is a unimodular *constant* $e^{i\theta}$ such that $\psi_1 = e^{i\theta}\psi_2$. Now we apply Zorn's Lemma to get a nonvanishing function ψ on Ω such that $K(x,y) = \psi(x)\, k(x,y)\, \overline{\psi(y)}$. If both kernels are holomorphic in their first variable x and antiholomorphic in their second variable y, then $\psi(x)\, \overline{\psi(y)} = \frac{K(x,y)}{k(x,y)}$

is holomorphic and nonvanishing in x, which proves that $\psi(x)$ is holomorphic and nonvanishing in x. Finally, the supremum bounds follow from the formula $|\psi(x)| = \frac{\sqrt{K(x,x)}}{\sqrt{k(x,x)}}$. □

Remark 15. *Condition (1) is equivalent to the equality of distance functions $d_k(x,y) = d_K(x,y)$, $x, y \in \Omega$, where for any kernel k on Ω, the distance function d_k is defined by*

$$d_k(x,y) \equiv \sqrt{1 - \frac{|k(x,y)|^2}{k(x,x)\,k(y,y)}}.$$

Note that $d_k(x,y) = \sin\theta_{x,y}$ where $\theta_{x,y}$ is the angle between k_x and k_y in the Hilbert function space $\mathcal{H}_k$.

3. The Bezout kernel multiplier characterization

The following definition will be used to characterize when we can solve Bezout's equation with a vector in the space $\oplus_{\ell=1}^{N} K_{\mathcal{H}}$ of kernel multipliers of $\mathcal{H}$.

Definition 16. *Let $\mathcal{H}$ be a Hilbert function space on a set Ω with nonvanishing kernel, and let $\mathcal{H}^a$ be the shifted Hilbert space for $a \in \Omega$. We say that a vector $\varphi \in \oplus_{\ell=1}^{N} L^\infty(\Omega)$ satisfies the $\mathcal{H}-$*convex Poisson condition *with positive constant C if for every finite collection of points $\mathbf{a} = (a_1, \ldots, a_M) \in \Omega^M$ and every collection of nonnegative numbers $\boldsymbol{\theta} = \{\theta_m\}_{m=0}^{M}$ summing to $1 = \sum_{m=0}^{M} \theta_m$, there is a vector $g^{\mathbf{a},\boldsymbol{\theta}} \in \oplus_{\ell=1}^{N} \mathcal{H}$ satisfying*

$$\begin{aligned} \varphi(z) \cdot g^{\mathbf{a},\boldsymbol{\theta}}(z) &= 1, \qquad z \in \Omega, \\ \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^{N} \mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} &= \theta_0 \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^{N} \mathcal{H}} + \sum_{m=1}^{M} \theta_m \left\| g^{\mathbf{a},\boldsymbol{\theta}} \right\|^2_{\oplus_{\ell=1}^{N} \mathcal{H}^{a_m}} \le C^2 . \end{aligned} \tag{3.1}$$

We denote the smallest such constant C by $\|\varphi\|_{cPc}$.

3.1. Kernel multiplier solutions.

Here now is our abstract characterization of solutions to Bezout's equation, which is of primary interest in those cases where the space of kernel multipliers $K_{\mathcal{H}}$ is an algebra. But first we require additional structure on our Hilbert function space $\mathcal{H}$ to substitute for Montel's theorem in complex analysis. This leads to the second main assumption needed for our alternate Toeplitz corona theorem.

Definition 17. *Let Ω be a topological space. A Hilbert function space $\mathcal{H}$ of continuous functions on Ω is said to be have the* Montel property *if there is a dense subset S of Ω with the property that for every sequence $\{f_n\}_{n=1}^{\infty}$ in the unit ball of $\mathcal{H}$, there are a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and a function g in the unit ball of $\mathcal{H}$, such that*

$$\lim_{k\to\infty} f_{n_k}(x) = g(x), \qquad x \in S.$$

A main ingredient in the proof is the following minimax lemma of von Neumann, proved for example in [Gam]. This lemma was introduced in this context by Amar [Amar], and used subsequently by Trent and Wick [TrWi] as well.

Lemma 18. *Suppose that M is a convex compact subset of a normed linear space, and that P is a convex subset of a vector space. Let $\mathcal{F} : M \times P \to [0,\infty)$ satisfy*

*(**1**) for each fixed $p \in P$, the section $\mathcal{F}^p$ given by $\mathcal{F}^p(m) = \mathcal{F}(m,p)$ is concave and continuous,*

*(**2**) for each fixed $m \in M$, the section $\mathcal{F}_m$ given by $\mathcal{F}_m(p) = \mathcal{F}(m,p)$ is convex.*

Then the following minimax equality holds:

$$\sup_{m\in M} \inf_{p\in P} \mathcal{F}(m,p) = \inf_{p\in P} \sup_{m\in M} \mathcal{F}(m,p).$$

Theorem 19. *Let $\mathcal{H}$ be a multiplier stable Hilbert function space of continuous functions on a separable topological space Ω, and assume that $\mathcal{H}$ has the Montel property. Suppose that $\varphi \in \oplus_{\ell=1}^{N} L^{\infty}(\Omega)$, and that $C>0$ is a positive constant. Then there is a vector function $f \in \oplus^{N} K_{\mathcal{H}}$ satisfying*

$$\begin{aligned} \varphi(z)\cdot f(z) &= 1, \qquad z\in\Omega, \\ \|f\|_{\oplus^{N} K_{\mathcal{H}}} &\leq C\ , \end{aligned} \tag{3.2}$$

if and only if φ *satisfies the* $\mathcal{H}-$*convex Poisson condition in Definition 16 with constant* $\|\varphi\|_{\mathcal{H}-cPc} \leq C$.

Porism: If we drop the lower semicontinuity assumption (2) in the definition of multiplier stability, then the proof below shows that if $\varphi \in \oplus_{\ell=1}^{N} L^{\infty}(\Omega)$ satisfies the $\mathcal{H}-$convex Poisson condition in Definition 16, then there is a vector function $f \in \oplus^{N}\mathcal{H}^{\infty}$ satisfying

$$\begin{aligned} \varphi(z)\cdot f(z) &= 1, \qquad z\in\Omega, \\ \|f\|_{\oplus^{N}\mathcal{H}^{\infty}} &\leq \|\varphi\|_{\mathcal{H}-cPc}\ . \end{aligned}$$

However, there is no converse assertion here in general.

Proof. Fix for the moment an integer $M\geq 1$ and a positive constant ε_M to be chosen later. Now fix $\mathbf{a}=\{a_m\}_{m=1}^{M}\subset\Omega$ and consider the simplex

$$\Sigma_M \equiv \left\{ \boldsymbol{\theta}=\{\theta_m\}_{m=0}^{M}\subset[0,1]^{M+1} : \sum_{j=0}^{M}\theta_j=1 \right\}.$$

For each $\boldsymbol{\theta}\in\Sigma_M$ pick $g^{\mathbf{a},\boldsymbol{\theta}}\in\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)$ as in the cPc Definition 16 for φ, i.e.

$$\begin{aligned} \varphi(z)\cdot g^{\mathbf{a},\boldsymbol{\theta}}(z) &= 1 \text{ in } \Omega, \\ \left\|g^{\mathbf{a},\boldsymbol{\theta}}\right\|_{\oplus_{\ell=1}^{N}\mathcal{K}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)} &\leq \|\varphi\|_{cPc}\ . \end{aligned}$$

Note that

$$\left\|g^{\mathbf{a},\boldsymbol{\theta}}\right\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}'}(\Omega)} < \infty, \qquad \text{for all } \boldsymbol{\theta},\boldsymbol{\theta}'\in\Sigma_M\ , \tag{3.3}$$

since as observed earlier, it follows from Lemma 9 that all of the interpolating spaces $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)$ are comparable with $\mathcal{H}$. It is here that we use the full force of our assumption that $\mathcal{H}$ is multiplier stable, as opposed to merely assuming that the kernel functions k_a are nonvanishing multipliers. Indeed, with only the latter assumption, an element of $\mathcal{H}^{a}$ has the form $\frac{1}{\widetilde{k_a}}g$ for $g\in\mathcal{H}$, and then $\|g\|_{\mathcal{H}^{a'}}^2=\left\|\frac{\widetilde{k_{a'}}}{\widetilde{k_a}}g\right\|_{\mathcal{H}}^2$ could be infinite.

Thus $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}\subset\mathcal{H}$, and we can define the set

$$\mathcal{C}_{\mathbf{a},M} \equiv \text{convex hull}\left\{g^{\mathbf{a},\boldsymbol{\theta}} : \boldsymbol{\theta}\in\Sigma_M(\varepsilon_M)\right\}$$

to be the convex hull in $\oplus_{\ell=1}^{N}\mathcal{H}$ of these Bezout solutions $g^{\mathbf{a},\boldsymbol{\theta}}$. We will apply the von Neumann minimax equality in Lemma 18 to the functional

$$\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},f) \equiv \|f\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}}^2 = \theta_0\|f\|_{\oplus_{\ell=1}^{N}\mathcal{H}}^2 + \sum_{m=1}^{M}\theta_m\|f\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{a_m}}^2\ ,$$

defined for $\boldsymbol{\theta}\in\Sigma_M$, and for $f\in\mathcal{C}_{\mathbf{a},M}$, noting that $\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},f)$ is then finite by (3.3).

Both Σ_M and $\mathcal{C}_{\mathbf{a},M}$ are convex, and in addition Σ_M is compact. The functional $\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},f)$ is linear and continuous in $\boldsymbol{\theta}$, hence also concave. It is also convex in f since if $\lambda_i\geq 0$ and $\sum_{i=1}^{L}\lambda_i=1$ and $A_i\in\mathcal{C}_{\mathbf{a},M}$, then

$$\begin{aligned} \mathcal{F}_{\mathbf{a}}\left(\boldsymbol{\theta},\sum_{i=1}^{L}\lambda_iA_i\right) &= \left\|\sum_{i=1}^{L}\lambda_iA_i\right\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)}^2 \leq \left(\sum_{i=1}^{L}\lambda_i\|A_i\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)}\right)^2 \\ &\leq \sum_{i=1}^{L}\lambda_i\|A_i\|_{\oplus_{\ell=1}^{N}\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)}^2 = \sum_{i=1}^{L}\lambda_i\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},A_i)\,. \end{aligned}$$

Thus the minimax equality in Lemma 18 applies to give

$$\begin{aligned}\inf_{f\in\mathcal{C}_{\mathbf{a},M}}\sup_{\boldsymbol{\theta}\in\Sigma_M}\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},f) &= \sup_{\boldsymbol{\theta}\in\Sigma_M}\inf_{f\in\mathcal{C}_{\mathbf{a},M}}\mathcal{F}_{\mathbf{a}}(\boldsymbol{\theta},f)\\ &\le \sup_{\boldsymbol{\theta}\in\Sigma_M}\mathcal{F}_{\mathbf{a}}\left(\boldsymbol{\theta},g^{\mathbf{a},\boldsymbol{\theta}}\right)\\ &= \sup_{\boldsymbol{\theta}\in\Sigma_M}\left\|g^{\mathbf{a},\boldsymbol{\theta}}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)}\le\|\varphi\|^2_{cPc}\ .\end{aligned}$$

So for each $M\ge 1$, we can pick $f^{(M)}\in\mathcal{C}_{\mathbf{a},M}\subset\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}(\Omega)$ so that $f^{(M)}$ is almost optimal for $\inf_{f\in\mathcal{C}_{\mathbf{a},M}}$ in the display above, more precisely,

$$\sup_{\boldsymbol{\theta}\in\Sigma_M}\mathcal{F}_{\mathbf{a}}\left(\boldsymbol{\theta},f^{(M)}\right)<(1+\varepsilon_M)\|\varphi\|^2_{cPc}.$$

We also have $\varphi(z)\cdot f^{(M)}(z)=1$. Let $\mathbf{e}_j=(\theta_0,\theta_1,\dots,\theta_M)\in\Sigma_M$ where $\theta_j=1$ and $\theta_k=0$ for $k\ne j$. Then we have

$$\left\|f^{(M)}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}(\Omega)}=\mathcal{F}_{\mathbf{a}}\left(\mathbf{e}_0,f^{(M)}\right)\le\sup_{\boldsymbol{\theta}\in\Sigma_M}\mathcal{F}_{\mathbf{a}}\left(\boldsymbol{\theta},f^{(M)}\right)\le(1+\varepsilon_M)\|\varphi\|^2_{cPc}\ , \tag{3.4}$$

and for each $\ell\ge 1$ we have,

$$\left\|f^{(M)}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{a_\ell}(\Omega)}=\left\|f^{(M)}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{a_\ell}(\Omega)}\le\mathcal{F}_{\mathbf{a}}\left(\widetilde{\mathbf{e}}_\ell,f^{(M)}\right)\le\sup_{\boldsymbol{\theta}\in\Sigma_M}\mathcal{F}_{\mathbf{a}}\left(\boldsymbol{\theta},f^{(M)}\right)\le(1+\varepsilon_M)\|\varphi\|^2_{cPc}\ . \tag{3.5}$$

Now we use the multiplier stable hypothesis together with the $\mathcal{H}$-Poisson reproducing formula (2.5) in Lemma 8, and then the Cauchy-Schwarz inequality for inner products, to obtain the pointwise estimates

$$\begin{aligned}\left|f^{(M)}(a_m)\right|^2 &= \left|\left\langle f^{(M)},1\right\rangle_{\oplus_{\ell=1}^N\mathcal{H}^{a_m}}\right|^2\le\left\langle f^{(M)},f^{(M)}\right\rangle_{\oplus_{\ell=1}^N\mathcal{H}^{a_m}}\langle 1,1\rangle_{\oplus_{\ell=1}^N\mathcal{H}^{a_m}}\\ &= \left\|f^{(M)}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{a_m}(\Omega)}\left\langle\widetilde{k_{a_m}},\widetilde{k_{a_m}}\right\rangle_{\oplus_{\ell=1}^N\mathcal{H}}\le(1+\varepsilon_M)\|\varphi\|^2_{cPc}\ .\end{aligned}$$

Now by separability of Ω, we can choose a dense set $S=\{a_m\}_{m=1}^\infty$ in Ω as in Definition 17, and then choose the positive constants ε_M so small that

$$\lim_{M\to\infty}\varepsilon_M=0.$$

From (3.4) and the fact that $\mathcal{H}$, and so also $\oplus_{\ell=1}^N\mathcal{H}$, has the Montel property, we conclude that there is a vector function $f\in\oplus_{\ell=1}^N\mathcal{H}$ with

$$\|f\|^2_{\oplus_{\ell=1}^N\mathcal{H}}\le\lim_{M\to\infty}(1+\varepsilon_M)\|\varphi\|^2_{cPc}=\|\varphi\|^2_{cPc}$$

and

$$f(a_m)=\lim_{M\to\infty}f^{(M)}(a_m),\qquad m\ge 1.$$

Thus we have

$$1=\lim_{M\to\infty}1=\lim_{M\to\infty}\varphi(a_m)\cdot f^{(M)}(a_m)=\varphi(a_m)\cdot\lim_{M\to\infty}f^{(M)}(a_m)=\varphi(a_m)\cdot f(a_m)$$

for all $m\ge 1$, and hence $f\in\oplus_{\ell=1}^N\mathcal{H}$ is a solution to the Bezout equation $\varphi(z)\cdot f(z)=1$ for $z\in\Omega$ since $\varphi\cdot f$ is continuous and S is dense. We also have from (3.5) the pointwise estimate

$$|f(a_m)|=\lim_{M\to\infty}\left|f^{(M)}(a_m)\right|\le\lim\sup_{M\to\infty}(1+\varepsilon_M)\|\varphi\|_{cPc}=\|\varphi\|_{cPc}$$

for each $m\ge 1$. Since $f\in\oplus_{\ell=1}^N\mathcal{H}$ satisfies $\|f\|_{\oplus_{\ell=1}^N\mathcal{H}}\le\|\varphi\|_{cPc}$ and is continuous in Ω, and since $S=\{a_m\}_{m=1}^\infty$ is dense in Ω, we thus have

$$\|f\|_{\oplus_{\ell=1}^N\mathcal{H}^\infty}=\max\left\{\|f\|_{\oplus_{\ell=1}^N\mathcal{H}},\|f\|_{L^\infty(\ell^2_N)}\right\}\le\|\varphi\|_{cPc}\ ,$$

since $\|f\|_{L^\infty(\ell^2_N)}=\sup_{z\in\Omega}|f(z)|$.

Finally, using the lower semicontinuity assumption (2) in the definition of multiplier stability (which has not been needed until now), we also have the stronger estimate

$$\|f\|_{\oplus_{\ell=1}^N K_{\mathcal{H}}} = \max\left\{\|f\|_{\mathcal{H}}, \sup_{a\in\Omega}\left\|f\widetilde{k_a}\right\|_{\mathcal{H}}\right\} = \max\left\{\|f\|_{\mathcal{H}}, \sup_{a\in\Omega}\|f\|_{\mathcal{H}^a}\right\} \leq \|\varphi\|_{cPc}. \tag{3.6}$$

Indeed, upon letting $M\to\infty$ in (3.5), we obtain

$$\max\left\{\|f\|_{\mathcal{H}}, \sup_{1\leq m<\infty}\|f\|_{\mathcal{H}^{a_m}}\right\} \leq \|\varphi\|_{cPc}.$$

The map $a\to\widetilde{k_a}$ is lower semicontinuous from Ω to $M_{\mathcal{H}}$ by assumption, and it follows that $\|f\|_{\mathcal{H}^a} = \left\|f\widetilde{k_a}\right\|_{\mathcal{H}} = \left\|\mathcal{M}_{\widetilde{k_a}}f\right\|_{\mathcal{H}}$ is a lower semicontinuous function of $a\in\Omega$. The proof of (3.6) is now completed using that $S=\{a_m\}_{m=1}^\infty$ is dense in Ω, so that $\|f\|_{\mathcal{H}^a}\leq \liminf_{a_m\to a}\|f\|_{\mathcal{H}^{a_m}}\leq\|\varphi\|_{cPc}$ for all $a\in\Omega$.

The converse assertion is straightforward. Suppose that $f\in\oplus_{\ell=1}^N K_{\mathcal{H}}$ solves $\varphi\cdot f=1$ in Ω. Then $\left\|f\widetilde{k_{a_m}}\right\|_{\oplus_{\ell=1}^N\mathcal{H}}\leq\|f\|_{\oplus_{\ell=1}^N K_{\mathcal{H}}}$ and so

$$\begin{aligned}
\|f\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} &= \theta_0\|f\|^2_{\oplus_{\ell=1}^N\mathcal{H}} + \sum_{m=1}^M\theta_m\|f\|^2_{\oplus_{\ell=1}^N\mathcal{H}^{a_m}} \\
&= \theta_0\|f\|^2_{\oplus_{\ell=1}^N\mathcal{H}} + \sum_{m=1}^M\theta_m\left\|f\widetilde{k_{a_m}}\right\|^2_{\oplus_{\ell=1}^N\mathcal{H}} \\
&\leq \theta_0\|f\|^2_{\oplus_{\ell=1}^N K_{\mathcal{H}}} + \sum_{m=1}^M\theta_m\|f\|^2_{\oplus_{\ell=1}^N K_{\mathcal{H}}} = \|f\|^2_{\oplus_{\ell=1}^N K_{\mathcal{H}}}.
\end{aligned}$$

□

4. The alternate Toeplitz corona theorem

We begin by establishing notation, in particular various corona properties.

4.1. **Corona Properties.** Fix $N\in\mathbb{N}$ throughout this subsection. Given a Hilbert function space $\mathcal{H}$ on a set Ω, denote by $\oplus^N K_{\mathcal{H}}$ and $\oplus^N M_{\mathcal{H}}$ the direct sum of the kernel multiplier spaces $K_{\mathcal{H}}$ and the multiplier algebras $M_{\mathcal{H}}$ respectively, equipped with the norms $\|\varphi\|_{\oplus^N K_{\mathcal{H}}}$ and $\|\varphi\|^{\text{column}}_{\oplus^N M_{\mathcal{H}}}$ respectively, as introduced in a previous subsection. We often write $\|\varphi\|_{\oplus^N M_{\mathcal{H}}} = \|\varphi\|^{\text{column}}_{\oplus^N M_{\mathcal{H}}}$ from now on. Next we define two 'baby' properties:

1. the first solves $\varphi\cdot f=k_a$ with $f\in\mathcal{H}$ for all $a\in\Omega$ and all $\varphi\in K_{\mathcal{H}}$ in the kernel multiplier space,
2. the second solves $\varphi\cdot f=h$ with $f\in\mathcal{H}$ for all $h\in\mathcal{H}$ and all $\varphi\in M_{\mathcal{H}}$ in the multiplier algebra.

Note that we typically define **conditions** that a vector φ of corona data might have, and we typically define **properties** that a space might have in terms of these conditions.

4.1.1. *Kernel Corona Property.*

Definition 20. *Suppose $\mathcal{H}$ is a Hilbert function space on a set Ω.*

(1) *Let $C>0$. We say that a vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N L^\infty(\Omega)$ satisfies the $\mathcal{H}-$*kernel corona condition *with constant C if for each $a\in\Omega$ there are $f_1,\dots,f_N\in\mathcal{H}$ satisfying*

$$\begin{aligned}
\|f\|^2_{\oplus^N\mathcal{H}} &= \|f_1\|^2_{\mathcal{H}}+\cdots+\|f_N\|^2_{\mathcal{H}}\leq C^2\|k_a\|^2_{\mathcal{H}}, \\
(\varphi\cdot f)(z) &= \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z)=k_a(z), \qquad z\in\Omega.
\end{aligned} \tag{4.1}$$

(2) *Let $c,C>0$. We say that $\mathcal{H}$ has the* Kernel Corona Property *with constants c,C if for every vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N K_{\mathcal{H}}$ satisfying both*

$$\|\varphi\|_{\oplus^N K_{\mathcal{H}}}\leq 1$$

and

$$|\varphi_1(z)|^2+\cdots+|\varphi_N(z)|^2\geq c^2>0, \qquad z\in\Omega, \tag{4.2}$$

the vector $\varphi=(\varphi_1,\dots,\varphi_N)$ satisfies the $\mathcal{H}$-kernel corona condition with constant C.

4.1.2. *Baby Corona Property.*

Definition 21. *Suppose $\mathcal{H}$ is a Hilbert function space on a set Ω.*

(1) *Let $C>0$. We say that a vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N L^\infty(\Omega)$ satisfies the $\mathcal{H}-$*baby corona condition *with constant C if for each $h\in\mathcal{H}$ there are $f_1,\dots,f_N\in\mathcal{H}$ satisfying*

$$\begin{aligned}\|f\|^2_{\oplus^N\mathcal{H}} &= \|f_1\|^2_{\mathcal{H}}+\cdots+\|f_N\|^2_{\mathcal{H}}\le C^2\|h\|^2_{\mathcal{H}},\\ (\varphi\cdot f)(z) &= \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z)=h(z), \qquad z\in\Omega.\end{aligned}\tag{4.3}$$

(2) *Let $c,C>0$. We say that $\mathcal{H}$ has the* Baby Corona Property *with constants c,C if for every vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N M_{\mathcal{H}}$ satisfying both*

$$\|\varphi\|_{\oplus^N M_{\mathcal{H}}}\le 1$$

and

$$|\varphi_1(z)|^2+\cdots+|\varphi_N(z)|^2\ge c^2>0, \qquad z\in\Omega,$$

the vector $\varphi=(\varphi_1,\dots,\varphi_N)$ satisfies the $\mathcal{H}$-baby corona condition with constant C.

4.1.3. *Corona Property.* Given a Hilbert function space $\mathcal{H}$ on Ω, and an algebra $\mathcal{A}$ contained in $\mathcal{H}^\infty=\mathcal{H}\cap L^\infty(\Omega)$, we define the Corona Property for the algebra $\mathcal{A}$ as follows. There is a small abuse of notation here since the definition we give will depend on N and the norm $\|\cdot\|_{\oplus^N\mathcal{A}}$ that we use for the direct sum $\oplus^N\mathcal{A}$. Thus we should really define the Corona Property for the triple $\left(\mathcal{A},N,\|\cdot\|_{\oplus^N\mathcal{A}}\right)$, but we will often suppress the dependence on N and $\|\cdot\|_{\oplus^N\mathcal{A}}$ and only specify them when needed.

Definition 22. *Suppose $N\ge 2$, $\mathcal{H}$ is a Hilbert function space on a set Ω, $\mathcal{A}$ is an algebra contained in $\mathcal{H}^\infty$, and the direct sum $\oplus^N\mathcal{A}$ is normed by $\|\cdot\|_{\oplus^N\mathcal{A}}$.*

(1) *Let $C>0$. We say that the vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N\mathcal{A}$ satisfies the $\mathcal{A}$-*corona condition *with constant C if for each $h\in\mathcal{A}$ there are $f_1,\dots,f_N\in\mathcal{A}$ satisfying*

$$\begin{aligned}\|f\|^2_{\oplus^N\mathcal{A}} &\le C^2\|h\|^2_{\mathcal{A}},\\ \varphi\cdot f(z) &= h(z), \qquad z\in\Omega.\end{aligned}\tag{4.4}$$

(2) *Let $c,C>0$. We say that $\mathcal{A}$ has the* Corona Property *with constants c,C if for every vector $\varphi=(\varphi_1,\dots,\varphi_N)\in\oplus^N\mathcal{A}$ satisfying both $\|\varphi\|_{\oplus^N\mathcal{A}}\le 1$ and (4.2), the vector $\varphi=(\varphi_1,\dots,\varphi_N)$ satisfies the $\mathcal{A}$-corona condition with constant C.*

Remark 23. *If there are $f_1,\dots,f_N\in\mathcal{A}$ satisfying (4.4) with $h=1$, then we can multiply the equation through by $h\in\mathcal{A}$, and use that $\mathcal{A}$ is an algebra to obtain $f_1h,\dots,f_Nh\in\mathcal{A}$, and hence that $\mathcal{A}$ satisfies the Corona Property.*

4.2. **Convex shifted spaces.** At this point we pause to note that (3.1) holds for all the *extreme* cases $\boldsymbol{\theta}=\mathbf{e}_m$ provided a slight strengthening of it holds for $\boldsymbol{\theta}=\mathbf{e}_0$. More precisely we have the following lemma.

Lemma 24. *Suppose that a vector $\varphi\in\oplus_{\ell=1}^N L^\infty(\Omega)$ satisfies the $\mathcal{H}$-kernel corona condition (4.1) with constant C. Then for every $a\in\Omega$, there is $g\in\oplus^N\mathcal{H}^a$ (depending on a) with $\|g\|^2_{\oplus^N\mathcal{H}^a}\le C^2$ such that $\varphi\cdot g=1$.*

Proof. Given $a\in\Omega$ there is by assumption a vector $f=(f_1,\dots,f_N)\in\oplus^N\mathcal{H}$ satisfying

$$\begin{aligned}\varphi\cdot f(z) &= \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z)=\widetilde{k_a}(z), \qquad z\in\Omega,\\ \|f\|^2_{\oplus^N\mathcal{H}} &= \|f_1\|^2_{\mathcal{H}}+\cdots+\|f_N\|^2_{\mathcal{H}}\le C^2.\end{aligned}$$

Now divide both sides of the Bezout equation above by $\widetilde{k_a}$ to obtain

$$\varphi_1(z)\frac{f_1(z)}{\widetilde{k_a}(z)}+\cdots+\varphi_N(z)\frac{f_N(z)}{\widetilde{k_a}(z)}=1, \qquad z\in\Omega.$$

Then with $g_\ell\equiv\frac{f_\ell}{\widetilde{k_a}}\in\mathcal{H}^a$ for $1\le\ell\le N$ (here we use the definition of membership in $\mathcal{H}^a$), we have

$$\|g_\ell\|^2_{\mathcal{H}^a}=\left\|g_\ell\widetilde{k_a}\right\|^2_{\mathcal{H}}=\left\|\frac{f_\ell}{\widetilde{k_a}}\widetilde{k_a}\right\|^2_{\mathcal{H}}=\|f_\ell\|^2_{\mathcal{H}}\ ,$$

and so

$$\begin{aligned} \varphi \cdot g(z) &= \varphi_1(z) g_1(z) + \cdots + \varphi_N(z) g_N(z) = 1, \qquad z \in \Omega, \\ \|g\|^2_{\oplus^N \mathcal{H}^a} &= \|g_1\|^2_{\mathcal{H}^a} + \cdots + \|g_N\|^2_{\mathcal{H}^a} = \|f_1\|^2_{\mathcal{H}} + \cdots + \|f_N\|^2_{\mathcal{H}} \le C^2. \end{aligned}$$

□

We do not know if the $\mathcal{H}$-kernel corona condition (4.1) with constant C implies the 'full' convex Poisson condition with constant C, or even with a larger positive constant C'. However, the $\mathcal{H}-$convex Poisson condition holds for every $\varphi \in \oplus_{\ell=1}^N H^\infty(\Omega)$ satisfying the $\mathcal{H}$-baby corona condition when $\mathcal{H} = H^2(\mathbb{D})$ is the classical Hardy space in the disk. This can be proved using an appropriate outer function in place of the reproducing kernel $\widetilde{k_a}$ used in the proof above, and is carried out in Lemma 40 in Section 6 below. The outer function in question is actually an invertible multiplier on $H^2(\mathbb{D})$, and this leads to the following definition.

Definition 25. *Let $\mathcal{H} = \mathcal{H}_k$ be a multiplier stable Hilbert function space on a set Ω with reproducing kernel k, and containing the constant functions. We say that the kernel k has the* Invertible Multiplier Property *if for every $(\mathbf{a}, \boldsymbol{\theta}) \in \Omega^M \times \Sigma_M(0)$, there is a normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} \in M_{\mathcal{H}}$ such that*

$$\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \left\langle \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} f, \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} g \right\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H}. \tag{4.5}$$

The normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ in (4.5) is uniquely determined up to a unimodular constant.

Lemma 26. *If two normalized invertible multipliers $\varphi_1, \varphi_2 \in M_{\mathcal{H}}$ satisfy*

$$\langle \varphi_1 f, \varphi_1 g\rangle_{\mathcal{H}} = \langle \varphi_2 f, \varphi_2 g\rangle_{\mathcal{H}} \text{ for all } f, g \in \mathcal{H}, \tag{4.6}$$

then $\varphi_1 = A\varphi_2$ for some unimodular constant A.

Proof. If equality holds in (4.6), then

$$\left\langle \frac{\varphi_1}{\varphi_2} f, \frac{\varphi_1}{\varphi_2} g \right\rangle_{\mathcal{H}} = \left\langle \varphi_1 \left(\frac{1}{\varphi_2} f\right), \varphi_1 \left(\frac{1}{\varphi_2} g\right) \right\rangle_{\mathcal{H}} = \left\langle \varphi_2 \left(\frac{1}{\varphi_2} f\right), \varphi_2 \left(\frac{1}{\varphi_2} g\right) \right\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}} ,$$

and so in particular,

$$\left\langle \frac{\varphi_1}{\varphi_2} f, \frac{\varphi_1}{\varphi_2} k_z \right\rangle_{\mathcal{H}} = \langle f, k_z\rangle_{\mathcal{H}} = f(z), \qquad z \in \Omega.$$

Thus the kernel $K_z(w) \equiv \frac{\varphi_1(w)}{\varphi_2(w)} \overline{\frac{\varphi_1(z)}{\varphi_2(z)}} k_z(w)$ satisfies

$$\langle f, K_z\rangle_{\mathcal{H}} = \frac{\varphi_1(z)}{\varphi_2(z)} \left\langle f, \frac{\varphi_1}{\varphi_2} k_z \right\rangle_{\mathcal{H}} = \frac{\varphi_1(z)}{\varphi_2(z)} \left\langle \frac{\varphi_1}{\varphi_2} \left(\frac{\varphi_2}{\varphi_1} f\right), \frac{\varphi_1}{\varphi_2} k_z \right\rangle_{\mathcal{H}} = \frac{\varphi_1(z)}{\varphi_2(z)} \left(\frac{\varphi_2}{\varphi_1} f\right)(z) = f(z)$$

for all $z \in \Omega$. By uniqueness of kernel functions, we obtain $\frac{\varphi_1(w)}{\varphi_2(w)} \overline{\frac{\varphi_1(z)}{\varphi_2(z)}} k_z(w) = k_z(w)$, hence $\frac{\varphi_1(w)}{\varphi_2(w)} \overline{\frac{\varphi_1(z)}{\varphi_2(z)}} = 1$, since $k_z(w)$ is nonvanishing by multiplier stability. Thus we conclude that there is a nonzero constant A such that $\frac{\varphi_1(w)}{\varphi_2(w)} = A$ for all $w \in \Omega$. Finally, since both multipliers are normalized we have $1 = \|\varphi_1\|_{\mathcal{H}} = \|A\varphi_2\|_{\mathcal{H}} = |A| \|\varphi_2\|_{\mathcal{H}} = |A|$. □

The identity (4.5) is equivalent to the assertion that the reproducing kernels $k^{\mathbf{a},\boldsymbol{\theta}}$ and k of the Hilbert spaces $\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}$ and $\mathcal{H}$ respectively, are rescalings of each other. Indeed, the reproducing kernel corresponding to the inner product $\left\langle \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} f, \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} g \right\rangle_{\mathcal{H}}$ is $\frac{k_w(z)}{\overline{\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(w)} \widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(z)}$. It turns out that the Invertible Multiplier Property is extremely rare - the 1-dimensional Szegö kernel has it, and this is essentially the only kernel we know with this property. See Section 6 below where we show that the Invertible Multiplier Property holds for the Hardy space on the disk, but fails for the Hardy space on both the ball and polydisc in higher dimensions.

We now show that if $\mathcal{H}$ satisfies the Invertible Multiplier Property, then the $\mathcal{H}$-baby corona condition is sufficient for the $\mathcal{H}$-convex Poisson corona condition.

Lemma 27. *Let $\mathcal{H}$ be a multiplier stable Hilbert function space on a set Ω, whose kernel k has the Invertible Multiplier Property. Then a vector $\varphi \in \oplus_{\ell=1}^N L^\infty(\Omega)$ satisfies the $\mathcal{H}-$convex Poisson condition in Definition 16 with positive constant C if $\varphi \in \oplus_{\ell=1}^N L^\infty(\Omega)$ satisfies the $\mathcal{H}$-baby corona condition (4.3) in Definition 21 with constant C.*

Proof. Repeat the proof of Lemma 24 using the invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ in place of $\widetilde{k_a}$ and use (4.5). □

4.3. **Formulation and proof.** In the case that $M_{\mathcal{H}}$ is the multiplier algebra of a Hilbert function space $\mathcal{H}$ with a *complete Pick* kernel k, there is a characterization of the Corona Property for $M_{\mathcal{H}}$ in terms of matrix-valued kernel positivity conditions involving k. This results in the Toeplitz corona theorem which asserts the equivalence of the Baby Corona Property for $\mathcal{H}$ and the Corona Property for its multiplier algebra $M_{\mathcal{H}}$, and with the **same** constants c, C - see [BaTrVi], [AmTi] and also [AgMc2, Theorem 8.57]. Here is a special case of the Toeplitz corona theorem as given in [AgMc2, Theorem 8.57].

Toeplitz corona theorem: Let $\mathcal{H}$ be a Hilbert function space in a set Ω with an irreducible complete Nevanlinna-Pick kernel. Let $C > 0$ and $N \in \mathbb{N}$ and let $\varphi = (\varphi_1, \ldots, \varphi_N) \in \oplus^N M_{\mathcal{H}}$. Then φ satisfies the $M_{\mathcal{H}}$-corona condition (4.4) with constant C in Definition 22 *if and only if* φ satisfies the $\mathcal{H}$-baby corona condition (4.3) with constant C in Definition 21.

In this paper we will use Theorem 19 to obtain an analogue of this theorem for the kernel multiplier space $K_{\mathcal{H}}$ when it is an algebra. The role of the Baby Corona Property for $\mathcal{H}$ will be played by the following property.

Definition 28. *Let $\mathcal{H}$ be a Hilbert function space with kernel k on a set Ω, and let $c, C > 0$. We say that the space $\mathcal{H}$ has the* Convex Poisson Property *with positive constants c, C if for all vectors $\varphi \in \oplus^N K_{\mathcal{H}}$ satisfying $\|\varphi\|_{\oplus^N K_{\mathcal{H}}} \leq 1$ and (4.2), the vector φ satisfies the $\mathcal{H}-$convex Poisson condition in Definition 16 with constant C.*

Here is our alternate Toeplitz corona theorem.

Theorem 29. *Suppose that $\mathcal{H}$ is a multiplier stable Hilbert function space of continuous functions on Ω that contains the constant functions, and enjoys the Montel property. Suppose further that the space of kernel multipliers $K_{\mathcal{H}}$ is an algebra.*

1. *Then $K_{\mathcal{H}}$, with the direct sum $\oplus^N K_{\mathcal{H}}$ normed by $\|\cdot\|_{\oplus^N K_{\mathcal{H}}}$, satisfies the Corona Property with positive constants c, C* if and only if *$\mathcal{H}$ satisfies the Convex Poisson Property with positive constants c, C.*
2. *Suppose in addition that $\mathcal{H}$ satisfies the Invertible Multiplier Property and that $M_{\mathcal{H}} = K_{\mathcal{H}}$ isometrically. Equip the direct sum $\oplus^N M_{\mathcal{H}}$ with the norm $\|\cdot\|_{\oplus^N M_{\mathcal{H}}}$.*
   (a) *Then $\mathcal{H}$ satisfies the Baby Corona Property with constants c, C if $M_{\mathcal{H}}$ satisfies the Corona Property with the constants c, C.*
   (b) *Conversely, $M_{\mathcal{H}}$ satisfies the Corona Property with constants $c, C\sqrt{N}$ if $\mathcal{H}$ satisfies the Baby Corona Property with constants c, C.*

Armed with Theorem 19, Lemma 27 and Corollary 11, it is now an easy matter to prove Theorem 29.

Proof of Theorem 29. The first assertion follows immediately from Theorem 19 and definitions. Now we turn to the second assertion. Clearly the Baby Corona Property with constants c, C holds for $\mathcal{H}$ if the multiplier algebra $M_{\mathcal{H}}$ satisfies the Corona Property with constants c, C. Conversely, assume the Baby Corona Property with constants c, C holds for $\mathcal{H}$ as above. We must show that the Corona Property holds for $M_{\mathcal{H}}$ with constants $c, C\sqrt{N}$. So fix a vector $\varphi \in M_{\mathcal{H}}$ with $\|\varphi\|_{\oplus_{\ell=1}^N M_{\mathcal{H}}} \leq 1$ and $|\varphi(z)| \geq c > 0$ for $z \in \Omega$. Then the Baby Corona Property for $\mathcal{H}$ with constants c, C implies that the vector φ satisfies the $\mathcal{H}$-baby corona condition with constant C, i.e. for each $h \in \mathcal{H}$ there is $f \in \oplus^N \mathcal{H}$ satisfying

$$\|f\|_{\oplus^N \mathcal{H}}^2 \leq C^2 \|h\|_{\mathcal{H}}^2 \text{ and } (\varphi \cdot f)(z) = h(z), \qquad z \in \Omega.$$

It follows from Lemma 27 that φ satisfies the $\mathcal{H}$-convex Poisson condition in Definition 16 with constant C. Now apply Theorem 19 to conclude that there is a vector function $f \in \oplus^N K_{\mathcal{H}}$ such that

$$\|f\|_{\oplus^N K_{\mathcal{H}}} \leq C \text{ and } (\varphi \cdot f)(z) = 1, \qquad z \in \Omega.$$

By (2.7) we thus conclude that $\|f\|_{\oplus^N M_{\mathcal{H}}} \leq C\sqrt{N}$, and this shows that $M_{\mathcal{H}}$ satisfies the Corona Property with constants $c, C\sqrt{N}$. □

Remark 30. *In the event that $\mathcal{H} = H^2(\Omega)$ is the Hardy space on a bounded domain with C^2 boundary in $\mathbb{C}^n$ (see Section 5 for definitions), then we have*

$$\begin{aligned}
\|\varphi\|^{\text{column}}_{\oplus_{\ell=1}^N M_{\mathcal{H}}} &= \|\mathbb{M}_\varphi\|_{\text{op}} \equiv \sup_{h\neq 0} \frac{\|\mathbb{M}_\varphi h\|_{\oplus^N \mathcal{H}}}{\|h\|_{\mathcal{H}}} = \sup_{h\neq 0} \frac{\sqrt{\sum_{\alpha=1}^N \left\|\mathcal{M}_{\varphi_\alpha} h\right\|^2_{H^2(\Omega)}}}{\|h\|_{H^2(\Omega)}} \\
&= \sup_{h\neq 0} \frac{\sqrt{\int_{\partial\Omega}\left(\sum_{\alpha=1}^N |\varphi_\alpha|^2\right)|h|^2 d\sigma}}{\sqrt{\int_{\partial\Omega}|h|^2 d\sigma}} = \left\|\sum_{\alpha=1}^N |\varphi_\alpha|^2\right\|_{L^\infty(\partial\Omega)} = \|\varphi\|_{L^\infty\left(\ell^2_N\right)} \\
&= \sup_{a\in\Omega} \frac{\sqrt{\int_{\partial\Omega}\left(\sum_{\alpha=1}^N |\varphi_\alpha|^2\right)\left|\widetilde{k_a}\right|^2 d\sigma}}{\sqrt{\int_{\partial\Omega}\left|\widetilde{k_a}\right|^2 d\sigma}} = \sup_{a\in\Omega}\sqrt{\int_{\partial\Omega}\left(\sum_{\alpha=1}^N |\varphi_\alpha|^2\right)\left|\widetilde{k_a}\right|^2 d\sigma} \leq \|\varphi\|_{\oplus_{\ell=1}^N K_{\mathcal{H}}}
\end{aligned}$$

and so $\|\varphi\|^{\text{column}}_{\oplus_{\ell=1}^N M_{\mathcal{H}}} = \|\varphi\|_{\oplus_{\ell=1}^N K_{\mathcal{H}}} = \|\varphi\|_{L^\infty\left(\ell^2_N\right)}$. Thus in the case $\mathcal{H} = H^2(\Omega)$, we can replace the constant $C\sqrt{N}$ in part (2) of Theorem 29 with the constant C, which shows that the Corona Property for $M_{\mathcal{H}}$ holds with the ***same*** *constants c, C for which the Baby Corona Property holds for $\mathcal{H}$.*

5. The alternate Toeplitz corona theorem for Bergman and Hardy spaces in $\mathbb{C}^n$

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ consists of those $f \in H(\Omega)$ such that

$$\int_\Omega |f|^2 d\widetilde{V} < \infty,$$

where $d\widetilde{V} = \frac{1}{|\Omega|} dV$ and dV is Lebesgue measure on Ω. We then have that $\|f\|_{A^2(\Omega)} \equiv \sqrt{\int_\Omega |f|^2 d\widetilde{V}}$ defines a norm on $A^2(\Omega)$. Now point evaluations are bounded, hence continuous, on $A^2(\Omega)$ by the mean value property,

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{|B(z, d_{\partial\Omega}(z))|}\int_{B(z,d_{\partial\Omega}(z))} f(w)\, dV(w)\right| \\
&\leq \frac{1}{c_n d_{\partial\Omega}(z)^n}\left(\int_\Omega |f(w)|^2 dV(w)\right)^{\frac{1}{2}} = c_n d_{\partial\Omega}(z)^{-n}\|f\|_{A^2(\Omega)}.
\end{aligned}$$

Thus the Bergman space $A^2(\Omega)$ is a reproducing kernel Hilbert space, and we denote by $k_z(w)$ the reproducing kernel for $A^2(\Omega)$ with respect to the inner product

$$\langle f, g\rangle_{A^2(\Omega)} \equiv \int_\Omega f\overline{g} d\widetilde{V}, \qquad f, g \in A^2(\Omega).$$

We then have

$$f(z) = \langle f, k_z\rangle_{A^2(\Omega)} = \int_\Omega f(w)\overline{k_z(w)} d\widetilde{V}(w), \qquad z \in \Omega,\ f \in A^2(\Omega).$$

Note also that $M_{A^2(\Omega)} = H^\infty(\Omega)$, the algebra of bounded analytic functions on Ω, and that in fact with $\mathcal{H} = A^2(\Omega)$, we have $M_{\mathcal{H}} = K_{\mathcal{H}} = \mathcal{H}^\infty = H^\infty(\Omega)$.

Remark 31. *If Ω is strictly pseudoconvex, then by Theorem 2 in C. Fefferman* [Fef] *(see also Boutet de Monvel and Sjöstrand* [BdMSj]*) we have that k_a is bounded for each $a \in \Omega$, and moreover that $\|k_a - k_b\|_\infty = \sup_{z\in\Omega}|k_a(z) - k_b(z)|$ tends to 0 as $b \to a$ for each $a \in \Omega$. Thus the map $a \to k_a$ is a* continuous *map from Ω to the multiplier algebra $H^\infty(\Omega) = M_{A^2(\Omega)}$. While the Bergman kernel functions are also nonvanishing in the ball, H. Boas has shown that in a generic sense, the Bergman kernels of strictly pseudoconvex domains have zeroes (*[Boas]*; see also Skwarczynski* [Skw, Skw]*, and Boas* [Boas2] *for a nice survey on Lu Qi-Keng's problem), and in a paper with Fu and Straube* [BFS]*, they constructed specific examples of such domains, including even some strictly convex smooth Reinhardt domains.*

Now we recall the definition of the Hardy spaces $H^2(\Omega)$ for a bounded domain Ω in $\mathbb{C}^n$ with C^2 boundary. The Hardy space $H^2(\Omega)$ consists of those $f \in H(\Omega)$ such that

$$\sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f|^2 \, d\sigma_\varepsilon < \infty,$$

where $\Omega_\varepsilon \equiv \{z \in \Omega : \rho(z) < -\varepsilon\}$ and ρ is an appropriate defining function for Ω, and where σ_ε is surface measure on $\partial\Omega_\varepsilon$. We then have that

$$\|f\|_{H^2(\Omega)} \equiv \sqrt{\int_{\partial\Omega} |f^*|^2 \, d\sigma}$$

defines a norm on $H^2(\Omega)$, where σ is surface measure on $\partial\Omega$, and where the nontangential boundary limits f^* exist a.e. $[\sigma]$ on $\partial\Omega$. We also note that

$$\int_\Omega |f|^2 \, dA \leq C \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f|^2 \, d\sigma_\varepsilon < \infty$$

for some constant C depending only on Ω, and this shows that the Bergman space norm $\|f\|_{A^2(\Omega)} \equiv \sqrt{\int_\Omega |f|^2 \, dA}$ is dominated by a multiple of the Hardy space norm $\|f\|_{H^2(\Omega)}$:

$$\|f\|_{A^2(\Omega)} \leq C_\Omega \|f\|_{H^2(\Omega)}.$$

Since point evaluations are continuous on $A^2(\Omega)$, it follows that they are also continuous on $H^2(\Omega)$. Thus the Hardy space $H^2(\Omega)$ is a reproducing kernel Hilbert space, and we denote by $k_z(w)$ the reproducing kernel for $H^2(\Omega)$ with respect to the inner product

$$\langle f, g \rangle_{H^2(\Omega)} \equiv \int_{\partial\Omega} f^* \overline{g^*} d\sigma, \qquad f, g \in H^2(\Omega).$$

We then have

$$f(z) = \langle f, k_z \rangle_{H^2(\Omega)} = \int_{\partial\Omega} f^*(w) \overline{k_z^*(w)} d\sigma(w), \qquad z \in \Omega, f \in H^2(\Omega).$$

We will typically suppress the star in the superscript and simply write $\langle f, g \rangle_{H^2(\Omega)} \equiv \int_{\partial\Omega} f\overline{g} d\sigma$ and $\langle f, k_z \rangle_{H^2(\Omega)} = \int_{\partial\Omega} f\overline{k_z} d\sigma$. In the case of the Hardy space on the polydisc $\mathbb{D}^n$, where $\partial\mathbb{D}^n$ is not C^2, integration is restricted to the distinguished boundary $\mathbb{T}^n$, a subset of measure zero of $\partial\Omega$.

The advantage of the Hardy space over the Bergman space is the equality of vector norms $\|\varphi\|^{\text{column}}_{\oplus^N H^2(\Omega)} = \|\varphi\|^{\text{row}}_{\oplus^N H^2(\Omega)} = \|\varphi\|_{L^2(\ell^2_N)}$, while the advantage of the Bergman space over the Hardy space is the greater generality of the domains Ω for which it is defined.

5.1. **The Bergman and Hardy space on the unit ball.** First note that the reproducing kernels $k_a(w) = \frac{1}{(1-\overline{a}\cdot w)^{n+1}}$ for the standard inner product on $A^2(\mathbb{B}_n)$ are invertible multipliers of $A^2(\mathbb{B}_n)$ (with possibly large multiplier norm $\|k_a\|_\infty = \frac{1}{(1-|a|)^{n+1}}$, but this norm plays no role here) depending continuously on $a \in \mathbb{B}_n$, that $M_{A^2(\mathbb{B}_n)} = H^\infty(\mathbb{B}_n)$, that $A^2(\mathbb{B}_n)$ contains the constants, and finally that $A^2(\mathbb{B}_n)$ satisfies the Montel property precisely by Montel's theorem for holomorphic functions. Similar considerations apply to the Hardy space $H^2(\mathbb{B}_n)$. From part (2) of Theorem 29 we obtain the following Corollary.

Corollary 32. *The Corona Property holds for $H^\infty(\mathbb{B}_n)$ if and only if $A^2(\mathbb{B}_n)$ satisfies the Convex Poisson Property if and only if $H^2(\mathbb{B}_n)$ satisfies the Convex Poisson Property.*

5.2. **The Hardy and Bergman space on the unit polydisc.** For the polydisc, we will again apply our alternate Toeplitz corona theorem to the Hardy space $H^2(\mathbb{D}^n)$ and the Bergman space $A^2(\mathbb{D}^n)$. First, we note that the multiplier algebra of $H^2(\mathbb{D}^n)$ is $H^\infty(\mathbb{D}^n)$, and that the reproducing kernels $k_a = \prod_{j=1}^n \frac{1}{(1-\overline{a}_j z_j)}$ for the standard inner product on $H^2(\mathbb{D}^n)$ are invertible multipliers of $H^2(\mathbb{D}^n)$ and that the map $a \to k_a$ is continuous from $\mathbb{D}^n$ to $H^\infty(\mathbb{D}^n)$. Similar considerations apply to the Bergman space $A^2(\mathbb{D}^n)$.

Corollary 33. *The corona theorem holds for $H^\infty(\mathbb{D}^n)$ if and only if $H^2(\mathbb{D}^n)$ satisfies the Convex Poisson Property if and only if $A^2(\mathbb{D}^n)$ satisfies the Convex Poisson Property.*

5.3. **General domains.** From part (2) of Theorem 29, we obtain the corona theorem for $H^\infty(\Omega)$ for any bounded domain Ω in $\mathbb{C}^n$ for which the Convex Poisson Property holds for the Bergman space $A^2(\Omega)$, and for which the Bergman space $A^2(\Omega)$ is multiplier stable (note that $K_{A^2(\Omega)} = H^\infty(\Omega)$, contains constants and satisfies the Montel property).

Greene and Krantz [GrKr] show that for domains Ω in a sufficiently small C^∞ neighbourhood of the unit ball $\mathbb{B}_n$ (or any other strictly pseudoconvex domain D with smooth boundary for which $\inf_{z,w\in D} k_D(z,w) > 0$), we have the lower bound $\inf_{z,w\in\Omega} k_\Omega(z,w) > 0$.

Corollary 34. *Let D be a strictly pseudoconvex domain with smooth boundary for which $\inf_{z,w\in D} k_D(z,w) > 0$, where k_D is the Bergman kernel for D. Then for Ω in a sufficiently small C^∞ neighbourhood of D, the corona theorem holds for $H^\infty(\Omega)$ if and only if the Convex Poisson Property holds for $A^2(\Omega)$.*

If $f \in \mathrm{Aut}(\Omega)$, then we have the Bergman kernel transformation law (see e.g. [Boas2])

$$K(z,w) = \det f'(z)\, K(f(z), f(w))\, \overline{\det f'(w)}, \qquad z, w \in \Omega.$$

If Ω is a complete circular domain, then $K(z,0)$ is a nonzero constant K_0 (see Bell [Bell]; see also [Boas2]). Now assume in addition that Ω is homogeneous and fix $\zeta \in \Omega$. Then with $f \in \mathrm{Aut}(\Omega)$ such that $f(\zeta) = 0$, the transformation law shows that

$$K(z,\zeta) = \det f'(z)\, K(f(z), f(\zeta))\, \overline{\det f'(\zeta)} = \det f'(z)\, K_0 \overline{\det f'(\zeta)}.$$

If $g = f^{-1}$ we have $\det f'(z) = \frac{1}{\det g'(z)}$, and from another application of the transformation law with $\eta = f(0)$,

$$\overline{K(z,\eta)} = K(\eta, z) = \det g'(\eta)\, K(g(\eta), g(z))\, \overline{\det g'(z)} = \det g'(\eta)\, \overline{K_0} \overline{\det g'(z)},$$

we see that

$$\begin{aligned} K(z,\zeta) &= \det f'(z)\, K_0 \overline{\det f'(\zeta)} = \frac{1}{\det g'(z)} K_0 \overline{\det f'(\zeta)} = \frac{\overline{\det g'(\eta)} K_0}{K(z,\eta)} K_0 \overline{\det f'(\zeta)} \\ &= K_0^2 \frac{\overline{\det g'(\eta) \det f'(\zeta)}}{K(z,\eta)}, \end{aligned}$$

and hence that

$$\inf_{z\in\Omega} |K(z,\zeta)| \geq |K_0|^2 \frac{|\det g'(\eta) \det f'(\zeta)|}{\|K_\eta\|_\infty} > 0$$

since $\|K_\eta\|_\infty < \infty$ if Ω is a strictly pseudoconvex domain. The continuity of the map $\zeta \to K(z,\zeta)$ also follows from these calculations. This gives the following corollary to Theorem 29.

Corollary 35. *Let Ω be a bounded strictly pseudoconvex homogeneous complete circular domain with smooth boundary in $\mathbb{C}^n$. Then the corona theorem holds for $H^\infty(\Omega)$ if and only if the Convex Poisson Property holds for $A^2(\Omega)$.*

Remark 36. *The above smoothness assumption on the boundaries of D and Ω can be relaxed, but we will not pursue that here.*

Remark 37. *In all of the above corollaries, the Baby Corona Property for $H^2(\Omega)$ and $A^2(\Omega)$ is known to hold - see* [KrLi] *for strictly pseudoconvex domains as above, and see* [Lin], [Li], [Lin2], [Tren], *and* [TrWi2] *for the polydisc.*

Remark 38. *For the pseudoconvex domain constructed by Sibony* [Sib]*, in which the Corona Property for $H^\infty(\Omega)$ fails, the Szegö kernel functions cannot be invertible multipliers depending in a lower semicontinuous way on the pole. Indeed, $M_{H^2(\Omega)} = H^\infty(\Omega)$ and $H^2(\Omega)$ contains constants and satisfies the Montel property. Moreover, by a result of Andersson and Carlsson* [AnCa]*, the baby corona theorem holds for $H^2(\Omega)$ on pseudoconvex domains with smooth boundary, and Theorem 29 now shows that the Szegö kernels cannot be bounded away from both 0 and ∞ and satisfy the semicontinuity assumption on the Sibony domain.*

Remark 39. *There is a partial result for certain 'polydomains' in $\mathbb{C}^n$. The Bergman space $A^2(\Omega)$ satisfies $K_{A^2(\Omega)} = M_{A^2(\Omega)} = H^\infty(\Omega)$, and clearly contains the constants and is a Montel space. If the Bergman spaces $A^2(\Omega_j)$, $\Omega_j \subset \mathbb{C}^{n_j}$, are multiplier stable for $1 \leq j \leq J$, then the Bergman space $A^2(\Omega)$ for the polydomain $\Omega = \prod_{j=1}^{J} \Omega_j \subset \mathbb{C}^n$, $n = \sum_{j=1}^{J} n_j$, is also multiplier stable (since the kernel function of $A^2(\Omega)$ is* *the product of the kernel functions for $A^2(\Omega_j)$, and the multiplier norm is the supremum norm). Thus in this case $M_{A^2(\Omega)}$ satisfies the Corona Property if and only if $A^2(\Omega)$ satisfies the Convex Poisson Property.*

6. Invertible Multiplier Property for Hardy spaces

Here we begin by showing that the Invertible Multiplier Property holds for the Hardy space $H^2(\mathbb{D})$ on the disk.

Lemma 40. *The Szegö kernel for the Hardy space $\mathcal{H} = H^2(\mathbb{D})$ has the Invertible Multiplier Property.*

Proof. Take $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ to be the outer function

$$\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(z) \equiv \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\ln\left(\sum_{m=0}^{M}\theta_m\left|\widetilde{k_{a_m}}(e^{it})\right|^2\right)dt\right\},$$

which by [Rud4, Theorem 17.16 (b)] satisfies $\left|\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(e^{it})\right|^2 = \sum_{m=0}^{M}\theta_m\left|\widetilde{k_{a_m}}(e^{it})\right|^2$ for almost every $0 \le t < 2\pi$ (the equality actually holds for all t by continuity of the $\widetilde{k_{a_m}}$ on the boundary). Here the normalized reproducing kernel $\widetilde{k_{a_m}}(z)$ is given by $\widetilde{k_{a_m}}(z) = \frac{\sqrt{1-|a_m|^2}}{1-\overline{a_m}z}$. Then $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ and $\frac{1}{\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}}$ are bounded in the disk and we have

$$\begin{aligned}
\left\langle\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}f,\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}g\right\rangle_{\mathcal{H}} &= \int_0^{2\pi}\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(e^{it})\,f^*(e^{it})\,\overline{\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(e^{it})\,g^*(e^{it})}\,dt\\
&= \int_0^{2\pi}f^*(e^{it})\,\overline{g^*(e^{it})}\,\left|\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}(e^{it})\right|^2dt = \sum_{m=0}^{M}\theta_m\int_0^{2\pi}f^*(e^{it})\,\overline{g^*(e^{it})}\,\left|\widetilde{k_{a_m}}(e^{it})\right|^2dt\\
&= \sum_{m=0}^{M}\theta_m\left\langle\widetilde{k_{a_m}}f,\widetilde{k_{a_m}}g\right\rangle_{H^2(\mathbb{D})} = \sum_{m=0}^{M}\theta_m\langle f,g\rangle_{\mathcal{H}^{a_m}} = \langle f,g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}}\ .
\end{aligned}$$

In particular $\left\|\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}\right\|_{H^2(\mathbb{D})} = \|1\|_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = 1$, and so we have shown that $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ is a normalized invertible multiplier satisfying (4.5). □

Now we show that the Invertible Multiplier Property *fails* for the Szegö kernel for the Hardy space on the ball in higher dimensions by showing that when at least two of the θ_m are positive, then there are no invertible multipliers whose real parts have boundary values equal to $\ln\left(\sum_{m=0}^{M}\theta_m\left|\widetilde{k_{a_m}}\right|^2\right)$ almost everywhere. This should be contrasted with results of Aleksandrov [Ale] and Løw [Low] that yield *nonvanishing* multipliers whose real parts *do* have boundary values equal to $\ln\left(\sum_{m=0}^{M}\theta_m\left|\widetilde{k_{a_m}}\right|^2\right)$ almost everywhere. In the proof of Theorem 19 above (see (3.3)) it was essential that $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}}$ was an *invertible* multiplier, as opposed to a more general *nonvanishing* multiplier. But the reciprocals of the Aleksandrov and Løw multipliers are not bounded in the ball (only their boundary values are bounded almost everywhere on the sphere), and in fact the reciprocals do not belong to any reasonable class of holomorphic functions on the ball. These nonvanishing multipliers can be thought of as ball analogues of products of *singular* inner functions with outer functions in the disk.

Lemma 41. *The Hardy space $\mathcal{H} = H^2(\mathbb{B}_n)$ on the unit ball in $\mathbb{C}^n$ fails to have the Invertible Multiplier Property when $n > 1$. The failure is spectacular in that for* any $(\mathbf{a},\boldsymbol{\theta}) \in \Omega^M \times \Sigma_M$ *with at least two of the θ_m positive, there is no normalized invertible multiplier $\widetilde{k_{\mathbf{a},\boldsymbol{\theta}}} \in M_{\mathcal{H}} = H^\infty(\mathbb{B}_n)$ satisfying (4.5).*

Proof. We prove the case when $M = 1$, $\theta_0 = \theta_1 = \frac{1}{2}$ and $a_1 = \alpha e_1 = (\alpha, 0, \ldots, 0) \in \mathbb{B}_n$, and leave the general case to the interested reader. We assume, in order to derive a contradiction, that that there is a normalized invertible multiplier $\varphi \in M_{\mathcal{H}}$ such that

$$\langle f,g\rangle_{\mathcal{H}} + \langle k_{a_1}f,k_{a_1}g\rangle_{\mathcal{H}} = 2\langle f,g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \langle\varphi f,\varphi g\rangle_{\mathcal{H}}\,, \qquad f,g\in\mathcal{H},$$

where we have absorbed the factor 2 into φ for convenience. Unraveling notation this becomes

$$\int_{\partial\mathbb{B}_n} f\overline{g}\left(1+\left|\widetilde{k_{a_1}}\right|^2\right)d\sigma=\int_{\partial\mathbb{B}_n} f\overline{g}\left|\varphi\right|^2 d\sigma,\qquad f,g\in H^2\left(\mathbb{B}_n\right).$$

In particular we obtain that

$$1+\left|\widetilde{k_{a_1}}\right|^2=\left|\varphi\right|^2\text{ on }\partial\mathbb{B}_n$$

by the Stone-Weierstrass theorem, since the algebra $\mathcal{A}$ generated by restrictions to $\partial\mathbb{B}_n$ of f and $\overline{g}$ for $f,g\in A\left(\mathbb{B}_n\right)$, is self-adjoint, separates points and contains the constants on $\partial\mathbb{B}_n$; and thus $\mathcal{A}$ is dense in $C\left(\partial\mathbb{B}_n\right)$. Since φ is an *invertible* multiplier, we claim it has a holomorphic logarithm $F=\log\varphi$ in $H^\infty\left(\mathbb{B}_n\right)$ whose boundary values satisfy

$$2\operatorname{Re}F=2\operatorname{Re}\log\varphi=\ln\left|\varphi\right|^2.$$

Indeed, to see this we consider the dilates $F_r\left(z\right)\equiv F\left(rz\right)$ of F which are clearly in $H^\infty\left(\mathbb{B}_n\right)$ (although not uniformly in $0<r<1$), and whose real parts are uniformly bounded in $0<r<1$:

$$\operatorname{Re}F_r\left(z\right)=\operatorname{Re}F\left(rz\right)=\operatorname{Re}\log\varphi\left(rz\right)=\ln\left|\varphi\left(rz\right)\right|\leq\left\|\ln\left|\varphi\right|\right\|_\infty<\infty,$$

where the boundedness of $\ln\left|\varphi\right|$ follows from the maximum principle since $\ln\left|\varphi\right|$ is continuous in $\overline{\mathbb{B}_n}$ and harmonic in $\mathbb{B}_n$. Then the Koranyi-Vagi theorem gives the uniform L^p estimate (see e.g. [Rud2, inequality (1) on page 125])

$$\int_{\partial\mathbb{B}_n}\left|F_r\right|^p d\sigma\leq M_{p,n}\int_{\partial\mathbb{B}_n}\left|\operatorname{Re}F_r\right|^p d\sigma\leq M_{p,n}\left\|\ln\left|\varphi\right|\right\|_\infty^p,\qquad 1<p<\infty.$$

Thus $F\in H^p\left(\mathbb{B}_n\right)$ for all $1<p<\infty$, and in particular F is in the Hardy space $H^2\left(\mathbb{B}_n\right)$. Thus the restriction h of $\ln\left|\varphi\right|^2=\ln\left(1+\left|\widetilde{k_{a_1}}\right|^2\right)$ to $\partial\mathbb{B}_n$ is the boundary value function of the real part $\operatorname{Re}F$ of a holomorphic function F in $H^2\left(\mathbb{B}_n\right)$. It follows that for almost every $\zeta\in\partial\mathbb{B}_n$, the slice function $F_\zeta\left(\lambda\right)=F\left(\lambda\zeta\right)$ defined on the slice $S_\zeta=\left\{z\in\mathbb{B}_n:z=\lambda\zeta,\lambda\in\mathbb{D}\right\}$, is in $H^2\left(S_\zeta\right)\approx H^2\left(\mathbb{D}\right)$ and has boundary values equal to $h\mid_{\partial S_\zeta}$ almost everywhere. In particular then, the integral of $h=\ln\left|\varphi\right|^2$ on the boundary of such a slice S_ζ, with respect to Haar measure dm on $\mathbb{T}=\partial S_\zeta$, must equal the value of h at the origin, i.e. the constant $\ln\left|\varphi\left(0\right)\right|^2$. Thus we have shown that the function

$$G\left(\zeta\right)\equiv\int_{\partial S_\zeta}\ln\left(1+\left|\widetilde{k_{a_1}}\right|^2\right)dm=\int_{\left|\lambda\right|=1}\ln\left(1+\left|\frac{\sqrt{1-\left|\alpha\right|^2}}{1-\alpha\lambda\zeta_1}\right|^{2n}\right)dm\left(\lambda\right)$$

equals the constant $\ln\left|\varphi\left(0\right)\right|^2$ almost everywhere on the sphere $\partial\mathbb{B}_n$. However, it is clear from the integral on the right hand side that the function G is continuous on the sphere $\partial\mathbb{B}_n$. In particular the map

$$g\left(z\right)\equiv G\left(z,0,\ldots,0,\sqrt{1-\left|z\right|^2}\right)=\int_{\left|\lambda\right|=1}\ln\left(1+\left|\frac{\sqrt{1-\left|\alpha\right|^2}}{1-\alpha\lambda z}\right|^{2n}\right)dm\left(\lambda\right)$$

is constant for $z\in\mathbb{D}$.

We now derive a contradiction by calculating that g is *not* constant provided that $\alpha\neq0$. In fact, g is clearly twice differentiable in the disk, and we will now show that g has nonvanishing Laplacian in the disk (this approach is suggested by a theorem of Forelli - see e.g. [Rud3, Theorem 4.4.4] - that shows $u:\mathbb{B}_n\to\mathbb{R}$ is the real part of a holomorphic function on the ball if and only if u is harmonic in each slice and smooth near the origin). We have

$$g\left(z\right)=\int_{\left|\lambda\right|=1}\ln A\left(\lambda z\right)dm\left(\lambda\right),$$

where

$$A\left(z\right)\equiv1+\left|\frac{\sqrt{1-\left|\alpha\right|^2}}{1-\alpha z}\right|^{2n}=1+\left(1-\left|\alpha\right|^2\right)^n\left|\left(1-\alpha z\right)^{-n}\right|^2\equiv1+c_{\alpha,n}\left|\left(1-\alpha z\right)^{-n}\right|^2.$$

Then we have

$$\begin{aligned}\frac{\partial}{\partial z}A(\lambda z) &= c_{\alpha,n}\overline{(1-\alpha\lambda z)^{-n}}(-n)(1-\alpha\lambda z)^{n+1}(-\alpha\lambda) = c_{\alpha,n}n\alpha\lambda\left|(1-\alpha\lambda z)^{-n}\right|^2(1-\alpha\lambda z)\\ &= c_{\alpha,n}n\alpha\lambda(1-\alpha\lambda z)(A(\lambda z)-1),\end{aligned}$$

and of course $\frac{\partial}{\partial \overline{z}}A(\lambda z) = \overline{\frac{\partial}{\partial z}A(\lambda z)}$ because A is real. Thus

$$\begin{aligned}&\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\ln A(\lambda z) = \frac{\partial}{\partial \overline{z}}\left\{c_{\alpha,n}n\alpha\lambda(1-\alpha\lambda z)\left(1-\frac{1}{A(\lambda z)}\right)\right\}\\ = &-c_{\alpha,n}n\alpha\lambda(1-\alpha\lambda z)\frac{\partial}{\partial \overline{z}}\frac{1}{A(\lambda z)} = c_{\alpha,n}n\alpha\lambda(1-\alpha\lambda z)\frac{1}{A(\lambda z)^2}\frac{\partial}{\partial \overline{z}}A(\lambda z)\\ = &\ c_{\alpha,n}n\alpha\lambda(1-\alpha\lambda z)\frac{1}{A(\lambda z)^2}\overline{\{n\alpha\lambda(1-\alpha\lambda z)(A(\lambda z)-1)\}} = c_{\alpha,n}\left|n\alpha\lambda(1-\alpha\lambda z)\right|^2\frac{A(z)-1}{A(\lambda z)^2},\end{aligned}$$

which is strictly positive for $|\lambda| = 1$ if $\alpha \neq 0$. This shows that

$$\frac{1}{4}\triangle g(z) = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}F(z) = \int_{|\lambda|=1}\left\{\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\ln A(\lambda z)\right\}dm(\lambda) > 0$$

for $z \in \mathbb{D}$, and this shows that g is not constant in the disk, providing the claimed contradiction. This completes the proof of Lemma 41. □

Essentially the same argument as above shows that the Hardy space $H^2(\mathbb{D}^n)$ on the polydisc $\mathbb{D}^n$ also fails to have the Invertible Multiplier Property when $n > 1$.

Lemma 42. *The Hardy space $\mathcal{H} = H^2(\mathbb{D}^n)$ on the polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$ fails to have the Invertible Multiplier Property when $n > 1$.*

Proof. We prove the Invertible Multiplier Property fails in the case when $M = 1$, $\theta_0 = \theta_1 = \frac{1}{2}$ and $a_1 = \alpha e_1 = (\alpha, \alpha, \dots, 0) \in \mathbb{D}^n$ with $\alpha \neq 0$. We assume, in order to derive a contradiction, that that there is a normalized invertible multiplier $\varphi \in M_{\mathcal{H}}$ such that

$$\langle f, g\rangle_{\mathcal{H}} + \langle k_{a_1}f, k_{a_1}g\rangle_{\mathcal{H}} = 2\langle f, g\rangle_{\mathcal{H}^{\mathbf{a},\boldsymbol{\theta}}} = \langle \varphi f, \varphi g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H},$$

where again we have absorbed the factor 2 into φ for convenience. Unraveling notation this becomes

$$\int_{\mathbb{T}^n} f\overline{g}\left(1 + \left|\widetilde{k_{a_1}}\right|^2\right)dm = \int_{\mathbb{T}^n} f\overline{g}\left|\varphi\right|^2 dm, \qquad f, g \in H^2(\mathbb{D}^n).$$

In particular we obtain that

$$1 + \left|\widetilde{k_{a_1}}\right|^2 = |\varphi|^2 \text{ on } \mathbb{T}^n$$

by the Stone-Weierstrass theorem, just as in the proof of Lemma 41. Since φ is an *invertible* multiplier, the argument used in the proof of Lemma 41 shows it has a holomorphic logarithm $F = \log\varphi$ in $H^2(\mathbb{D}^n)$ with boundary values $\ln|\varphi|^2 = 2\operatorname{Re}\log\varphi = 2\operatorname{Re}F$. Thus the restriction h of $\ln\left(1 + \left|\widetilde{k_{a_1}}\right|^2\right)$ to $\mathbb{T}^n$ is the distinguished boundary value function of the real part $\operatorname{Re}F$ of a holomorphic function F in $H^2(\mathbb{D}^n)$. It follows that for almost every $\zeta \in \mathbb{T}^n$, the slice function $F_\zeta(\lambda) = F(\lambda\zeta)$ defined on the slice $S_\zeta = \{z \in \mathbb{D}^n : z = \lambda\zeta, \lambda \in \mathbb{D}\}$, is in $H^2(S_\zeta) \approx H^2(\mathbb{D})$ and has boundary values equal to $h\,|_{\mathbb{T}^n}$. In particular then, the integral of h on the boundary of such a slice S_ζ, with respect to Haar measure dm on $\mathbb{T} = \partial S_\zeta$, must equal the constant $\ln|\varphi(0)|^2$. Thus we have shown that the function

$$G(\zeta) \equiv \int_{\partial S_\zeta}\ln\left(1 + \left|\widetilde{k_{a_1}}\right|^2\right)dm = \int_{|\lambda|=1}\ln\left(1 + \left|\frac{\sqrt{1-|\alpha|^2}}{1-\alpha\lambda\zeta_1}\frac{\sqrt{1-|\alpha|^2}}{1-\alpha\lambda\zeta_2}\right|^2\right)dm(\lambda)$$

equals the constant $\ln|\varphi(0)|^2$ almost everywhere on the distinguished boundary $\mathbb{T}^n$. In particular the map

$$g(z) \equiv G(z, z, 0, \dots, 0) = \int_{|\lambda|=1}\ln\left(1 + \left|\left(\frac{\sqrt{1-|\alpha|^2}}{1-\alpha\lambda z}\right)^2\right|^2\right)dm(\lambda)$$

is the constant $\ln |\varphi(0)|^2$ almost everywhere on $\mathbb{D}$, and since g is clearly continuous in $\mathbb{D}$, it is constant in $\mathbb{D}$. But this function g is the same as the function g obtained in the previous proof for $n = 2$. We showed there that this function is not constant, and this completes the proof of Lemma 42. □

We emphasize that $H^2(\Omega)$ for certain bounded finitely connected planar domains are essentially the only spaces we currently know to have the Invertible Multiplier Property ($H^2(\Omega)$ has this property if Ω is simply connected by the Riemann mapping theorem).

7. Smoothness spaces and nonholomorphic spaces

In the first subsection, we show that if $\mathcal{H}$ is a Besov-Sobolev space $B_2^\sigma(\mathbb{B}_n)$ on the ball for $\sigma > 0$ and $n \geq 1$, then the kernel multiplier space $K_{\mathcal{H}}$ is an algebra. Moreover, the reproducing kernels satisfy the hypotheses of the alternate Toeplitz corona theorem 29, so that as a consequence of this subsection we obtain the following corollary.

Corollary 43. *Let $\sigma > 0$ and $n \geq 1$. The kernel multiplier algebra $K_2^\sigma(\mathbb{B}_n)$ satisfies the Corona Property if and only if the Convex Poisson Property holds for $B_2^\sigma(\mathbb{B}_n)$.*

Then in the second subsection we make the obvious extension of kernel multiplier algebras for the Banach spaces of analytic functions $B_p^\sigma(\mathbb{D})$ and then prove the Corona Property for the kernel multiplier algebras $K_{B_p^\sigma(\mathbb{D})}$ on the disk when $0 \leq \sigma < \frac{1}{p}$. We recall here that the algebra $B_2^0(\mathbb{D})^\infty = H^\infty(\mathbb{D}) \cap B_2^0(\mathbb{D})$ of bounded Dirichlet space functions was shown to have the Corona Property by Nicolau [Nic], who used the difficult theory of best approximation in VMO due to Peller and Hruscev [PeHr].

Finally, in the third subsection, we demonstrate that there are many Hilbert function spaces to which our alternate Toeplitz corona theorem applies, that are *not* spaces of holomorphic functions.

7.1. **The kernel multiplier algebras for Besov-Sobolev spaces.**

We begin by extending some of the background material developed in [ArRoSa2] for the Besov spaces $B_p(\mathbb{B}_n)$ to the Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$, $\sigma \geq 0$. Some of this material appears in [ArRoSa3].

7.1.1. *Besov-Sobolev spaces.* Recall the invertible "radial" operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [Zhu] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. If the inverse of $R^{\gamma,t}$ is denoted $R_{\gamma,t}$, then Proposition 1.14 of [Zhu] yields

$$\begin{aligned} R^{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}}\right) &= \frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}}, \\ R_{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}}\right) &= \frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}}, \end{aligned} \tag{7.1}$$

for all $w \in \mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t. From Theorem 6.1 and Theorem 6.4 of [Zhu] we have that the derivatives $R^{\gamma,m} f(z)$ are "L^p norm equivalent" to $\sum_{k=0}^{m-1} \left|f^{(k)}(0)\right| + f^{(m)}(z)$ for m large enough.

Proposition 44. *(analogue of Theorem 6.1 and Theorem 6.4 of* [Zhu]*) Suppose that $0 < p < \infty$, $0 \leq \sigma < \infty$, $n + \gamma$ is not a negative integer, and $f \in H(\mathbb{B}_n)$. Then the following four conditions are equivalent:*

$$\begin{aligned} \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) &\in L^p(d\lambda_n) \text{ for some } m+\sigma > \frac{n}{p}, m \in \mathbb{N}, \\ \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) &\in L^p(d\lambda_n) \text{ for all } m+\sigma > \frac{n}{p}, m \in \mathbb{N}, \\ \left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) &\in L^p(d\lambda_n) \text{ for some } m+\sigma > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N}, \\ \left(1-|z|^2\right)^{m+\sigma} R^{\gamma,m} f(z) &\in L^p(d\lambda_n) \text{ for all } m+\sigma > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N}. \end{aligned}$$

Moreover, with $\rho(z) = 1 - |z|^2$*, we have for* $1 < p < \infty$*,*
(7.2)

$$C^{-1} \left\| \rho^{m_1+\sigma} R^{\gamma,m_1} f \right\|_{L^p(d\lambda_n)} \leq \sum_{k=0}^{m_2-1} \left| f^{(k)}(0) \right| + \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m_2+\sigma} f^{m_2}(z) \right|^p d\lambda_n(z) \right)^{\frac{1}{p}} \leq C \left\| \rho^{m_1+\sigma} R^{\gamma,m_1} f \right\|_{L^p(d\lambda_n)}$$

for all $m_1 + \sigma, m_2 + \sigma > \frac{n}{p}$*,* $m_1 + n + \gamma \notin -\mathbb{N}$*,* $m_2 \in \mathbb{N}$*, and where the constant* C *depends only on* σ*,* m_1*,* m_2*,* n*,* γ *and* p*.*

Definition 45. *We define the analytic Besov-Sobolev spaces* $B_p^\sigma(\mathbb{B}_n)$ *on the ball* $\mathbb{B}_n$ *by taking* $\gamma = 0$ *and* $m = \frac{n+1}{p}$ *and setting*

$$B_p^\sigma = B_p^\sigma(\mathbb{B}_n) = \left\{ f \in H(\mathbb{B}_n) : \left\| \rho^{m+\sigma} R^{0,m} f \right\|_{L^p(d\lambda_n)} < \infty \right\}. \tag{7.3}$$

We will indulge in the usual abuse of notation by using $\|f\|_{B_p^\sigma(\mathbb{B}_n)}$ to denote any of the norms appearing in (7.2).

7.1.2. *Reproducing kernels.* For $\alpha > -1$, let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product for the weighted Bergman space A_α^2:

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \overline{g(z)} d\nu_\alpha(z), \quad f, g \in A_\alpha^2,$$

where $d\nu_\alpha(z) = \left(1 - |z|^2\right)^\alpha dV(z)$. Recall that $K_w^\alpha(z) = K^\alpha(z,w) = (1 - \overline{w} \cdot z)^{-n-1-\alpha}$ is the reproducing kernel for A_α^2 (Theorem 2.7 in [Zhu]):

$$f(w) = \langle f, K_w^\alpha \rangle_\alpha = \int_{\mathbb{B}_n} f(z) \overline{K_w^\alpha(z)} d\nu_\alpha(z), \quad f \in A_\alpha^2.$$

This formula continues to hold as well for $f \in A_\alpha^p$, $1 < p < \infty$, since the polynomials are dense in A_α^p.

The proof of Corollary 6.5 of [Zhu] shows that $R^{\gamma, \frac{n+1+\alpha}{p} - \sigma}$ is a bounded invertible operator from B_p^σ onto A_α^p, provided that neither $n + \gamma$ nor $n + \gamma + \frac{n+1+\alpha}{p} - \sigma$ is a negative integer. It turns out to be convenient to take $\gamma = \alpha - \frac{n+1+\alpha}{p} + \sigma$ here (with this choice we can explicitly compute certain formulas - see (9) below), and thus we single out the special operators

$$\mathcal{R}_t^\alpha = R^{\alpha-t,t}.$$

Note that the operators $\mathcal{R}_t^\alpha$ and their inverses $(\mathcal{R}_t^\alpha)^{-1} = R_{\alpha-t,t}$ are self-adjoint with respect to $\langle \cdot, \cdot \rangle_\alpha$ since the monomials are orthogonal with respect to $\langle \cdot, \cdot \rangle_\alpha$ (see (1.21) and (1.23) in [Zhu]), and the operators act on the homogeneous expansion of f by multiplying the homogeneous coefficients of f by certain positive constants. The next definition is motivated by the fact that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p} - \sigma}$ is a bounded invertible operator from B_p^σ onto A_α^p, and that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'} - \sigma}$ is a bounded invertible operator from $B_{p'}^\sigma$ onto $A_\alpha^{p'}$, provided that neither $n + \alpha$, $n + \alpha - \frac{n+1+\alpha}{p} + \sigma$ nor $n + \alpha - \frac{n+1+\alpha}{p'} + \sigma$ is a negative integer. Note that this proviso holds in particular for $\alpha > -1$, $\sigma \geq 0$.

Definition 46. *For* $\alpha > -1$*,* $\sigma \geq 0$ *and* $1 < p < \infty$*, we define a pairing* $\langle \cdot, \cdot \rangle_{\alpha,p}^\sigma$ *for* B_p^σ *and* $B_{p'}^\sigma$ *using* $\langle \cdot, \cdot \rangle_\alpha$ *as follows:*

$$\begin{aligned} \langle f, g \rangle_{\alpha,p}^\sigma &= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p} - \sigma} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'} - \sigma} g \right\rangle_\alpha = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p} - \sigma} f(z) \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'} - \sigma} g(z)} d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\{ \left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p} - \sigma} f(z) \right\} \overline{\left\{ \left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'} - \sigma} g(z) \right\}} d\lambda_n(z). \end{aligned}$$

Clearly we have

$$\left| \langle f, g \rangle_{\alpha,p}^\sigma \right| \leq \|f\|_{B_p^\sigma} \|g\|_{B_{p'}^\sigma}$$

by Hölder's inequality. By Theorem 2.12 of [Zhu], we also have that every continuous linear functional Λ on B_p^σ is given by $\Lambda f = \langle f, g\rangle_{\alpha,p}^\sigma$ for a unique $g \in B_{p'}^\sigma$ satisfying

$$\|g\|_{B_{p'}^\sigma} = \sup_{\|f\|_{B_p^\sigma}=1} \left|\langle f, g\rangle_{\alpha,p}^\sigma\right|. \tag{7.4}$$

Indeed, if $\Lambda \in \left(B_p^\sigma\right)^*$, then $\Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} \in \left(A_\alpha^p\right)^*$, and by Theorem 2.12 of [Zhu], there is $G \in A_\alpha^{p'}$ with $\|G\|_{A_\alpha^{p'}} = \|\Lambda\|$ such that $\Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} F = \langle F, G\rangle_\alpha$ for all $F \in A_\alpha^p$. If we set $g = \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma}\right)^{-1} G$, then we have $\|g\|_{B_{p'}^\sigma} = \|G\|_{A_\alpha^{p'}} = \|\Lambda\|$ and with $F = \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f$, we also have

$$\Lambda f = \Lambda \circ \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} F = \langle F, G\rangle_\alpha = \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma} g \right\rangle_\alpha = \langle f, g\rangle_{\alpha,p}^\sigma$$

for all $f \in B_p^\sigma$. Then (7.4) follows from

$$\|g\|_{B_{p'}^\sigma} = \|\Lambda\| = \sup_{\|f\|_{B_p^\sigma}=1} |\Lambda(f)| = \sup_{\|f\|_{B_p^\sigma}=1} \left|\langle f, g\rangle_{\alpha,p}^\sigma\right|.$$

With $K_w^\alpha(z)$ the reproducing kernel for A_α^2, we now claim that the kernel

$$k_w^{\sigma,\alpha,p}(z) = \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} K_w^\alpha(z) \tag{7.5}$$

satisfies the following reproducing formula for B_p^σ:

$$f(w) = \langle f, k_w^{\sigma,\alpha,p}\rangle_{\alpha,p}^\sigma = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f(z)\, \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma} k_w^{\sigma,\alpha,p}(z)}\, d\nu_\alpha(z), \qquad f \in B_p^\sigma. \tag{7.6}$$

Indeed, for f a polynomial, we have

$$\begin{aligned} f(w) &= \langle f, K_w^\alpha\rangle_\alpha = \left\langle \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f, K_w^\alpha \right\rangle_\alpha = \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f, \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} K_w^\alpha \right\rangle_\alpha \\ &= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} K_w^\alpha \right\rangle_\alpha \\ &= \left\langle f, \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} K_w^\alpha \right\rangle_{\alpha,p}^\sigma . \end{aligned}$$

We now obtain the claim since the polynomials are dense in B_p^σ and the kernels $k_w^{\sigma,\alpha,p}$ are in $B_{p'}^\sigma$ for each fixed $w \in \mathbb{B}_n$. Thus we have proved the following theorem.

Theorem 47. *Let $1 < p < \infty$, $\sigma \geq 0$ and $\alpha > -1$. Then the dual space of B_p^σ can be identified with $B_{p'}^\sigma$ under the pairing $\langle \cdot, \cdot\rangle_{\alpha,p}^\sigma$, and the reproducing kernel $k_w^{\sigma,\alpha,p}$ for this pairing is given by (7.5).*

From (7.5) and (7.1) we have

$$\begin{aligned} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}-\sigma} k_w^{\sigma,\alpha,p}(z) &= \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}-\sigma}\right)^{-1} K_w^\alpha(z) \\ &= R_{\alpha-\frac{n+1+\alpha}{p}+\sigma, \frac{n+1+\alpha}{p}-\sigma} \left((1-\overline{w}\cdot z)^{-(n+1+\alpha)} \right) \\ &= (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}-\sigma} . \end{aligned} \tag{7.7}$$

7.1.3. *Kernel multiplier spaces.* Define the Banach space $K_p^\sigma(\mathbb{B}_n)$ of kernel multipliers of $B_p^\sigma(\mathbb{B}_n)$ to consist of those holomorphic functions in the ball $\mathbb{B}_n$ such that

$$\|\varphi\|_{K_p^\sigma(\mathbb{B}_n)} \equiv \sup_{a\in\mathbb{B}_n} \frac{\left\|\varphi k_a^{\sigma,p'}\right\|_{B_p^\sigma(\mathbb{B}_n)}}{\left\|k_a^{\sigma,p'}\right\|_{B_p^\sigma(\mathbb{B}_n)}} < \infty,$$

where $k_a^{\sigma,p'}(z)$ is a reproducing kernel for $B_{p'}^{\sigma}(\mathbb{B}_n)$, e.g., $k_w^{\sigma,\alpha,p'}(z)$ - any admissible choice of α can be used here. Standard arguments using the reproducing kernel $k_a^{\sigma,p'}(z)$ show that $K_p^{\sigma}(\mathbb{B}_n)$ embeds in $H^{\infty}(\mathbb{B}_n)$. Indeed,

$$|\varphi(a)| = \frac{1}{k_a^{\sigma,\alpha,p'}(a)} \left| \left\langle \varphi k_a^{\sigma,\alpha,p'}, k_a^{\sigma,\alpha,p} \right\rangle_{\alpha} \right| \approx \left| \left\langle \varphi \widetilde{k_a^{\sigma,\alpha,p'}}, \widetilde{k_a^{\sigma,\alpha,p}} \right\rangle_{\alpha} \right| \leq \|\varphi\|_{K_{\mathcal{H}}} ,$$

Above we have used (see (7.8) below):

$$\left\| k_a^{\sigma,\alpha,p'} \right\|_{B_{\sigma}^{p}(\mathbb{B}_n)} \approx \frac{1}{(1-|a|^2)^{\sigma}} \approx \sqrt{k_a^{\sigma,\alpha,p'}(a)}.$$

Let $WC_p^{\sigma}(\mathbb{B}_n)$ consist of those holomorphic functions in $f \in B_p^{\sigma}(\mathbb{B}_n)$ that satisfy the weak or one-box σ-Carleson condition:

$$\begin{aligned} WC_p^{\sigma}(\mathbb{B}_n) &\equiv \left\{ f \in B_p^{\sigma}(\mathbb{B}_n) : \|f\|_{WC_p^{\sigma}(\mathbb{B}_n)} < \infty \right\}; \\ \|f\|_{WC_p^{\sigma}(\mathbb{B}_n);m}^{p} &\equiv \sup_{Q} \frac{1}{|Q|^{\frac{p\sigma}{n}}} \int_{S(Q)} \left| \left(1-|z|^2\right)^{m+\sigma} \nabla^m f(z) \right|^p d\lambda_n(z), \end{aligned}$$

and where Q is a nonisotropic ball on the sphere (see e.g. [Rud2, page 65]) and $|Q|$ is its surface measure. Here $m > \frac{n}{p} - \sigma$ and as usual, we will see below that the norms $\|f\|_{WC_p^{\sigma}(\mathbb{B}_n);m}^{p}$ are equivalent provided $p(m+\sigma) > n$, and so we can drop the dependence on m. But first we establish the following standard equivalence for one-box Carleson measures.

Lemma 48. *For $d\mu$ a positive Borel measure on the ball $\mathbb{B}_n$, $\sigma > 0$ and $1 < p < \infty$, we have*

$$\|\mu\|_{WCM_p^{\sigma}} \equiv \sup_{a \in \mathbb{B}_n} \left(\int_{\mathbb{B}_n} \left| \widetilde{k_a^{\sigma,p'}(z)} \right|^p d\mu(z) \right)^{\frac{1}{p}} \approx \sup_{Q \subset \partial\mathbb{B}_n} \frac{\left(\int_{S(Q)} d\mu(z) \right)^{\frac{1}{p}}}{|Q|^{\frac{\sigma}{n}}}.$$

We will call measures μ that satisfy Lemma 48 weak Carleson measures and norm them via either expression. This can be contrasted with standard Carleson measures which are normed by:

$$\|\mu\|_{CM_p^{\sigma}} = \sup_{\|f\|_{B_p^{\sigma}(\mathbb{B}_n)}=1} \left(\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}}.$$

Proof. The proof of Lemma 48 is standard using the pointwise bounds

$$\begin{aligned} \left| k_a^{\sigma,p'}(z) \right| &\gtrsim \left(\frac{1}{1-|a|^2} \right)^{2\sigma}, \qquad z \in S_a , \\ \left| k_a^{\sigma,p'}(z) \right| &\lesssim \left| \frac{1}{1-\overline{a}\cdot z} \right|^{2\sigma}, \qquad z \in \mathbb{B}_n , \end{aligned} \tag{7.8}$$

which follow from (7.7), $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma} k_w^{\sigma,\alpha,p'}(z) = \frac{1}{(1-\overline{w}\cdot z)^{\frac{n+1+\alpha}{p}+\sigma}}$, since $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma}$ is essentially differentiation of order $t = \frac{n+1+\alpha}{p} - \sigma$, and so

$$k_w^{\sigma,\alpha,p'}(z) = \left(\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma} \right)^{-1} \frac{1}{(1-\overline{w}\cdot z)^{\frac{n+1+\alpha}{p}+\sigma}} \approx \frac{1}{(1-\overline{w}\cdot z)^{\frac{n+1+\alpha}{p}+\sigma-t}} = \frac{1}{(1-\overline{w}\cdot z)^{2\sigma}}.$$

From this we also have the use approximation that:

$$\widetilde{k_a^{\sigma,p'}}(z) \approx \frac{(1-|a|^2)^{\sigma}}{(1-\overline{a}\cdot z)^{2\sigma}}.$$

To show the inequality $\gtrsim$ we simply use the first inequality in (7.8). Conversely, to show the inequality $\lesssim$, we break up the integral

$$\left\| \widetilde{k_a^{\sigma,p'}} \right\|_{L^p(\mu)}^{p} = \int_{\mathbb{B}_n} \left| \widetilde{k_a^{\sigma,p'}}(z) \right|^p d\mu(z)$$

into geometric annuli $2^{\ell+1}S_a \setminus 2^{\ell}S_a$, where S_a is the usual Carleson box associated with $a \in \mathbb{B}_n$. Then we complete the proof using the one box condition on the Carleson boxes $2^{\ell+1}S_a$ together with the second inequality in (7.8), and then summing up a geometric series. $\square$

Now we show that $K_p^\sigma(\mathbb{B}_n) = H^\infty(\mathbb{B}_n) \cap WC_p^\sigma(\mathbb{B}_n)$ with comparable norms. The corresponding result for the multiplier algebra $M_{B_p^\sigma(\mathbb{B}_n)}$ is due to Ortega and Fabrega [OrFa, Theorem 3.7], and the proof there carries over almost verbatim for weak Carleson measures in place of Carleson measures, which we provide for the ease of the reader. It follows as a corollary of this result that the weak Carleson measure condition is independent of $m > \frac{n}{p} - \sigma$.

Proposition 49. *Let $\varphi \in H^\infty(\mathbb{B}_n) \cap B_p^\sigma(\mathbb{B}_n)$, $\sigma > 0$ and $m + \sigma > \frac{n}{p}$. Then $\varphi \in K_p^\sigma(\mathbb{B}_n)$ if and only if $\varphi \in WC_p^\sigma(\mathbb{B}_n)$ i.e.*

$$d\mu(z) \equiv \left| \left(1 - |z|^2\right)^{m+\sigma} \varphi^{(m)}(z) \right|^p d\lambda_n(z)$$

is a weak $B_p^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.

Proof. Suppose that $\varphi \in H^\infty(\mathbb{B}_n) \cap WC_p^\sigma(\mathbb{B}_n)$. In [OrFa], Ortega and Fabrega use the notation $A_{\delta,k}^p(\mathbb{B}_n)$ to describe a space of holomorphic functions equivalent to $B_p^\sigma(\mathbb{B}_n)$ when $\sigma = \frac{n+\delta}{p} - k$. They define the norm on $A_{\delta,k}^p(\mathbb{B}_n)$ by

$$\|f\|_{A_{\delta,k}^p(\mathbb{B}_n)} \equiv \left(\sum_{|\alpha| \le k} \int_{\mathbb{B}_n} |D^\alpha f(z)|^p \left(1 - |z|^2\right)^{\delta - 1} dV(z) \right)^{\frac{1}{p}}.$$

They show (bottom of page 66 of [OrFa]) that with R the radial derivative,

$$\left\| (I+R)^k \left(\varphi k_a^{\sigma,p'}\right) \right\|_{A_{\delta,0}^p(\mathbb{B}_n)} \lesssim \left\| k_a^{\sigma,p'} (I+R)^k \varphi \right\|_{A_{\delta,0}^p(\mathbb{B}_n)} + \sum_{\substack{m+\ell \le k \\ m>0}} \left\| \left(R^\ell \varphi\right) \left(R^m k_a^{\sigma,p'}\right) \right\|_{A_{\delta,0}^p(\mathbb{B}_n)}. \tag{7.9}$$

Now the first term on the right side of (7.9) is controlled by the weak Carleson norm $\|\varphi\|_{WC_p^\sigma(\mathbb{B}_n)}$ of φ. The needed estimates for the other terms on the right hand side of (7.9) are obtained from the argument used to prove Theorem 3.5 in [OrFa] since only the weak Carleson condition is needed here. Indeed we quote from the bottom of page 66 in [OrFa]:

"The estimates of the other terms can be obtained following the same argument used to prove Theorem 3.5. Observe that the properties of φ used in this proof were that φ was a bounded holomorphic function and that for $\delta - 1 - N - kp < 0$

$$\int_{\mathbb{B}_n} \frac{\left|(I+R)^k \varphi(w)\right|^p \left(1 - |w|^2\right)^{\delta-1}}{|1 - \overline{w} \cdot z|^{n+1+N}} dV(w) \lesssim \left(1 - |z|^2\right)^{\delta - 1 - N - kp}.$$

It is clear that they are consequences of $\varphi \in H^\infty(\mathbb{B}_n)$ and just testing the measure $d\mu$ on the function $\left(\frac{1}{1-\overline{w}\cdot z}\right)^{\frac{n+1+N}{p}}$."

From this quote it is clear that we need only use the weak Carleson condition when testing the measure $d\mu$ for the other terms on the right side of (7.9). $\square$

Now we turn to showing that $K_p^\sigma(\mathbb{B}_n)$ is an algebra for $\sigma > 0$ and $1 < p < \infty$.

Theorem 50. *Let $\sigma > 0$ and $1 < p < \infty$. Then $K_p^\sigma(\mathbb{B}_n)$ is an algebra.*

Proof. Fix $m > \frac{2n}{p}$. Note this choice is twice as large as need be, and this will play a role in the proof. To show that $K_p^\sigma(\mathbb{B}_n)$ is an algebra, it is enough by polarization, $2fg = (f+g)^2 - f^2 - g^2$, to show that $\left\|\varphi^2\right\|_{K_p^\sigma(\mathbb{B}_n)} \le \|\varphi\|_{K_p^\sigma(\mathbb{B}_n)}^2$. Now $\left\|\varphi^2\right\|_\infty \le \|\varphi\|_\infty^2$ for $\varphi \in K_p^\sigma(\mathbb{B}_n)$ and using Lemma 48 it remains to show that

$$\left\|\varphi^2\right\|_{WC_p^\sigma(\mathbb{B}_n)} \lesssim \|\varphi\|_{K_p^\sigma(\mathbb{B}_n)}^2. \tag{7.10}$$

We have

$$\left\|\varphi^2\right\|_{WC_p^\sigma(\mathbb{B}_n)} = \sum_{k=0}^{m-1}\left|\nabla^k\left(\varphi^2\right)(0)\right| + \left(\sup_Q \frac{1}{|Q|^{\frac{p\sigma}{n}}}\int_{S(Q)}\left|\left(1-|z|^2\right)^{m+\sigma}\nabla^m\left(\varphi^2\right)(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}},$$

and

$$\nabla^m\left(\varphi^2\right)(z) = \sum_{k=0}^{m} c_{m,k}\left(\nabla^{m-k}\varphi(z)\right)\left(\nabla^k\varphi(z)\right).$$

Now

$$\begin{aligned}
&\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^m\nabla^m\left(\varphi^2\right)(z)\right|^p\left(1-|z|^2\right)^{\sigma p}d\lambda_n(z)\right)^{\frac{1}{p}}\\
\leq\;& C\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^m\nabla^m\varphi(z)\right|^p|\varphi(z)|^p\left(1-|z|^2\right)^{\sigma p}d\lambda_n(z)\right)^{\frac{1}{p}}\\
&+C\sum_{k=1}^{m-1}\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right|^p\left|\left(1-|z|^2\right)^k\nabla^k\varphi(z)\right|^p\left(1-|z|^2\right)^{\sigma p}d\lambda_n(z)\right)^{\frac{1}{p}}\\
&+C\left(\int_{S(Q)}|\varphi(z)|^p\left|\left(1-|z|^2\right)^m\nabla^m\varphi(z)\right|^p\left(1-|z|^2\right)^{\sigma p}d\lambda_n(z)\right)^{\frac{1}{p}} \equiv I+II+III.
\end{aligned}$$

Terms $I = III$ are easily controlled by

$$I \leq C\|\varphi\|_\infty\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^{m+\sigma}\nabla^m\varphi(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \leq C\|\varphi\|_\infty\|\varphi\|_{WC_p^\sigma(\mathbb{B}_n)}|Q|^{\frac{\sigma}{n}}.$$

As for term II, fix $1 \leq k \leq m-1$ for the moment, and assume without loss of generality that $k \geq m-k$. Then we can use Cauchy's estimate

$$\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right| \lesssim \|\varphi\|_\infty$$

to obtain

$$\begin{aligned}
&\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^{m-k}\nabla^{m-k}\varphi(z)\right|^p\left|\left(1-|z|^2\right)^k\nabla^k\varphi(z)\right|^p\left(1-|z|^2\right)^{\sigma p}d\lambda_n(z)\right)^{\frac{1}{p}}\\
\leq\;& \|\varphi\|_\infty\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^{k+\sigma}\nabla^k\varphi(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \leq \|\varphi\|_\infty\|\varphi\|_{WC_p^\sigma(\mathbb{B}_n)}|Q|^{\frac{\sigma}{n}}
\end{aligned}$$

since $k \geq m-k$ implies $k \geq \frac{m}{2} > \frac{n}{p} > \frac{n}{p}-\sigma$ (this is where we use $m > \frac{2n}{p}$). Altogether then we have

$$\left(\int_{S(Q)}\left|\left(1-|z|^2\right)^{m+\sigma}\nabla^m\left(\varphi^2\right)(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \lesssim \|\varphi\|_\infty\|\varphi\|_{WC_p^\sigma(\mathbb{B}_n)}|Q|^{\frac{\sigma}{n}},$$

which gives (7.10):

$$\left\|\varphi^2\right\|_{WC_p^\sigma(\mathbb{B}_n)} \lesssim \|\varphi\|_\infty\|\varphi\|_{WC_p^\sigma(\mathbb{B}_n)} \lesssim \|\varphi\|^2_{K_p^\sigma(\mathbb{B}_n)}.$$

□

In particular, when $\mathcal{H} = B_2^\sigma(\mathbb{B}_n)$ and $\sigma > 0$, the kernel multiplier space $K_{\mathcal{H}} = K_2^\sigma(\mathbb{B}_n)$ is an algebra.

7.2. **The Corona Property for kernel multiplier algebras on the disk.** Here we prove the Corona Property for the one-dimensional algebras of kernel multipliers $K_p^\sigma(\mathbb{D})$ for $0<\sigma<\frac{1}{p}$, $1<p<\infty$.

Theorem 51. *Let $N\geq 2$, $1<p<\infty$ and $0<\sigma<\frac{1}{p}$ and suppose that $\varphi_1,\dots,\varphi_N\in K_p^\sigma(\mathbb{D})$ with norm at most one satisfy*

$$\max\left\{\left|\varphi_1(z)\right|^2,\dots,\left|\varphi_N(z)\right|^2\right\}\geq c>0,\qquad z\in\mathbb{D}.$$

Then there are a positive constant C and $f_1,\dots,f_N\in K_p^\sigma(\mathbb{D})$ satisfying

$$\begin{aligned}\max\left\{\|f_1\|_{K_p^\sigma(\mathbb{D})},\dots,\|f_N\|_{K_p^\sigma(\mathbb{D})}\right\} &\leq C, \qquad z\in\mathbb{D},\\ \varphi_1(z)f_1(z)+\cdots+\varphi_N(z)f_N(z) &= 1,\qquad z\in\mathbb{D}.\end{aligned}$$

The Corona Property for the multiplier algebras $M_{B_p^\sigma(\mathbb{D})}$ was obtained by Arcozzi, Blasi and Pau in [ArBlPa] using the Peter Jones solution to the $\overline{\partial}$-equation, and we now adapt these methods to prove the Corona Property for the algebras $K_p^\sigma(\mathbb{D})$ of kernel multipliers. The key point is that in the arguments in [ArBlPa], which in turn are a generalization of the results from [Xia2] for the case $p=2$, we are always able to substitute the weak or one-box Carleson condition for the actual Carleson condition.

As in [ArBlPa] let $dA_s(z)=\left(1-|z|^2\right)^s dA(z)$ where $dA(z)$ is normalized area measure on the disk $\mathbb{D}$ (the connection between s there and σ here is $s=p\sigma$). If $|g(z)|^p\,dA_{p(1+\sigma)-2}(z)$ is a weak Carleson measure for $B_p^\sigma(\mathbb{D})$ with $0<\sigma<\frac{1}{p}$, then Lemma 6.3 in [ArBlPa] shows that $|g(z)|\,dA(z)$ is a Carleson measure for $H^2(\mathbb{D})=B_2^{\frac{1}{2}}(\mathbb{D})$ (or in fact $H^p(\mathbb{D})$ since Carleson measures for Hardy space are all the same!). Indeed,

$$\begin{aligned}\int_{S(I)}|g(z)|\,dA(z) &= \int_{S(I)}|g(z)|\left(1-|z|^2\right)^{\sigma+1-\frac{2}{p}}\left(1-|z|^2\right)^{-\sigma-1+\frac{2}{p}}dA(z)\\ &\leq \left(\int_{S(I)}|g(z)|^p\left(1-|z|^2\right)^{p(\sigma+1)-2}dA(z)\right)^{\frac{1}{p}}\left(\int_{S(I)}\left(1-|z|^2\right)^{-(\sigma+1)\frac{p}{p-1}+\frac{2}{p-1}}dA(z)\right)^{\frac{p-1}{p}}\\ &\lesssim \left(|I|^{p\sigma}\right)^{\frac{1}{p}}\left(|I|^{(1-\sigma)\frac{p}{p-1}}\right)^{\frac{p-1}{p}}=|I|.\end{aligned}$$

Now we obtain from the Peter Jones solution to the $\overline{\partial}$-equation the following 'weak' version of Theorem 6.4 in [ArBlPa].

Theorem 52. *Let $1<p<\infty$ and $0<\sigma<\frac{1}{p}$. If $|g(z)|^p\,dA_{p(\sigma+1)-2}(z)$ is a weak Carleson measure, then there is $f\in C(\mathbb{D})$ satisfying*

$$\frac{\partial f}{\partial\overline{z}}=g \text{ on } \mathbb{D} \text{ in the sense of distributions,}$$

and such that f has radial limits f^* a.e. on $\mathbb{T}$ with $f^*\in K\left(L_{p\sigma}^p\right)$.

Here $L_p^{p\sigma}$ is the Sobolev space on the circle $\mathbb{T}$: for $f\in L^p(\mathbb{T})$,

$$\|f\|^2_{L^p_{p\sigma}(\mathbb{T})}=\|f\|^p_{L^p(\mathbb{T})}+\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\left|f\left(e^{it}\right)-f\left(e^{iu}\right)\right|^p}{\left|e^{it}-e^{iu}\right|^{2-p\sigma}}dudt\approx\|f\|^p_{L^p(\mathbb{T})}+\int\int_{\mathbb{D}}|\nabla Pf(z)|^p\left(1-|z|^2\right)^{p(\sigma+1)-2}dA(z),$$

where Pf is the Poisson extension of f to the disk, and where the second summands on each line are actually comparable. The space $K\left(L^p_{p\sigma}\right)$ is the subspace of $L^\infty(\mathbb{T})$ for which $|\nabla Pf(z)|^p\left(1-|z|^2\right)^{p(\sigma+1)-2}dA(z)$ is a weak Carleson measure on $B_p^\sigma(\mathbb{D})$ normed by

$$\|f\|_{K(L^p_{p\sigma})}=\|f\|_{L^\infty(\mathbb{T})}+\left\||\nabla Pf(z)|^p\left(1-|z|^2\right)^{p(\sigma+1)-2}dA(z)\right\|_{WCM_p^\sigma}.$$

A straightforward 'weak' modification of Lemma 4.5 in [ArBlPa] shows that if $F\in C^1(\mathbb{D})\cap L^\infty(\mathbb{D})$ has radial limits F^* existing a.e. on $\mathbb{T}$, then

$$\|F^*\|_{K(L^p_{p\sigma})}\lesssim\|F^*\|_{L^\infty(\mathbb{T})}+\left\||\nabla F(z)|^p\left(1-|z|^2\right)^{p(\sigma+1)-2}dA(z)\right\|_{WCM_p^\sigma}. \tag{7.11}$$

Indeed, Lemma 4.5 proved the analogue

$$\|F^*\|_{M(L^p_{p\sigma})} \lesssim \|F^*\|_{L^\infty(\mathbb{T})} + \left\| |\nabla F(z)|^p \left(1-|z|^2\right)^{p(\sigma+1)-2} dA(z) \right\|_{CM_p^\sigma}$$

where the norm on the left is the multiplier norm of F^*, which is equivalent to

$$\|F^*\|_{L^\infty(\mathbb{T})} + \left\| |\nabla PF^*(z)|^p \left(1-|z|^2\right)^{p(\sigma+1)-2} dA(z) \right\|_{CM_p^\sigma}.$$

The passage from Carleson norms to weak Carleson norms here is routine because one simply repeats the proof with a reproducing kernel in place of the generic function used in their proof.

Proof. To prove Theorem 52, we note that since $d\mu(z) \equiv g(z)\, dA(z)$ is a Carleson measure for $H^2(\mathbb{D})$, we can invoke the Jones solution with $\nu = \frac{\mu}{\|\mu\|_{H^2(\mathbb{D})-Car}}$:

$$\begin{aligned} u(z) &= \int\int_{\mathbb{D}} K(\nu, z, \zeta)\, d\mu(\zeta), \qquad z \in \overline{\mathbb{D}}, \\ K(\nu, z, \zeta) &\equiv \frac{2i}{\pi} \frac{1-|\zeta|^2}{(z-\zeta)\left(1-\overline{\zeta}z\right)} \exp\left\{ \int\int_{|\omega|\geq|\zeta|} \left[-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta} \right] d\nu(\omega) \right\}. \end{aligned}$$

This solution $u(z)$ to $\frac{\partial f}{\partial \overline{z}} = g$ on $\mathbb{D}$ satisfies $u^* \in L^\infty(\mathbb{T})$. We now show that in addition $u^* \in K\left(L^p_{p\sigma}\right)$. For this purpose we consider the function

$$v(z) = \frac{2i}{\pi} \int\int_{\mathbb{D}} \frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2} \exp\left\{ \int\int_{|\omega|\geq|\zeta|} \left[-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta} \right] |g(\omega)|\, dA(\omega) \right\} g(\zeta)\, dA(\zeta),$$

which has the same boundary values as $zu(z)$. Since $e^{-i\theta} \in K\left(L^p_{p\sigma}(\mathbb{T})\right)$, it suffices by (7.11) to show that $v^* \in L^\infty(\mathbb{T})$ and that $|\nabla v(z)|^p \left(1-|z|^2\right)^{p(\sigma+1)-2} dA(z)$ is a weak Carleson measure for $B_p^\sigma(\mathbb{D})$.

We begin with an estimate of A. Nicolau and J. Xiao for this particular function v, see for example [NiXi]

$$|\nabla v(z)| \leq C \int\int_{\mathbb{D}} \frac{|g(\omega)|}{|1-\overline{\omega}z|^2} dA(\omega). \tag{7.12}$$

Since $|g(z)|^p\, dA_{p(\sigma+1)-2}(z)$ is a weak Carleson measure for $B_2^\sigma(\mathbb{D})$, it now follows from (7.12) and a 'weak' modification of Lemma E in [ArBlPa], which is a Carleson spreading lemma of Arcozzi, Blasi and Pau, that $|\nabla v(z)|^p \left(1-|z|^2\right)^{p(\sigma+1)-2} dA(z)$ is a weak Carleson measure for $B_p^\sigma(\mathbb{D})$ (see for example the analogous lemmas in [ArRoSa2] and [CoSaWi2] and references given there). So it only remains to show that $v^* \in L^\infty(\mathbb{T})$. But this is a consequence of the Jones trick that uses first

$$\mathrm{Re}\left\{ \int\int_{|\omega|\geq|\zeta|} \left[-\frac{1+\overline{\omega}z}{1-\overline{\omega}z} + \frac{1+\overline{\omega}\zeta}{1-\overline{\omega}\zeta} \right] |g(\omega)|\, dA(\omega) \right\} \leq 2 \int\int_{\mathbb{D}} \frac{1-|\zeta|^2}{\left|1-\overline{\zeta}z\right|^2} |g(\omega)|\, dA(\omega) \leq C,$$

and then for $z \in \mathbb{T}$ that

$$|v(z)| \lesssim \int\int_{\mathbb{D}} \frac{1-\left|\overline{\zeta}z\right|^2}{\left|1-\overline{\zeta}z\right|^2} e^{-\int\int_{|\omega|\geq|\zeta|} \frac{1-|\overline{\omega}z|^2}{|1-\overline{\omega}z|^2} |g(\omega)| dA(\omega)} |g(\zeta)|\, dA(\zeta) \lesssim 1,$$

which completes the proof of Theorem 52. □

With Theorem 52 in hand, it is now a routine matter to use the Koszul complex, or the simplified version in dimension $n=1$, to prove the Corona Theorem 51 for the algebras $K_p^\sigma(\mathbb{D})$ when $0 < \sigma < \frac{1}{p}$. This proof strategy for weak Carleson measures can be seen in [CoSaWi2].

7.3. **Bergman spaces of solutions to generalized Cauchy-Riemann equations.** Let $A(z)$ and $B(z)$ be two smooth real-valued symmetric invertible $n\times n$ matrices defined on a domain Ω in $\mathbb{C}^n=\mathbb{R}^{2n}$, where we identify $z=(z_j)_{j=1}^n\in\mathbb{C}^n$ with $\left((x_j)_{j=1}^n,(y_j)_{j=1}^n\right)\in\mathbb{R}^n\times\mathbb{R}^n$ under the correspondence $z_j\leftrightarrow x_j+iy_j$. Then the complex first order partial differential operator $P\equiv A(z)\nabla_x+iB(z)\nabla_y$ on $\mathbb{C}^n$ is elliptic, and we can consider the solution space $\mathcal{N}_P$ of complex-valued functions (that are necessarily smooth by ellipticity):

$$\mathcal{N}_P\equiv\{f\in C^\infty(\Omega;\mathbb{C}):Pf=0\text{ in }\Omega\}.$$

Remark 53. *If $A=B$ is the $n\times n$ identity matrix, then $P=\overline{\partial}$. Thus in this case $Pf=0$ is the system of Cauchy-Riemann equations on Ω, and $\mathcal{N}_P$ is the linear space of holomorphic functions in Ω.*

The linear space $\mathcal{N}_P$ is actually an algebra since if $f,g\in\mathcal{N}_P$, then $P(\alpha f+\beta g)=\alpha P(f)+\beta P(g)=0$ and $P(fg)=fP(g)+gP(g)=0$. The real and imaginary parts u of the functions in $\mathcal{N}_P$ are also solutions of the following real elliptic smooth coefficient divergence form second order equation in $\mathbb{R}^{2n}$:

$$P^*Pu=\operatorname{div}\begin{bmatrix}A^*A & 0\\ 0 & B^*B\end{bmatrix}\operatorname{grad}u=0.$$

In particular, from the Schauder interior estimates (see e.g. Theorem 6.9 in [GiTr]), solutions u to $P^*Pu=0$ satisfy a local Lipschitz inequality

$$|u(a)-u(b)|\le C_{K,A,B}\left(\int_\Omega|u(x)|^2\,dx\right)^{\frac{1}{2}},\qquad a,b\in K,$$

for every compact subset K of Ω.

Now we define the P-Bergman space $A_P^2(\Omega)$ on Ω by

$$A_P^2(\Omega)\equiv\left\{f\in\mathcal{N}_P:\frac{1}{|\Omega|}\int_\Omega|f(x)|^2\,dx<\infty\right\},$$

with norm $\|f\|_{A_P^2(\Omega)}=\sqrt{\frac{1}{|\Omega|}\int_\Omega|f(x)|^2\,dx}$. The local Lipschitz inequality for real and imaginary parts of solutions shows that point evaluations on $A_P^2(\Omega)$ are continuous, and so $A_P^2(\Omega)$ is a pre-Hilbert function space. Moreover the Montel property for $A_P^2(\Omega)$ is a standard consequence of the local Lipschitz inequality and the Arzela-Ascoli theorem, and we thus see that $A_P^2(\Omega)$ is a Hilbert function space on Ω with the Montel property. Using that $\mathcal{N}_P$ is an algebra, it is now easy to see with $\mathcal{H}=A_P^2(\Omega)$, that the multiplier algebra $M_{\mathcal{H}}$ of $\mathcal{H}$ is isometrically isomorphic to $\mathcal{H}^\infty(\Omega)$ equipped with the sup norm $\|h\|_\infty\equiv\sup_{a\in\Omega}|h(a)|$. Thus $\mathcal{H}=A_P^2(\Omega)$ will be multiplier stable if the kernel functions k_a for $\mathcal{H}$ are bounded away from 0 and ∞ and lower semicontinuous in a, i.e.

$$\|k_a\|_\infty\text{ is lower semicontinuous in }a,\left\|\frac{1}{k_a}\right\|_\infty<\infty,\qquad a\in\Omega. \tag{7.13}$$

Thus provided (7.13) holds, our alternate Toeplitz corona theorem reduces the Corona Property for the Banach algebra $\mathcal{H}^\infty(\Omega)$ of bounded solutions f to $Pf=0$, to the Convex Poisson Property for the P-Bergman space $A_P^2(\Omega)$. This generalizes the notion of holomorphicity for which such corona theorems may hold. We do not pursue this direction any further here.

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