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Catherine Bénéteau

Greg Knese
Washington University in St. Louis, geknese@wustl.edu

Łukasz Kosiński

Constanze Liaw

Daniel Seco

See next page for additional authors

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Authors

Catherine Bénéteau, Greg Knese, Łukasz Kosiński, Constanze Liaw, Daniel Seco, and Alan Sola

CYCLIC POLYNOMIALS IN TWO VARIABLES

CATHERINE BÉNÉTEAU, GREG KNESE, ŁUKASZ KOSIŃSKI,
CONSTANZE LIAW, DANIEL SECO, AND ALAN SOLA

ABSTRACT. We give a complete characterization of polynomials in two complex variables that are cyclic with respect to the coordinate shifts acting on Dirichlet-type spaces in the bidisk, which include the Hardy space and the Dirichlet space of the bidisk. The cyclicity of a polynomial depends on both the size and nature of the zero set of the polynomial on the distinguished boundary. The techniques in the proof come from real analytic function theory, determinantal representations for polynomials, and harmonic analysis on curves.

CONTENTS

1. INTRODUCTION

In groundbreaking and classical work, V.I. Smirnov [20] and independently A. Beurling [4] characterized the cyclic vectors for the shift operator, multiplication by z, in the Hardy space of the unit disk $\mathbb{D} \subset \mathbb{C}$: a function $f \in H^2(\mathbb{D})$ is cyclic precisely if it is *outer*, meaning

$$\log|f(0)| = \int_0^{2\pi} \log|f(e^{i\theta})|\frac{d\theta}{2\pi} \text{ and } f(0) \neq 0.$$

We recall that a vector is cyclic for an operator (or operators) if the invariant subspace generated by the vector is dense. In this paper, we generalize in two directions—we move to functions on the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$$

and we study a whole scale of Hilbert spaces—Bergman, Hardy, and Dirichlet spaces on the bidisk among them. At the same time, we also confine ourselves to characterizing cyclic *polynomials* for multiplication by the coordinates z_1, z_2.

There is an intriguing motivation for generalizing to polydisks. As the title of [17] suggests, characterizing cyclic vectors of the Hardy space on the infinite polydisk is "in the shadow of the Riemann hypothesis," because of the connection between Nyman's dilation completeness problem (equivalent to the RH) and cyclicity on the Hardy space of the infinite polydisk. Characterizing the cyclic polynomials in the Hardy space of the polydisk $H^2(\mathbb{D}^n)$ is straightforward. Neuwirth, Ginsberg, and Newman [16] proved that a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is cyclic for multiplication by the coordinates $z_1, z_2, \ldots, z_n$ exactly when f has no zeros in $\mathbb{D}^n$; see also [7]. However, not much is known with regard to the cyclicity problem for spaces of functions on the polydisk beyond the Hardy space even when we restrict to the problem for polynomials. Here we address this problem on the bidisk for so-called Dirichlet-type spaces. We will see that the answer both depends on the size and nature of the zero sets on the *distinguished boundary*

$$\mathbb{T}^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$$

of the bidisk.

Let $\alpha \in \mathbb{R}$ be fixed. A holomorphic function $f \colon \mathbb{D}^2 \to \mathbb{C}$ belongs to the *Dirichlet-type space* $\mathfrak{D}_\alpha$ if the coefficients $\{a_{k,l}\}_{k,l=0}^\infty$ in its Taylor series expansion

$$f(z_1, z_2) = \sum_{k=0}^\infty \sum_{l=0}^\infty a_{k,l} z_1^k z_2^l$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^\infty \sum_{l=0}^\infty (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty. \tag{1.1}$$

It follows directly from this definition that polynomials in two variables form a dense subset of $\mathfrak{D}_\alpha$ for each α.

The spaces $\mathfrak{D}_\alpha$ generalize the one-variable Dirichlet-type spaces D_α, $\alpha \in \mathbb{R}$, that consist of functions $f(z) = \sum_{k=0}^\infty a_k z^k$ that are analytic in $\mathbb{D}$ and satisfy

$$\|f\|_{D_\alpha}^2 = \sum_{k=0}^\infty (k+1)^\alpha |a_k|^2 < \infty.$$

These spaces of functions in the unit disk are discussed in the textbook [6]. Returning to the two-dimensional setting and putting $\alpha = 0$, we recover

the *Hardy space* of the bidisk (see [19]). The choice $\alpha = -1$ leads to the *Bergman space* of square area-integrable functions on the bidisk, while $\alpha = 1$ produces the *Dirichlet space* of the bidisk. A partial list of works addressing $\mathfrak{D}_\alpha$ for other values of α includes [9, 11, 10, 3]. As is pointed out in these references, for $\alpha < 2$, these spaces have the equivalent norm

$$\|f\|_{\alpha,*}^2 = |f(0,0)|^2$$
$$+ \int_{\mathbb{D}} |\partial_{z_1} f(z_1,0)|^2 w(z_1)^{1-\alpha} dA(z_1) + \int_{\mathbb{D}} |\partial_{z_2} f(0,z_2)|^2 w(z_2)^{1-\alpha} dA(z_2)$$
$$+ \int_{\mathbb{D}^2} |\partial_{z_1 z_2} f(z_1,z_2)|^2 w(z_1)^{1-\alpha} w(z_2)^{1-\alpha} dA(z_1) dA(z_2)$$

where $w(z) := (1 - |z|^2)$ and dA is area measure on $\mathbb{D}$.

Evaluation at a point of $\mathbb{D}^2$ is a bounded linear functional on the Dirichlet-type spaces, and hence the $\mathfrak{D}_\alpha$ are *reproducing kernel Hilbert spaces* for all α. When $\alpha > 1$, the spaces $\mathfrak{D}_\alpha$ are in fact *algebras* contained as subsets of the *bidisk algebra* $A^2(\mathbb{D}^2)$ (see [19]).

1.1. **Cyclic vectors.** Inspecting the definition of the norm in (1.1), we see that the *shift operators* S_1 and S_2 defined via

$$S_1 f(z_1, z_2) = z_1 f(z_1, z_2) \quad \text{and} \quad S_2 f(z_1, z_2) = z_2 f(z_1, z_2)$$

act boundedly on $\mathfrak{D}_\alpha$ as commuting linear operators. Coordinate shifts acting on Dirichlet-type spaces represent a concrete and natural model of multivariable operator theory on Hilbert space, and the *lattice of invariant subspaces* of the shift operators acting on $\mathfrak{D}_\alpha$ exhibits a rich structure that is to a large extent still not well understood.

In this paper we are primarily interested in invariant subspaces generated by a given function $f \in \mathfrak{D}_\alpha$. These subspaces are of the form

$$[f] = \overline{\text{span}}\{z_1^k z_2^l f \colon k = 0, 1, \ldots; l = 0, 1, \ldots\}$$

where $\overline{\text{span}}$ denotes the closed span of the given set. The set $[f]$ is, by definition, the smallest closed subspace that contains f and is invariant under the action of the shift operators S_1 and S_2. A function $f \in \mathfrak{D}_\alpha$ for which $[f]$ coincides with $\mathfrak{D}_\alpha$ is called *cyclic*. In this paper, as in [3], we address the *cyclicity problem* for $\mathfrak{D}_\alpha$—the problem of determining the $f \in \mathfrak{D}_\alpha$ for which $[f] = \mathfrak{D}_\alpha$.

Cyclicity of the constant functions $f = c$, $c \in \mathbb{C}\backslash\{0\}$, follows from the fact that polynomials are dense in $\mathfrak{D}_\alpha$. As point evaluations are bounded linear functionals, cyclicity of f is then equivalent to the existence of a sequence of polynomials (p_n) such that

$$p_n(z_1, z_2) f(z_1, z_2) - 1 \longrightarrow 0, \quad (z_1, z_2) \in \mathbb{D}^2,$$

and

$$\|p_n f - 1\|_\alpha \leq C.$$

In view of this, it is clear that functions that have zeros inside the bidisk cannot be cyclic. When $\alpha > 1$, the spaces $\mathfrak{D}_\alpha$ are algebras, and functions

are cyclic if and only if they are non-vanishing on $\overline{\mathbb{D}^2}$, the closure of the bidisk, but when $\alpha \leq 1$ there exist cyclic functions that vanish on $\partial\mathbb{D}^2$.

In contrast to the Smirnov-Beurling result on $H^2(\mathbb{D})$, the direct two variable analogue of outer function fails to characterize cyclicity.

Rudin discovered that any cyclic function in $H^2(\mathbb{D}^2)$ must be outer, but there exist non-cyclic outer functions in the bidisk [19]. As mentioned in the introduction, a polynomial is cyclic in $H^2(\mathbb{D}^2)$, and hence in $\mathfrak{D}_\alpha$ for $\alpha \leq 0$, if and only if it has no zeros in the bidisk [16], [7]. On the other hand, in [3] the authors exhibited polynomials without zeros in $\mathbb{D}^2$ that are *not* cyclic in $\mathfrak{D}_\alpha$ when $\alpha > 1/2$. Non-cyclicity of certain polynomials has also been observed by Stefan Richter and Carl Sundberg [18] in the context of the Drury-Arveson space in the unit ball of $\mathbb{C}^n$ when $n \geq 4$.

In [3], certain classes of cyclic functions were identified using previous work on Dirichlet spaces in the unit disk in [2]. Indeed, three examples which actually illustrate our main theorem were discussed. The polynomials depending on one variable $z_1 - 1$ or $z_2 - 1$ are cyclic exactly when $\alpha \leq 1$ (exactly as in one variable). The polynomial $2 - z_1 - z_2$ which vanishes at a single point $(1, 1)$ of $\mathbb{T}^2$ is cyclic exactly when $\alpha \leq 1$. Finally, the polynomial $1 - z_1 z_2$, which does not depend on one variable and vanishes on a curve in $\mathbb{T}^2$, is cyclic exactly when $\alpha \leq 1/2$. The problem of characterizing the cyclic polynomials for each $\mathfrak{D}_\alpha$, $\alpha \in (0, 1]$, was stated as an open problem, see [3, Problem 5.2].

1.2. **Main result and plan of the paper.** In the present paper, we solve this problem and provide a complete characterization of the cyclic polynomials in Dirichlet-type spaces of the bidisk. Note that since polynomials are multipliers, a polynomial is cyclic if and only if each of its irreducible factors is cyclic; see Section 2.1. Therefore, we may state the main theorem as follows. Below $\mathcal{Z}(f) = \{z \in \mathbb{C}^2 : f(z) = 0\}$.

Theorem (Cyclicity of polynomials in the bidisk)**.** *Let $f(z_1, z_2)$ be an irreducible polynomial with no zeros in the bidisk.*

(1) *If $\alpha \leq 1/2$, then f is cyclic in $\mathfrak{D}_\alpha$.*
(2) *If $1/2 < \alpha \leq 1$, then f is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathcal{Z}(f) \cap \mathbb{T}^2$ is an empty or finite set, or f is a constant multiple of $\zeta - z_1$ or of $\zeta - z_2$ for some $\zeta \in \mathbb{T}$.*
(3) *If $\alpha > 1$, then f is cyclic in $\mathfrak{D}_\alpha$ if and only if $\mathcal{Z}(f) \cap \mathbb{T}^2$ is empty.*

The new and non-trivial content of the theorem is the case $0 < \alpha \leq 1$ and for polynomials f with $\mathcal{Z}(f) \cap \partial\mathbb{D}^2 \neq \varnothing$. Recall that any polynomial that vanishes *inside* the bidisk cannot be cyclic for any α, a polynomial that does not vanish on $\overline{\mathbb{D}}^2$ is cyclic for all α, and polynomials that do not vanish in the bidisk are known to be cyclic for $\alpha \leq 0$, see [16]. In addition, it is known that a function $f \in \mathfrak{D}_\alpha$ for $\alpha > 1$ is cyclic if and only if it does not vanish on the closure of the bidisk, since in that case $\mathfrak{D}_\alpha$ is an algebra.

The analogous problem of characterizing cyclic polynomials for Dirichlet-type spaces in one variable was solved by Brown and Shields [5]. In that context, a polynomial with no zeros inside the disk is always cyclic for $\alpha \leq 1$. The problem without the assumption of f being a polynomial is already quite difficult in one dimension, and the so-called *Brown-Shields conjecture*—a proposed characterization of cyclicity in the Dirichlet space—still remains open. Generalizing our theorem to three or more variables also remains open, as much of our approach in item (1) of the main theorem relies heavily on an essentially two variable result: the existence of determinantal representations for polynomials with no zeros on the bidisk. See [1, 8, 12, 22, 23].

We now present a plan of the paper. Section 2 contains some preliminary material concerning polynomials in two variables, including determinantal representation formulas, as well as estimates on Fourier coefficients of measures supported on curves in the torus, and a brief discussion of the notion of α-capacity and its relevance to the cyclicity problem.

In Section 3 we prove that a polynomial that is non-zero in the bidisk and vanishes at only finitely many points in the distinguished boundary is cyclic for *all* $\alpha \leq 1$, see Theorem 3.1. The proof uses Łojasiewicz's inequality and results on cyclicity of product functions.

Next, in Section 4, we show that any polynomial with $\mathcal{Z}(f) \cap \mathbb{D}^2 = \varnothing$, whether vanishing at finitely many points or along a curve, is cyclic whenever $\alpha \leq 1/2$. This is the content of Theorem 4.1. The argument relies on the determinantal formulas mentioned above.

Finally, in Section 5, we use the notion of α-capacity, along with estimates on Fourier coefficients of measures supported on subvarieties of the torus, to give conditions on zero sets that imply non-cyclicity in the Dirichlet spaces $\mathfrak{D}_\alpha$ in the remaining range $1/2 < \alpha \leq 1$. Thus, Theorem 5.2 completes the classification enunciated in the Main Theorem. In fact, some of the methods in Section 5 apply to more general functions not just polynomials.

2. Preliminaries

2.1. **Function space properties.** We summarize certain basic facts concerning multipliers and Möbius invariance we shall need later on.

First, we note that all polynomials in two variables whose zero sets do not intersect the bidisk are outer, hence are natural candidates for being cyclic in $\mathfrak{D}_\alpha$ when $\alpha \in [0, 1]$. Furthermore, if f is a polynomial, then f belongs to the *multiplier space*

$$M(\mathfrak{D}_\alpha) = \{\varphi \colon \mathbb{D}^2 \to \mathbb{C} \colon \varphi f \in \mathfrak{D}_\alpha, \text{ for all } f \in \mathfrak{D}_\alpha\},$$

for all α (see [10] for more on multipliers). In addition, holomorphic functions that extend to a bigger polydisk also furnish multipliers, as can be seen by approximating such a function by a finite truncation of its power series.

We have already seen that constant functions are cyclic in $\mathfrak{D}_\alpha$. Moreover, since all polynomials are multipliers, standard techniques (as those used for example in [5, Proposition 8]) show that it is enough to check cyclicity for *irreducible polynomials* that do not vanish on the bidisk: a product of multipliers is cyclic if and only if each factor is cyclic. In particular, multiplicity does not matter: f^2 is cyclic whenever the polynomial f is cyclic.

The automorphism group of the bidisk consists of the Möbius transformations

$$m_{a,b}\colon (z_1, z_2) \mapsto \left(\frac{a - z_1}{1 - \bar{a}z_1}, \frac{b - z_2}{1 - \bar{b}z_2} \right), \tag{2.1}$$

where $a, b \in \mathbb{D}$, as well as rotations of individual variables and permutation of variables. Using the integral norm for $\mathfrak{D}_\alpha$, one can show that composition with any $m_{a,b} \in \mathrm{Aut}(\mathbb{D}^2)$ defines a bounded invertible operator, and hence

$$\|f \circ m_{a,b}\|_\alpha \asymp \|f\|_\alpha, \quad f \in \mathfrak{D}_\alpha.$$

Just as in one variable, *equality* and not just equivalence of norms for all automorphisms actually characterizes the Dirichlet space $\mathfrak{D} = \mathfrak{D}_1$ among Hilbert spaces of analytic functions (see [6] and [11] for details).

2.2. **General remarks on polynomials and their zero varieties.** Here we discuss of polynomials in several variables, their zero sets

$$\mathcal{Z}(f) = \{(z_1, z_2) \in \mathbb{C}^2 \colon f(z_1, z_2) = 0\},$$

and factorization properties.

It is well-known from commutative algebra that $\mathbb{C}[z_1, z_2]$, the ring of polynomials in two complex variables, is a unique factorization domain, meaning any polynomial can be written as a finite product of irreducible polynomials and this factorization is unique up to units and ordering of irreducible factors. Whereas the fundamental theorem of algebra asserts that irreducible polynomials over $\mathbb{C}$ are linear, irreducible polynomials in two variables come in many different forms.

Let us now closely examine the structure of polynomials that do not vanish in the bidisk, but have zeros on its boundary. It suffices to examine irreducible polynomials which do not depend on one variable alone, because these polynomials are already addressed in the earlier work [3]. So, let $f \in \mathbb{C}[z_1, z_2]$ be irreducible with no zeros in $\mathbb{D}^2$ and assume f depends on both variables.

The set $\mathcal{Z}(f) \cap \mathbb{T}^2$ can be described in terms of zeros of real polynomials (by considering real and imaginary parts of f), and this implies in particular that $\mathcal{Z}(f) \cap \mathbb{T}^2$ is a *semi-algebraic set*. By a result by Stanisław Łojasiewicz (see [15, pp.1584-1585] for a precise statement, and [15, 14] for background material), such sets split into finitely many connected sets which are again

semi-algebraic. It follows that the intersection of the zero set of a polynomial with $\mathbb{T}^2$ consists of finitely many points and a union of finitely many subvarieties of $\mathbb{T}^2$. We can say much more about the zero set of f.

First, our irreducible polynomial f cannot vanish on $\overline{\mathbb{D}}^2 \setminus \mathbb{T}^2$. In fact, a zero on $\mathbb{D} \times \mathbb{T}$ forces f to vanish on a line $z_2 = \zeta$ for some $\zeta \in \mathbb{T}$ and hence $z_2 - \zeta$ would divide f, which we have ruled out. This is because $g_\zeta(z_1) := f(z_1, \zeta)$ is non-vanishing in $\mathbb{D}$ as a function of z_1 for fixed $\zeta \in \mathbb{D}$, and by Hurwitz's theorem as ζ goes to a point on $\mathbb{T}$, g_ζ is either non-vanishing in $\mathbb{D}$ or identically zero.

Next, if f has bidegree (n, m)—degree n in z_1 and m in z_2—then we define
$$\tilde{f}(z) = z_1^n z_2^m \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}.$$
There are now two cases depending on whether or not f divides $\tilde{f}$. If f does not divide $\tilde{f}$, then f has only finitely many zeros on $\mathbb{T}^2$. Indeed, zeros of f on $\mathbb{T}^2$ will be common zeros of f with $\tilde{f}$ and two bivariate polynomials have infinitely many common zeros if and only if they have a common factor by Bézout's theorem. If f divides $\tilde{f}$, then $f = \lambda \tilde{f}$ for some $\lambda \in \mathbb{T}$ since f and $\tilde{f}$ have the same degree and $|f| = |\tilde{f}|$ on $\mathbb{T}^2$. Because of this symmetry, if we set $\mathbb{E} = \mathbb{C} \setminus \overline{\mathbb{D}}$, then
$$\mathcal{Z}(f) \subset (\mathbb{D} \times \mathbb{E}) \cup \mathbb{T}^2 \cup (\mathbb{E} \times \mathbb{D}).$$

It turns out that our polynomial f possesses the following determinantal representation
$$f(z) = c \det \left[I_{n+m} - U \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix} \right] \tag{2.2}$$
where c is a constant, U is an $(n+m) \times (n+m)$ unitary matrix, I_n, I_m, I_{n+m} are identity matrices. This representation easily extends to all polynomials non-vanishing on the bidisk all of whose irreducible factors intersect $\mathbb{T}^2$ on an infinite set. This formula and the techniques used to prove it from [12, 13] will play a fundamental role in our proof in Section 4. The determinantal formula is also useful for generating concrete examples as in Section 5.

We mention in passing that these polynomials are closely related to a special class of varieties studied by Agler and McCarthy [1] called *distinguished varieties*. Such varieties are given as the zero set $\mathcal{Z}(g)$ of a polynomial g satisfying
$$\mathcal{Z}(g) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$
This definition is equivalent to the original definition of Agler and McCarthy and is sometimes easier to work with (see [12]). The requirement can be phrased as requiring that $\mathcal{Z}(g)$ exits the bidisk through the distinguished boundary. Since these varieties must intersect $\mathbb{D}^2$, they are tangential to the topic of this paper; however, the formula (2.2) was discovered in [12] while giving a new proof of a similar formula for distinguished varieties from [1].

2.3. **Fourier analysis and measures supported on curves.** We will use a generalization of van der Corput's lemma from the theory of oscillatory integrals (see Stein [21, Section VIII.3.2]) that provides estimates on Fourier coefficients of measures supported on submanifolds. Such estimates will be useful in Section 5 to establish existence of measures of finite energy.

Let S be a smooth curve in $\mathbb{T}^2$, which we identify with $[0, 2\pi) \times [0, 2\pi)$ in this section. Consider a smooth parametrization $\varphi : I \to \mathbb{T}^2$ where I is an interval in $\mathbb{R}$ (and $d\varphi/dt \neq 0$ on I). We define the *type* of a point $\xi = \varphi(x) \in \varphi(I)$ as the smallest τ such that, for all unit vectors $\eta \in \mathbb{R}^2$ there exists an integer $k \leq \tau$ such that we have

$$\left[\frac{d^k \varphi}{dt^k} \cdot \eta \right]_{t=x} \neq 0. \tag{2.3}$$

A curve S is said to have type τ if the maximum of the types of its points is τ. In particular, a curve is of type 2 if it has everywhere non-vanishing curvature.

In Section 5, we discuss an example in detail, however it is useful to discuss the type of a curve which is simply a piece of a graph $\varphi(t) = (t, \psi(t))$ with $\psi'(0) \neq 0$. If $\eta \in \mathbb{R}^2$ is a unit vector with $\varphi'(t) \cdot \eta = \eta_1 + \psi'(0)\eta_2 = 0$, then $\eta_2 \neq 0$. If we also had $0 = \varphi''(0) \cdot \eta = \psi''(0)\eta_2$, then necessarily $\psi''(0) = 0$. Thus, the curve has type 2 at $t = 0$ if $\psi''(0) \neq 0$ and it will not have type 2 if $\psi''(0) = 0$. It will be useful later to notice that if we do not have type 2 at a point we can apply a Möbius transformation to get a point of type 2. What we have in mind is if φ parametrizes a piece of the zero set of f on $\mathbb{T}^2$, then $(\gamma(t), \psi(t))$ parametrizes a piece of the zero set of $f \circ m_{a,0}$ where $\gamma(t) = \arg m_a(e^{it})$. But, this new curve has type 2 at $t = 0$. One can compute that $\gamma'(0) > 0$ and $\gamma''(0) \neq 0$ as long as $\text{Im}(a) \neq 0$. So, if $\psi''(0) = 0$, the two equations $\eta_1\gamma'(0) + \eta_2\psi'(0) = 0$ and $\eta_1\gamma''(0) = 0$ cannot hold simultaneously. The type of a curve will give us control on the growth of Fourier coefficients of a measure supported on the curve.

The Fourier coefficients of a finite Borel measure on $\mathbb{T}^2$ are given by

$$\hat{\mu}(k, l) = \int_{\mathbb{T}^2} e^{-i(k\theta_1 + l\theta_2)} d\mu(\theta_1, \theta_2), \quad k, l \in \mathbb{Z}. \tag{2.4}$$

The measures we work with will be of the form

$$d\mu(x) = \psi(x)d\sigma(x), \quad x \in S \subset \mathbb{T}^2, \tag{2.5}$$

where S is a curve in $\mathbb{T}^2$ of finite type, $\psi \in C_0^\infty(S)$ is non-negative, and σ is the measure on S induced by pulling back to Lebesgue measure on the line using the parametrization of S. For such measures, we have the following estimate (cf. Stein [21], Theorem VIII.2).

Theorem 2.1 (Decay of Fourier coefficients of measures on varieties)**.** *If S is of finite type $\tau \in \mathbb{N}$, and μ is of the form* (2.5), *then there exists a constant $C > 0$ such that*

$$|\hat{\mu}(k, l)| \leq C(k^2 + l^2)^{-1/(2\tau)}, \quad k, l \in \mathbb{Z}.$$

The constant C may depend on S and ψ, but not on k and l.

2.4. **Riesz α-capacity.** In Section 5, we will use an analogue of logarithmic capacity in one variable to analyze the cyclicity properties of polynomials whose zero sets meet the torus along curves. We define the following product capacity on $\mathbb{T}^2$, which was considered in [3].

Definition 2.2. Let $E \subset \mathbb{T}^2$ be a Borel set. We say that a probability measure μ supported in E has *finite (Riesz) α-energy* $(0 < \alpha < 1)$ if

$$I_\alpha[\mu] = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \frac{1}{|e^{i\theta_1} - e^{i\eta_1}|^{1-\alpha}} \frac{1}{|e^{i\theta_2} - e^{i\eta_2}|^{1-\alpha}} d\mu(\theta_1, \theta_2) d\mu(\eta_1, \eta_2) < \infty.$$

We define the *(Riesz) α-capacity* by taking the infimum

$$\mathrm{cap}_\alpha(E) := 1/\inf\{I_\alpha[\mu] : \mu \in \mathcal{P}(E)\},$$

where $\mathcal{P}(E)$ denotes the set of all probability measures with support contained in E. If E supports no such measure, then $\mathrm{cap}_\alpha(E) = 0$.

When $\alpha = 1$, we use the kernel $\log(e/|e^{i\theta_1} - e^{i\eta_1}|) \log(e/|e^{i\theta_2} - e^{i\eta_2}|)$ in the definitions of energy and capacity.

In [3, Section 4], it was shown that functions $f \in \mathfrak{D}_\alpha$ with $\mathrm{cap}_{\alpha_0}(\mathcal{Z}(f) \cap \mathbb{T}^2) > 0$ for some $0 < \alpha_0 < 1$ are not cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \geq \alpha_0$. Here, $\mathcal{Z}(f) \cap \mathbb{T}^2$ refers to the zero set of f on $\mathbb{T}^2$. (As explained in [3], the non-tangential boundary values of f on $\mathbb{T}^2$ exist quasi-everywhere with respect to α-capacity so that we can make sense of the capacity of $\mathcal{Z}(f) \cap \mathbb{T}^2$.) In brief, the proof involves identifying the dual space of $\mathfrak{D}_\alpha$ with $\mathfrak{D}_{-\alpha}$ and considering the Cauchy integral of a measure μ supported on $\mathcal{Z}(f) \cap \mathbb{T}^2$ having finite α-energy. The finiteness of the energy then guarantees the membership of the Cauchy transform in $\mathfrak{D}_{-\alpha}$, and the fact that its generating measure lives on the zero set of f implies that the dual pairing annihilates the elements of $[f]$.

For two Borel sets $E, F \subset \mathbb{T}^2$ with $E \subset F$ we have $\mathcal{P}(E) \subset \mathcal{P}(F)$, and so $\mathrm{cap}_\alpha(E) \leq \mathrm{cap}_\alpha(F)$. This means that if we want to show that a given zero set has positive capacity, it is enough to prove that some subset of the given zero set has this property. It is often useful to express the α-energy of μ in terms of its Fourier coefficients (2.4):

$$I_\alpha[\mu] \asymp 1 + \sum_{k=1}^\infty \frac{|\hat{\mu}(k,0)|^2}{k^\alpha} + \sum_{l=1}^\infty \frac{|\hat{\mu}(0,l)|^2}{l^\alpha} + \frac{1}{2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{l=1}^\infty \frac{|\hat{\mu}(k,l)|^2}{|k|^\alpha l^\alpha}. \quad (2.6)$$

3. POLYNOMIALS WITH FINITELY MANY ZEROS

We now turn to the main result. We begin by tackling the case when f is a polynomial whose zero set satisfies $0 < \#[\mathcal{Z}(f) \cap \mathbb{T}^2] < \infty$.

Building on the work of Håkan Hedenmalm on closed ideals in $\mathfrak{D}_2$ in [9], the authors in [3] proved that polynomials (and in fact, certain functions in $\mathfrak{D}_2$) with no zeros in the bidisk that vanish at a single point of the torus are cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \leq 1$. Hedenmalm's proof does not seem to extend

directly to the case of finitely many zeros. Instead, we resolve the question of cyclicity for polynomials with finitely many zeros on the distinguished boundary using a different argument.

Theorem 3.1. *Let $f \in \mathbb{C}[z_1, z_2]$ have no zeros in $\mathbb{D}^2$ and finitely many zeros on $\mathbb{T}^2$. Then, f is cyclic in $\mathfrak{D}_\alpha$ for $\alpha \leq 1$.*

We shall use Łojasiewicz's inequality as recorded in [14].

Theorem 3.2 (Łojasiewicz's inequality)**.** *Let f be a nonzero real analytic function on an open set $U \subset \mathbb{R}^n$. Assume the zero set $\mathcal{Z}(f)$ of f in U is nonempty. Let E be a compact subset of U. Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E, such that*

$$|f(x)| \geq C \cdot \mathrm{dist}(x, \mathcal{Z}(f))^q$$

for every $x \in E$.

To prove our theorem we will essentially compare our polynomial f to polynomials which we know to be cyclic.

Lemma 3.3. *Assume $f \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$ and finitely many zeros on $\mathbb{T}^2$. Then, for any positive integer k there exist $g \in \mathbb{C}[z_1]$ and $h \in \mathbb{C}[z_2]$ with zeros only on $\mathbb{T}$ such that Q defined by*

$$Q(z_1, z_2) = \frac{g(z_1)h(z_2)}{f(z_1, z_2)}$$

is k-times continuously differentiable on $\mathbb{T}^2$.

Proof. Let $r(x) = r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2$, a function which is evidently real analytic throughout $\mathbb{R}^2$, and let $\mathcal{Z}(r)$ be the set of zeros of r. Set $E = [0, 2\pi]^2$. By Łojasiewicz's inequality there is a constant $C > 0$ and positive integer q so that

$$r(x) \geq C \, \mathrm{dist}(x, \mathcal{Z}(r))^q$$

for $x \in E$.

By the assumption on f, $\mathcal{Z}(r) \cap E$ is finite and thus there is a constant $c > 0$ so that for $x \in E$

$$\mathrm{dist}(x, \mathcal{Z}(r))^2 \geq c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2.$$

But $|x - y|^2 = |x_1 - y_1|^2 + |x_2 - y_2|^2 \geq |e^{ix_1} - e^{iy_1}|^2 + |e^{ix_2} - e^{iy_2}|^2 \geq 2|(e^{ix_1} - e^{iy_1})(e^{ix_2} - e^{iy_2})|$. Replacing (e^{ix_1}, e^{ix_2}) by (z_1, z_2) and (e^{iy_1}, e^{iy_2}) by (ζ_1, ζ_2), we get that

$$\frac{\prod_{\zeta \in \mathcal{Z}(f) \cap \mathbb{T}^2} |(z_1 - \zeta_1)(z_2 - \zeta_2)|^{q/2}}{|f(z_1, z_2)|^2}$$

is bounded on $\mathbb{T}^2 \setminus \mathcal{Z}(f)$. If we increase the power in the numerator (say to $4q$) we get a function which is continuous on $\mathbb{T}^2$ and set equal to zero at the

zeros of f. Thus, for

$$Q_0(z) = \prod_{\zeta \in \mathcal{Z}(f) \cap \mathbb{T}^2} (z_1 - \zeta_1)^q (z_2 - \zeta_2)^q$$

we have that Q_0/f is bounded and continuous on $\mathbb{T}^2$ (where again we set the function equal to zero at zeros of f). Therefore, for a large enough power N, Q_0^N/f is k-times continuously differentiable on $\mathbb{T}^2$. By construction, $Q_0^N(z_1, z_2) = g(z_1)h(z_2)$ for some one variable polynomials g, h which only vanish on the circle. $\qquad \square$

Proof of Theorem 3.1. By Lemma 3.3, we have one variable polynomials g, h which vanish only on the circle such that $Q(z_1, z_2) = g(z_1)h(z_2)/f(z_1, z_2)$ is 2-times continuously differentiable on $\mathbb{T}^2$. Then, the Fourier coefficients of Q satisfy

$$\sum_{k,l} |\hat{Q}(k,l)|^2 (k+1)^2 (l+1)^2 < \infty$$

which puts $Q \in \mathfrak{D}_\alpha$ for $\alpha \le 2$. Hence, $g(z_1)h(z_2) \in f\mathfrak{D}_\alpha$ for $\alpha \le 2$. Since f is a multiplier, $f\mathfrak{D}_\alpha = [f]$. But $g(z_1)h(z_2)$ is cyclic for α with $\alpha \le 1$ (by [3]), and therefore f is cyclic for $\alpha \le 1$. $\qquad \square$

4. Cyclicity for small parameter values

In this section, we prove that any polynomial that does not vanish in the bidisk is cyclic for $\alpha \le 1/2$, regardless of the size of its zero set. As was mentioned previously, the case $\alpha \le 0$ follows from [16].

Theorem 4.1. *Let $0 < \alpha \le 1/2$. Then any polynomial that does not vanish in $\mathbb{D}^2$ is cyclic in $\mathfrak{D}_\alpha$.*

The proof presented here relies heavily on the machinery developed in [12, 13]. As discussed in Section 2.2, we can assume $f \in \mathbb{C}[z_1, z_2]$ is

    (1) non-vanishing in $\mathbb{D}^2$,
    (2) irreducible,
    (3) a function of both variables,
    (4) of bidegree (n, m), and
    (5) $f = \lambda \tilde{f}$ for some $\lambda \in \mathbb{T}$.

Here $\tilde{f}(z, w) = z_1^n z_2^m \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$. The determinantal formula (2.2) is a consequence of the following key result which we explain at the end of this section.

Proposition 4.2. *Assume $f \in \mathbb{C}[z_1, z_2]$ satisfies (1)-(5) above. Then, there exists an $(n + m) \times (n + m)$ unitary U and a column vector polynomial $\vec{B} \in \mathbb{C}^{n+m}[z_1, z_2]$ such that*

$$\left(I - U \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix} \right) \vec{B}(z) \in f(z)\mathbb{C}^{n+m}[z_1, z_2].$$

Furthermore, there exists a row vector polynomial $\vec{a} \in \mathbb{C}^{n+m}[z_2]$ *such that* $p(z_2) := \vec{a}(z_2)\vec{B}(z_1, z_2)$ *is a one variable polynomial with no zeros on* $\mathbb{D}$.

Armed with this proposition, the proof of cyclicity is straightforward.

Proof of Theorem 4.1. Assume $f, \vec{B}, U$ are as in Proposition 4.2 and let $\alpha \leq 1/2$. Suppose $g \in \mathfrak{D}_\alpha$ is orthogonal to $[f]$ in $\mathfrak{D}_\alpha$. We shall show first that for every row vector polynomial $\vec{v} \in \mathbb{C}^{n+m}[z_1, z_2]$

$$\langle \vec{v}\vec{B}, g \rangle_\alpha = 0.$$

For simplicity, set

$$A(z) = U \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix}$$

The important thing to notice about A is that it is homogeneous of degree 1.

Since $(I - A(z))\vec{B}(z) \in f\mathbb{C}^{n+m}[z_1, z_2]$ it follows that $(I - A(z)^k)\vec{B}(z) \in f\mathbb{C}^{n+m}[z_1, z_2]$ for any $k \geq 1$. So, if $g \perp [f]$ in $\mathfrak{D}_\alpha$, then

$$\langle \vec{v}\vec{B}, g \rangle_\alpha = \langle \vec{v}A^k\vec{B}, g \rangle_\alpha \tag{4.1}$$

for any $k \geq 0$.

Let $d = \deg \vec{v} + \deg \vec{B} + 1$ where the degree of a vector polynomial is the maximum of the degrees of each entry. We emphasize we are using total degree and not bidegree. Then, the monomials appearing in $\vec{v}A^k\vec{B}$ have total degree between k and $k + d - 1$. So, $\{\vec{v}A^{kd}\vec{B}\}$ forms a sequence of pairwise orthogonal elements of $\mathfrak{D}_\alpha$ since the degrees of the monomials in $\vec{v}A^{kd}\vec{B}$ lie in the interval $[kd, (k+1)d)$.

By Bessel's inequality

$$\|g\|_\alpha^2 \geq \sum_{k \geq 0} \frac{|\langle \vec{v}A^{kd}\vec{B}, g \rangle_\alpha|^2}{\|\vec{v}A^{kd}\vec{B}\|_\alpha^2} = \sum_{k \geq 0} \frac{|\langle \vec{v}\vec{B}, g \rangle_\alpha|^2}{\|\vec{v}A^{kd}\vec{B}\|_\alpha^2}.$$

Notice if one of the denominators happened to be zero, then our goal $\langle \vec{v}\vec{B}, g \rangle_\alpha = 0$ holds automatically by (4.1). So, we can assume $\vec{v}A^k\vec{B} \neq 0$ for all $k \geq 0$ and the goal now is to show the denominator grows too slowly for the above sum to be finite without having $\langle \vec{v}\vec{B}, g \rangle_\alpha = 0$.

Since $\vec{v}A^{kd}\vec{B}$ has degree at most $(k+1)d$,

$$\|\vec{v}A^{kd}\vec{B}\|_\alpha^2 \leq \max\{(i+1)^\alpha(j+1)^\alpha : i + j \leq (k+1)d\}\|\vec{v}A^{kd}\vec{B}\|_{H^2}^2$$
$$\leq ((k+1)d/2 + 1)^{2\alpha}\|\vec{v}A^{kd}\vec{B}\|_{H^2}^2.$$

But $\|\vec{v}A^{kd}\vec{B}\|_{H^2} \leq \|\vec{v}\|_{H^\infty}\|\vec{B}\|_{H^2}$; since A is unitary valued on $\mathbb{T}^2$ we have $\|A^{kd}\|_{H^\infty} = 1$ for all k. We are implicitly using vector-valued H^2 and matrix valued H^∞ as well as the fact that H^∞ is the multiplier algebra of H^2. Therefore,

$$\|\vec{v}A^{kd}\vec{B}\|_\alpha^2 \leq C(k+1)^{2\alpha}$$

for some constant C depending on $d, \vec{v}, \vec{B}$. Thus,

$$\sum_{k \geq 0} \frac{|\langle \vec{v}\vec{B}, g \rangle_\alpha|^2}{\|\vec{v}A^{kd}\vec{B}\|_\alpha^2} \geq \sum_{k \geq 0} \frac{|\langle \vec{v}\vec{B}, g \rangle_\alpha|^2}{C(k+1)^{2\alpha}}.$$

The only way this can be finite when $\alpha \leq 1/2$ is if $\langle \vec{v}\vec{B}, g \rangle_\alpha = 0$.

Now we can finish the proof. Setting $\vec{v}(z) = z_1^j z_2^k \vec{a}(z_2)$ where $\vec{a}$ is as in Proposition 4.2, we see that for all $j, k \geq 0$

$$0 = \langle z_1^j z_2^k \vec{a}\vec{B}, g \rangle_\alpha = \langle z_1^j z_2^k p, g \rangle_\alpha$$

where $p = \vec{a}\vec{B} \in \mathbb{C}[z_2]$ from Proposition 4.2 has no zeros in $\mathbb{D}$. Thus, g is orthogonal to $[p]$, but since since p is cyclic for $\alpha \leq 1$ we have $[p] = \mathfrak{D}_\alpha$. Hence, $g = 0$ and we conclude f is cyclic. $\qquad\square$

We now turn to explaining Proposition 4.2. This result can be essentially proven by quoting results from [12] but some further explanation is necessary.

First, we can assume $f = \tilde{f}$ by replacing f with a unimodular multiple. The condition $f = \tilde{f}$ is what is termed "$\mathbb{T}^2$-symmetric" in [12]. Many of the results we use from [12] require this of f. By Theorem 2.9 of [12], if we set

$$h = z_1 \frac{\partial f}{\partial z_1} + z_2 \frac{\partial f}{\partial z_2}$$

then $\tilde{h}$ has no zeros in $\mathbb{D}^2$. Furthermore, by computation or by Lemma 2.7 of [12], $h + \tilde{h} = (n+m)f$. Since f is irreducible, h and $\tilde{h}$ have no factors in common unless they are multiples of one another by looking at degrees. This possibility is disallowed by $h(0,0) = 0$. Thus, $\tilde{h}$ has no zeros in $\mathbb{D}^2$ and no factors in common with h.

By the main result of [13], there exist vector polynomials $\vec{P} \in \mathbb{C}^n[z_1, z_2]$ and $\vec{Q} \in \mathbb{C}^m[z_1, z_2]$ such that

$$\tilde{h}(z)\overline{\tilde{h}(w)} - h(z)\overline{h(w)} = (1 - z_1\bar{w}_1)\vec{P}(w)^*\vec{P}(z) + (1 - z_2\bar{w}_2)\vec{Q}(w)^*\vec{Q}(z). \quad (4.2)$$

Furthermore, we may choose $\vec{P}$ to be of degree $n-1$ in z_1 and when we write

$$\vec{P}(z_1, z_2) = P(z_2) \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_1^{n-1} \end{pmatrix} \quad (4.3)$$

the $n \times n$ matrix $P(z_2)$ is invertible for $z_2 \in \mathbb{D}$. Lemma 2.8 of [12], or a computation, shows the left hand side of (4.2) is equal to

$$(n+m)^2 f(z)\overline{f(w)} - (n+m)(h(z)\overline{f(w)} + f(z)\overline{h(w)})$$

—the main point here is that this vanishes on $\mathcal{Z}(f)$. Thus, for $z, w \in \mathcal{Z}(f)$ equation (4.2) rearranges into

$$\vec{P}(w)^*\vec{P}(z) + \vec{Q}(w)^*\vec{Q}(z) = z_1\bar{w}_1\vec{P}(w)^*\vec{P}(z) + z_2\bar{w}_2\vec{Q}(w)^*\vec{Q}(z)$$

and so the map

$$\begin{pmatrix} z_1\vec{P}(z) \\ z_2\vec{Q}(z) \end{pmatrix} \mapsto \begin{pmatrix} \vec{P}(z) \\ \vec{Q}(z) \end{pmatrix}$$

defined for $z \in \mathcal{Z}(f)$ extends linearly to an isometry which in turn can be extended to a mapping given by an $(n+m) \times (n+m)$ unitary matrix U:

$$U \begin{pmatrix} z_1\vec{P}(z) \\ z_2\vec{Q}(z) \end{pmatrix} = \begin{pmatrix} \vec{P}(z) \\ \vec{Q}(z) \end{pmatrix}$$

for $z \in \mathcal{Z}(f)$. Thus, if we set $\vec{B} = \begin{pmatrix} \vec{P} \\ \vec{Q} \end{pmatrix}$, then

$$\left(I - U \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix} \right) \vec{B}(z) \tag{4.4}$$

vanishes on $\mathcal{Z}(f)$ and since f is irreducible its entries must belong to the polynomial ideal generated by f. Therefore, (4.4) belongs to $f\mathbb{C}^{n+m}[z_1, z_2]$.

To finish we must find a row vector polynomial $\vec{a} \in \mathbb{C}^{n+m}[z_2]$ such that $\vec{a}(z_2)\vec{B}(z_1, z_2)$ is a one variable polynomial in z_2 with no zeros in $\mathbb{D}$. Set $\vec{a}(z_2) = (e_1 \mathrm{adj}(P(z_2)), 0_m)$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^n$ and $0_m \in \mathbb{C}^m$ is the zero (row) vector. Here adj denotes the adjugate matrix or the transpose of the matrix of cofactors. Then, by (4.3)

$$\vec{a}(z_2)\vec{B}(z) = e_1 \mathrm{adj}(P(z_2))\vec{P}(z) = e_1 \mathrm{adj}(P(z_2))P(z_2) \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_1^{n-1} \end{pmatrix} = \det(P(z_2))$$

which has no zeros in $\mathbb{D}$.

5. Non-cyclicity and curves in the distinguished boundary

We now turn to the remaining case where $\alpha > 1/2$ and f is a polynomial whose zero set meets the distinguished boundary along curves. Here, we find that local curvature properties determine the parameters α for which the function is not cyclic in $\mathfrak{D}_\alpha$. Our methods are analytic in nature, and do not rely on the algebraic properties of polynomials to the same extent as in previous sections. In particular, the next theorem holds for a general $f \in \mathfrak{D}_\alpha$ which is not necessarily a polynomial. Recall that for $\alpha \geq 0$, any $f \in \mathfrak{D}_\alpha$ has non-tangential boundary values on $\mathbb{T}^2$ quasi-everywhere with respect to α-capacity and thus the zero set $\mathcal{Z}(f) \cap \mathbb{T}^2$ of f restricted to $\mathbb{T}^2$ is well-defined up to sets of α-capacity zero; see [11].

Theorem 5.1. *Assume that $f \in \mathfrak{D}_\alpha$ is such that the intersection $\mathcal{Z}(f) \cap \mathbb{T}^2$ contains a locally smooth curve S of type τ. Then f is not cyclic in $\mathfrak{D}_\alpha$ for any $\alpha > 1 - 1/\tau$.*

Proof. Since $S \subset \mathcal{Z}(f) \cap \mathbb{T}^2$, we have $\text{cap}_\alpha(S) \leq \text{cap}_\alpha(\mathcal{Z}(f) \cap \mathbb{T}^2)$. Our goal is to show that $\text{cap}_\alpha(S) > 0$ for $\alpha > 1 - 1/\tau$ by proving the existence of a measure μ of the form (2.5) that has finite energy. This is accomplished by estimating the Fourier coefficients $\hat{\mu}(k, l)$ of such a measure, and plugging these estimates into the Fourier formula for the energy given in (2.6):

$$I_\alpha[\mu] \asymp 1 + \sum_{k=1}^\infty \frac{|\hat{\mu}(k,0)|^2}{k^\alpha} + \sum_{l=1}^\infty \frac{|\hat{\mu}(0,l)|^2}{l^\alpha} + \frac{1}{2} \sum_{k \in \mathbb{Z} \backslash \{0\}} \sum_{l=1}^\infty \frac{|\hat{\mu}(k,l)|^2}{|k|^\alpha l^\alpha}.$$

Applying the estimate in Theorem 2.1 to the sum of diagonal terms ($k = l \geq 1$) we obtain

$$\sum_{k=1}^\infty \frac{|\hat{\mu}(k,k)|^2}{(k+1)^{2\alpha}} \leq C \sum_{k=1}^\infty \frac{1}{(2k^2)^{1/\tau}(k+1)^{2\alpha}} \leq C \sum_{k=1}^\infty \frac{1}{k^{2\alpha+2/\tau}}.$$

The latter series converges when $\alpha > 1/2 - 1/\tau$. Moreover, we have

$$\sum_{k=1}^\infty \frac{|\hat{\mu}(k,0)|^2}{(k+1)^\alpha} \leq C \sum_{k=1}^\infty \frac{1}{k^{\alpha+1/\tau}},$$

and similarly for the series involving $\hat{\mu}(0,l)$, and having $\alpha > 1 - 1/\tau$ forces convergence.

We perform a similar computation for the remaining terms. We now have to sum over both k and l; however, using the symmetry in Theorem 2.1, we obtain

$$\sum_{k=2}^\infty \sum_{l=1}^{k-1} \frac{|\hat{\mu}(k,l)|^2}{(k+1)^\alpha(l+1)^\alpha} + \sum_{l=2}^\infty \sum_{k=1}^{l-1} \frac{|\hat{\mu}(k,l)|^2}{(k+1)^\alpha(l+1)^\alpha}$$

$$\leq C \sum_{k=2}^\infty \sum_{l=1}^{k-1} \frac{1}{(k+l)^{2/\tau}(k+1)^\alpha(l+1)^\alpha}$$

$$\leq C \sum_{k=2}^\infty \frac{1}{k^{2/\tau}(k+1)^\alpha} \sum_{l=1}^{k-1} \frac{1}{(1+l/k)^{2/\tau}(l+1)^\alpha}$$

$$\leq C \sum_{k=2}^\infty \frac{1}{k^{\alpha+2/\tau}} \sum_{l=1}^{k-1} \frac{1}{(l+1)^\alpha}$$

$$\leq C \sum_{k=1}^\infty \frac{1}{k^{\alpha+2/\tau}} \cdot k^{1-\alpha} \qquad \text{(since } 1 - \alpha > 0\text{)}$$

$$\leq C \sum_{k=1}^\infty \frac{1}{k^{2\alpha-1+2/\tau}}.$$

The latter sum is finite if and only if $2\alpha - 1 + 2/\tau > 1$, which happens precisely if $\alpha > 1 - 1/\tau$.

In conclusion, the total energy of μ is finite if $\alpha > 1 - 1/\tau$, and non-cyclicity of f follows. $\qquad \square$

We illustrate the content of the theorem, and shed new light on the results obtained in [3], by looking at a family of examples.

Example 1. For $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$, we consider the unitary matrix

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}.$$

Taking $n = m = 1$ in the formula (2.2), and plugging in this choice of U, we obtain the family of polynomials

$$f_a(z_1, z_2) = 1 - az_1 - \bar{a}z_2 + z_1z_2. \tag{5.1}$$

Setting $a = 0$, we obtain $f_0(z_1, z_2) = 1 + z_1z_2$, an irreducible polynomial whose zero set and cyclic properties were studied in [3, Section 4]. In particular, it was established that f_0 is cyclic in $\mathfrak{D}_\alpha$ precisely when $\alpha \leq 1/2$. The choice $a = 1$ leads to the polynomial $f_1(z_1, z_2) = 1 - z_1 - z_2 + z_1z_2 = (1 - z_1)(1 - z_2)$, which factors into one-variable polynomials; such polynomials were shown to be cyclic for all $\alpha \leq 1$ in [3, Section 2].

Solving the equation $f_a(z_1, z_2) = 0$ for z_2, we obtain

$$z_2 = \frac{az_1 - 1}{z_1 - \bar{a}}.$$

Direct computation (or the observation that the formula for $1/z_2$ describes a Möbius transformation) shows that $|z_2| = 1$ when $|z_1| = 1$. This leads to a parametric representation of $\mathcal{Z}(f_a) \cap \mathbb{T}^2$:

$$t \mapsto (e^{it}, e^{im(t)}), \quad \text{where} \quad m(t) = \arg\left(\frac{ae^{it} - 1}{e^{it} - \bar{a}}\right), \tag{5.2}$$

or, if the torus is identified with $[0, 2\pi) \times [0, 2\pi)$, $t \mapsto (t, m(t))$. Restricting to $a \in (0, 1)$ for simplicity, we see that $\mathcal{Z}(f_a)$ "interpolates" between

$$\mathcal{Z}(f_0) = \{(e^{it}, -e^{-it})\} \quad \text{and} \quad \mathcal{Z}(f_1) = (\mathbb{T} \times \{1\}) \cup (\{1\} \times \mathbb{T}).$$

In [3], it was shown that $\mathcal{Z}(f_0) \cap \mathbb{T}^2$ has positive $1/2$-capacity, while $\mathcal{Z}(f_1) \cap \mathbb{T}^2$ has α-capacity 0 for all $\alpha \leq 1$. We will now show that the α-capacity of $\mathcal{Z}(f_a) \cap \mathbb{T}^2$ for intermediate $a \in (0, 1)$ is positive for all $1/2 < \alpha \leq 1$, so that f_a is not cyclic in $\mathfrak{D}_\alpha$ for this range of parameter values.

We show that $\mathcal{Z}(f_a)$ is of type $\tau = 2$ at $t = \pi/2$ when $a \in (0, 1)$. Computations yield

$$m(t) = \pi + \arctan\left(\frac{(1 - a^2)\sin t}{2a - (1 + a^2)\cos t}\right),$$

$$m'(t) = \frac{1 - a^2}{2a\cos t - 1 - a^2},$$

$$\text{and } m''(t) = \frac{2a(1 - a^2)\sin t}{(2a\cos t - 1 - a^2)^2}.$$

If $\eta = (\eta_1, \eta_2)^\top \in \mathbb{R}^2$ is a unit vector such that

$$\left[\frac{d}{dt}(t, m(t))\right]_{t=\pi/2} \cdot \eta = \eta_1 - \frac{1-a^2}{1+a^2} \eta_2 = 0,$$

then η is determined up to sign. If $a \neq 0, 1$, then we cannot simultaneously have

$$\left[\frac{d^2}{dt^2}(t, m(t))\right]_{t=\pi/2} \cdot \eta = -\frac{2a(1-a^2)}{(1+a^2)^2} \eta_2 = 0,$$

and hence $\mathcal{Z}(f_a)$ is of type 2 at this point.

By smoothness of $\mathcal{Z}(f_a)$, for fixed a, there exists $\varepsilon > 0$ such that $S = \varphi([\pi/2 - \epsilon, \pi/2 + \epsilon])$ is of type 2. (A computation reveals that $\mathcal{Z}(f_a)$ is actually of type 3 at $(1, 1)$.) By virtue of Theorem 2.1, we now have $|\hat{\mu}(k, l)| \leq C(k^2 + l^2)^{-1/4}$ for measures supported on the zero set in the torus, so that the right hand side of (2.6) converges for $\alpha > 1/2$, as in the proof of Theorem 5.1. In conclusion, f_a is not cyclic in $\mathfrak{D}_\alpha$ for all $\alpha > 1/2$.

To address the remaining range $\alpha \in (0, 1/2)$, we recall that $f(z_1, z_2) = 1 + z_1 z_2$ was proved to be cyclic in $\mathfrak{D}_\alpha$ when $\alpha \leq 1/2$ using restriction arguments in [3]. We now note that

$$1 - az_1 - \bar{a}z_2 + z_1 z_2 = (1 - az_1) \cdot f \circ m_{a,0}(z_1, z_2),$$

where $m_{a,0}$ denotes a Möbius automorphism of the bidisk of the form (2.1). This means that f_a is cyclic if $f \circ m_{a,0}$ is. But since the $\mathfrak{D}_\alpha$-norm of a function precomposed with a Möbius map is comparable to the norm of the function itself (see Section 2.1), the cyclicity of f implies that f_a is cyclic in $\mathfrak{D}_\alpha$ for $\alpha \leq 1/2$. This is because the approximating polynomials for f transform into rational functions with no poles in the closed bidisk, and such functions can be approximated in multiplier norm by polynomials. An analogous argument shows that f_a is cyclic for $\alpha \leq 1/2$; this of course is in agreement with the result in Section 4.

In the case where $a = 0$ or $a = 1$, the zero set can be viewed as straight lines in the torus: a local parametric representation can be given as

$$t \mapsto t(p, q), \quad t \in [0, 2\pi),$$

where $p, q \in \mathbb{Q}$. (We have $(p, q) = (1, -1)$ in the case $a = 0$, while $(p, q) = (1, 0)$ and $(0, 1)$ for $a = 1$.) In particular, both $\mathcal{Z}(f_0)$ and $\mathcal{Z}(f_1)$ are of infinite type. In this case, we consider the measures induced by the integration current associated with $\mathcal{Z}(f_a) \cap \mathbb{T}^2$, as in [3, Section 4.2], and note that their Fourier coefficients are constant along indices (k, l) that satisfy

$$(k, l) \cdot (p, q) = kp + lq = 0,$$

and equal to zero otherwise. Depending on whether both p and q are non-zero or not, the series in (2.6) will then converge for $\alpha > 1/2$ or $\alpha > 1$. While the local curvatures of the zero sets are the same in these cases, the cyclicity properties of the polynomials differ because the norm in $\mathfrak{D}_\alpha$ is of

product type, and this is reflected in the fact that measures supported on lines aligned along the coordinate axes are assigned greater energy. ♦

Returning to polynomials, we note that a component of $\mathcal{Z}(f) \cap \mathbb{T}^2$ aligns with one of the coordinate axes precisely when f contains a factor that is a polynomial in one variable only. If f is a polynomial in one variable, then f is cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \leq 1$.

Bearing this in mind, we formulate our main result for polynomials that vanish along curves.

Theorem 5.2. *Let $f \in \mathbb{C}[z_1, z_2]$ have no zeros in $\mathbb{D}^2$, and suppose $\mathcal{Z}(f)$ meets $\mathbb{T}^2$ along a curve. If f is not a polynomial in one variable only, then f is not cyclic in $\mathfrak{D}_\alpha$ for $\alpha > 1/2$.*

Proof. Suppose f is not a polynomial in one of the variables only. If $\mathcal{Z}(f) \cap \mathbb{T}^2$ contains a subset of type 2, then f is not cyclic by Theorem 5.1. If $\mathcal{Z}(f) \cap \mathbb{T}^2$ contains no subset of type 2, then we can find a Möbius transformation $m_{a,b}$ that maps a curve contained in $\mathcal{Z}(f) \cap \mathbb{T}^2$ to a curve of type 2. To see this we note that since the zero set of f cannot be aligned along the coordinate axes in $\mathbb{T}^2$, $\mathcal{Z}(f) \cap \mathbb{T}^2$ must contain a curve with a tangent vector which is neither horizontal nor vertical. Then, it is possible to parametrize a piece of $\mathcal{Z}(f) \cap \mathbb{T}^2$ via $\varphi(t) = (t, \psi(t))$ where $\psi'(0) \neq 0$. In Section 2.3 we explained how to apply a Möbius transformation to force the curve to be type 2 at a point (and hence in a neighborhood of the point on the curve).

By Theorem 5.1 again, the function $F = f \circ m_{a,b}^{-1}$ is not cyclic for $\alpha > 1/2$. But this implies the non-cyclicity of f. For if there existed a sequence of polynomials $(p_n)_{n=1}^\infty$ such that

$$\|p_n f - 1\|_\alpha \to 0 \quad n \to \infty,$$

then by precomposing with the Möbius map $m_{a,b}^{-1}$, we would obtain

$$\|p_n \circ m_{a,b}^{-1} \cdot F - 1\|_\alpha \leq C\|p_n f - 1\|_\alpha \to 0,$$

contradicting the non-cyclicity of F (as can be seen by approximating the sequence $p_n \circ m_{a,b}^{-1}$ in multiplier norm by polynomials).

It follows that f is not cyclic, as claimed. □

To complete the capacity picture, we point out that the cyclicity result in Section 4 shows that the α-capacities of $\mathcal{Z}(f) \cap \mathbb{T}^2$ for any polynomial f that does not vanish in $\mathbb{D}^2$ are equal to 0 for all $\alpha \leq 1/2$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, 4202 E. FOWLER AVENUE, TAMPA, FL 33620-5700, USA.
  E-mail address: cbenetea@usf.edu

WASHINGTON UNIVERSITY IN ST. LOUIS, DEPARTMENT OF MATHEMATICS, ONE BROOK-INGS DRIVE, CAMPUS BOX 1146, ST. LOUIS, MO 63130-4899, USA.
  E-mail address: geknese@math.wustl.edu

INSTITUTE OF MATHEMATICS, JAGIELLONIAN UNIVERSITY, LOJASIEWICZA 6, 30-348 KRAKÓW, POLAND.
DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, PAVILLON ALEXANDREVA-CHON, 1045 AV. DE LA MÉDECINE, UNIVERSITÉ LAVAL, QUÉBEC (QUÉBEC), CANADA, G1V 0A6.
  E-mail address: lukasz.kosinski@gazeta.pl

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, ONE BEAR PLACE #97328, WACO, TX 76798-7328, USA.
  E-mail address: Constanze_Liaw@baylor.edu

MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UK.
  E-mail address: D.Seco@warwick.ac.uk

CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK.
  E-mail address: a.sola@statslab.cam.ac.uk