Spring 4-16-2003

Newtonian Cosmology

Ronald Lovett
Washington University in St. Louis

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A. The Cosmological Principle

This is a philosophical bias: While we observe the universe from *our* home, we will not see something different from what other observers see from *their* home.

The meaning of this depends upon the meaning of *home*. We assume that "home" means "a local inertial frame," a frame that moves with the same velocity as the average velocity of the matter around the observer. Alternatively, the average velocity near an observer in a local frame is zero.

As the universe we see is expanding, there is some conflict with the assertion of conventional mechanics that all inertial frames are related by static rotations and translations. The local frame of an observer located some distance from us is *moving* relative to our frame. This is a cosmological extension of what we mean by an inertial frame.

B. The Galilean Application

Suppose an observer $\mathcal{A}$ observes $\boldsymbol{v}(\boldsymbol{r}, t)$, the velocity of local frames located a distance $\boldsymbol{r}$ from $\mathcal{A}$ at time t. Of course

$$\boldsymbol{v}(0, t) = 0$$

Suppose that an observer $\overline{\mathcal{A}}$ located at $\boldsymbol{r}$ in $\mathcal{A}$'s frame looks at a position $\overline{\boldsymbol{a}}$ to see the velocity of the local frame there. $\overline{\mathcal{A}}$ sees $\overline{\boldsymbol{v}}(\overline{\boldsymbol{a}}, \bar{t})$, of course. If $\mathcal{A}$ calls this same position $\boldsymbol{a}$ in his frame, he will identify the velocity of the local frame there as $\boldsymbol{v}(\boldsymbol{a}, t)$.

The Galilean connection between these two observations is

$$\begin{aligned} t &= \bar{t} \\ \boldsymbol{a} &= \overline{\boldsymbol{a}} + \boldsymbol{r} \\ \boldsymbol{v}(\boldsymbol{a}, t) &= \overline{\boldsymbol{v}}(\overline{\boldsymbol{a}}, \bar{t}) + \boldsymbol{v}(\boldsymbol{r}, t) \end{aligned} \tag{1}$$

But the Cosmological principle says that

$$\overline{\boldsymbol{v}}(\overline{\boldsymbol{r}}, \bar{t}) = \boldsymbol{v}(\overline{\boldsymbol{r}}, \bar{t}), \qquad \text{all} \quad \overline{\boldsymbol{r}}, \bar{t}, \tag{2}$$

so (1) collapses to

$$\boldsymbol{v}(\boldsymbol{a}, t) = \boldsymbol{v}(\boldsymbol{a} - \boldsymbol{r}, t) + \boldsymbol{v}(\boldsymbol{r}, t), \tag{3}$$

a functional equation for $\boldsymbol{v}(\boldsymbol{r},t)$ that can only be satisfied (it must be satisfied for *all* $\boldsymbol{r}$ and $\boldsymbol{a}$) with functions $\boldsymbol{v}(\boldsymbol{r},t)$ that vary *linearly* with $\boldsymbol{r}$.

If we extend the cosmological principle to the assertion that there is no preferred direction in the universe, the universe must be isotropic at all points and the only possible form for $\boldsymbol{v}(\boldsymbol{r},t)$ is

$$\boldsymbol{v}(\boldsymbol{r},t) = \mathcal{F}(t)\,\boldsymbol{r} \tag{4}$$

The only possible dynamics is a uniform expansion or contraction of the universe.

C. Hydrodynamics

To go further, we need a dynamical model for the universe. A simple dynamical model is provided by hydrodynamics. Such a model is appropriate when

1) the one-particle distribution function $\rho(\boldsymbol{r},t)$ adequately characterizes the distribution of matter in a system.

   This always holds on observational length scales $\gg$ any correlation length in a system. We assume that there is a coarse-grained view of the universe which satisfies this condition.

2) the local velocity distribution at each $\boldsymbol{r},t$ has a Maxwell-Boltzmann form.

   This always holds after some (usually quite short) relaxation time in a system.

   Such a distribution is completely characterized by the center of the velocity distribution $\boldsymbol{u}(\boldsymbol{r},t)$ (the *average* velocity at $\boldsymbol{r},t$) and the local temperature $T(\boldsymbol{r},t)$ (the width of the distribution).

A system is described hydrodynamically by giving five time-evolving fields, $\rho(\boldsymbol{r},t)$, $T(\boldsymbol{r},t)$ and $\boldsymbol{u}(\boldsymbol{r},t)$.

D. Eulerian Dynamics

To describe the system's dynamics, we need five equations of motion for the five fields $\rho(\boldsymbol{r},t)$, $T(\boldsymbol{r},t)$ and $\boldsymbol{u}(\boldsymbol{r},t)$. The physical basis for these are the conservation laws associated with mass, energy and momentum. If $\rho(\boldsymbol{r},t)$ is taken to be the *mass* density, then the conservation of mass $\Rightarrow$

$$\frac{\partial\rho(\boldsymbol{r},t)}{\partial t} + \nabla\cdot[\rho(\boldsymbol{r},t)\,\boldsymbol{u}(\boldsymbol{r},t)] = 0 \tag{5}$$

The time evolution of $\rho(\boldsymbol{r},t)$ is completely fixed by $\boldsymbol{u}(\boldsymbol{r},t)$.

The *Eulerian* approximation corresponds to neglecting all thermodynamically irreversible processes. Usually, the validity of this approximation grows in time in any particular system. With this approximation, the hydrodynamic rules for the dynamical evolution of a system only depend on the thermodynamic properties of the system. This provides a good description of a system as long as the hydrodynamic velocity is small compared to molecular velocities. While this doesn't describe the universe at all times, it does describe the current state of the universe and forms the basis for a "standard" model of the dynamics of the universe. Corrections are basically treated as perturbations to this model.

In the Eulerian approximation, the only forces present arise from a spatial variation of the pressure and a coupling to some external force field. We assume that there is an equation of state

$$p = p(\rho, T)$$

that applies locally at each $\boldsymbol{r}, t$. Thus p varies with $\boldsymbol{r}, t$ because ρ and T vary with $\boldsymbol{r}, t$. Force balance (the conservation of momentum relation) is

$$\rho(\boldsymbol{r}, t)\left[\frac{\partial \boldsymbol{u}(\boldsymbol{r}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla \boldsymbol{u}(\boldsymbol{r}, t)\right] = \boldsymbol{f}(\boldsymbol{r}, t) - \nabla p(\boldsymbol{r}, t) \tag{6}$$

with $\boldsymbol{f}(\boldsymbol{r}, t)$ the force per unit volume due to the *external* forces acting on the system.

The Eulerian approximation also means that there is no entropy *production* in time. Most simply,

$$\frac{\partial s(\boldsymbol{r}, t)}{\partial t} + \nabla \cdot [\boldsymbol{u}(\boldsymbol{r}, t)\, s(\boldsymbol{r}, t)] = 0 \tag{7}$$

with $s(\boldsymbol{r}, t)$ the entropy per unit mass in the system. Since $s = s(\rho, T)$, (7) provides the fifth dynamical equation needed by hydrodynamics.

E. Cosmological Dynamics

The hydrodynamic field $\boldsymbol{u}(\boldsymbol{r}, t)$ is just the velocity of local frames. Thus, in the cosmological application,

$$\boldsymbol{u}(\boldsymbol{r}, t) = \mathcal{F}(t)\, \boldsymbol{r} \tag{8}$$

We will use hydrodynamics to determine the possible forms for $\mathcal{F}(t)$.

Using (8) in (5) gives

$$\frac{\partial \rho(\boldsymbol{r}, t)}{\partial t} + \mathcal{F}(t)\, \nabla \cdot [\boldsymbol{r}\, \rho(\boldsymbol{r}, t)] = 0$$

If this is viewed as an initial value problem with $\rho(\boldsymbol{r}, t_0)$ prescribed, then the solution is

$$\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}\, e^{-\int_{t_0}^{t} \mathcal{F}(t')\, dt'}, t_0)\, e^{-3\int_{t_0}^{t} \mathcal{F}(t')\, dt'}$$

At the present time, the universe appears to be spatially uniform. If

$$\rho(\boldsymbol{r}, t_0) = \rho_0,$$

then

$$\rho(\boldsymbol{r}, t) = \rho(t) = \rho_0\, e^{-3\int_{t_0}^{t} \mathcal{F}(t')\, dt'} \tag{9}$$

If the density is spatially uniform at some time, it will remain so at all times. The local density changes in time only because of the expansion or contraction of the universe.

If we assume that $s(\boldsymbol{r}, t_0) = s_0$, another uniform value, then (7') gives

$$s(\boldsymbol{r}, t) = s(t) = s_0 \tag{10}$$

The relations (9) and (10) follow from (5) and (7). Since the mass density and entropy density only vary with time,

$$p(\boldsymbol{r}, t) = p(t) \tag{11}$$

and the $\nabla p(\boldsymbol{r}, t)$ term in (6) is zero.

The gravitational force is left out of thermodynamic descriptions of systems because gravity operates on a completely different spatial scale than the forces responsible for the equation of state of matter. It is natural to regard gravity as an *external* force operating on a system. Of course, it isn't really external. In fact it is determined by the $\rho(\boldsymbol{r}, t)$ through

$$\boldsymbol{f}(\boldsymbol{r}, t) = -\rho(\boldsymbol{r}, t)\, \nabla \phi_{grav}(\boldsymbol{r}, t), \qquad \phi_{grav}(\boldsymbol{r}, t) = G \int d\boldsymbol{R}\, \frac{\rho(\boldsymbol{R}, t)}{|\boldsymbol{R} - \boldsymbol{r}|}, \tag{12}$$

with G the gravitational constant. Together, (12) and (6) determine $\boldsymbol{f}(\boldsymbol{r}, t)$ and $\rho(\boldsymbol{r}, t)$.

In the cosmological application, however, $\rho(\boldsymbol{r}, t) \to \rho(t)$ and the integral in (12) over all space doesn't converge. In the cases that the integral does converge, however, (12) implies that

$$\nabla^2 \phi_{grav}(\boldsymbol{r}, t) = -4\pi G\, \rho(\boldsymbol{r}, t) \tag{13}$$

and we assume that (13) continues to apply in the cosmological case. Taking the divergence of (6) with $\rho(\boldsymbol{r},t) = \rho(t)$ and (13) gives

$$\rho(t)\nabla\cdot\left[\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{r},t)\right] = -4\pi G\rho(t)^2$$

Using (8) to express $\boldsymbol{u}(\boldsymbol{r},t)$ in terms of $\mathcal{F}(t)$, leads to

$$\dot{\mathcal{F}}(t) + \mathcal{F}(t)^2 = -\frac{4\pi G}{3}\rho(t) \tag{14}$$

This result together with (9) provides the hydrodynamic constraint on $\mathcal{F}(t)$.

In practice, the universe is characterized by a dimensionless scale factor $\mathcal{R}(t)$. If $\mathcal{R}(t_0) = 1$ describes the present situation, a current distance $\boldsymbol{r}_0$ will become $\mathcal{R}(t)\,\boldsymbol{r}_0$ at time t. If ρ_0 is the current density,

$$\rho(t) = \frac{\rho_0}{\mathcal{R}(t)^3} \tag{15}$$

Comparison with (9) shows that

$$\frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} = \mathcal{F}(t) \tag{16}$$

Interpreting (14) as an equation of motion for $\mathcal{R}(t)$ gives

$$\mathcal{R}(t)^2\,\ddot{\mathcal{R}}(t) + \frac{4\pi G}{3}\rho_0 = 0 \tag{17}$$

The time evolution of the universe is completely determined by the single parameter,

$$\gamma = \frac{4\pi G}{3}\rho_0$$

with dimensions time^{-2} and the initial conditions,

$$\begin{aligned}\mathcal{R}(t_0) &= 1,\\ \dot{\mathcal{R}}(t_0) &= H.\end{aligned}$$

F. The Differential Equation

We want to solve this system of equations,

$$\begin{aligned} \mathcal{R}(t)^2\,\ddot{\mathcal{R}}(t) + \gamma &= 0, \\ \mathcal{R}(t_0) &= 1, \\ \dot{\mathcal{R}}(t_0) &= H. \end{aligned} \tag{18}$$

If $\mathcal{P} = \dot{\mathcal{R}}(t)$ is taken to be a function of $\mathcal{R}$, (18) can be reduced to

$$\begin{aligned} &\mathcal{R}^2\,\mathcal{P}(R)\frac{d\mathcal{P}(R)}{dR} + \gamma = 0, \\ &\mathcal{P}(1) = H, \end{aligned}$$

which can be integrated to

$$\mathcal{P}(R)^2 = H^2 - 2\gamma + \frac{2\gamma}{R} \tag{19}$$

In this relation, $\gamma > 0$ and we can assume $H > 0$ since the $H < 0$ solutions just correspond to running the time of the $H > 0$ solutions backwards.

To get a starting image of the time evolution, suppose that $H^2 = 2\gamma$. In this special case, (19) gives

$$\frac{d\mathcal{R}(t)}{dt} = \sqrt{\frac{2\gamma}{\mathcal{R}(t)}}$$

which can be integrated to

$$\mathcal{R}(t) = \left[1 + \sqrt{\frac{9\gamma}{2}}(t - t_0)\right]^{2/3}$$

At large times, the universe expands with

$$\mathcal{R}(t) \sim \left(\frac{9\gamma}{2}\right)^{1/3} t^{2/3}$$

At a time sufficiently $< t_0$, however, $\mathcal{R}(t) \to 0$. This is the "big bang" and it occurs at a time

$$t_{bb} = t_0 - \sqrt{\frac{2}{9\gamma}}$$

The fact that $\mathcal{R}(t) \to 0$ at early times is present even if $H^2 \neq 2\gamma$. The behavior at long times, however, depends upon the *sign* of $H^2 - 2\gamma$. If $H^2 > 2\gamma$, $\mathcal{R}(t)$ grows without limit, assuming the form

$$\mathcal{R}(t) \sim \sqrt{H^2 - 2\gamma} \quad t$$

at long times. If $H^2 < 2\gamma$, however, continued growth of $\mathcal{R}(t)$ eventually forces $\mathcal{P}(R) \to 0$. $\mathcal{R}(t)$ can not grow larger than

$$\mathcal{R}^* = \frac{2\gamma}{2\gamma - H^2}$$

If $\Delta(t) = \mathcal{R}^* - \mathcal{R}(t)$, this will be small when $\mathcal{R}(t)$ is near this maximum value. In this case

$$\dot{\Delta}(t)^2 = \frac{2\gamma}{\mathcal{R}^{*2}}\Delta(t) + \mathcal{O}(\Delta^2) \tag{20}$$

If $t^\dagger$ is the time at which $\mathcal{R} = \mathcal{R}^*$, (20) can be integrated to

$$\Delta(t) = \frac{\gamma}{2\mathcal{R}^{*2}} \quad (t - t^\dagger)^2 + \mathcal{O}(t - t^\dagger)^3$$

showing that $\mathcal{R}(t)$ approaches $\mathcal{R}^*$ from below and then *turns around.*

The qualitative picture is this:

1) if the universe is currently expanding, we can follow the expansion backwards in time to a singular point (where hydrodynamics, classical mechanics, ..., fails)

2) the future will be continued expansion indefinitely if $H^2 > 2\gamma$ and a reversal of the expansion if $H^2 < 2\gamma$. The $H^2 = 2\gamma$ is a boundary case. One can interpret $H^2 > 2\gamma$ as saying that the kinetic energy of the current expansion exceeds the binding energy of the gravitational field.

G. The Relativistic Case

In fact, the first "equation of motion" for the universe was given shortly after the general theory of relatively was published. The cosmological principle still gives only uniform expansion and contraction as possible.

The Newtonian picture was constructed shortly thereafter in the wake of concerns about whether the universe could be static. Experimental observation was taken to show that the universe was static. Equation (18) cannot have a static solution. But the *form* of the equation of motion for the scale factor à la Newton turns out to be identical to that of the relativistic description.