5-16-2016

A remark on the multipliers on spaces of Weak Products of functions

Stefan Richter

Brett D. Wick
Washington University in St. Louis, bwick@wustl.edu

Follow this and additional works at: https://openscholarship.wustl.edu/math_facpubs

Part of the Analysis Commons

Recommended Citation
Richter, Stefan and Wick, Brett D., "A remark on the multipliers on spaces of Weak Products of functions" (2016). Mathematics Faculty Publications. 33.
https://openscholarship.wustl.edu/math_facpubs/33

This Article is brought to you for free and open access by the Mathematics and Statistics at Washington University Open Scholarship. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of Washington University Open Scholarship. For more information, please contact digital@wumail.wustl.edu.
A remark on the multipliers on spaces of Weak Products of functions

Stefan Richter* and Brett D. Wick

DOI 10.1515/conop-2016-0004
Received October 30, 2015; accepted January 27, 2016.

Abstract: If \( \mathcal{H} \) denotes a Hilbert space of analytic functions on a region \( \Omega \subseteq \mathbb{C}^d \), then the weak product is defined by

\[
\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \| f_n \|_{\mathcal{H}} \| g_n \|_{\mathcal{H}} < \infty \right\}.
\]

We prove that if \( \mathcal{H} \) is a first order holomorphic Besov Hilbert space on the unit ball of \( \mathbb{C}^d \), then the multiplier algebras of \( \mathcal{H} \) and of \( \mathcal{H} \odot \mathcal{H} \) coincide.

Keywords: Dirichlet space, Drury-Arveson space, Weak product, Multiplier

MSC: 47B37

1 Introduction

Let \( d \) be a positive integer and let \( R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i} \) denote the radial derivative operator. For \( s \in \mathbb{R} \) the holomorphic Besov space \( B_s \) is defined to be the space of holomorphic functions \( f \) on the unit ball \( \mathbb{B}_d \) of \( \mathbb{C}^d \) such that for some nonnegative integer \( k > s \)

\[
\| f \|_{k,s}^2 = \int_{\mathbb{B}_d} |(1 + R)^k f(z)|^2 (1 - |z|^2)^{2(k-s)-1} dV(z) < \infty.
\]

Here \( dV \) denotes Lebesgue measure on \( \mathbb{B}_d \). It is well-known that for any \( f \in \text{Hol}(\mathbb{B}_d) \) and any \( s \in \mathbb{R} \) the quantity \( \| f \|_{k,s} \) is finite for some nonnegative integer \( k > s \) if and only if it is finite for all nonnegative integers \( k > s \), and that for each \( k > s \) \( \| \cdot \|_{k,s} \) defines a norm on \( B_s \), and that all these norms are equivalent to one another, see [2]. For \( s < 0 \) one can take \( k = 0 \) and these spaces are weighted Bergman spaces. In particular, \( B_{-1/2} = L^2_{\mathbb{B}_d}(\mathbb{B}_d) \) is the unweighted Bergman space. For \( s = 0 \) one obtains the Hardy space of \( \mathbb{B}_d \) and one has that for each \( k \geq 1 \) \( \| f \|_{k,0}^2 \) is equivalent to \( \int_{\partial \mathbb{B}_d} |f|^2 d\sigma \), where \( \sigma \) is the rotationally invariant probability measure on \( \partial \mathbb{B}_d \). We also note that for \( s = (d - 1)/2 \) we have \( B_s = H^2_d \), the Drury-Arveson space. If \( d = 1 \) and \( s = 1/2 \), then \( B_s = D \), the classical Dirichlet space of the unit disc.

Let \( \mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d) \) be a reproducing kernel Hilbert space such that \( 1 \in \mathcal{H} \). The weak product of \( \mathcal{H} \) is denoted by \( \mathcal{H} \odot \mathcal{H} \) and it is defined to be the collection of all functions \( h \in \text{Hol}(\mathbb{B}_d) \) such that there are sequences \( \{f_i\}_{i \geq 1}, \{g_i\}_{i \geq 1} \subseteq \mathcal{H} \) with \( \sum_{i=1}^{\infty} \| f_i \|_{\mathcal{H}} \| g_i \|_{\mathcal{H}} < \infty \) and for all \( z \in \mathbb{B}_d \), \( h(z) = \sum_{i=1}^{\infty} f_i(z) g_i(z) \).

*Corresponding Author: Stefan Richter: Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA,
E-mail: richter@math.utk.edu

Brett D. Wick: Department of Mathematics, Washington University – St. Louis, St. Louis, MO 63130, USA and School of Mathematics, Georgia Institute of Technology, Atlanta GA 30332-0160, USA, E-mail: wick@math.wustl.edu
We define a norm on $\mathcal{H} \odot \mathcal{H}$ by
\[
\|h\|_s = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_H \|g_i\|_H : h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z) \text{ for all } z \in \mathbb{B}_d \right\}.
\]
In what appears below we will frequently take $\mathcal{H} = B_t$, and will use the same notation for this weak product.

Weak products have their origin in the work of Coifman, Rochberg, and Weiss [5]. In the framework of the Hilbert space $\mathcal{H}$ one may consider the weak product to be an analogue of the Hardy $H^1$-space. For example, one has $H^2(\partial \mathbb{B}_d) \odot H^2(\partial \mathbb{B}_d) = H^1(\partial \mathbb{B}_d)$ and $L^2_1(\mathbb{B}_d) \odot L^2_1(\mathbb{B}_d) = L^1_1(\mathbb{B}_d)$, see [5]. For the Dirichlet space $D$ the weak product $D \circ D$ has recently been considered in [1, 3, 6, 7, 9]. The space $H^2_0 \odot H^2$ was used in [10]. For further motivation and general background on weak products we refer the reader to [1] and [9].

Let $B$ be a Banach space of analytic functions on $\mathbb{B}_d$ such that point evaluations are continuous and such that $1 \in B$. We use $M(B)$ to denote the multiplier algebra of $B$,
\[
M(B) = \{ \psi : \varphi f \in B \text{ for all } \varphi \in B \}.
\]
The multiplier norm $\|\varphi\|_M$ is defined to be the norm of the associated multiplication operator $M_\varphi : B \to B$. It is easy to check that for all $s \leq 0$ we have $M(B_s) = H^\infty(\mathbb{B}_d)$. For $s > d/2$ the space $B_s$ is an algebra [2], hence $B_s = M(B_s)$, and for $0 < s < d/2$ one has $M(B_s) \subset B_s \cap H^\infty(\partial \mathbb{B}_d)$. For those cases $M(B_s)$ has been described by a certain Carleson measure condition, see [4, 8].

It is easy to see that $M(H) \subset M(H \odot H) \subset H^\infty$ (see Proposition 3.1). Thus, if $s \leq 0$, then $M(B_s) = M(B_s \odot B_s) = H^\infty$. Furthermore, if $s > d/2$, then $B_s = B_s \odot B_s = M(B_s)$ since $B_s$ is an algebra. This raises the question whether $M(B_s)$ and $M(B_s \odot B_s)$ always agree. We prove the following:

**Theorem 1.1.** Let $s \in \mathbb{R}$ and $d \in \mathbb{N}$. If $s \leq 1$ or $d \leq 2$, then $M(B_s) = M(B_s \odot B_s)$.

Note that when $d \leq 2$, then $B_s$ is an algebra for all $s > 1$. Thus for each $d \in \mathbb{N}$ the nontrivial range of the Theorem is $0 < s \leq 1$. If $d = 1$ then the theorem applies to the classical Dirichlet space of the unit disc and for $d < 3$ it applies to the Drury-Arveson space.

## 2 Preliminaries

For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and $t \in \mathbb{R}$ we write $e^{it}z = (e^{it}z_1, \ldots, e^{it}z_d)$ and we write $(z, w)$ for the inner product in $\mathbb{C}^d$. Furthermore, if $h$ is a function on $\mathbb{B}_d$, then we define $T_t f$ by $(T_t f)(z) = f(e^{it}z)$. We say that a space $\mathcal{H} \subset \text{Hol}(\mathbb{B}_d)$ is radially symmetric, if each $T_t$ acts isometrically on $\mathcal{H}$ and for all $t_0 \in \mathbb{R}$, $T_t \to T_{t_0}$ in the strong operator topology as $t \to t_0$, i.e. if $\|T_t f\|_H = \|f\|_H$ and $\|T_t f - T_{t_0} f\|_H \to 0$ for all $f \in \mathcal{H}$. For example, for each $s \in \mathbb{R}$ the holomorphic Besov space $B_s$ is radially symmetric when equipped with any of the norms $\| \cdot \|_{k,s}$, $k > s$.

It is elementary to verify the following lemma.

**Lemma 2.1.** If $\mathcal{H} \subset \text{Hol}(\mathbb{B}_d)$ is radially symmetric, then so is $\mathcal{H} \odot \mathcal{H}$.

Note that if $h$ and $\varphi$ are functions on $\mathbb{B}_d$, then for every $t \in \mathbb{R}$ we have $(T_t \varphi)h = T_t(\varphi T_{-t} h)$, hence if a space is radially symmetric, then $T_t$ acts isometrically on the multiplier algebra. For $0 < r < 1$ we write $f_r(z) = f(r z)$.

**Lemma 2.2.** If $\mathcal{H} \subset \text{Hol}(\mathbb{B}_d)$ is radially symmetric, and if $\varphi \in M(\mathcal{H} \odot \mathcal{H})$, then for all $0 < r < 1$ we have $\|\varphi_r\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})}$.

**Proof.** Let $\varphi \in M(\mathcal{H} \odot \mathcal{H})$ and $h \in \mathcal{H} \odot \mathcal{H}$, then for $0 < r < 1$ we have
\[
\varphi_r h = \int_0^{\pi} \frac{1 - r^2}{|1 - r e^{it}|^2} (T_t \varphi) h \frac{dt}{2\pi}.
\]
This implies
\[ \|\varphi_r h\|_* \leq \frac{\pi}{2} \frac{1 - r^2}{|1 - re^{it}|^2} \| (T_{\varphi} h) \|_* \leq \frac{\pi}{2} \| \varphi \|_{M(\mathcal{H} \otimes \mathcal{H})} \| h \|_* . \]

Thus, \[ \| \varphi_r \|_{M(\mathcal{H} \otimes \mathcal{H})} \leq \| \varphi \|_{M(\mathcal{H} \otimes \mathcal{H})} . \]

3 Multipliers

The following Proposition is elementary.

**Proposition 3.1.** We have \( M(\mathcal{H}) \subseteq M(\mathcal{H} \otimes \mathcal{H}) \subseteq H^\infty \) and if \( \varphi \in M(\mathcal{H}) \), \( \| \varphi \|_{M(\mathcal{H} \otimes \mathcal{H})} \leq \| \varphi \|_{M(\mathcal{H})} \).

As explained in the Introduction, the following will establish Theorem 1.1.

**Theorem 3.2.** Let \( 0 < s \leq 1 \). Then \( M(B_s) = M(B_s \ominus B_s) \) and there is a \( C_s > 0 \) such that
\[ \| \varphi \|_{M(B_s \ominus B_s)} \leq \| \varphi \|_{M(B_s)} \leq C_s \| \varphi \|_{M(B_s \ominus B_s)} \]
for all \( \varphi \in M(B_s) \).

Here for each \( s \) we have the norm on \( B_s \) to be \( \| \cdot \|_{k,s} \), where \( k \) is the smallest natural number \( > s \).

**Proof.** We first do the case \( 0 < s < 1 \). Then \( k = 1 \), and \( \| f \|_{B_1}^2 = \int_{B_1} |(I + R)f(z)|^2 dV(z) \), where \( dV(z) = (1 - |z|^2)^{1-2s} dV(z) \). For later reference we note that a short calculation shows that \( \int_{B_1} |Rf|^2 dV \leq \| f \|_{B_1}^2 \).

We write \( \| R\varphi \|_{C_{a}(B_s)} \) for the Carleson measure norm of \( |R\varphi|^2 \), i.e.
\[ \| R\varphi \|_{C_{a}(B_s)}^2 = \inf \left\{ C > 0 : \int_{B_1} |f|^2 |R\varphi|^2 dV \leq C \| f \|_{B_1}^2 \right\} . \]

Since \( \| f \|_{B_1}^2 = \int_{B_1} |\varphi(z)(I + R)f(z) + f(z)R\varphi(z)|^2 dV(z) \) it is clear that \( \| \varphi \|_{M(B_s)} \) is equivalent to \( \| \varphi \|_{\infty} + \| R\varphi \|_{C_{a}(B_s)} \). Thus, it suffices to show that there is a \( C > 0 \) such that \( \| R\varphi \|_{C_{a}(B_s)} \leq C \| \varphi \|_{M(B_s \ominus B_s)} \) for all \( \varphi \in M(B_s \ominus B_s) \).

First we note that if \( b \) is holomorphic in a neighborhood of \( \overline{B}_d \) and \( h = \sum_{i=1}^{\infty} f_i g_i \in B_s \ominus B_s \), then
\[ \int_{\overline{B}_d} |(Rh)b| dV \leq \sum_{i=1}^{\infty} \int_{\overline{B}_d} |(R(f_i)g_i)b| dV + \int_{\overline{B}_d} |(Rg_i)f_i b| dV \]
\[ \leq \sum_{i=1}^{\infty} \| f_i \|_{B_s} \left( \int_{\overline{B}_d} |g_i b|^2 dV \right)^{1/2} + \| g_i \|_{B_s} \left( \int_{\overline{B}_d} |f_i b|^2 dV \right)^{1/2} \]
\[ \leq 2 \sum_{i=1}^{\infty} \| f_i \|_{B_s} \| g_i \|_{B_s} \| Rh \|_{C_{a}(B_s)} . \]

Hence
\[ \int_{\overline{B}_d} |(Rh)b| dV \leq 2 \| h \|_{*} \| Rh \|_{C_{a}(B_s)} , \]
where we have continued to write \( \| \cdot \|_s \) for \( \| \cdot \|_{B_s \ominus B_s} \).

Let \( \varphi \in M(B_s \ominus B_s) \) and let \( 0 < r < 1 \). Then for all \( f \in B_s \) we have \( f^2, \varphi_r f^2 \in B_s \ominus B_s \), hence
\[ \int_{\overline{B}_d} |f|^2 |R\varphi_r|^2 dV = \int_{\overline{B}_d} |R(\varphi_r f^2) - \varphi_r R(f^2)| |R\varphi_r| dV . \]
Next we take the sup of the left hand side of this expression over all $f$ with $\|f\|_{B_r} = 1$ and we obtain $\|R\varphi_R\|^2_{C_{a}(B_1)} \leq 4\|\varphi\|_{M(B_r)B_1} \|f\|_{B_1}^2 \|R\varphi_R\|_{C_{a}(B_1)}$ which implies that $\|R\varphi_R\|_{C_{a}(B_1)} \leq 4\|\varphi\|_{M(B_r)B_1}$ holds for all $0 < r < 1$. Thus, for $0 < s < 1$ the result follows from Fatou’s lemma as $r \to 1$.

If $s = 1$, then $\|f\|_{B_1}^2 \sim \int_{B_1} (I + R)(z) \|f(z)\|^2 d\sigma(z)$ and the argument proceeds as above.

Acknowledgement: B.D. Wick’s research supported in part by National Science Foundation DMS grant #1603246 and #1560955.

This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program “Investissements d’Avenir” (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

References