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# A remark on the multipliers on spaces of Weak Products of functions

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**Abstract:** If  $\mathcal{H}$  denotes a Hilbert space of analytic functions on a region  $\Omega \subseteq \mathbb{C}^d$ , then the weak product is defined by

$$\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{H}} \|g_n\|_{\mathcal{H}} < \infty \right\}.$$

We prove that if  $\mathcal{H}$  is a first order holomorphic Besov Hilbert space on the unit ball of  $\mathbb{C}^d$ , then the multiplier algebras of  $\mathcal{H}$  and of  $\mathcal{H} \odot \mathcal{H}$  coincide.

**Keywords:** Dirichlet space, Drury-Arveson space, Weak product, Multiplier

**MSC:** 47B37

## 1 Introduction

Let  $d$  be a positive integer and let  $R = \sum_{i=1}^d z_i \frac{\partial}{\partial z_i}$  denote the radial derivative operator. For  $s \in \mathbb{R}$  the holomorphic Besov space  $B_s$  is defined to be the space of holomorphic functions  $f$  on the unit ball  $\mathbb{B}_d$  of  $\mathbb{C}^d$  such that for some nonnegative integer  $k > s$

$$\|f\|_{k,s}^2 = \int_{\mathbb{B}_d} |(I + R)^k f(z)|^2 (1 - |z|^2)^{2(k-s)-1} dV(z) < \infty.$$

Here  $dV$  denotes Lebesgue measure on  $\mathbb{B}_d$ . It is well-known that for any  $f \in \text{Hol}(\mathbb{B}_d)$  and any  $s \in \mathbb{R}$  the quantity  $\|f\|_{k,s}$  is finite for some nonnegative integer  $k > s$  if and only if it is finite for all nonnegative integers  $k > s$ , and that for each  $k > s$   $\|\cdot\|_{k,s}$  defines a norm on  $B_s$ , and that all these norms are equivalent to one another, see [2]. For  $s < 0$  one can take  $k = 0$  and these spaces are weighted Bergman spaces. In particular,  $B_{-1/2} = L_a^2(\mathbb{B}_d)$  is the unweighted Bergman space. For  $s = 0$  one obtains the Hardy space of  $\mathbb{B}_d$  and one has that for each  $k \geq 1$   $\|f\|_{k,0}^2$  is equivalent to  $\int_{\partial\mathbb{B}_d} |f|^2 d\sigma$ , where  $\sigma$  is the rotationally invariant probability measure on  $\partial\mathbb{B}_d$ . We also note that for  $s = (d-1)/2$  we have  $B_s = H_d^2$ , the Drury-Arveson space. If  $d = 1$  and  $s = 1/2$ , then  $B_s = D$ , the classical Dirichlet space of the unit disc.

Let  $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$  be a reproducing kernel Hilbert space such that  $1 \in \mathcal{H}$ . The weak product of  $\mathcal{H}$  is denoted by  $\mathcal{H} \odot \mathcal{H}$  and it is defined to be the collection of all functions  $h \in \text{Hol}(\mathbb{B}_d)$  such that there are sequences  $\{f_i\}_{i \geq 1}, \{g_i\}_{i \geq 1} \subseteq \mathcal{H}$  with  $\sum_{i=1}^{\infty} \|f_i\|_{\mathcal{H}} \|g_i\|_{\mathcal{H}} < \infty$  and for all  $z \in \mathbb{B}_d$ ,  $h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z)$ .

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We define a norm on  $\mathcal{H} \odot \mathcal{H}$  by

$$\|h\|_* = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_{\mathcal{H}} \|g_i\|_{\mathcal{H}} : h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z) \text{ for all } z \in \mathbb{B}_d \right\}.$$

In what appears below we will frequently take  $\mathcal{H} = B_s$ , and will use the same notation for this weak product.

Weak products have their origin in the work of Coifman, Rochberg, and Weiss [5]. In the frame work of the Hilbert space  $\mathcal{H}$  one may consider the weak product to be an analogue of the Hardy  $H^1$ -space. For example, one has  $H^2(\partial\mathbb{B}_d) \odot H^2(\partial\mathbb{B}_d) = H^1(\partial\mathbb{B}_d)$  and  $L_a^2(\mathbb{B}_d) \odot L_a^2(\mathbb{B}_d) = L_a^1(\mathbb{B}_d)$ , see [5]. For the Dirichlet space  $D$  the weak product  $D \odot D$  has recently been considered in [1, 3, 6, 7, 9]. The space  $H_d^2 \odot H_d^2$  was used in [10]. For further motivation and general background on weak products we refer the reader to [1] and [9].

Let  $\mathcal{B}$  be a Banach space of analytic functions on  $\mathbb{B}_d$  such that point evaluations are continuous and such that  $1 \in \mathcal{B}$ . We use  $M(\mathcal{B})$  to denote the multiplier algebra of  $\mathcal{B}$ ,

$$M(\mathcal{B}) = \{\varphi : \varphi f \in \mathcal{B} \text{ for all } f \in \mathcal{B}\}.$$

The multiplier norm  $\|\varphi\|_M$  is defined to be the norm of the associated multiplication operator  $M_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ . It is easy to check and is well-known that  $M(\mathcal{B}) \subseteq H^\infty(\mathbb{B}_d)$ , and that for  $s \leq 0$  we have  $M(B_s) = H^\infty(\mathbb{B}_d)$ . For  $s > d/2$  the space  $B_s$  is an algebra [2], hence  $B_s = M(B_s)$ , but for  $0 < s \leq d/2$  one has  $M(B_s) \subsetneq B_s \cap H^\infty(\partial\mathbb{B}_d)$ . For those cases  $M(B_s)$  has been described by a certain Carleson measure condition, see [4, 8].

It is easy to see that  $M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq H^\infty$  (see Proposition 3.1). Thus, if  $s \leq 0$ , then  $M(B_s) = M(B_s \odot B_s) = H^\infty$ . Furthermore, if  $s > d/2$ , then  $B_s = B_s \odot B_s = M(B_s)$  since  $B_s$  is an algebra. This raises the question whether  $M(B_s)$  and  $M(B_s \odot B_s)$  always agree. We prove the following:

**Theorem 1.1.** *Let  $s \in \mathbb{R}$  and  $d \in \mathbb{N}$ . If  $s \leq 1$  or  $d \leq 2$ , then  $M(B_s) = M(B_s \odot B_s)$ .*

Note that when  $d \leq 2$ , then  $B_s$  is an algebra for all  $s > 1$ . Thus for each  $d \in \mathbb{N}$  the nontrivial range of the Theorem is  $0 < s \leq 1$ . If  $d = 1$  then the theorem applies to the classical Dirichlet space of the unit disc and for  $d \leq 3$  it applies to the Drury-Arveson space.

## 2 Preliminaries

For  $z = (z_1, \dots, z_d) \in \mathbb{C}^d$  and  $t \in \mathbb{R}$  we write  $e^{it}z = (e^{it}z_1, \dots, e^{it}z_d)$  and we write  $\langle z, w \rangle$  for the inner product in  $\mathbb{C}^d$ . Furthermore, if  $h$  is a function on  $\mathbb{B}_d$ , then we define  $T_t f$  by  $(T_t f)(z) = f(e^{it}z)$ . We say that a space  $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$  is radially symmetric, if each  $T_t$  acts isometrically on  $\mathcal{H}$  and if for all  $t_0 \in \mathbb{R}$ ,  $T_t \rightarrow T_{t_0}$  in the strong operator topology as  $t \rightarrow t_0$ , i.e. if  $\|T_t f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$  and  $\|T_t f - T_{t_0} f\|_{\mathcal{H}} \rightarrow 0$  for all  $f \in \mathcal{H}$ . For example, for each  $s \in \mathbb{R}$  the holomorphic Besov space  $B_s$  is radially symmetric when equipped with any of the norms  $\|\cdot\|_{k,s}$ ,  $k > s$ .

It is elementary to verify the following lemma.

**Lemma 2.1.** *If  $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$  is radially symmetric, then so is  $\mathcal{H} \odot \mathcal{H}$ .*

Note that if  $h$  and  $\varphi$  are functions on  $\mathbb{B}_d$ , then for every  $t \in \mathbb{R}$  we have  $(T_t \varphi)h = T_t(\varphi T_{-t} h)$ , hence if a space is radially symmetric, then  $T_t$  acts isometrically on the multiplier algebra. For  $0 < r < 1$  we write  $f_r(z) = f(rz)$ .

**Lemma 2.2.** *If  $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$  is radially symmetric, and if  $\varphi \in M(\mathcal{H} \odot \mathcal{H})$ , then for all  $0 < r < 1$  we have  $\|\varphi_r\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})}$ .*

*Proof.* Let  $\varphi \in M(\mathcal{H} \odot \mathcal{H})$  and  $h \in \mathcal{H} \odot \mathcal{H}$ , then for  $0 < r < 1$  we have

$$\varphi_r h = \int_{-\pi}^{\pi} \frac{1-r^2}{|1-re^{it}|^2} (T_t \varphi) h \frac{dt}{2\pi}.$$

This implies

$$\|\varphi_r h\|_* \leq \int_{-\pi}^{\pi} \frac{1-r^2}{|1-re^{it}|^2} \|(T_t \varphi)h\|_* \frac{dt}{2\pi} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \|h\|_*.$$

Thus,  $\|\varphi_r\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})}$ . □

### 3 Multipliers

The following Proposition is elementary.

**Proposition 3.1.** *We have  $M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq H^\infty$  and if  $\varphi \in M(\mathcal{H})$ ,  $\|\varphi\|_{M(\mathcal{H} \odot \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H})}$ .*

As explained in the Introduction, the following will establish Theorem 1.1.

**Theorem 3.2.** *Let  $0 < s \leq 1$ . Then  $M(B_s) = M(B_s \odot B_s)$  and there is a  $C_s > 0$  such that*

$$\|\varphi\|_{M(B_s \odot B_s)} \leq \|\varphi\|_{M(B_s)} \leq C_s \|\varphi\|_{M(B_s \odot B_s)}$$

for all  $\varphi \in M(B_s)$ .

Here for each  $s$  we have the norm on  $B_s$  to be  $\|\cdot\|_{k,s}$ , where  $k$  is the smallest natural number  $> s$ .

*Proof.* We first do the case  $0 < s < 1$ . Then  $k = 1$ , and  $\|f\|_{B_s}^2 = \int_{\mathbb{B}_d} |(I + R)f(z)|^2 dV_s(z)$ , where  $dV_s(z) = (1 - |z|^2)^{1-2s} dV(z)$ . For later reference we note that a short calculation shows that  $\int_{\mathbb{B}_d} |Rf|^2 dV_s \leq \|f\|_{B_s}^2$ .

We write  $\|R\varphi\|_{C\alpha(B_s)}$  for the Carleson measure norm of  $|R\varphi|^2$ , i.e.

$$\|R\varphi\|_{C\alpha(B_s)}^2 = \inf \left\{ C > 0 : \int_{\mathbb{B}_d} |f|^2 |R\varphi|^2 dV_s \leq C \|f\|_{B_s}^2 \text{ for all } f \in B_s \right\}.$$

Since  $\|\varphi f\|_{B_s}^2 = \int_{\mathbb{B}_d} |\varphi(z)(I + R)f(z) + f(z)R\varphi(z)|^2 dV_s(z)$  it is clear that  $\|\varphi\|_{M(B_s)}$  is equivalent to  $\|\varphi\|_\infty + \|R\varphi\|_{C\alpha(B_s)}$ . Thus, it suffices to show that there is a  $c > 0$  such that  $\|R\varphi\|_{C\alpha(B_s)} \leq c \|\varphi\|_{M(B_s \odot B_s)}$  for all  $\varphi \in M(B_s \odot B_s)$ .

First we note that if  $b$  is holomorphic in a neighborhood of  $\overline{\mathbb{B}_d}$  and  $h = \sum_{i=1}^\infty f_i g_i \in B_s \odot B_s$ , then

$$\begin{aligned} \int_{\mathbb{B}_d} |(Rh)Rb| dV_s &\leq \sum_{i=1}^\infty \int_{\mathbb{B}_d} |(Rf_i)g_i Rb| dV_s + \int_{\mathbb{B}_d} |(Rg_i)f_i Rb| dV_s \\ &\leq \sum_{i=1}^\infty \|f_i\|_{B_s} \left( \int_{\mathbb{B}_d} |g_i Rb|^2 dV_s \right)^{1/2} + \|g_i\|_{B_s} \left( \int_{\mathbb{B}_d} |f_i Rb|^2 dV_s \right)^{1/2} \\ &\leq 2 \sum_{i=1}^\infty \|f_i\|_{B_s} \|g_i\|_{B_s} \|Rb\|_{C\alpha(B_s)}. \end{aligned}$$

Hence

$$\int_{\mathbb{B}_d} |(Rh)Rb| dV_s \leq 2 \|h\|_* \|Rb\|_{C\alpha(B_s)},$$

where we have continued to write  $\|\cdot\|_*$  for  $\|\cdot\|_{B_s \odot B_s}$ .

Let  $\varphi \in M(B_s \odot B_s)$  and let  $0 < r < 1$ . Then for all  $f \in B_s$  we have  $f^2, \varphi_r f^2 \in B_s \odot B_s$ , hence

$$\int_{\mathbb{B}_d} |f|^2 |R\varphi_r|^2 dV_s = \int_{\mathbb{B}_d} |R(\varphi_r f^2) - \varphi_r R(f^2)| |R\varphi_r| dV_s$$

$$\begin{aligned}
&\leq 2(\|\varphi_r f^2\|_* + \|\varphi\|_\infty \|f^2\|_*) \|R\varphi_r\|_{Ca(B_s)} \\
&\leq 2(\|\varphi\|_{M(B_s \odot B_s)} \|f^2\|_* + \|\varphi\|_\infty \|f^2\|_*) \|R\varphi_r\|_{Ca(B_s)} \\
&\leq 4\|\varphi\|_{M(B_s \odot B_s)} \|f\|_{B_s}^2 \|R\varphi_r\|_{Ca(B_s)}.
\end{aligned}$$

Next we take the sup of the left hand side of this expression over all  $f$  with  $\|f\|_{B_s} = 1$  and we obtain  $\|R\varphi_r\|_{Ca(B_s)}^2 \leq 4\|\varphi\|_{M(B_s \odot B_s)} \|R\varphi_r\|_{Ca(B_s)}$  which implies that  $\|R\varphi_r\|_{Ca(B_s)} \leq 4\|\varphi\|_{M(B_s \odot B_s)}$  holds for all  $0 < r < 1$ . Thus, for  $0 < s < 1$  the result follows from Fatou's lemma as  $r \rightarrow 1$ .

If  $s = 1$ , then  $\|f\|_{2,1}^2 \sim \int_{\partial\mathbb{B}_d} |(I + R)f(z)|^2 d\sigma(z)$  and the argument proceeds as above.  $\square$

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