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# UNIONS OF LEBESGUE SPACES AND $A_1$ MAJORANTS

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**We study two questions. When does a function belong to the union of Lebesgue spaces, and when does a function have an $A_1$ majorant? We provide a systematic study of these questions and show that they are fundamentally related. We show that the union of $L^p_w(\mathbb{R}^n)$ spaces with $w \in A_p$ is equal to the union of all Banach function spaces for which the Hardy–Littlewood maximal function is bounded on the space itself and its associate space.**

## 1. Introduction and statement of the main results

While the $L^p$ spaces are considered fundamental spaces of interest in analysis, the weighted $L^p$ spaces and the related study of $A_p$ weights are perhaps part of a more specialized area of analysis. It is the goal of this article to show that the $L^p$ spaces considered in aggregate are intimately linked to these latter topics and to the notion of an $A_1$ majorant. By recent developments our results indicate that weighted Lebesgue spaces with $A_p$ weights may be good candidates for ambient spaces for operators in harmonic analysis.

We begin with the following question.

**Question 1.1.** When does a function belong to the union of $L^p$ spaces?

Question 1.1 is vaguely stated on purpose. By union, we mean either the union of $L^p$ as $p$ varies or the union of $L^p_w$ as $w$ varies with $p$ fixed. The union of $L^p$ spaces often arises when considering a general domain to define operators in harmonic analysis. Several such operators are bounded on $L^p$ for all $1 < p < \infty$, and hence take functions from $\bigcup_{p>1} L^p$ into itself.

It turns out Question 1.1 is closely related to the theory of weighted Lebesgue spaces and the action of the Hardy–Littlewood maximal operator on these spaces.

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For our purposes, a weight is a positive locally integrable function. An $A_1$ weight is one that satisfies

$$Mw \le Cw \quad \text{a.e.}$$

Here $M$ denotes the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|\,dx.$$

We exclude the weight $w \equiv 0$ from belonging to $A_1$, and in this case we see that if $w \in A_1$ then $w > 0$ a.e. The $A_1$ class of weights characterizes when $M$ maps $L^1_w$ into $L^{1,\infty}_w$. When $1 < p < \infty$, $M$ is bounded on $L^p_w$ exactly when $w \in A_p$:

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for all cubes $Q$. At the other endpoint the $A_\infty$ class is defined to be the union of all $A_p$ for $p \ge 1$. We now come to our second question.

**Question 1.2.** Given a measurable function $f$, when does there exist an $A_1$ weight $w$ such that

(1) $$|f| \le w?$$

We call a weight satisfying (1) an $A_1$ majorant of $f$ and write $\mathcal{M}_{A_1}$ for the set of measurable functions possessing an $A_1$ majorant. As stated, Question 1.2 does not seem to have been considered before. As far as we can tell, the first notion of an $A_1$ majorant appeared in an article by Rutsky [2011]. In Rutsky's paper, however, a different definition of an $A_1$ majorant is given — one which requires the function and the weight to a priori belong to a more restrictive class of functions.

If we examine weights locally, say on the interval [0, 1], then our problem has a remarkably simple answer which reveals a close connection between traditional $L^p$ spaces, weighted $L^p$ spaces, and $A_1$ majorants:

(2) $$\mathcal{M}_{A_1}([0,1]) = \bigcup_{p>1} L^p([0,1]) = \bigcup_{w \in A_2} L^2_w([0,1]).$$

The proof of (2) is a synthesis of known important results for Muckenhoupt weights. This equivalence reinforces the saying attributed to Antonio Córdoba, "There are no $L^p$ spaces, only weighted $L^2$ spaces."

The local theory has several extensions including an application to Hardy spaces on the unit disk. In [McCarthy 1990], while studying the range of Toeplitz operators, the second author showed that the Smirnov class, $N^+$, can be realized as a union of weighted Hardy spaces:

$$N^+ = \bigcup_{w \in \mathcal{W}} H^2_w$$

where $\mathscr{W}$ is the Szegő class of weights (see Section 2 for relevant definitions). The class $A_\infty(\mathbb{T})$ is a proper subset of $\mathscr{W}$ (as $\bigcup_{p>0} H^p$ is a proper subspace of $N^+$). Using our techniques we are able to give a characterization of $\bigcup_{p>0} H^p$ in terms of weighted $H^2$ spaces:

$$\bigcup_{p>0} H^p = \bigcup_{w\in A_\infty} H^2_w. \tag{3}$$

We refer the reader to Section 4 for more on the local case.

For functions on $\mathbb{R}^n$, the theory is not as nice. In the local case the $L^p([0,1])$ spaces are nested in $p$, whereas the $L^p(\mathbb{R}^n)$ spaces are not. We are not able to obtain equality of $\bigcup_{p>1} L^p(\mathbb{R}^n)$ and $\mathcal{M}_{A_1}(\mathbb{R}^n)$. Remarkably, even the much larger union over weak-$L^p(\mathbb{R}^n)$ spaces is not equal to $\mathcal{M}_{A_1}(\mathbb{R}^n)$. As a consequence of our results, if $p_0$ is any exponent satisfying $1 < p_0 < \infty$ then

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subsetneq \bigcup_{w\in A_{p_0}} L^{p_0}_w(\mathbb{R}^n) \subsetneq \mathcal{M}_{A_1}(\mathbb{R}^n). \tag{4}$$

The class $\mathcal{M}_{A_1}(\mathbb{R}^n)$ can be thought of as a generalization of $L^\infty(\mathbb{R}^n)$ — i.e., functions that are majorized by constants, which are $A_1$ weights — while $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ is a generalization of $L^1(\mathbb{R}^n)$. With this in mind we obtain the following theorem.

**Theorem 1.3.** *Suppose* $1 < p < \infty$. *Then*

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \left( \bigcup_{w\in A_1} L^1_w(\mathbb{R}^n) \right).$$

Considering the basic fact

$$L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset \bigcap_{1<p<\infty} L^p(\mathbb{R}^n),$$

Theorem 1.3 shows that if we enlarge both $L^\infty(\mathbb{R}^n)$ to $\mathcal{M}_{A_1}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ to $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ and intersect the two, then we pick up an even bigger class of functions, one that by (4) properly contains the union of all $L^p(\mathbb{R}^n)$ for $p > 1$. As a consequence to Theorem 1.3, we see that for all $1 < p, q < \infty$,

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) = \bigcup_{u\in A_q} L^q_u(\mathbb{R}^n).$$

The proof of Theorem 1.3 uses the extrapolation theory of Rubio de Francia [1984; 1987] (see also the book [Cruz-Uribe et al. 2011]).

The union $\bigcup_{p>1} L^p$ is a good candidate for a natural collection of functions on which to iterate the Hardy–Littlewood maximal function. Rutsky [2014, Theorem 1] showed that Banach function spaces $\mathscr{X}$ on $\mathbb{R}^n$ (see Section 2) for which the Hardy–Littlewood maximal function is bounded on both the space $\mathscr{X}$ and the associate space $\mathscr{X}'$ act as a natural domain for the set of all Calderón–Zygmund operators. We end the introduction with our main result which says a function belongs to a function space $\mathscr{X}$ for which the Hardy–Littlewood maximal function is bounded on $\mathscr{X}$ and $\mathscr{X}'$ if and only if $f \in L^p_w(\mathbb{R}^n)$ for some $p > 1$ and $w \in A_p(\mathbb{R}^n)$.

**Theorem 1.4.** *Suppose* $1 < p < \infty$. *Then*

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \bigcup \{\mathscr{X} : M \in \mathscr{B}(\mathscr{X}) \cap \mathscr{B}(\mathscr{X}')\},$$

*where the second union is over all Banach function spaces such that the Hardy–Littlewood maximal operator is bounded on* $\mathscr{X}$ *and* $\mathscr{X}'$.

Banach function spaces for which $M \in \mathscr{B}(\mathscr{X}) \cap \mathscr{B}(\mathscr{X}')$ are also related to the Fefferman–Stein inequality. Define the sharp maximal function $M^\#$ by

$$M^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx,$$

where $f_Q = \frac{1}{|Q|} \int_Q f \, dx$. Lerner [2010] proved that if $M \in \mathscr{B}(\mathscr{X})$, then the Fefferman–Stein inequality

$$\|f\|_{\mathscr{X}} \le c \|M^\# f\|_{\mathscr{X}} \tag{5}$$

holds for all nice functions in $\mathscr{X}$ if and only if $M \in \mathscr{B}(\mathscr{X}')$. In particular, Theorem 1.4 shows that if $f$ belongs to a Banach function space for which $M \in \mathscr{B}(\mathscr{X})$ and the Fefferman–Stein inequality (5) holds on $\mathscr{X}$, then for any $1 < p < \infty$, there exists $w \in A_p$ for which $f \in L^p_w(\mathbb{R}^n)$.

The outline of this paper is as follows. In Section 2 we state preliminary results that are necessary for the rest of the paper. In Section 3 we study the classes of functions with $A_1$ and $A_p$ majorants. In Section 4 we give a treatise of local theory with applications to Hardy spaces on the unit disk. Section 5 is devoted to the theory on $\mathbb{R}^n$, in particular the proofs of Theorems 1.3 and 1.4. We finish the article with some open questions in Section 6.

## 2. Preliminaries

In this section, $\Omega$ denotes either $\mathbb{R}^n$ or a cube $Q$ with sides parallel to the coordinate planes in $\mathbb{R}^n$. For $0 < p < \infty$, $L^p(\Omega)$ is the set of measurable functions such that

$$\|f\|^p_{L^p} = \int_\Omega |f|^p \, dx < \infty.$$

Given $p$ with $1 \le p \le \infty$, we use $p'$ to denote the dual exponent defined by the equation $1/p + 1/p' = 1$. A weight defined on a cube $Q$ is a positive function in $L^1(Q)$. A weight on $\mathbb{R}^n$ is a positive function in $L^1_{\text{loc}}(\mathbb{R}^n)$. Given a weight, $w$, define $L^p_w(\Omega)$ to be the collection of functions satisfying

$$\|f\|^p_{L^p_w} = \int_\Omega |f|^p w\, dx < \infty.$$

We define $L^\infty_w(\Omega)$ to be the space of functions for which $f/w \in L^\infty(\Omega)$. This space is normed by

$$\|f\|_{L^\infty_w} = \|f/w\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \frac{|f(x)|}{w(x)}.$$

If $\mathbb{T}$ is the unit circle in the complex plane, then $L^p(\mathbb{T})$ and $L^p_w(\mathbb{T})$ are identified as the space of $2\pi$ periodic functions that belong to $L^p([0, 2\pi])$ and $L^p_w([0, 2\pi])$, respectively.

We also examine the "complex analyst's Hardy space", as opposed to the real analyst's Hardy space defined in terms of maximal functions. Let $\mathbb{D}$ denote the unit disk in the plane with boundary $\mathbb{T}$. Given $p$ with $0 < p < \infty$, let $H^p = H^p(\mathbb{D})$ be the space of analytic functions "normed" by

$$\|f\|_{H^p} = \sup_{0<r<1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

"Norm" is in quotes since this is not a norm for $0 < p < 1$, but we use norm notation $\|\cdot\|$ nonetheless. The Nevanlinna class, denoted $N$, is the collection of analytic functions on $\mathbb{D}$ such that

$$\|f\|_N = \sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty.$$

Functions in $N$ have nontangential limits almost everywhere on the boundary, so we may treat them as functions on the disk or the circle. The Smirnov class $N^+$ consists of functions $f \in N$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f(e^{i\theta})| \frac{d\theta}{2\pi}.$$

It is well known that

$$\bigcup_{p>0} H^p \subsetneqq N^+ \subsetneqq N$$

(see, e.g., the books by Duren [1970] or Rudin [1964]). The Smirnov class is often considered a natural limit of $H^p$ as $p \to 0$.

The weighted Hardy space $H^p_w = H^p_w(\mathbb{D})$ is the closure of analytic polynomials in $L^p_w(\mathbb{T})$. While there are real variable definitions of weighted Hardy spaces, this classical definition has an intuitive appeal.

Let $M_\Omega$ be the Hardy–Littlewood maximal operator restricted to $\Omega$, i.e.,

$$M_\Omega f(x) = \sup_{\substack{Q\subset\Omega \\ x\in Q}} \frac{1}{|Q|}\int_Q |f|\,dy.$$

When $\Omega = \mathbb{R}^n$ we write $M_{\mathbb{R}^n} f = Mf$.

We define $A_1(\Omega)$ to be the class of all weights on $\Omega$ such that $M_\Omega w(x) \le Cw(x)$ a.e. $x \in \Omega$. For $p > 1$, $A_p(\Omega)$ is the class of all weights on $\Omega$ such that

$$\sup_{Q\subset\Omega}\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} < \infty.$$

Given an $A_p$ weight $w$ we refer to the weight $\sigma = w^{1-p'}$ as the dual weight. For the endpoint, $p = \infty$, we use the definition

$$A_\infty(\Omega) = \bigcup_{p\ge 1} A_p(\Omega).$$

There are several other definitions of $A_\infty$, e.g., weights satisfying a reverse Jensen inequality, a reverse Hölder inequality, or a fairness condition with respect to Lebesgue measure [Duoandikoetxea 2001; Grafakos 2008].

A weight on the torus is a positive function in $L^1(\mathbb{T})$. The classes $A_1(\mathbb{T})$, $A_p(\mathbb{T})$, and $A_\infty(\mathbb{T})$ are defined analogously on $\mathbb{T}$. The Szegő class of weights, denoted $\mathcal{W}$, are weights on $\mathbb{T}$ satisfying

$$\int_{\mathbb{T}} \log w\,d\theta > -\infty.$$

We notice that if $w \in A_\infty(\mathbb{T})$, then we have

$$\left(\int_{\mathbb{T}} w\,\frac{d\theta}{2\pi}\right)\exp\left(-\int_{\mathbb{T}} \log w\,\frac{d\theta}{2\pi}\right) < \infty.$$

In particular, $A_\infty(\mathbb{T}) \subset \mathcal{W}$.

**Example 2.1.** Let $x_0 \in \Omega$, $1 \le p \le \infty$, and $w_{x_0}(x) = |x - x_0|^\alpha$. Then $w_{x_0} \in A_p(\Omega)$ if and only if $-n < \alpha < n(p-1)$.

We will need some elementary properties of $A_p$ weights, most of which follow from the definition (see [Duoandikoetxea 2001, Proposition 7.2]).

**Theorem 2.2.** *The following hold*:

(i) $A_1 \subset A_p \subset A_q \subset A_\infty$ *if* $1 < p < q < \infty$.

(ii) *For* $1 < p < \infty$, $w \in A_p$ *if and only if* $\sigma = w^{1-p'} \in A_{p'}$.

(iii) *If* $0 < s \le 1$ *and* $w \in A_p$, *then* $w^s \in A_p$.

(iv) *If* $u, v \in A_1$, *then* $uv^{1-p} \in A_p$.

It is interesting to note that the converse of (iv) also holds, but the proof is much more intricate. This was shown by Jones [1980] and later by Rubio de Francia [1982]. We emphasize that we do not need this converse statement, only the statement (iv).

We also need the following deeper property of $A_\infty$ weights known as the reverse Hölder inequality. See [Hytönen et al. 2012] for a simple proof with nice constants.

**Theorem 2.3.** *If $w \in A_\infty(\Omega)$, then there exists $s > 1$ such that for every cube $Q \subset \Omega$,*

$$\frac{1}{|Q|}\int_Q w^s\,dx \le \left(\frac{2}{|Q|}\int_Q w\,dx\right)^s.$$

As a corollary to Theorem 2.3 we have the following openness properties of $A_p$ classes.

**Theorem 2.4.** *Let $1 \le p \le \infty$. The following hold*:

(i) *If $p > 1$ then $A_p(\Omega) = \bigcup_{1\le q<p} A_q(\Omega)$.*

(ii) *If $w \in A_p(\Omega)$ then $w^s \in A_p(\Omega)$ for some $s > 1$.*

For the results on $\mathbb{R}^n$ we need the notion of a Banach function space. We refer the reader to the book by Bennett and Sharpley [1988, Chapter 1] for an excellent reference on the subject. A mapping $\rho$, defined on the set of nonnegative $\mathbb{R}^n$-measurable functions and taking values in $[0, \infty]$, is said to be a Banach function norm if it satisfies the following properties:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$ for $a > 0$, $\rho(f+g) \le \rho(f)+\rho(g)$;

(ii) if $0 \le f \le g$ a.e., then $\rho(g) \le \rho(f)$;

(iii) if $f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$;

(iv) if $B \subset \mathbb{R}^n$ is bounded, then $\rho(\chi_B) < \infty$;

(v) if $B \subset \mathbb{R}^n$ is bounded, then

$$\int_B f\,dx \le C_B\rho(f)$$

for some constant $C_B$ with $0 < C_B < \infty$.

We note that our definition of a Banach function space is slightly different from that found in [Bennett and Sharpley 1988]. In particular, in the axioms (iv) and (v) we assume that the set $B$ is a bounded set, whereas it is sometimes assumed that $B$ merely satisfy $|B| < \infty$. We do this so that the spaces $L^p_w(\mathbb{R}^n)$ with $w \in A_p$ satisfy items (iv) and (v). (See also the discussion at the beginning of Chapter 1 on page 2 of [Bennett and Sharpley 1988].)

Given Banach function norm $\rho$, $\mathscr{X} = \mathscr{X}(\mathbb{R}^n, \rho)$ is the collection of measurable functions such that $\rho(|f|) < \infty$. In this case we may equip $\mathscr{X}$ with the norm

$$\|f\|_{\mathscr{X}} = \rho(|f|).$$

The associate space $\mathscr{X}'$ is the set of all measurable functions $g$ such that $fg \in L^1(\mathbb{R}^n)$ for all $f \in \mathscr{X}$. This space is normed by

$$\|g\|_{\mathscr{X}'} = \sup\left\{\int_{\mathbb{R}^n} |fg|\,dx : \|f\|_{\mathscr{X}} \le 1\right\}. \tag{6}$$

Equipped with this norm $\mathscr{X}'$ is also a Banach function space and

$$\int_{\mathbb{R}^n} |fg|\,dx \le \|f\|_{\mathscr{X}}\|g\|_{\mathscr{X}'}.$$

Typical examples of Banach function spaces are $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, whose associate spaces are $L^{p'}(\mathbb{R}^n)$. Other Banach spaces include weak type spaces $L^{p,\infty}(\mathbb{R}^n)$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$, and Orlicz spaces $L^\Phi(\mathbb{R}^n)$ defined for a Young function $\Phi$ (see [Bennett and Sharpley 1988; Cruz-Uribe et al. 2011]). When $w \in A_p(\mathbb{R}^n)$ and $1 \le p \le \infty$, the spaces $L^p_w(\mathbb{R}^n)$ are also Banach function spaces with respect to Lebesgue measure. To see this, it suffices to check property (v). Suppose $f \ge 0$, $1 < p < \infty$, and $B$ is bounded. Then $B \subset Q$ for some cube $Q$ so $\sigma(B) < \infty$, and Hölder's inequality implies

$$\int_B f\,dx = \int_B f w^{1/p} w^{-1/p}\,dx \le \sigma(B)^{1/p'}\left(\int_B f^p w\,dx\right)^{1/p} \le \sigma(B)^{1/p'}\|f\|_{L^p_w}.$$

To see that $L^1_w(\mathbb{R}^n)$ is a Banach function space when $w \in A_1(\mathbb{R}^n)$, note that

$$\int_B f\,dx = \int_B f w w^{-1}\,dx \le (\inf_B w)^{-1}\|f\|_{L^1_w}. \tag{7}$$

Finally, if $f \in L^\infty_w$, then

$$\int_B f\,dx = \int_B (f/w) w\,dx \le w(B)\|f\|_{L^\infty_w},$$

showing $L^\infty_w$ is a Banach function space.

When $1 < p < \infty$ and $w \in A_p$, the associate space of $L^p_w(\mathbb{R}^n)$ defined by the pairing in (6) is given not by $L^{p'}_w(\mathbb{R}^n)$ but by $L^{p'}_\sigma(\mathbb{R}^n)$ for $\sigma = w^{1-p'}$. When $p = 1$ and $w \in A_1$, the associate space of $L^1_w$ is given by $L^\infty_w(\mathbb{R}^n)$. We are particularly interested in Banach function spaces $\mathscr{X}$ for which

$$\|Mf\|_{\mathscr{X}} \le C\|f\|_{\mathscr{X}},$$

in which case we write $M \in \mathscr{B}(\mathscr{X})$.

We end this section with the classical result of Coifman and Rochberg [1980] (see also [García-Cuerva and Rubio de Francia 1985, Theorem 3.4, p. 158]). This result requires a definition.

**Definition 2.5.** We say that a function $f(x)$ belongs to $\mathcal{M}_F(\Omega)$ if

$$M_\Omega f(x) < \infty \quad \text{for a.e. } x \in \Omega.$$

If $f$ belongs to a Banach function space for which $M \in \mathcal{B}(\mathcal{X})$, then $f \in \mathcal{M}_F$.

**Theorem 2.6.** *If $f \in \mathcal{M}_F(\Omega)$ and $0 < \delta < 1$, then $(M_\Omega f)^\delta \in A_1(\Omega)$.*

We leave the reader with a table of the notation used throughout the article.

| | |
|---|---|
| $\Omega$ | Domain of interest, either $\mathbb{R}^n$ or a cube $Q \subset \mathbb{R}$; |
| $M_\Omega$ | Hardy–Littlewood maximal operator restricted to $\Omega$; |
| $A_p(\Omega)$ | class of $A_p$ weights on $\Omega$; |
| $\mathcal{M}^r_{A_p}(\Omega)$ | functions on $\Omega$ with $\lvert f\rvert^r$ majorized by an $A_p$ weight; |
| $\mathcal{M}_F(\Omega)$ | functions on $\Omega$ such that $M_\Omega f < \infty$ a.e.; |
| $A^F_p(\Omega)$ | $A_p(\Omega) \cap \mathcal{M}_F(\Omega)$; |
| $\mathcal{M}_{A^F_p}(\Omega)$ | functions majorized by $A^F_p(\Omega)$ weights. |

## 3. The classes $\mathcal{M}^r_{A_p}$

Let us now define a general class of functions majorized by $A_p$ weights and establish some properties of such classes. We remind the reader that a domain $\Omega$ will denote throughout either all of $\mathbb{R}^n$ or a cube $Q$ in $\mathbb{R}^n$.

**Definition 3.1.** Let $r$ and $p$ satisfy $0 < r < \infty$ and $1 \le p \le \infty$. Define $\mathcal{M}^r_{A_p}(\Omega)$ to be the collection of all measurable functions $f$ on $\Omega$ such that

$$|f(x)|^r \le w(x) \quad \text{for a.e. } x \in \Omega$$

for some $w \in A_p(\Omega)$. When $r = 1$ we simply write $\mathcal{M}_{A_p}(\Omega)$.

Theorem 2.4 implies the following general facts about the $\mathcal{M}^r_{A_p}$ classes.

**Theorem 3.2.** *Suppose $r$ and $p$ satisfy $0 < r < \infty$ and $1 \le p \le \infty$. Then*

$$\mathcal{M}^r_{A_p}(\Omega) = \bigcup_{s>r} \mathcal{M}^s_{A_p}(\Omega) \tag{8}$$

*and if $p > 1$,*

$$\mathcal{M}^r_{A_p}(\Omega) = \bigcup_{1\le q<p} \mathcal{M}^r_{A_q}(\Omega). \tag{9}$$

*Proof.* We first prove (8). It is clear from (iii) of Theorem 2.2 that the union $\bigcup_{r<s} \mathcal{M}^s_{A_p}(\Omega) \subset \mathcal{M}^r_{A_p}(\Omega)$. On the other hand, if $f \in \mathcal{M}^r_{A_p}(\Omega)$ then $|f|^r \le w \in A_p$. By (ii) of Theorem 2.4, there exists $t > 1$ such that $w^t \in A_p(\Omega)$. But then, taking $s = rt > r$ and $u = w^t$, we have $|f|^s \le u \in A_p$, so $f \in \bigcup_{r<s} \mathcal{M}^s_{A_p}(\Omega)$. The proof of equality (9) follows directly from (i) of Theorem 2.4. □

Our next theorem shows that for a function to have an $A_1$ majorant it is equivalent for its maximal function to have an $A_1$ majorant.

**Theorem 3.3.** *We have $f \in \mathcal{M}_{A_1}(\Omega)$ if and only if $M_\Omega f \in \mathcal{M}_{A_1}(\Omega)$.*

*Proof.* If $f \in \mathcal{M}_{A_1}(\Omega)$, then $M_\Omega f \le M_\Omega w \le Cw$ since $w \in A_1(\Omega)$, which is to say $M_\Omega f \in \mathcal{M}_{A_1}(\Omega)$. The converse statement follows from the fact that $|f| \le M_\Omega f$. □

Using the exact same reasoning it is easy to prove that $f \in \mathcal{M}^r_{A_1}(\Omega)$ if and only if $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$. However, there is a better result when $r \ge 1$.

**Theorem 3.4.** *If $r \ge 1$ then the following are equivalent*:

(i) $f \in \mathcal{M}^r_{A_1}(\Omega)$.

(ii) $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$.

(iii) $M_\Omega f \in \mathcal{M}^r_{A_1}(\Omega)$.

*Proof.* The equivalence (i) ⇔ (ii) follows from Theorem 3.3. We will prove (ii) ⇒ (iii) and (iii) ⇒ (i).

Suppose that $w \in A_1(\Omega)$ and $M_\Omega(|f|^r) \le w$. Since $r \ge 1$, we know that $(M_\Omega f)^r \le M_\Omega(|f|^r) \le w$, which is to say that $M_\Omega f \in \mathcal{M}^r_{A_1}$.

On the other hand if $(M_\Omega f)^r \le w \in A_1(\Omega)$, then $M_\Omega f < \infty$ a.e., and hence $f$ is locally integrable on $\Omega$. By the Lebesgue differentiation theorem we have

$$|f|^r \le (M_\Omega f)^r \le w. \qquad \square$$

In the case $0 < r < 1$, we still have $f \in \mathcal{M}^r_{A_1}(\Omega)$ if and only if $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$. However, it is not true that this is equivalent to $(M_\Omega f)^r \in \mathcal{M}_{A_1}(\Omega)$. Consider the following simple example.

**Example 3.5.** Let $f(x) = |x|^{-n}$ on $Q = [-1, 1]^n$. If $0 < r < 1$, then $f \in M^r_{A_1}(Q)$ but $M_Q f \equiv \infty$.

Of course, if $0 < r < 1$ and $M_\Omega f < \infty$ a.e., then $(M_\Omega f)^r \in A_1(\Omega)$ (and hence $M_\Omega f \in \mathcal{M}^r_{A_1}(\Omega)$) automatically by Theorem 2.6.

We now study the class $\mathcal{M}_{A_p}$. Since the $A_p$ classes are nested, we have

$$\mathcal{M}_{A_1} \subset \mathcal{M}_{A_p} \subset \mathcal{M}_{A_q} \subset \mathcal{M}_{A_\infty}$$

for $1 \le p \le q \le \infty$. In the local case we have the following characterization.

**Theorem 3.6.** *If $Q$ is a cube in $\mathbb{R}^n$ then*

$$\mathcal{M}_{A_1}(Q) = \mathcal{M}_{A_\infty}(Q).$$

*Proof.* It suffices to show $\mathcal{M}_{A_\infty}(Q) \subset \mathcal{M}_{A_1}(Q)$. Suppose that $f \in \mathcal{M}_{A_\infty}(Q)$, so that there exists $w \in A_\infty(Q)$ with

$$|f| \leq w.$$

Since $w \in A_\infty(Q)$, the reverse Hölder inequality implies that there exists $s > 1$ such that

$$(M_Q w^s)^{1/s} \leq 2M_Q w \leq 2(M_Q w^s)^{1/s}.$$

Moreover, since $w \in L^1(Q)$, we have $M_Q w < \infty$ a.e. By Theorem 2.6, $M_Q w$ is bounded above and below by an $A_1(Q)$ weight, and hence is in $A_1(Q)$ itself. $\square$

In the global case we have $\mathcal{M}_{A_1}(\mathbb{R}^n) \subsetneq \mathcal{M}_{A_p}(\mathbb{R}^n)$ for any $p > 1$, as the following example indicates.

**Example 3.7.** Let $p > 1$ and $0 < \alpha < n(p-1)$. Now consider the function $f(x) = |x|^\alpha$. Then $f \in A_p(\mathbb{R}^n) \subset \mathcal{M}_{A_p}(\mathbb{R}^n)$, but $f \notin \mathcal{M}_F(\mathbb{R}^n)$ so in particular, $f \notin \mathcal{M}_{A_1}(\mathbb{R}^n)$. To see this, notice that for every $x \in \mathbb{R}^n$ and $r > |x|$,

$$Mf(x) \geq \frac{c}{r^n} \int_{|x| \leq r} |x|^\alpha \, dx \simeq r^\alpha$$

so $Mf \equiv \infty$.

To obtain positive results on $\mathbb{R}^n$ for the classes $\mathcal{M}_{A_p}(\mathbb{R}^n)$ and $\mathcal{M}_{A_\infty}(\mathbb{R}^n)$ similar to Theorem 3.6, we must restrict to $A_p$ majorants whose maximal function is finite. Given $w \in A_\infty$, a simple way to create a weight in $A_\infty^F$ is to take a truncation: let $w_\lambda = \min(w, \lambda)$ for $\lambda > 0$. Then $w_\lambda \in A_\infty \cap L^\infty \subset A_\infty^F$. We end our study of the class $\mathcal{M}_{A_1}$ with the following characterizations.

**Theorem 3.8.** $\mathcal{M}_{A_1}(\mathbb{R}^n) = \mathcal{M}_{A_\infty^F}(\mathbb{R}^n)$.

*Proof.* Since $A_1(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n) \subset \mathcal{M}_F(\mathbb{R}^n)$, we have the inclusion $\mathcal{M}_{A_1}(\mathbb{R}^n) \subset \mathcal{M}_{A_\infty^F}(\mathbb{R}^n)$. On the other hand, if $f$ is dominated by a weight $w$ in $A_\infty^F(\mathbb{R}^n) = A_\infty(\mathbb{R}^n) \cap \mathcal{M}_F(\mathbb{R}^n)$, then by Theorem 2.3 we have

$$M(w^s)^{1/s} \leq 2Mw < \infty \quad \text{a.e.}$$

for some $s > 1$. So in particular, $|f| \leq M(|f|^s)^{1/s} \leq M(w^s)^{1/s} \in A_1(\mathbb{R}^n)$. $\square$

**Theorem 3.9.** *A function $f$ belongs to $\mathcal{M}_{A_1}(\mathbb{R}^n)$ if and only if there is an $s > 1$ such that $|f|^s \in \mathcal{M}_F(\mathbb{R}^n)$.*

**Remark 3.10.** Given $r > 0$, if one defines $\mathcal{M}^r_F(\mathbb{R}^n)$ to be the class of functions such that $M(|f|^r) < \infty$ a.e. (equivalently $|f|^r \in \mathcal{M}_F(\mathbb{R}^n)$), then Theorem 3.9 can be stated as

$$\mathcal{M}_{A_1}(\mathbb{R}^n) = \bigcup_{r>1} \mathcal{M}^r_F(\mathbb{R}^n).$$

*Proof of Theorem 3.9.* Let $w$ be an $A_1(\mathbb{R}^n)$ majorant of $f$. Since $w \in A_1(\mathbb{R}^n)$, $w^s \in A_1(\mathbb{R}^n)$ for some $s > 1$, which implies $|f|^s \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. By Theorem 3.4 we have $M(|f|^s) \in \mathcal{M}_{A_1}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. On the other hand, if there exists $s > 1$ such that $M(|f|^s) < \infty$ a.e., then $M(|f|^s)^{1/s} \in A_1(\mathbb{R}^n)$ by Theorem 2.6, and $|f| \le M(|f|^s)^{1/s}$. □

## 4. The local case

For this section $Q$ will be a fixed cube in $\mathbb{R}^n$. We begin with the following extension of the equivalences in (2).

**Theorem 4.1.** *Let $Q$ be a cube in $\mathbb{R}^n$ and $r$, $p_0$ satisfy $0 < r < p_0 < \infty$. Then*

$$\mathcal{M}^r_{A_1}(Q) = \bigcup_{p>r} L^p(Q) = \bigcup_{w \in A_{p_0/r}} L^{p_0}_w(Q).$$

*Proof.* We will prove the chain of containments

$$\bigcup_{w \in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{p>r} L^p(Q) \subset \mathcal{M}^r_{A_1}(Q) \subset \bigcup_{w \in A_{p_0/r}} L^{p_0}_w(Q).$$

- $\big(\bigcup_{w \in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{p>r} L^p(Q)\big)$: Suppose we have $f \in L^{p_0}_w(Q)$ for some $w \in A_{p_0/r}(Q)$. Set $q_0 = p_0/r$. By (ii) of Theorem 2.2, $\sigma = w^{1-q_0'} \in A_{q_0'}(Q)$. By Theorem 2.3, $\sigma$ satisfies a reverse Hölder inequality:

$$\left(\frac{1}{|Q'|}\int_{Q'} \sigma^s\,dx\right)^{1/s} \le \frac{2}{|Q'|}\int_{Q'} \sigma\,dx$$

  for some $s > 1$ and all $Q' \subseteq Q$. This implies that $\sigma \in L^s(Q)$. Define $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{sq_0'}$ so that $q > 1$, and let $p = rq > r$. Then

$$\begin{aligned}\left(\int_Q |f|^p\,dx\right)^{1/p} &= \left(\int_Q |f|^{rq} w^{q/q_0} w^{-q/q_0}\,dx\right)^{1/p} \\ &\le \left(\int_Q |f|^{p_0} w\,dx\right)^{1/p_0} \left(\int_Q \sigma^s\,dx\right)^{1/(sq_0')}.\end{aligned}$$

- $\big(\bigcup_{p>r} L^p(Q) \subset \mathcal{M}^r_{A_1}(Q)\big)$: If $f \in L^p(Q)$ for some $p > r$, then Theorem 2.6 implies $|f|^r \le M_Q(|f|^p)^{r/p} \in A_1(\Omega)$.

- $\big(\mathcal{M}^r_{A_1}(Q) \subset \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q)\big)$: Set $q_0 = p_0/r > 1$ and suppose we have $g = |f|^r \le w \in A_1(Q)$. Then $w^{1-q_0} \in A_{q_0}(Q)$ by (iv) of Theorem 2.2 and

$$\int_Q |f|^{p_0} w^{1-q_0}\,dx = \int_Q g^{q_0} w^{1-q_0}\,dx \le \int_Q w\,dx < \infty. \qquad \square$$

Next, we extend Theorem 4.1 to $A_\infty$ weights.

**Theorem 4.2.** *Let $Q$ be a cube in $\mathbb{R}^n$ and $p_0$ be an exponent with $0 < p_0 < \infty$. Then*

$$\bigcup_{r>0} \mathcal{M}^r_{A_1}(Q) = \bigcup_{p>0} L^p(Q) = \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$

*Proof.* We first prove

$$\bigcup_{r>0} \mathcal{M}^r_{A_1}(Q) = \bigcup_{p>0} L^p(Q).$$

- $(\subset)$: If $f \in \mathcal{M}^r_{A_1}(Q)$ for some $r > 0$, and $w \in A_1(Q)$ is such that $|f|^r \le w$, then $f \in L^r(Q) \subset \bigcup_{p>0} L^p(Q)$.
- $(\supset)$: If $f \in L^p(Q)$ for some $p > 0$, let $r$ be such that $0 < r < p$. Then $|f|^r \le M_Q(|f|^p)^{r/p} \in A_1(Q)$.

Next we show

$$\bigcup_{p>0} L^p(Q) = \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$

- $(\subset)$: Suppose $f \in L^p(Q)$ for some $0 < p < \infty$. Then if $r < \min(p, p_0)$ we have

$$f \in L^p(Q) \subset \bigcup_{r<p} L^p(Q) = \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$

- $(\supset)$: Suppose $f \in L^{p_0}_w(Q)$ for some $w \in A_\infty$. Then $w \in A_q$ for some $q > 1$. Set $p = p_0/q$ and notice that $p < p_0$. Then

$$\int_Q |f|^p\,dx = \int_Q |f|^p w^{1/q} w^{-1/q}\,dx \le \left(\int_Q |f|^{p_0} w\,dx\right)^{1/q} \left(\int_Q w^{1-q'}\,dx\right)^{1/q'}. \qquad \square$$

**Example 4.3.** The function

$$f(x) = x^{-1}(\log x)^{-2}\chi_{(0,1/2)}(x) \tag{10}$$

does not belong to $\mathcal{M}_{A_1}([0,1])$. This follows from Theorem 4.1 since it can be readily checked that

$$f \in L^1([0,1]) \setminus \left(\bigcup_{p>1} L^p([0,1])\right).$$

However, $f \in \mathcal{M}_F([0,1])$ since $f \in L^1([0,1])$.

**Remark 4.4.** Suppose $0 < p < \infty$. Then

$$L^p(Q) = \bigcup_{w \in A_1} L^p_w(Q).$$

The proof of the equality in Remark 4.4 follows from the fact that $1 \in A_1$ and from inequality (7) with $B = Q$.

We define $H_{A_1}(\mathbb{T})$ as the set of functions in $N^+$ whose boundary function is majorized by an $A_1(\mathbb{T})$ weight. Since we may identify the torus $\mathbb{T}$ with $Q = [0, 2\pi]$, it is obvious that Theorems 4.1 and 4.2 hold for $L^p(\mathbb{T})$ and $L^p_w(\mathbb{T})$ spaces. We have the following analogs for Hardy spaces.

**Theorem 4.5.** *If $p_0$ is an exponent satisfying $1 < p_0 < \infty$, then*

$$H_{A_1}(\mathbb{T}) = \bigcup_{p>1} H^p = \bigcup_{w \in A_{p_0}} H^{p_0}_w.$$

**Theorem 4.6.** *If $p_0$ is an exponent satisfying $0 < p_0 < \infty$, then*

$$\bigcup_{p>0} H^p = \bigcup_{w \in A_\infty} H^{p_0}_w.$$

*Proof of Theorems 4.5 and 4.6.* Since $N^+ \cap L^p(\mathbb{T}) = H^p$ for $p > 0$ [Duren 1970, Theorem 2.11], we see that

$$H_{A_1}(\mathbb{T}) = N^+ \cap \mathcal{M}_{\mathcal{A}_1}(\mathbb{T}) = N^+ \cap \bigcup_{p>1} L^p(\mathbb{T}) = \bigcup_{p>1} H^p.$$

This is the first part of Theorem 4.5.

To go from equality of the analogous $L^p$ spaces to the Hardy spaces is a matter of using two facts for $0 < p_0 < \infty$:

(a) $\int_{\mathbb{T}} \log w \, d\theta > -\infty$ and $w \in L^1(\mathbb{T})$ implies that $w = |h|^{p_0}$ for some outer function $h \in H^{p_0}$.

(b) If $h \in H^{p_0}$ is outer, then the set $h\mathbb{C}[z] = \vee\{z^j h : j \geq 0\}$ is dense in $H^{p_0}$.

Item (a) comes from the standard construction of an outer function [Duren 1970, Section 2.5]. As for item (b), when $1 \leq p_0 < \infty$ this is a standard generalization of Beurling's theorem [Duren 1970, Theorem 7.4]. When $0 < p_0 < 1$, this is a less well known result that can be found in Gamelin [1966, Theorem 4].

For Theorem 4.5 we must show for $1 < p_0 < \infty$ that

$$\bigcup_{p>1} H^p = \bigcup_{w \in A_{p_0}} H^{p_0}_w.$$

Now, for $f \in H^p \subset L^p$, we know there exists $w \in A_{p_0}(\mathbb{T})$ such that $f \in L^{p_0}_w(\mathbb{T})$ by (2). Factor $w = |h|^{p_0}$ with outer $h \in H^{p_0}$. Then, $fh \in N^+ \cap L^{p_0}(\mathbb{T}) = H^{p_0}$

while $h\mathbb{C}[z]$ is dense in $H^{p_0}$ so that there exist polynomials $Q_n$ satisfying

$$\int |fh - Q_n h|^{p_0}\, d\theta = \int |f - Q_n|^{p_0} w\, d\theta \to 0$$

as $n \to \infty$. This shows $f \in H^{p_0}_w$ (since it is initially defined as the closure of the analytic polynomials in $L^{p_0}_w(\mathbb{T})$).

Conversely, we have seen that if $f \in H^{p_0}_w$, then $f \in L^p(\mathbb{T})$ for some $p > 1$. Factor $w = |h|^{p_0}$ as before. Then, $fh \in H^{p_0}$ and $1/h$ is outer, so that $f = fh(1/h) \in N^+$. Since $f \in L^p(\mathbb{T})$, we can then conclude that $f \in H^p$.

The proof of Theorem 4.6, which claims for $0 < p_0 < \infty$ that

$$\bigcup_{p>0} H^p = \bigcup_{w \in A_\infty} H^{p_0}_w,$$

is similar once we know the corresponding fact for $L^p(\mathbb{T})$ spaces. Indeed, take $f \in H^p$ for some $p > 0$. There exists $w \in A_\infty$ such that $f \in L^{p_0}_w(\mathbb{T})$ by Theorem 4.2. Factor $w = |h|^{p_0}$ with outer $h \in H^{p_0}$. Then, $f \in H^{p_0}_w$ as above using Gamelin's result. The converse is similar to the previous proof. □

## 5. The global case

In this section we address the case when our functions are defined on all of $\mathbb{R}^n$. Let us first prove Theorem 1.3, which states that for any $1 < p < \infty$,

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n).$$

*Proof of Theorem 1.3.* First we show

$$\mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) \subset \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

Suppose $w$ is an $A_1$ majorant of $f$ and $f \in L^1_u(\mathbb{R}^n)$ for some $u \in A_1(\mathbb{R}^n)$. By Theorem 2.2, $uw^{1-p} \in A_p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f|^p w^{1-p} u\, dx \le \int_{\mathbb{R}^n} |f| u\, dx.$$

To see the reverse containment suppose that $f \not\equiv 0$ belongs to $L^p_w(\mathbb{R}^n)$ for some $w \in A_p(\mathbb{R}^n)$. We will use the fact that $w \in A_p(\mathbb{R}^n)$ implies $M \in \mathcal{B}(L^p_w)$ to apply the Rubio de Francia algorithm:

$$Rf = \sum_{k=0}^{\infty} \frac{M^k f}{2^k \|M\|^k_{\mathcal{B}(L^p_w)}}.$$

Then $Rf$ is an $A_1$ majorant of $f$ so $f \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. Also let $g$ be any function in $L^{p'}_\sigma(\mathbb{R}^n)$ where $\sigma = w^{1-p'}$ satisfying $\|g\|_{L^{p'}_\sigma(\mathbb{R}^n)} = 1$. Again, since $\sigma \in A_{p'}(\mathbb{R}^n)$, we apply the Rubio de Francia algorithm

$$Rg = \sum_{k=0}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathcal{B}(L^{p'}_\sigma)}},$$

so that $Rg$ is in $A_1(\mathbb{R}^n)$ and $\|Rg\|_{L^{p'}_\sigma(\mathbb{R}^n)} \le 2$. Hence

$$\int_{\mathbb{R}^n} |f| Rg\, dx = \int_{\mathbb{R}^n} |f| w^{1/p} Rg w^{-1/p}\, dx \le \|f\|_{L^p_w(\mathbb{R}^n)} \|Rg\|_{L^{p'}_\sigma(\mathbb{R}^n)} \le 2\|f\|_{L^p_w(\mathbb{R}^n)},$$

showing that $f \in \bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ as well. □

Before moving on, we remark that the intersection of $\mathcal{M}_{A_1}(\mathbb{R}^n)$ and $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ is necessary for the result on $\mathbb{R}^n$. We did not encounter this phenomenon in the local case since for a fixed cube, $\mathcal{M}_{A_1}(Q) \subset L^1(Q)$. To see that the intersection is necessary, notice that the function in Example 4.3 viewed as a function on $\mathbb{R}$ belongs to $L^1(\mathbb{R}) \subset \bigcup_{w\in A_1} L^1_w(\mathbb{R})$, but does not belong to $L^p_w(\mathbb{R})$ for any $p > 1$ and $w \in A_p(\mathbb{R})$ since it is not in $L^p_{\mathrm{loc}}(\mathbb{R})$ for any $p > 1$. Theorem 1.3 shows that for $1 < p < \infty$,

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) \subset \mathcal{M}_{A_1}(\mathbb{R}^n).$$

Below we will show this containment is proper (see Example 5.2).

We now prove Theorem 1.4.

*Proof of Theorem 1.4.* By Theorem 1.3 it suffices to show

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) \subset \bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')\} \tag{11}$$

and

$$\bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')\} \subset \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w\in A_1} L^1_w(\mathbb{R}^n). \tag{12}$$

However, the containment (11) is immediate, since

$$M \in \mathcal{B}(L^p_w(\mathbb{R}^n)) \Leftrightarrow w \in A_p(\mathbb{R}^n) \Leftrightarrow \sigma \in A_{p'}(\mathbb{R}^n) \Leftrightarrow M \in \mathcal{B}(L^{p'}_\sigma(\mathbb{R}^n)).$$

On the other hand, for containment (12), if $f \not\equiv 0$, then $f \in \mathcal{X}$ for some Banach function space $\mathcal{X}$ such that $M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')$. Then we may use the Rubio de Francia algorithm to construct an $A_1(\mathbb{R}^n)$ majorant:

$$Rf = \sum_{k=0}^{\infty} \frac{M^k f}{2^k \|M\|^k_{\mathcal{B}(\mathcal{X})}}.$$

Then $Rf \in A_1$ and $|f| \le Rf$, so $f \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. Given $g \in \mathscr{X}'$ let

$$Rg = \sum_{k=0}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathscr{B}(\mathscr{X}')}},$$

so that $Rg \in A_1(\mathbb{R}^n) \cap \mathscr{X}'$ and $\|Rg\|_{\mathscr{X}'} \le 2\|g\|_{\mathscr{X}'}$. Then

$$\int_{\mathbb{R}^n} |f| Rg\, dx \le \|f\|_{\mathscr{X}} \|Rg\|_{\mathscr{X}'} \le 2\|f\|_{\mathscr{X}} \|g\|_{\mathscr{X}'},$$

which yields $f \in \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$. □

When $p > 1$, $L^{p,\infty}(\mathbb{R}^n)$ is a Banach function space on which $M$ is bounded (see [Grafakos 2008]), and likewise, its associate $(L^{p,\infty}(\mathbb{R}^n))' = L^{p',1}(\mathbb{R}^n)$, the Lorentz space with exponents $p'$ and 1, is also a Banach function space on which $M$ is bounded (see [Ariño and Muckenhoupt 1990]).

**Corollary 5.1.** *Suppose* $1 < p_0 < \infty$. *Then*

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subset \bigcup_{w \in A_{p_0}} L^{p_0}_w(\mathbb{R}^n).$$

From Corollary 5.1 we see that the analogous version of the equivalences in (2) are not true on $\mathbb{R}^n$. This follows since

$$\bigcup_{p>1} L^p(\mathbb{R}^n) \subsetneqq \bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n).$$

For example, $f(x) = |x|^{-n/2} \in L^{2,\infty}(\mathbb{R}^n)$ but $f \notin \bigcup_{p>0} L^p(\mathbb{R}^n)$.

We also remark that the techniques required for $\mathbb{R}^n$ are completely different than the local case. For example, to prove the containment

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subset \mathcal{M}_{A_1}(\mathbb{R}^n)$$

it is not enough to simply dominate $|f|$ by $M(|f|^p)^{1/p}$. However, for $f \in L^{p,\infty}(\mathbb{R}^n)$, $M(|f|^p)$ may not be finite (take $f(x) = |x|^{-n/p}$, in which case $M(|f|^p) \equiv \infty$). Instead we must refine our construction of an $A_1$ majorant using the techniques of Rubio de Francia [1984].

We now provide examples to show that the inclusions in (4) are proper. We first show that the second inclusion is proper, i.e.,

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) \subsetneqq \mathcal{M}_{A_1}(\mathbb{R}^n).$$

Since

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n),$$

it suffices to find a function in $\mathcal{M}_{A_1}(\mathbb{R}^n) \setminus \big(\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)\big)$.

**Example 5.2.** The function $f(x) = 1$ belongs to $\mathcal{M}_{A_1}(\mathbb{R}^n) \setminus \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$. To prove this we need the fact that if $w \in A_\infty$ then $w \notin L^1(\mathbb{R}^n)$. One way to see this (pointed out by the referee) is to notice that $A_\infty$ weights are doubling, and doubling measures have infinite total mass. We can also give an ad hoc argument using the reverse Hölder inequality. If $w$ satisfies

$$\left(\frac{1}{|Q|}\int_Q w^s\,dx\right)^{1/s} \le \frac{2}{|Q|}\int_Q w\,dx$$

for some $s > 1$ and all cubes $Q$, then by taking $Q_N = [-N, N]^n$, we have

$$\left(\frac{1}{|Q_N|}\int_{Q_1} w^s\,dx\right)^{1/s} \le \left(\frac{1}{|Q_N|}\int_{Q_N} w^s\,dx\right)^{1/s} \le \frac{2}{|Q_N|}\int_{Q_N} w\,dx \le \frac{2}{|Q_N|}\|w\|_{L^1(\mathbb{R}^n)}.$$

Letting $N \to \infty$ we arrive at a contradiction. Finally, to see $1 \notin \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$, notice that $1 \in L^1_w(\mathbb{R}^n)$ if and only if $w \in L^1(\mathbb{R}^n)$.

Next we show that

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subsetneq \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

For this example we need the following lemma.

**Lemma 5.3.** *Suppose $u, v \in A_1(\mathbb{R}^n)$. Then*

$$\max(u, v) \in A_1(\mathbb{R}^n) \quad \textit{and} \quad \min(u, v) \in A_1(\mathbb{R}^n).$$

*Proof.* To see that $\max(u, v)$ is in $A_1(\mathbb{R}^n)$ note that $\max(u, v) \le u + v \le 2\max(u, v)$, and hence

$$M(\max(u, v)) \le Mu + Mv \le C(u + v) \le 2C\max(u, v).$$

To prove $\min(u, v) \in A_1(\mathbb{R}^n)$ we use the equivalent definition of $A_1(\mathbb{R}^n)$:

$$w \in A_1(\mathbb{R}^n) \Leftrightarrow \frac{1}{|Q|}\int_Q w\,dx \le C \inf_Q w \quad \forall Q \subset \mathbb{R}^n$$

where the infimum is the essential infimum of $w$ over the cube $Q$. Set $w = \min(u, v)$ and let $Q$ be a cube. Notice that $\inf_Q u > \inf_Q v$ implies $\inf_Q w = \inf_Q v$ and hence

$$\frac{1}{|Q|}\int_Q w\,dx \le \frac{1}{|Q|}\int_Q v\,dx \le C \inf_Q v = C \inf_Q w.$$

On the other hand, if $\inf_Q u \le \inf_Q v$ then $\inf_Q w = \inf_Q u$ and so

$$\frac{1}{|Q|}\int_Q w\,dx \le \frac{1}{|Q|}\int_Q u\,dx \le C \inf_Q u = C \inf_Q w.$$

So $w \in A_1(\mathbb{R}^n)$. ∎

**Example 5.4.** Consider $f(x) = \max(|x|^{-\alpha n}, |x|^{-\beta n})$. If $0 < \alpha < \beta < 1$ then $f \notin \bigcup_{p>0} L^{p,\infty}(\mathbb{R}^n)$. However,

$$|f(x)| \leq w(x)$$

where $w(x) = \max(|x|^{-\beta n}, 1)$, and $f \in L^1_u(\mathbb{R}^n)$ where $u(x) = \min(|x|^{-\gamma n}, 1)$ when $1 - \alpha < \gamma < 1$. By Lemma 5.3, both $u$ and $w$ belong to $A_1(\mathbb{R}^n)$. Thus

$$f \in \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) = \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

Finally, we end with brief descriptions of $\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$ and $\mathcal{M}_{A_1}(\mathbb{R}^n)$ in terms of Banach function spaces.

**Theorem 5.5.** $\displaystyle\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) = \bigcup \{\mathscr{X} : M \in \mathscr{B}(\mathscr{X}')\} = \bigcup \{\mathscr{X} : \mathscr{X}' \cap A_1(\mathbb{R}^n) \neq \varnothing\}.$

*Proof.* It is clear that

$$\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) \subset \bigcup \{\mathscr{X} : M \in \mathscr{B}(\mathscr{X}')\},$$

since the associate space of $L^1_w(\mathbb{R}^n)$ is

$$L^\infty_w(\mathbb{R}^n) = \{f : f/w \in L^\infty\}$$

with norm $\|f\|_{L^\infty_w} = \|f/w\|_{L^\infty}$. For any cube $Q$,

$$\frac{1}{|Q|} \int_Q |f| \, dx \leq \|f/w\|_{L^\infty} \frac{1}{|Q|} \int_Q w \, dx.$$

Hence if $w \in A_1$, then

$$Mf \leq \|f\|_{L^\infty_w} Mw \leq C \|f\|_{L^\infty_w} w,$$

and dividing through by $w$ we obtain $M \in \mathscr{B}(L^\infty_w)$.

The associate space is always a closed subspace of the dual space [Bennett and Sharpley 1988; Rubio de Francia 1987]. Suppose $\mathscr{X}$ is such that $M \in \mathscr{B}(\mathscr{X}')$. Given $g \in \mathscr{X}'$ with $g \not\equiv 0$ (notice Banach function spaces always contain nonzero functions by property (iv) of Banach function norms), let

$$w = \sum_{k=1}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathscr{B}(\mathscr{X}')}}$$

so that $w \in A_1(\mathbb{R}^n)$ and $\|w\|_{\mathscr{X}'} \leq \|g\|_{\mathscr{X}'}$. Thus $w \in \mathscr{X}' \cap A_1(\mathbb{R}^n)$, showing that

$$M \in \mathscr{B}(\mathscr{X}') \Rightarrow \mathscr{X} \cap A_1(\mathbb{R}^n) \neq \varnothing.$$

Finally, suppose $f \in \mathcal{X}$ for some $\mathcal{X}$ such that $\mathcal{X}'$ contains an $A_1$ weight. Let $w \in \mathcal{X}' \cap A_1(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} |f| w \, dx \le \|f\|_{\mathcal{X}} \|w\|_{\mathcal{X}'},$$

so that $f \in L^1_w(\mathbb{R}^n)$. □

Finally we refer to a result of Chu [2013] which gives the final characterization of $\mathcal{M}_{A_1}(\mathbb{R}^n)$.

**Theorem 5.6** [Chu 2013]. $\mathcal{M}_{A_1}(\mathbb{R}^n) = \bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X})\}$.

## 6. Questions

We leave the reader with some open questions.

**1.** Let $A_p^* = \bigcap_{q>p} A_q$. Is there a characterization of the union

$$\bigcup_{w \in A_p^*} L^p_w?$$

In general $A_p \subsetneq A_p^*$. For example $w(x) = \max((\log |x|^{-1})^{-1}, 1)$ belongs to $A_1^*$ but not $A_1$. Moreover,

$$\{w : w, 1/w \in A_1^*\} = \mathrm{clos}_{BMO} L^\infty$$

(see [García-Cuerva and Rubio de Francia 1985; Johnson and Neugebauer 1987]). In the local case we have

$$\bigcup_{w \in A_p^*} L^p_w(Q) \subset \bigcap_{s<p} \bigcup_{r>s} L^r(Q) = \limsup_{r \to p^-} L^r(Q).$$

Are these two sets equal?

**2.** It is well known that

$$L^1 \cap L^\infty \subset \bigcap_{1<p<\infty} L^p \subset \bigcup_{1<p<\infty} L^p \subset L^1 + L^\infty.$$

When can we write a function as the sum of a function in $\mathcal{M}_{A_1}$ and $\bigcup_{w \in A_1} L^1_w$? That is, what conditions on a function guarantee it belongs to $\mathcal{M}_{A_1} + \bigcup_{w \in A_1} L^1_w$?

**3.** What can one say about

$$\bigcup_{w \in A_p} L^{p,\infty}_w?$$

If $w \in A_1$ and $p > 1$ then $M \in \mathcal{B}(L^{p,\infty}_w)$, so for $p > 1$,

$$\bigcup_{w \in A_1} L^{p,\infty}_w \subset \mathcal{M}_{A_1}.$$

**4.** Do these results transfer to more general domains? It is possible to consider a general open set $\Omega$ as our domain of interest. We may define the $A_p(\Omega)$ classes, $\mathcal{M}_{A_1}(\Omega)$, and the Hardy–Littlewood maximal operator $M_\Omega$ exactly as before. However, the openness results, Theorems 2.3 and 2.4, may not hold for $\Omega$, even if it is bounded [Cruz-Uribe et al. 2011]. In the local case we assume that weights belong to $L^1(\Omega)$. What happens if we only assume $L^1_{\mathrm{loc}}(\Omega)$?

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