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Integrability and regularity of rational functions

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INTEGRABILITY AND REGULARITY OF RATIONAL FUNCTIONS

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Abstract. Motivated by recent work in the mathematics and engineering literature, we study integrability and non-tangential regularity on the two-torus for rational functions that are holomorphic on the bidisk. One way to study such rational functions is to fix the denominator and look at the ideal of polynomials in the numerator such that the rational function is square integrable. A concrete list of generators is given for this ideal as well as a precise count of the dimension of the subspace of numerators with a specified bound on bidegree. The dimension count is accomplished by constructing a natural pair of commuting contractions on a finite dimensional Hilbert space and studying their joint generalized eigenspaces.

Non-tangential regularity of rational functions on the polydisk is also studied. One result states that rational inner functions on the polydisk have non-tangential limits at *every* point of the n-torus. An algebraic characterization of higher non-tangential regularity is given. We also make some connections with the earlier material and prove that rational functions on the bidisk which are square integrable on the two-torus are non-tangentially bounded at every point. Several examples are provided.

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Contents

1. Introduction

This paper is about integrability and boundary regularity properties of rational functions in several variables. Although this sounds like well-traveled territory, the questions we are interested in seem to have no general theory for systematically addressing them. The paper focuses on rational functions that are holomorphic on the bidisk

$$\mathbb{D}^2 \overset{\text{def}}{=} \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| < 1\}$$

and their behavior on or near the distinguished boundary

$$\mathbb{T}^2 \overset{\text{def}}{=} \{z \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}.$$

Our questions are:

Question A. *For fixed $p \in \mathbb{C}[z_1, z_2]$ with no zeros on $\mathbb{D}^2$, is there an algebraic characterization of the ideal*

$$\mathcal{I}_p \overset{def}{=} \{q \in \mathbb{C}[z_1, z_2] : q/p \in L^2(\mathbb{T}^2)\}?$$

Namely, can a finite list of generators be explicitly described?

Question B. *For fixed $p \in \mathbb{C}[z_1, z_2]$ with no zeros in $\mathbb{D}^2$, what is the dimension of*

$$\mathcal{P}_{j,k} \overset{def}{=} \{q \in \mathbb{C}[z_1, z_2] : q/p \in L^2(\mathbb{T}^2), \deg q \leq (j, k)\}?$$

Here $\deg q$ *refers to the bidegree of* q.

Question C. *When does a rational function q/p on $\mathbb{D}^2$ possess a limit as $z \in \mathbb{D}^2 \to \zeta \in \mathbb{T}^2$ non-tangentially? When does q/p possess higher non-tangential regularity?*

Readers can certainly imagine many other natural variations on these questions—change the domain, change the regularity or integrability conditions—but already these questions are rich. After applying a Cayley transform, many of these questions can be converted to questions about rational functions on a product of upper half-planes where the boundary of interest is now simply $\mathbb{R}^2$. For local issues this makes little difference, but for more global questions having a compact boundary is important and in particular makes Question A sensible ($\mathcal{I}_p$ is not an ideal if we replace $\mathbb{T}^2$ with $\mathbb{R}^2$).

Although these questions are certainly fundamental in nature, why are they worthy of in-depth study? There are several reasons.

In the engineering literature, there is interest in understanding "non-essential singularities of the second kind" of rational functions. These are singularities on $\mathbb{T}^2$ where both the numerator and denominator vanish (assuming they have no factors in common). An early influential paper on this was Goodman [22] which studied when the Fourier coefficients of q/p are in ℓ^1 (bounded-input-bounded-output stability) or ℓ^2 (square-summable impulse response) or ℓ^∞ (bounded impulse response). In particular, a detailed study was given of the examples

$$G_1(z) = \frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2} \quad G_2(z) = \frac{(1-z_1)(1-z_2)}{2-z_1-z_2}$$

$$G_3(z) = \frac{2}{2-z_1-z_2}.$$

As shown by computations in [22], G_1, G_2 are bounded in $\mathbb{D}^2$; the Fourier coefficients of G_1 are in ℓ^1; the Fourier coefficients of G_2 are in $\ell^2 \setminus \ell^1$; the Fourier coefficients of G_3 are in $c_0 \setminus \ell^2$. Using the techniques presented here it is possible to prove these facts more systematically. The recent paper [28] studies certain 2D linear systems where singularities on $\mathbb{T}^2$ are forced by the structure at hand; an example is given to vehicle platooning. See [28] for further references in the engineering literature.

In the mathematics literature, rational functions on $\mathbb{D}^2$ with singularities on $\mathbb{T}^2$ play a role in several places in complex analysis, essentially as important extremal functions. In [23] they appear as the functions satisfying equality in a certain version of the Schwarz lemma on the polydisk. In [1] they appear as solutions of a three point interpolation problem for bounded holomorphic functions on $\mathbb{D}^2$. In [9], polynomials with no zeros on $\mathbb{D}^2$ and some zeros on $\mathbb{T}^2$ appear in a characterization of cyclic polynomials for Dirichlet type spaces on the bidisk. A major impetus for the present paper is our previous study of a certain class of rational functions called rational inner functions on the bidisk [25] where the goal was to understand all rational inner functions and not just the regular ones (those extending analytically past $\overline{\mathbb{D}}^2$) as in the important work [20]. This distinction plays a role in [37], a paper about interpolation problems on the polydisk, where certain theorems are only proven for regular rational inner functions. Question C is related to the work in [4] where non-tangential convergence is studied for general bounded analytic functions on the bidisk, and rational functions appear as important examples. The paper [3] is also relevant.

Although it is something of an aside, these issues are also relevant in some problems in dynamics, specifically in the study of algebraic $\mathbb{Z}^d$-actions as in [30, 31]. While the requirement of non-vanishing in $\mathbb{D}^d$ does not seem to be relevant in this context, the integrability properties of rational functions on $\mathbb{T}^d$ do seem to be of interest. The example $p(z) = 2 - z_1 - z_2$ makes an appearance as Example 7.2 of [30] and Example 4.3 of [31]. They point out that G_3 (or just $1/p$) is in $L^1(\mathbb{T}^2)$ and $(z_1 - 1)^3/p(z)$ has absolutely convergent Fourier series.

All of this serves to point out that regularity/integrability of rational functions on the torus and on the polydisk plays an important role in a number of contexts, yet there does not seem to be an associated theory for addressing it. We shall give a sampling of our answers to Questions A,B,C here in the introduction, and leave more complete answers to later sections.

To answer Question A, let $d\sigma$ denote normalized Lebesgue measure on $\mathbb{T}^2$ and define the following orthogonal complements using the inner product on $L^2(\frac{d\sigma}{|p|^2})$:

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \mathcal{P}_{n-1,m} \ominus z_2\mathcal{P}_{n-1,m-1}, \quad \mathcal{F}_1 \stackrel{\text{def}}{=} \mathcal{P}_{n-1,m} \ominus \mathcal{P}_{n-1,m-1},$$

$$\mathcal{E}_2 \stackrel{\text{def}}{=} \mathcal{P}_{n,m-1} \ominus z_1\mathcal{P}_{n-1,m-1}, \quad \mathcal{F}_2 \stackrel{\text{def}}{=} \mathcal{P}_{n,m-1} \ominus \mathcal{P}_{n-1,m-1}.$$

If $p \in \mathbb{C}[z_1, z_2]$ has bidegree (n, m), its reflection is given by

$$\tilde{p}(z) \stackrel{\text{def}}{=} z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}.$$

Theorem A. *Assume $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$ and assume p and $\tilde{p}$ have no common factors. The ideal $\mathcal{I}_p$ is generated by bases for $\mathcal{E}_1, \mathcal{F}_1, \mathcal{F}_2$.*

These spaces can all be explicitly constructed using the one variable matrix Fejér-Riesz lemma. The asymmetry in the theorem is explained in Section 7. The theorem is restated as Theorem 7.1.

We also study the ideal

$$\mathcal{I}_p^\infty \stackrel{\text{def}}{=} \{q \in \mathbb{C}[z_1, z_2] : q/p \in L^\infty(\mathbb{T}^2)\}$$

and construct one variable polynomials $g(z_1), h(z_2)$ such that $gh\mathcal{I}_p \subset \mathcal{I}_p^\infty$ in Section 8.

Question B is answered as follows.

Theorem B. *Let $p \in \mathbb{C}[z_1, z_2]$ have no zeros in $\mathbb{D}^2$ and assume p and $\tilde{p}$ have no common factors. Let $N_{\mathbb{T}^2}(p, \tilde{p})$ denote the number of common zeros of p and $\tilde{p}$ on $\mathbb{T}^2$ where zeros are counted with appropriate multiplicities, as in Bézout's theorem. Then,*

$$\dim \mathcal{P}_{j,k} = (j + 1)(k + 1) - \frac{1}{2} N_{\mathbb{T}^2}(p, \tilde{p}).$$

The assumption that p and $\tilde{p}$ have no common factors is no serious reduction since common factors divide every element of $\mathcal{I}_p$. It is intuitively clear that common zeros of p and $\tilde{p}$ on $\mathbb{T}^2$ should occur with even multiplicity by a perturbation argument, however we give a proof using Puiseux series in Appendix C.

Question C can actually be answered in more than two variables and it has an especially clean answer for rational inner functions, which are a generalization of finite Blaschke products to several variables.

Theorem C. *If $\phi : \mathbb{D}^d \to \overline{\mathbb{D}}$ is a rational inner function, then for* every *$\zeta \in \mathbb{T}^d$, the limit*

$$\lim_{z \to \zeta} \phi(z)$$

exists as $z \in \mathbb{D}^d$ goes to ζ non-tangentially.

A rational function $\phi = q/p$, holomorphic on $\mathbb{D}^d$, is *inner* if $|q| = |p|$ on $\mathbb{T}^d$. By the maximum principle ϕ maps $\mathbb{D}^d$ to $\overline{\mathbb{D}}$ and $q(z)$ must be of the form $\mu z^\alpha \tilde{p}(z)$ where $\mu \in \mathbb{T}$, α is a multi-index, and $\tilde{p}$ is the *reflection* of p just as in two variables:

$$\tilde{p}(z_1, \dots, z_d) \stackrel{\text{def}}{=} z_1^{n_1} \cdots z_d^{n_d} \overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_d)}$$

assuming the multidegree of p is $(n_1, \dots, n_d)$; see [36], Theorem 5.2.5.

To say $z \to \zeta$ non-tangentially means the quantities $|z_j - \zeta_j|$ for $j = 1, \dots, d$ and $1 - |z_j|$ for $j = 1, 2, \dots, d$ are all comparable as $z = (z_1, \dots, z_d) \to \zeta = (\zeta_1, \dots, \zeta_d)$. This result is perhaps surprising because rational inner functions need not be continuous up to $\mathbb{D}^d$.

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2. Overview of the paper

Question A is addressed by studying the Hilbert space $L^2(\frac{d\sigma}{|p|^2})$, where $d\sigma$ is normalized Lebesgue measure on $\mathbb{T}^2$, and certain special orthogonal decompositions in $L^2(\frac{d\sigma}{|p|^2})$. These make it possible to construct generators for the ideal $\mathcal{I}_p$, thus answering Question A. Sections 5-7 are occupied with this.

The special orthogonal decompositions of $L^2(\frac{d\sigma}{|p|^2})$ are used in [11, 25] to establish an important sums of squares formula. If $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{D}^2$ and bidegree (n, m) then

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2) \sum_{j=1}^{n} |A_j(z)|^2 + (1 - |z_2|^2) \sum_{j=1}^{m} |B_j(z)|^2 \tag{2.1}$$

for some $A_1, A_2, \dots, A_n, B_1, \dots, B_m \in \mathbb{C}[z_1, z_2]$. This formula has several applications: Agler's Pick interpolation theorem, two variable matrix monotone functions, and determinantal formulas for distinguished varieties, polynomials with no zeros on $\mathbb{D}^2 \cup (\mathbb{C}^2 \setminus \overline{\mathbb{D}}^2)$, and hyperbolic polynomials; see [5, 12, 24, 25, 27]. Thus, it should pay off to understand it better. The Hilbert space approach for proving this formula produces the A_j and B_j as elements of $\mathcal{I}_p$, and understanding this approach in depth is the key to addressing Question B. A method adapted from Ball-Sadosky-Vinnikov [8] shows that minimal sums of squares formulas for p are in correspondence with joint invariant subspaces of a pair of commuting truncated shift operators on a finite dimensional subspace of $L^2(\frac{d\sigma}{|p|^2})$. The joint eigenvalues of this pair of operators are directly related to common zeros of p and $\tilde{p}$ and this enables us to compute the dimension of $\mathcal{P}_{j,k}$ in terms of common zeros of p and $\tilde{p}$, thus answering Question B. Sections 9-13 are occupied with this. We include a background section on intersection multiplicities.

Section 14 is devoted to addressing Question C. The beginning of this section is actually independent of the rest of paper and hinges on a proposition stating that the bottom term in the homogeneous expansion of $p \in \mathbb{C}[z_1, \dots, z_d]$ at a boundary zero has no zeros in a product of half-planes. After addressing Theorem C we make some connections to earlier material. Namely, if a rational inner function on $\mathbb{D}^2$ has higher regularity at a boundary point where p vanishes then this forces a larger intersection multiplicity of this common zero of p and $\tilde{p}$. We get the interesting conclusion that the number of points with a certain amount of regularity but no higher is finite and can be explicitly bounded.

We have attempted to make this paper as accessible as possible. Consequently, there are several background sections and appendices which experts in one area or another should be able to skim. There is a section with notation at the end of the paper. For further background reading we recommend: [2, 12] for reproducing kernels and bounded analytic functions on $\mathbb{D}^2$, [15] for positive semi-definite polynomials, [39] for the study of measures of the form $\frac{1}{|p(e^{i\theta})|^2}d\theta$ in one variable (Bernstein-Szegő measures), [13, 17, 18] for algebraic curves and intersection multiplicities.

We begin with an example.

3. Example: $p(z) = 2 - z_1 - z_2$

The polynomial $p(z) = 2 - z_1 - z_2$ is the simplest non-trivial example that can be used to illustrate many of the theorems of this paper.

Note $\tilde{p}(z) = 2z_1z_2 - z_1 - z_2$ and

$$f(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$$

is a rational inner function which does not extend continuously to $\mathbb{T}^2$. To see this consider the path in $\mathbb{D}$ given by $z_\epsilon(t) = (1 - \epsilon e^{it}\cos t, 1 - \epsilon e^{-it}\cos t)$ where $t \nearrow \pi/2$ and $\epsilon > 0$ is small. Then, for $t \in (0, \frac{\pi}{2})$

$$f(z_\epsilon(t)) = -1 + \epsilon$$

but $z_\epsilon(\frac{\pi}{2}) = (1, 1)$. Theorem C tells us that despite this discontinuity, f has a limit along any non-tangential path to $\mathbb{T}^2$. Of course, z_ϵ approaches tangentially, so there is no contradiction. The key observation to proving non-tangential convergence at $(1, 1)$ is to expand

$$f(1 - \zeta_1, 1 - \zeta_2) = -1 + \frac{2\zeta_1\zeta_2}{\zeta_1 + \zeta_2}.$$

If $z \to (1, 1)$ non-tangentially in $\mathbb{D}^2$, then $\zeta \to (0, 0)$ non-tangentially in RHP^2; $RHP =$ the right half plane. This means $|\zeta_1|, |\zeta_2|, \mathrm{Re}\zeta_1, \mathrm{Re}\zeta_2$ are all comparable quantities in a non-tangential approach region and so $|\zeta_1 + \zeta_2| \geq c|\zeta_1|$. This is enough to show $f(z) \to -1$ as $z \to (1, 1)$ non-tangentially. A similar estimate will hold for more general rational inner functions. Specifically, the lowest order homogeneous term of $p(1 - \zeta_1, 1 - \zeta_2)$ will be non-vanishing in RHP^2.

It is also worth pointing out that a function can be bounded non-tangentially at every point even though it is globally unbounded. Let

$$g(z) = \frac{1 - z_1}{2 - z_1 - z_2}.$$

Then, $g(1-\zeta_1, 1-\zeta_2) = \zeta_1/(\zeta_1+\zeta_2)$ which is bounded in any non-tangential approach region to $(0,0)$ in RHP^2. At the same time, if we let $z(\theta) = (1-\theta^2)(e^{i\theta}, e^{-i\theta})$ then for θ close to 0

$$\begin{aligned} |g(z(\theta))| &= \frac{|1-(1-\theta^2)e^{i\theta}|}{2-2(1-\theta^2)\cos\theta} \\ &\geq C\frac{|\theta|}{1-\cos\theta+\theta^2\cos\theta} \\ &\geq C\frac{1}{|\theta|} \end{aligned}$$

which is unbounded.

The only common zero of p and $\tilde{p}$ on $\mathbb{T}^2$ is the point $(1,1)$, and this zero occurs with multiplicity 2. Therefore, by Theorem B, the space $\mathcal{P}_{0,0}$ is trivial which just means that

$$\frac{1}{2-z_1-z_2}$$

is not in $L^2(\mathbb{T}^2)$. Of course, this could be checked by direct computation but we emphasize that Theorem B lets us show this algebraically. Also, Theorem B tells us that $\mathcal{P}_{j,k} = (j+1)(k+1)-1$ so that the space $\mathcal{I}_p$ has co-dimension one among all polynomials, and by Theorem A for all $q \in \mathcal{I}_p$, $q(1,1) = 0$. Thus, $q/p \in L^2(\mathbb{T}^2)$ iff $q(1,1) = 0$. So, for example, we automatically know g above is in $L^2(\mathbb{T}^2)$.

As mentioned in the overview section, these results are proven by examining a sums of squares formula which in this case is

$$|p|^2 - |\tilde{p}|^2 = (1-|z_1|^2)2|1-z_2|^2 + (1-|z_2|^2)2|1-z_1|^2.$$

It follows from later work that we can multiply elements of $\mathcal{I}_p$ by some specific one variable polynomials $p_1(z_1), p_2(z_2)$ to force $p_1p_2\mathcal{I}_p \subset \mathcal{I}_p^\infty$. In this example, $(1-z_1)(1-z_2) \in \mathcal{I}_p^\infty$ so that G_2 from the introduction is in $L^\infty(\mathbb{T}^2)$. From this it is not hard to reason that G_1 is four times continuously differentiable and so if G_1 has Fourier coefficients $\{a_{n,m}\}$ then $\sum_{n,m}(n+1)^2(m+1)^2|a_{n,m}|^2 < \infty$ and therefore $\{a_{n,m}\} \in \ell^1$ by Cauchy-Schwarz. This shows we can recover many of the details of [22] from our theorems.

4. Background: Vector polynomials and matrix functions

In this section we make a few general observations about vectors and vector polynomials as well as vector-valued Hardy spaces and reproducing kernels. Let

$$\begin{aligned} \mathbb{C}^n &= \text{The space of } n\text{-dimensional column vectors} \\ \mathbb{C}^{1\times n} &= \text{The space of } n\text{-dimensional row vectors} \\ \mathbb{C}^{n\times m} &= \text{The space of } n \times m \text{ matrices with entries in } \mathbb{C} \\ V[z_1, z_2] &= \text{The space of two variable polynomials with coefficients in } V \end{aligned}$$

where V is some vector space such as $\mathbb{C}^n, \mathbb{C}^{1\times n}, \mathbb{C}^{m\times n}$.

A theorem which is useful for dealing with vector polynomial equations is the polarization theorem for holomorphic functions. See [14] for a proof.

Theorem 4.1 (Polarization Theorem)**.** *Suppose $F : \Omega \times \Omega^* \to \mathbb{C}$ is holomorphic where $\Omega \subset \mathbb{C}^n$ is a domain and $\Omega^* = \{\bar{z} : z \in \Omega\}$. If $F(z, \bar{z}) = 0$ for all $z \in \Omega$, then $F(z, w) = 0$ for all $(z, w) \in \Omega \times \Omega^*$.*

An important instance of the polarization theorem is the following proposition.

Proposition 4.2. *If $\vec{A} \in \mathbb{C}^n[z_1, z_2], \vec{B} \in \mathbb{C}^m[z_1, z_2]$ and if $|\vec{A}(z)|^2 = |\vec{B}(z)|^2$, then there exists an $m \times n$ isometric matrix U such that $U\vec{A}(z) = \vec{B}(z)$. If $\vec{A}$ and $\vec{B}$ have linearly independent entries, then $m = n$ and U is a unitary.*

Proof. By the polarization theorem

$$\vec{A}(w)^*\vec{A}(z) = \vec{B}(w)^*\vec{B}(z),$$

in this case $F(z,w) = \vec{A}(\bar{w})^*\vec{A}(z) - \vec{B}(\bar{w})^*\vec{B}(z)$ and by assumption $F(z, \bar{z}) \equiv 0$.

In this situation, $\vec{A}$ and $\vec{B}$ are related by an isometric matrix. Indeed, the map

$$\vec{A}(z) \mapsto \vec{B}(z)$$

extends linearly to an isometry from the span of the vectors on the left to the span of the vectors on the right. Indeed, if $a : \mathbb{C}^2 \to \mathbb{C}$ is a finitely supported function, then for $v_1 = \sum_{z\in\mathbb{C}^2} a(z)\vec{A}(z)$ and $v_2 = \sum_{z\in\mathbb{C}^2} a(z)\vec{B}(z)$ we have

$$v_1^*v_1 = \sum_{z,w\in\mathbb{C}^2} \overline{a(w)}a(z)\vec{A}(w)^*\vec{A}(z) = \sum_{z,w\in\mathbb{C}^2} \overline{a(w)}a(z)\vec{B}(w)^*\vec{B}(z) = v_2^*v_2.$$

Thus, $v_1 \mapsto v_2$ is at once well-defined ($|v_1| = 0$ iff $|v_2| = 0$), linear, and isometric.

This isometry is initially defined on $\text{span}\{\vec{A}(z) : z \in \mathbb{C}^2\}$, but it can be extended to all of $\mathbb{C}^n$ by standard linear algebra and can then be realized via an $m \times n$ isometric matrix V: $V\vec{A}(z) = \vec{B}(z)$.

If the entries of $\vec{A}$ and $\vec{B}$ form a linearly independent set of polynomials, then $\text{span}\{\vec{A}(z) : z \in \mathbb{C}^2\} = \mathbb{C}^n$ and $\text{span}\{\vec{B}(z) : z \in \mathbb{C}^2\} = \mathbb{C}^m$, and $m = n$ because these spaces are related by an isometry. □

Often in this paper, we break apart a vector polynomial $\vec{A} \in \mathbb{C}^N[z_1, z_2]$ into one variable pieces. For instance, if $\vec{A}$ has degree at most $n-1$ in z_1 then it is possible to write

$$\vec{A}(z) = A(z_2)\Lambda_n(z_1)$$

where $A \in \mathbb{C}^{N\times n}[z_2]$ is a matrix polynomial and

$$\Lambda_n(z_1) \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ z_1 \\ \vdots \\ z_1^{n-1} \end{pmatrix} \in \mathbb{C}^n[z_1]. \tag{4.1}$$

This is simply a way of extracting the coefficients of powers of z_1 into a matrix.

Vector polynomials appear most often in this paper in relation to reproducing kernels. If $\mathcal{H}$ is a finite dimensional Hilbert space of polynomials and if $\vec{H}$ is a vector polynomial whose entries form an orthonormal basis for $\mathcal{H}$, then $k_w(z) = \vec{H}(w)^*\vec{H}(z)$ is a reproducing kernel for $\mathcal{H}$ in the sense that

$$\langle f, k_w\rangle_{\mathcal{H}} = f(w)$$

for all $f \in \mathcal{H}$. This formula can be proven by first verifying it for the entries of $\vec{H}$; the general formula then follows by linearity.

If the entries of $\vec{H}$ are not an orthonormal basis, then $\vec{H}(w)^*\vec{H}(z)$ need not be a reproducing kernel for $\mathcal{H}$. Nevertheless, an expression of this form can be characterized as being a *positive semi-definite kernel function* which abstractly refers to a function $k : \Omega \times \Omega \to \mathbb{C}$ with the property that for any finitely supported function $a : \Omega \to \mathbb{C}$ we have

$$\sum_{z,w\in\Omega} a(z)\overline{a(w)}k(z,w) \geq 0.$$

Here Ω is just a set, but if Ω is actually a domain and $k(z,w)$ is a polynomial in $z, \bar{w}$ then we get the following.

Proposition 4.3. *Suppose Ω is a domain in $\mathbb{C}^n$ and k is a positive semi-definite kernel function such that $k(z,w)$ is a polynomial in $z, \bar{w}$. Then, there exists a vector polynomial $\vec{H}$ such that $k(z,w) = \vec{H}(w)^*\vec{H}(z)$.*

Proof. We build a Hilbert space $\mathcal{H}$ and an inner product such that $k(z,w)$ is the reproducing kernel. Indeed, let $\mathcal{H}$ be the finite dimensional vector space $\text{span}\{k_w : w \in \Omega\}$ where $k_w(z) \stackrel{\text{def}}{=} k(z,w)$. Declare that $\langle k_w, k_z\rangle_{\mathcal{H}} = k(z,w)$ and extend by linearity to all of $\mathcal{H}$. This is well-defined because for any finitely supported function $a : \Omega \to \mathbb{C}$, if the polynomial $f(z) = \sum_{w\in\Omega} \overline{a(w)}k_w(z)$ is identically zero then $\langle f, g\rangle_{\mathcal{H}} = 0$ simply because if $g = \sum_{z\in\Omega} \overline{b(z)}k_z$ then

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{z\in\Omega} b(z)f(z) = 0.$$

This is also a bona fide inner product because if $\langle f, f\rangle_{\mathcal{H}} = 0$, then $|f(z)|^2 = |\langle f, k_z\rangle_{\mathcal{H}}|^2 \leq \langle f, f\rangle_{\mathcal{H}} k(z,z) = 0$ for all z so that $f = 0$. This inequality follows from Cauchy-Schwarz for an a priori semi-definite inner product.

Thus, $\mathcal{H}$ is a finite dimensional reproducing kernel Hilbert space (consisting of polynomials) and with reproducing kernel $k(z,w)$. Any vector polynomial $\vec{H}$ whose entries form an orthonormal basis for $\mathcal{H}$ will yield the formula for the reproducing kernel we seek:

$$k(z,w) = \vec{H}(w)^*\vec{H}(z).$$

□

As pointed out by the referee, the above proof is a special case of the construction of a Hilbert space from a positive kernel due to Aronszajn [7], who in turn attributes it to E.H. Moore.

Next, we turn to some background on vector-valued Hardy spaces. Let $H^2_{1\times n}$ denote the row-vector valued Hardy space on the unit circle: $H^2_{1\times n} \stackrel{\text{def}}{=} H^2(\mathbb{T}) \otimes \mathbb{C}^{1\times n}$. Row vectors end up being natural for what follows because we chose column representations for vector polynomials. An $n \times n$ matrix function Φ whose entries are rational functions of $z \in \mathbb{C}$ with no poles in $\overline{\mathbb{D}}$ is a *matrix rational inner function* if Φ is unitary valued on $\mathbb{T}$:

$$\Phi(z)^*\Phi(z) = I \qquad z \in \mathbb{T}.$$

By the maximum principle, $\|\Phi(z)\| \leq 1$ for all $z \in \mathbb{D}$—actually, by the maximum principle applied to $v_1^*\Phi(z)v_2$ for all $v_1, v_2 \in \mathbb{C}^n$. Note $H^2_{1\times n}\Phi \subset H^2_{1\times n}$; multiplication on right by Φ looks odd, but since our space consists of row vector valued functions it is correct.

Proposition 4.4. *Let Φ be a $n\times n$ matrix rational inner function. The space $H^2_{1\times n}\ominus H^2_{1\times n}\Phi$ is finite dimensional, consists of rational vector-valued functions with no poles in $\overline{\mathbb{D}}$, and has reproducing kernel*

$$K_w(z) = \frac{I-\Phi(w)^*\Phi(z)}{1-\bar{w}z}.$$

The last statement means that for any $\vec{f}\in H^2_{1\times n}\ominus H^2_{1\times n}\Phi$ and for $w\in\mathbb{D}, v\in\mathbb{C}^{1\times n}$

$$\langle \vec{f}, vK_w\rangle_{H^2_{1\times n}} = \langle \vec{f}(w), v\rangle_{\mathbb{C}^{1\times n}}.$$

Because of this reproducing property K is a positive semidefinite kernel function which for matrix valued kernels means that for any finitely supported function $\vec{a}:\mathbb{D}\to\mathbb{C}^n$ we have

$$\sum_{z,w\in\mathbb{D}} \vec{a}(w)^* K(z,w)\vec{a}(z)\geq 0.$$

More generally though, if Φ is an analytic $n\times m$ matrix valued function such that $\|\Phi(z)\|\leq 1$ for all $z\in\mathbb{D}$, then K defined as above will still be a positive semidefinite kernel. See [2] for instance.

Proof of Proposition. Let $\operatorname{adj}(\Phi)$ be the adjugate or "classical adjoint" of Φ. Then, $\Phi\operatorname{adj}(\Phi)=\det(\Phi)I_n$. Note that $b\overset{\text{def}}{=}\det(\Phi)$ is a finite Blaschke product.

Now, $H^2_{1\times n}\det(\Phi)I_n\subset H^2_{1\times n}\Phi$ so that $H^2_{1\times n}\ominus H^2_{1\times n}\Phi\subset H^2_{1\times n}\ominus H^2_{1\times n}\det(\Phi)I_n$, and the latter space is a direct sum of the scalar spaces $H^2\ominus bH^2$. This last space is finite dimensional and consists of rational functions with no poles in $\overline{\mathbb{D}}$. To see this, write $b=\mu\frac{\tilde{p}}{p}$ where $|\mu|=1$, $p\in\mathbb{C}[z]$ and $\tilde{p}(z)=z^N\overline{p(1/\bar{z})}$ for some N. Then, $f\perp bH^2$ iff $f\bar{z}^Np=\bar{z}\bar{p}\bar{g}$ for some $g\in H^2$. Or, $fp=\bar{z}\tilde{p}\bar{g}\in H^2\cap Z^{N-1}\overline{H^2}$ which means fp is a polynomial of degree at most $N-1$ and therefore f is a rational function with denominator p and numerator with degree at most $N-1$.

The formula for the reproducing kernel is a basic calculation depending on the fact that Φ is unitary valued on $\mathbb{T}$ and bounded and holomorphic in $\mathbb{D}$. We omit the details. □

A useful construction of matrix rational inner functions is given below.

Lemma 4.5. *If U is a unitary matrix written in block form $U=\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}$ and if $\Xi(z)\overset{\text{def}}{=} U_{11}+zU_{12}(I-zU_{22})^{-1}U_{21}$, then Ξ is a matrix valued rational inner function.*

Proof. Observe

$$\begin{pmatrix}U_{11} & U_{12}\\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix} I\\ z(I-zU_{22})^{-1}U_{21}\end{pmatrix} = \begin{pmatrix}\Xi(z)\\ (I-zU_{22})^{-1}U_{21}\end{pmatrix}.$$

Since U is a unitary

$$I+|z|^2U_{21}^*(I-\bar{z}U_{22}^*)^{-1}(I-zU_{22})^{-1}U_{21} = \Xi(z)^*\Xi(z)+U_{21}^*(I-\bar{z}U_{22}^*)^{-1}(I-zU_{22})^{-1}U_{21}$$

which rearranges to

$$I-\Xi(z)^*\Xi(z) = (1-|z|^2)U_{21}^*(I-\bar{z}U_{22}^*)^{-1}(I-zU_{22})^{-1}U_{21}.$$

This shows $\Xi(z)$ is unitary valued on $\mathbb{T}$ except possibly at points where $\det(I-zU_{22})=0$. But, there can only be finitely many such points and since Ξ is bounded near these points any singularities (which are at worst poles) must be removable. Thus, Ξ is unitary valued on all of $\mathbb{T}$ and holomorphic on $\overline{\mathbb{D}}$. $\square$

5. Background: the Hilbert space $L^2(\frac{d\sigma}{|p|^2})$

Definition 5.1. Let $p \in \mathbb{C}[z_1, z_2]$ have degree (n, m) and set

$$\tilde{p}(z) \stackrel{\text{def}}{=} z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}, \text{ the } \textit{reflection} \text{ of } p.$$

We shall say p is *scattering stable* if p has no zeros in $\mathbb{D}^2$ and if p and $\tilde{p}$ have no common factors.

Scattering stable polynomials have at most finitely many zeros on $\mathbb{T}^2$ by Bézout's theorem since zeros on $\mathbb{T}^2$ are shared with the reflected polynomial.

Let p be scattering stable. We will work in the Hilbert space $L^2(\frac{d\sigma}{|p|^2})$ where $d\sigma$ is normalized Lebesgue measure on $\mathbb{T}^2$. All orthogonal complements and orthogonal direct sums below are taken with respect to this space.

Let

$$\mathcal{P}_{j,k} \stackrel{\text{def}}{=} \{f \in \mathbb{C}[z_1, z_2] \cap L^2(\frac{d\sigma}{|p|^2}) : \deg f \leq (j, k)\}$$

where $\deg f$ denotes the bidegree of f—the ordered pair consisting of the degree in z_1, the degree in z_2.

Notation 5.2. We define a number of spaces using orthogonal complements.

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \mathcal{P}_{n-1,m} \ominus z_2\mathcal{P}_{n-1,m-1}, \quad \mathcal{F}_1 \stackrel{\text{def}}{=} \mathcal{P}_{n-1,m} \ominus \mathcal{P}_{n-1,m-1},$$

$$\mathcal{E}_2 \stackrel{\text{def}}{=} \mathcal{P}_{n,m-1} \ominus z_1\mathcal{P}_{n-1,m-1}, \quad \mathcal{F}_2 \stackrel{\text{def}}{=} \mathcal{P}_{n,m-1} \ominus \mathcal{P}_{n-1,m-1},$$

and

$$\mathcal{G} \stackrel{\text{def}}{=} \mathcal{P}_{n-1,m-1}.$$

By Lemma 6.7 and Theorem 7.4 of [25],

$$\dim \mathcal{E}_1 = \dim \mathcal{F}_1 = n \text{ and } \dim \mathcal{E}_2 = \dim \mathcal{F}_2 = m.$$

The computation of $\dim \mathcal{G}$ is a main result of this paper. It is clear that $\dim \mathcal{G} \leq nm$.

Notation 5.3. We let $\vec{E}_1, \vec{F}_1 \in \mathbb{C}^n[z_1, z_2], \vec{E}_2, \vec{F}_2 \in \mathbb{C}^m[z_1, z_2]$ be vector polynomials whose entries form an orthonormal basis for $\mathcal{E}_1, \mathcal{F}_1, \mathcal{E}_2, \mathcal{F}_2$, resp. Let $\vec{G}$ be a vector polynomial whose entries form an orthonormal basis for $\mathcal{G}$. Note that these vector polynomials are unique up to multiplication by a unitary matrix on the left.

Proposition 5.4. *Let p be scattering stable. Using Notation 5.3, there exist choices of orthonormal bases for $\mathcal{E}_1, \mathcal{E}_2$ so that*

$$\vec{E}_1(z) = z_1^{n-1} z_2^m \overline{\vec{F}_1(1/\bar{z}_1, 1/\bar{z}_2)} \textit{ and } \vec{E}_2(z) = z_1^n z_2^{m-1} \overline{\vec{F}_2(1/\bar{z}_1, 1/\bar{z}_2)}.$$

Throughout the paper, we will assume $\vec{F}_1, \vec{F}_2$ satisfy the relationship above.

Proof. The map on $L^2(\frac{d\sigma}{|p|^2})$ given by

$$Tf = z_1^{n-1} z_2^m \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$$

is an anti-unitary meaning

$$\langle Tf, Tg\rangle_{L^2(\frac{d\sigma}{|p|^2})} = \langle g, f\rangle_{L^2(\frac{d\sigma}{|p|^2})}$$

for all $f, g \in L^2(\frac{d\sigma}{|p|^2})$. So, it preserves orthogonality and maps an orthonormal basis to an orthonormal basis. Because of this, T maps $\mathcal{F}_1$ to $\mathcal{E}_1$ and therefore the entries of $z_1^{n-1} z_2^m \overline{\vec{F}_1(1/\bar{z}_1, 1/\bar{z}_2)}$ are an orthonormal basis for $\mathcal{E}_1$. Hence, this vector polynomial is a valid choice for $\vec{E}_1$. The formula for $\vec{E}_2$ is similar. □

Definition 5.5. If vector polynomials $\vec{A}_1, \vec{A}_2$ satisfy

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{2} (1 - |z_j|^2)|\vec{A}_j(z)|^2$$

the above formula will be called an *Agler decomposition* and $(\vec{A}_1, \vec{A}_2)$ will be called an *Agler pair* for p.

Note that this formula can be polarized. The following key theorem says, among other things, that $(\vec{E}_1, \vec{F}_2)$ and $(\vec{F}_1, \vec{E}_2)$ are both Agler pairs. It is proven as Corollary 7.5 and Proposition 5.5 of [25]. It can also be extracted from [11].

Theorem 5.6. *Let p be scattering stable. Then, using Notation 5.3*

$$\begin{aligned}
\overline{p(w)}p(z) - \overline{\tilde{p}(w)}\tilde{p}(z) =& (1 - \bar{w}_1 z_1)\vec{E}_1(w)^*\vec{E}_1(z) + (1 - \bar{w}_2 z_2)\vec{F}_2(w)^*\vec{F}_2(z) \\
=& (1 - \bar{w}_1 z_1)\vec{F}_1(w)^*\vec{F}_1(z) + (1 - \bar{w}_2 z_2)\vec{E}_2(w)^*\vec{E}_2(z) \\
=& \sum_{j=1}^{2} (1 - \bar{w}_j z_j)\vec{F}_j(w)^*\vec{F}_j(z) \\
& + (1 - \bar{w}_1 z_1)(1 - \bar{w}_2 z_2)\vec{G}(w)^*\vec{G}(z)
\end{aligned}$$

and

$$\vec{G}(w)^*\vec{G}(z) = \frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{F}_1(w)^*\vec{F}_1(z)}{1 - \bar{w}_2 z_2} = \frac{\vec{E}_2(w)^*\vec{E}_2(z) - \vec{F}_2(w)^*\vec{F}_2(z)}{1 - \bar{w}_1 z_1}.$$

Example 5.7. To get a feel for the theorem, it helps to look at a trivial example $p(z) = 1$ thought of as a polynomial of degree $(1, 2)$ so that $\tilde{p}(z) = z_1 z_2^2$. Then, $\mathcal{E}_1 = \mathbb{C}, \mathcal{F}_1 = \mathbb{C}z_2^2, \mathcal{E}_2 = \text{span}\{1, z_2\}, \mathcal{F}_2 = \text{span}\{z_1, z_1 z_2\}, \mathcal{G} = \text{span}\{1, z_2\}$. And, $\vec{E}_1(z) = 1, \vec{F}_1(z) = z_2^2, \vec{E}_2(z) = \begin{pmatrix} 1 \\ z_2 \end{pmatrix}, \vec{F}_2(z) = z_2\vec{E}_2(z), \vec{G}(z) = \vec{E}_2(z)$. The resulting formulas (evaluated on the diagonal $z = w$) are

$$\begin{aligned}
1 - |z_1 z_2^2|^2 &= (1 - |z_1|^2) + (1 - |z_2|^2)|z_1|^2(1 + |z_2|^2) \\
&= (1 - |z_1|^2)|z_2^2|^2 + (1 - |z_2|^2)(1 + |z_2|^2)
\end{aligned}$$

and so on. Of course everything is so easy in this case because $L^2(\frac{d\sigma}{|p|^2}) = L^2(\mathbb{T}^2)$. ◆

Definition 5.8. Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable and $\deg p = (n, m)$. Define $\vec{E}_j, \vec{F}_j$ as above. We will refer to $(\vec{E}_1, \vec{F}_2)$ as the *max-min Agler pair* of p and $(\vec{F}_1, \vec{E}_2)$ as the *min-max Agler pair* of p. We will refer to either of these pairs as a *canonical Agler pair*.

In general, there are important orthogonality relations which hold in $L^2(\frac{d\sigma}{|p|^2})$. Note that $L^2(\frac{d\sigma}{|p|^2}) \subset L^2(d\sigma) = L^2(\mathbb{T}^2)$ and therefore Fourier coefficients, $\hat{f}(j,k) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} f \bar{z}_1^j \bar{z}_2^k d\sigma$, are well-defined for elements of $L^2(\frac{d\sigma}{|p|^2})$. Let $\text{supp}\hat{f} = \{(j,k) \in \mathbb{Z}^2 : \hat{f}(j,k) \neq 0\}$.

Theorem 5.9. *Let p be scattering stable with $\deg p = (n, m)$. Using Notation 5.2 we have that in $L^2(\frac{d\sigma}{|p|^2})$*

$$\mathcal{F}_1 \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j \geq 0), (k < m)\}\}$$

$$\mathcal{E}_1 \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j < n), (k > 0)\}\}$$

$$\mathcal{F}_2 \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j < n), (k \geq 0)\}\}$$

$$\mathcal{E}_2 \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j > 0), (k < m)\}\}$$

$$p \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (k > 0) \text{ or } (k = 0 \text{ and } j > 0)\}\}$$

$$p \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j > 0) \text{ or } (j = 0 \text{ and } k > 0)\}\}$$

$$\tilde{p} \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (k < m) \text{ or } (k = m \text{ and } j < n)\}\}$$

$$\tilde{p} \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : supp\hat{f} \subset \{(j,k) : (j < n) \text{ or } (j = n \text{ and } k < m)\}\}$$

We recommend drawing pictures of the various support sets above. In Appendix A, we explain how this follows from the work in [11, 25]. One direct consequence we use later is

$$\begin{aligned} p &\perp \bar{z}_1^j z_2^k \mathcal{G} \text{ for } j \geq 0 \text{ and } k > 0 \\ \tilde{p} &\perp \bar{z}_1^j z_2^k \mathcal{G} \text{ for } j \geq 0 \text{ and } k \geq 0 \text{ (sic).} \end{aligned} \tag{5.1}$$

An important corollary of the above orthogonality conditions is the following.

Corollary 5.10. *Using the setup of the previous theorem, if $N \geq n-1, M \geq m-1$, then*

$$\mathcal{P}_{N,M} = \mathcal{G} \oplus \bigoplus_{j=0}^{N-n} z_1^j \mathcal{F}_2 \oplus \bigoplus_{k=0}^{M-m} z_2^k \mathcal{F}_1 \oplus \bigoplus_{\substack{0 \leq j \leq N-n \\ 0 \leq k \leq M-m}} z_1^j z_2^k \mathbb{C}\tilde{p}$$

A direct sum of the form $\bigoplus_{j=0}^{-1}$ is to be interpreted as the trivial subspace. Before we prove the corollary we discuss a few special cases and variations. The case $N = n, M = m$ is

$$\mathcal{P}_{n,m} = \mathcal{G} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathbb{C}\tilde{p}. \tag{5.2}$$

If we apply the anti-unitary reflection operation $f \mapsto z_1^n z_2^m \overline{f(1/\bar{z}_1 1/\bar{z}_2)}$, we get the decomposition

$$\mathcal{P}_{n,m} = z_1 z_2 \mathcal{G} \oplus z_1 \mathcal{E}_1 \oplus z_2 \mathcal{E}_2 \oplus \mathbb{C}p. \tag{5.3}$$

A couple other useful variations of the corollary are

$$\mathcal{P}_{N,M} = z_2^{M-m+1}\mathcal{G} \oplus \bigoplus_{j=0}^{N-n} z_1^j \mathcal{F}_2 \oplus \bigoplus_{k=0}^{M-m} z_2^k \mathcal{E}_1 \oplus \bigoplus_{\substack{0\leq j\leq N-n\\ 0\leq k\leq M-m}} z_1^j z_2^k \mathbb{C}\tilde{p} \tag{5.4}$$

$$\mathcal{P}_{N,M} = z_2^{M-m+1}\mathcal{G} \oplus \bigoplus_{j=0}^{N-n} z_2^{M-m+1} z_1^j \mathcal{F}_2 \oplus \bigoplus_{k=0}^{M-m} z_1^{N-n+1} z_2^k \mathcal{E}_1 \oplus \bigoplus_{\substack{0\leq j\leq N-n\\ 0\leq k\leq M-m}} z_1^j z_2^k \mathbb{C}p \tag{5.5}$$

The formula (5.4) follows from applying the reflection operation $f \mapsto z_1^{n-1} z_2^M \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$ to $\mathcal{P}_{n-1,M} = \mathcal{G} \oplus \bigoplus_{k=0}^{M-m} z_2^k \mathcal{F}_1$ to get $\mathcal{P}_{n-1,M} = z_2^{M-m+1}\mathcal{G} \oplus \bigoplus_{k=0}^{M-m} z_2^k \mathcal{E}_1$. The formula (5.5) follows from reflecting formula (5.4) and then interchanging the roles of z_1 and z_2.

Remark 5.11. A consequence of (5.4) or (5.5) and Theorem 5.9 is that $\mathcal{P}_{N,M} \ominus z_2^{M-m+1}\mathcal{G}$ is orthogonal to the entire quadrant

$$\{f \in L^2(\frac{d\sigma}{|p|^2}) : \text{supp}\hat{f} \subset \{(j,k) : j < n \text{ and } k > M-m\}\}.$$

Another way to say this is if $g \in L^2(\frac{d\sigma}{|p|^2})$ has finite Fourier support (i.e. is a Laurent polynomial) and the Fourier support is contained in

$$\{(j,k) : j \geq 0 \text{ and } k < m\}$$

and if $g \perp \mathcal{G}$, then in fact g is orthogonal to

$$\{f \in L^2(\frac{d\sigma}{|p|^2}) : \text{supp}\hat{f} \subset \{(j,k) : j < n \text{ and } k \geq 0\}\}.$$

This follows by multiplying everything in the previous statement by a power of $\bar{z}_2$.

Proof of Corollary 5.10. The fact that all of the spaces involved are pairwise orthogonal follows directly from Theorem 5.9.

Since by definition $\mathcal{P}_{n-1,m-1} = \mathcal{G}$, by induction it is enough to show for $j,k \geq 0$

$$\mathcal{P}_{n+j,m-1+k} = \mathcal{P}_{n-1+j,m-1+k} \oplus z_1^j \mathcal{F}_2 \oplus \bigoplus_{\alpha=0}^{k-1} z_1^j z_2^\alpha \mathbb{C}\tilde{p}$$

and by symmetry a similar relation holds for $\mathcal{P}_{n-1+j,m+k}$.

Our orthogonality relations directly show

$$z_1^j \mathcal{F}_2 \oplus \bigoplus_{\alpha=0}^{k-1} z_1^j z_2^\alpha \mathbb{C}\tilde{p} \subset \mathcal{P}_{n+j,m-1+k} \ominus \mathcal{P}_{n-1+j,m-1+k}. \tag{5.6}$$

The space on the left is $m+k$ dimensional. The space on the right is at most $m+k$ dimensional. Indeed, more than $m+k$ elements in this space would necessarily be linearly dependent as some combination of them would have no Fourier support on $\{(n+j,0),(n+j,1),\dots,(n+j,m-1+k)\}$ and hence would be orthogonal to itself.

Since the dimensions of both sides of (5.6) are equal we must have equality and not just inclusion. $\square$

There is more we can say about $\vec{E}_j, \vec{F}_j$. Recalling (4.1), we may write

$$\begin{aligned} \vec{E}_1(z) &= E_1(z_2)\Lambda_n(z_1) & \vec{F}_1(z) &= F_1(z_2)\Lambda_n(z_1) \\ \vec{E}_2(z) &= E_2(z_1)\Lambda_m(z_2) & \vec{F}_2(z) &= F_2(z_1)\Lambda_m(z_2) \end{aligned} \tag{5.7}$$

for some matrix polynomials $E_1, F_1 \in \mathbb{C}^{n\times n}[z_2]$, $E_2, F_2 \in \mathbb{C}^{m\times m}[z_1]$. **Warning**: E_1, F_1 are functions of z_2, while E_2, F_2 are functions of z_1!

By Proposition 5.4,

$$E_1(z_2) = z_2^m \overline{F_1(1/\bar{z}_2)} X_n \qquad E_2(z_1) = z_1^n \overline{F_2(1/\bar{z}_1)} X_m \tag{5.8}$$

where $X_N \in \mathbb{C}^{N\times N}$ is the matrix

$$X_N = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} = (\delta_{j,N-k})_{j,k=1,\dots,N} \tag{5.9}$$

which appears due to the fact that $z^{N-1}\Lambda_N(1/z) = X_N \Lambda_N(z)$. We emphasize that we are taking entrywise complex conjugates of the above matrices F_1, F_2.

Proposition 5.12. *With the above definitions,* $\det E_1(z_2), \det E_2(z_1)$ *are non-vanishing for* $z_1, z_2 \in \mathbb{D}$*, while all zeros of* $\det F_1(z_2), \det F_2(z_1)$ *are in* $\overline{\mathbb{D}}$*.*

This proposition follows from Proposition 5.3 and Lemma 6.7 of [25] and (5.8) above.

In Appendix B, we discuss how E_1 and E_2 can be constructed using the one variable matrix Fejér-Riesz lemma.

6. General Agler pairs

In this section, we examine how general Agler pairs relate to the canonical Agler pairs constructed earlier. Along the way, we relate the spaces $\mathcal{E}_j, \mathcal{F}_j$ to certain spaces of one variable vector valued functions and this combined with Corollary 5.10 produces a list of generators for $\mathcal{I}_p$.

Our first observation is that every Agler decomposition leads to what is known in systems engineering terminology as a transfer function representation or realization. This will show the canonical pairs $(\vec{E}_1, \vec{F}_2), (\vec{F}_1, \vec{E}_2)$ are minimal in a certain sense.

Lemma 6.1. *Assume p is scattering stable and* $\deg p = (n, m)$*. Let $(\vec{A}_1, \vec{A}_2)$ be an Agler pair for p with $\vec{A}_1 \in \mathbb{C}^N[z_1, z_2], \vec{A}_2 \in \mathbb{C}^M[z_1, z_2]$. Then, there exists a $(1+N+M)\times(1+N+M)$ unitary matrix $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the block decomposition corresponds to the direct sum decomposition $\mathbb{C}^{1+N+M} = \mathbb{C} \oplus \mathbb{C}^{N+M}$, such that*

$$\frac{\tilde{p}(z)}{p(z)} = A + B\Delta(z)(I - D\Delta(z))^{-1}C$$

where $\Delta(z) = \begin{pmatrix} z_1 I_N & 0 \\ 0 & z_2 I_M \end{pmatrix}$.

Although the Agler pair does not appear directly in the conclusion of this lemma, the sizes N and M *do* appear and that is the main purpose of the lemma. The proof (which does involve the Agler pair) is a standard argument, so we relegate it to Appendix A for the curious reader.

Lemma 6.2. *Let p be scattering stable and $\deg p = (n, m)$. Let $\vec{A}_1 \in \mathbb{C}^N[z_1, z_2]$ and $\vec{A}_2 \in \mathbb{C}^M[z_1, z_2]$ and let $(\vec{A}_1, \vec{A}_2)$ be an Agler pair for p. Then $N \geq n$, $M \geq m$, $\vec{A}_1$ has bidegree at most $(n-1, m)$, and $\vec{A}_2$ has bidegree at most $(n, m-1)$.*

Proof. The bounds $N \geq n, M \geq m$ follow from Lemma 6.1 because the numerator and denominator of the transfer function realization for $\tilde{p}/p$ have bidegree at most (N, M) and since p and $\tilde{p}$ have no common factors this bidegree bound holds for p and $\tilde{p}$ as well. The bounds on bidegrees for $\vec{A}_1, \vec{A}_2$ follow from Theorem 2.10 of [26]. □

Using the lemma we can write

$$\vec{A}_1(z) = A_1(z_2)\Lambda_n(z_1) \quad \vec{A}_2(z) = A_2(z_1)\Lambda_m(z_2) \tag{6.1}$$

for one variable matrix polynomials $A_1 \in \mathbb{C}^{N\times n}[z_2], A_2 \in \mathbb{C}^{M\times m}[z_1]$.

One of the main results of [25] was a characterization of canonical Agler pairs. The following is a direct result of Theorem 1.3 of [25] and Proposition 5.12 above.

Lemma 6.3. *Assume the setup of the previous lemma. If $A_1(z_2)$ is invertible for all $z_2 \in \mathbb{D}$, then $(\vec{A}_1, \vec{A}_2) = (\vec{E}_1, \vec{F}_2)$ up to unitary multiplication. If $A_2(z_1)$ is invertible for all $z_1 \in \mathbb{D}$, then $(\vec{A}_1, \vec{A}_2) = (\vec{F}_1, \vec{E}_2)$ up to unitary multiplication.*

Lemma 6.4. *Assume the setup of the previous lemma. If we define A_1, A_2 as in* (6.1) *then*

$$A_1(z_2)E_1(z_2)^{-1} \qquad A_2(z_1)E_2(z_1)^{-1} \tag{6.2}$$

are holomorphic in $\mathbb{D}$ and extend to be holomorphic and isometry-valued on $\mathbb{T}$.

If $N = n$, then $A_1E_1^{-1}$ is rational inner and $F_1A_1^{-1}$ is also a well-defined matrix rational inner function. Similarly, if $M = m$, then $A_2E_2^{-1}$ and $F_2A_2^{-1}$ are matrix rational inner functions.

In particular,

$$\Phi_1(z_2) := F_1(z_2)E_1(z_2)^{-1} \quad \Phi_2(z_1) := F_2(z_1)E_2(z_1)^{-1}$$

are both one variable matrix rational inner functions on $\mathbb{D}$.

Proof. Since $(\vec{A}_1, \vec{A}_2)$ is an Agler pair,

$$\sum_{j=1}^{2}(1-\bar{w}_j z_j)\vec{A}_j(w)^*\vec{A}_j(z) = (1-\bar{w}_1 z_1)\vec{E}_1(w)^*\vec{E}_1(z) + (1-\bar{w}_2 z_2)\vec{F}_2(w)^*\vec{F}_2(z). \tag{6.3}$$

If we set $z_2 = w_2 \in \mathbb{T}$ we get $\vec{A}_1(w)^*\vec{A}_1(z) = \vec{E}_1(w)^*\vec{E}_1(z)$ and if we rewrite in terms of matrices we get

$$\Lambda_n(w_1)^*E_1(z_2)^*E_1(z_2)\Lambda_n(z_1) = \Lambda_n(w_1)^*A_1(z_2)^*A_1(z_2)\Lambda_n(z_1)$$

for $z_1, w_1 \in \mathbb{C}$ and $z_2 \in \mathbb{T}$. So,

$$E_1(z_2)^*E_1(z_2) = A_1(z_2)^*A_1(z_2) \tag{6.4}$$

for $z_2 \in \mathbb{T}$. This implies $\Phi(z_2) \stackrel{\text{def}}{=} A_1(z_2)E_1(z_2)^{-1}$ is isometry-valued on $\mathbb{T}$—in particular any singularities on $\mathbb{T}$ are removable—and extends to be holomorphic on $\overline{\mathbb{D}}$. An analogous argument holds for $A_2 E_2^{-1}$.

As discussed in Section 4

$$\frac{I - \Phi(w_2)^*\Phi(z_2)}{1 - \bar{w}_2 z_2}$$

is positive semi-definite and after multiplying on the left by $\Lambda_n(w_1)^* E_1(w_2)^*$ and the right by $E_1(z_2)\Lambda_n(z_1)$ we see that

$$\frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{A}_1(w)^*\vec{A}_1(z)}{1 - \bar{w}_2 z_2}$$

is also positive semi-definite. By (6.3),

$$(1 - \bar{w}_1 z_1)(\vec{E}_1(w)^*\vec{E}_1(z) - \vec{A}_1(w)^*\vec{A}_1(z)) = (1 - \bar{w}_2 z_2)(\vec{A}_2(w)^*\vec{A}_2(z) - \vec{F}_2(w)^*\vec{F}_2(z)),$$

and so $(1 - \bar{w}_1 z_1)$ divides the right hand side and $(1 - \bar{w}_2 z_2)$ divides the left. So,

$$\frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{A}_1(w)^*\vec{A}_1(z)}{1 - \bar{w}_2 z_2} = \frac{\vec{A}_2(w)^*\vec{A}_2(z) - \vec{F}_2(w)^*\vec{F}_2(z)}{1 - \bar{w}_1 z_1} \tag{6.5}$$

is a positive semi-definite polynomial and similarly so is

$$\frac{\vec{E}_2(w)^*\vec{E}_2(z) - \vec{A}_2(w)^*\vec{A}_2(z)}{1 - \bar{w}_1 z_1} = \frac{\vec{A}_1(w)^*\vec{A}_1(z) - \vec{F}_1(w)^*\vec{F}_1(z)}{1 - \bar{w}_2 z_2}. \tag{6.6}$$

Now assume $N = n$. Then, A_1 is square and hence $A_1 E_1^{-1}$ will be rational inner. We now show $\Psi(z_2) \stackrel{\text{def}}{=} F_1(z_2)A_1(z_2)^{-1}$ is a matrix rational inner function. By Proposition 4.3 we can factor (6.6) as $\vec{H}(w)^*\vec{H}(z)$ for some vector polynomial $\vec{H} \in \mathbb{C}^K[z_1, z_2]$ and then

$$\vec{A}_1(w)^*\vec{A}_1(z) + \bar{w}_2 z_2 \vec{H}(w)^*\vec{H}(z) = \vec{F}_1(w)^*\vec{F}_1(z) + \vec{H}(w)^*\vec{H}(z).$$

By Proposition 4.2, there exists an $(n + K) \times (n + K)$ unitary $\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ such that

$$\begin{pmatrix} \vec{F}_1(z) \\ \vec{H}(z) \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \vec{A}_1(z) \\ z_2 \vec{H}(z) \end{pmatrix} = \begin{pmatrix} U_{11}\vec{A}_1(z) + U_{12} z_2 \vec{H}(z) \\ U_{21}\vec{A}_1(z) + U_{22} z_2 \vec{H}(z) \end{pmatrix}.$$

Solve for $\vec{H}$ using the second component and insert the result into the first component to get $\Xi(z_2)\vec{A}_1(z) = \vec{F}_1(z)$ where

$$\Xi(z_2) = U_{11} + z_2 U_{12}(I - z_2 U_{22})^{-1} U_{21}. \tag{6.7}$$

In terms of matrices $\Xi(z_2)A_1(z_2) = F_1(z_2)$, and since $\det A_1$ is not identically zero by (6.4) we see that $\Xi = \Psi$. In particular, Ψ is analytic in $\mathbb{D}$. By Lemma 4.5, Ξ is a matrix-valued rational inner function. ☐

The following is an important corollary of the proof of Lemma 6.4. This was first proven in [25].

Corollary 6.5. *Let p be scattering stable. Then, p has a unique Agler pair (up to unitary multiplication) iff $\mathcal{E}_j = \mathcal{F}_j$ for $j = 1$ or 2 iff $\mathcal{G} = \{0\}$.*

Proof. If p has a unique Agler pair, then $\vec{E}_j, \vec{F}_j$ are unitary multiples of each other for $j = 1, 2$ by Theorem 5.6. This is equivalent to equality of the spaces $\mathcal{E}_j = \mathcal{F}_j$. By the last equation of Theorem 5.6, this is equivalent to $\vec{G} = 0$ as well as $\mathcal{G} = \{0\}$. Also, by this equation $\mathcal{E}_1 = \mathcal{F}_1$ iff $\mathcal{E}_2 = \mathcal{F}_2$.

Finally, if $\vec{E}_j, \vec{F}_j$ are unitary multiples, then the positive semidefinite expressions in (6.5) and (6.6) must equal zero, meaning $|\vec{A}_j|^2 = |\vec{E}_j|^2 = |\vec{F}_j|^2$ for $j = 1, 2$. Hence, Agler pairs are unique in this case. $\square$

Our next goal is to show that $\mathcal{G}$ can be viewed as two different one variable vector valued Hardy spaces.

Lemma 6.6. *Using the notation of Lemma 6.4, given $f \in \mathcal{G}$ we may write*

$$f(z) = \vec{f}_1(z_2)\Lambda_n(z_1) = \vec{f}_2(z_1)\Lambda_m(z_2)$$

for $\vec{f}_1 \in \mathbb{C}^{1\times n}[z_2], \vec{f}_2 \in \mathbb{C}^{1\times m}[z_1]$. The map $f \mapsto \vec{f}_1 E_1^{-1}$ is a unitary from $\mathcal{G}$ onto $H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1$ and the map $f \mapsto \vec{f}_2 E_2^{-1}$ is a unitary from $\mathcal{G}$ onto $H^2_{1\times m} \ominus H^2_{1\times m}\Phi_2$.

The inverses of the maps are given by $\vec{f} \mapsto \vec{f}\vec{E}_j$. Consequently, any $f \in \mathcal{G}$ is of the form $\vec{f}_1(z_2)\vec{E}_1(z) = \vec{f}_2(z_1)\vec{E}_2(z)$ where $\vec{f}_1(z_2), \vec{f}_2(z_1)$ are rational functions with no poles in $\overline{\mathbb{D}}$

The space $H^2_{1\times n}$ above is row vector $\mathbb{C}^{1\times n}$-valued Hardy space on $\mathbb{D}$ as explained in Section 4.

Proof. The space $\mathcal{G}$ is a reproducing kernel Hilbert space with reproducing kernel

$$k_w(z) = \frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{F}_1(w)^*\vec{F}_1(z)}{1 - \bar{w}_2 z_2} = \vec{E}_1(w)^* \frac{I - \Phi_1(w_2)^*\Phi_1(z_2)}{1 - \bar{w}_2 z_2} \vec{E}_1(z)$$

by Theorem 5.6. The space $H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1$ is a vector-valued reproducing kernel Hilbert space with reproducing kernel

$$K_{w_2}(z_2) = \frac{I - \Phi_1(w_2)^*\Phi_1(z_2)}{1 - \bar{w}_2 z_2}$$

by Proposition 4.4.

Let T be the map in question: $Tf = \vec{f}_1 E_1^{-1}$. Then,

$$Tk_w = \vec{E}_1(w)^* K_{w_2}.$$

Since reproducing kernels span $\mathcal{G}$, this shows that $\mathcal{G}$ maps into $H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1$. The map T is onto because if $\vec{f} \in H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1$ is orthogonal to the range of T, then

$$\langle \vec{f}, \vec{E}_1(w)^* K_{w_2} \rangle_{H^2_{1\times n}} = \vec{f}(w_2)\vec{E}_1(w) = 0 \text{ for all } w \in \mathbb{D}^2.$$

But, this implies $\vec{f}(w_2)E_1(w_2)\Lambda_n(w_1) \equiv 0$, which implies $\vec{f}(w_2)E_1(w_2) \equiv 0$, which implies $\vec{f} = 0$ since E_1 is invertible except at finitely many points.

The map is a unitary because

$$\langle \vec{E}_1(w)^* K_{w_2}, \vec{E}_1(z)^* K_{z_2} \rangle_{H^2_{1\times n}} = \vec{E}_1(w)^* K_{w_2}(z_2)\vec{E}_1(z) = k_w(z) = \langle k_w, k_z \rangle_{\mathcal{G}}$$

is enough to show T is isometric on linear combinations of reproducing kernels.

It is clear that the inverse of T is given by $\vec{f}(z_2) \mapsto \vec{f}(z_2)\vec{E}_1(z)$. Proposition 4.4 states that $\vec{f}(z_2)$ is rational with no poles in $\overline{\mathbb{D}}$. $\square$

7. Generators for $\mathcal{I}_p$ and Theorem A

Question A from the introduction asks for a list of generators of $\mathcal{I}_p$. We could settle for saying $\mathcal{I}_p$ is generated by bases for $\mathcal{G}, \mathcal{F}_1$, and $\mathcal{F}_2$ as well as $\tilde{p}$ by Corollary 5.10. Since the space $\mathcal{G}$ is in some ways more elusive (e.g. a main theorem of this paper is a formula for $\dim \mathcal{G}$), it is worth pointing out that we can replace $\mathcal{G}$ with $\mathcal{E}_1$. We can also remove $\tilde{p}$ from the list. The following theorem is a restatement of Theorem A.

Theorem 7.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable. The ideal $\mathcal{I}_p$ is generated by the entries of $\vec{E}_1, \vec{F}_1$, and $\vec{F}_2$.*

Proof. By Lemma 6.6, if $f \in \mathcal{G}$, then $f = \vec{f}(z_2)\vec{E}_1(z)$ where $\vec{f}(z_2)$ is a $\mathbb{C}^{1\times n}$ valued rational function with no poles in $\overline{\mathbb{D}}$. So, there exists a polynomial $g(z_2)$ with no zeros in $\overline{\mathbb{D}}$ such that $g(z_2)f(z)$ is a multiple of $\vec{E}_1$. The same argument applies to $\tilde{f}(z) = z_1^{n-1}z_2^{m-1}\overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$ so that there is an $h(z_2) \in \mathbb{C}[z_2]$ with no zeros in $\overline{\mathbb{D}}$ such that $h(z_2)\tilde{f}(z)$ is a multiple of $\vec{E}_1$. If we reflect this at an appropriate degree, we get $\tilde{h}(z_2)f(z)$ is a multiple of $\vec{F}_1$. Note that $\tilde{h}$ will have all zeros in $\mathbb{D}$. Since g and $\tilde{h}$ have no common zeros, there exist $A(z_2), B(z_2)$ such that $Ag + B\tilde{h} = 1$. Thus, $(Ag + B\tilde{h})f = f$ is in the ideal generated by the entries of $\vec{F}_1, \vec{E}_1$.

By Corollary 5.10, any $f \in \mathcal{I}_p$ can be written as a combination of polynomial multiples of $\vec{F}_1, \vec{F}_2$ and $\tilde{p}$ and an element of $\mathcal{G}$. Thus, the entries of $\vec{E}_1, \vec{F}_1, \vec{F}_2$ and $\tilde{p}$ generate the ideal $\mathcal{I}_p$.

Finally, $\tilde{p}$ is in the ideal generated by $\mathcal{E}_1, \mathcal{F}_2$. To see this simply let w be a zero of p which is not a zero of $\tilde{p}$ and insert this value for w into Theorem 5.6. This immediately exhibits $\tilde{p}$ as a combination of elements of $\mathcal{E}_1, \mathcal{F}_2$. $\square$

Remark 7.2. If one wants to construct generators of $\mathcal{I}_p$ it is only necessary to construct $\vec{E}_1$: $\vec{F}_1$ is just a reflection of $\vec{E}_1$ (as in Proposition 5.4) and $\vec{F}_2(w)^*\vec{F}_2(z)$ can then be solved for using Theorem 5.6. Once $\vec{F}_2(w)^*\vec{F}_2(z)$ is known, we can extract coefficients of powers of $z_2, \bar{w}_2$ to get $F_2(w_1)^*F_2(z_1)$ and if we further extract coefficients of powers of $z_1, \bar{w}_1$ we can write

$$F_2(w_1)^*F_2(z_1) = (I_m, \bar{w}_1 I_m, \dots, \bar{w}_1^{n-1} I_m) H \begin{pmatrix} I_m \\ z_1 I_m \\ \vdots \\ z_1^{n-1} I_m \end{pmatrix}$$

for some $nm \times nm$ positive semi-definite matrix H. We can factor $H = J^*J$ for some $m \times nm$ matrix J since H necessarily has rank m. Then, $F_2(z_1) = J(I_m, z_1 I_m, \dots, z_1^{n-1} I_m)^t$. This is the approach taken in Example 15.3.

Notice that the common zeros of $\mathcal{I}_p$ are all on $\mathbb{T}^2$ as one would expect. This is because $\vec{F}_1$ has no zeros in $\mathbb{C} \times (\mathbb{C} \setminus \overline{\mathbb{D}})$ and $\vec{E}_1$ has no zeros in $\mathbb{C} \times \mathbb{D}$ by Lemma 5.12 and this leaves any common zeros in $\mathbb{C} \times \mathbb{T}$. By symmetry any common zeros must also be in $\mathbb{T} \times \mathbb{C}$ and this leaves $\mathbb{T}^2$.

The ideal $\mathcal{I}_p$ can also be characterized using an inequality. We need the following lemma.

Lemma 7.3. *Suppose p is scattering stable and $\deg p = (n, m)$. Let $(\vec{A}_1, \vec{A}_2)$ be an Agler pair for p. Then, for $z \in \mathbb{T}^2$*

$$|\vec{A}_1(z)|^2 = n|p(z)|^2 - 2Re(\overline{p(z)}(z_1\partial_1 p(z)))$$

$$|\vec{A}_2(z)|^2 = m|p(z)|^2 - 2Re(\overline{p(z)}(z_2\partial_2 p(z))).$$

Proof. We can compute $|\vec{A}_1(z)|^2$ for $z \in \mathbb{T}^2$ directly as follows.

For $z_2 \in \mathbb{T}$,

$$|\vec{A}_1(z)|^2 = \frac{|p(z)|^2 - |\tilde{p}(z)|^2}{1 - |z_1|^2}$$

and for $r \in (0,1)$ and $z_1 \in \mathbb{T}$

$$|\vec{A}_1(rz_1, z_2)|^2 = \frac{|p(rz_1, z_2)|^2 - r^{2n}|p(z_1/r, z_2)|^2}{1 - r^2}.$$

Letting $r \to 1$ we get for $z \in \mathbb{T}^2$

$$\begin{aligned}|\vec{A}_1(z)|^2 &= \frac{4\mathrm{Re}(\overline{p(z)}(\partial_1 p(z) z_1)) - 2n|p(z)|^2}{-2} \\ &= n|p(z)|^2 - 2\mathrm{Re}(\overline{p(z)}(z_1\partial_1 p(z))).\end{aligned}$$

The proof for $\vec{A}_2$ is similar. $\square$

Corollary 7.4. *Suppose $p \in \mathbb{C}[z_1, z_2]$ is scattering stable and $\deg p = (n, m)$. Let $f \in \mathbb{C}[z_1, z_2]$. Then, $f \in \mathcal{I}_p$ iff there is a constant $c > 0$ such that for $z \in \mathbb{T}^2$*

$$|f(z)|^2 \leq c((n+m)|p(z)|^2 - 2Re(\overline{p(z)}(z_1\partial_1 p + z_2\partial_2 p))).$$

Proof. By Theorem 7.1 any f can be written in terms of $\vec{E}_1, \vec{F}_1, \vec{F}_2$. Since $|\vec{E}_1| = |\vec{F}_2|$ on $\mathbb{T}^2$, it follows that on $\mathbb{T}^2$ we have

$$|f| \leq c(|\vec{E}_1| + |\vec{F}_2|)$$

Since $(\vec{E}_1, \vec{F}_2)$ is an Agler pair, Lemma 7.3 gives the estimate as claimed after applying Cauchy-Schwarz.

On the other hand, if the inequality holds then f is bounded by elements of $L^2(\frac{d\sigma}{|p|^2})$ and therefore must belong to this space. $\square$

8. The ideal $\mathcal{I}_p^\infty$

Recall $\mathcal{I}_p^\infty$ is the the set of q such that q/p is essentially bounded on $\mathbb{T}^2$. One way for q/p to be bounded on $\mathbb{T}^2$ is if $q \in \langle p, \tilde{p}\rangle$, the ideal generated by $p, \tilde{p}$. However this cannot be all elements of $\mathcal{I}_p^\infty$ since elements of $\langle p, \tilde{p}\rangle$ vanish at all common zeros of p and $\tilde{p}$, which could include zeros not on $\mathbb{T}^2$ and these should not affect boundedness of q/p on $\mathbb{T}^2$. It turns out that we can explicitly construct one variable polynomials $g(z_1), h(z_2)$ such that $gh\mathcal{I}_p \subset \mathcal{I}_p^\infty$. It is not surprising that such polynomials exist but the actual choice of g, h may be of some interest.

If $g \in \mathbb{C}[z_1]$ is a one variable polynomial, we can factor $g = g_1 g_2$ where g_1 has no zeros on $\mathbb{T}$ and g_2 has all of its zeros on $\mathbb{T}$. We will refer to g_2 as the $\mathbb{T}$-factor of g. The $\mathbb{T}$-factor is unique up to constant multiples.

The following theorem identifies a large subset of $\mathcal{I}_p^\infty$. We leave the search for a complete characterization of $\mathcal{I}_p^\infty$ for future work.

Theorem 8.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable, $\deg p = (n, m)$, and we will use E_1, E_2 as defined in* (5.7). *Let $g \in \mathbb{C}[z_2]$ be the $\mathbb{T}$-factor of $\det E_1(z_2)$ and let $h \in \mathbb{C}[z_1]$ be the $\mathbb{T}$-factor of $\det E_2(z_1)$. Then, $g(z_2)h(z_1)\mathcal{I}_p \subset \mathcal{I}_p^\infty$.*

Lemma 8.2. *For an arbitrary $p \in \mathbb{C}[z_1, z_2]$ of degree (n, m), let*

$$L_{w_1}(z) = L(z_1, z_2; w_1) = z_2^m \frac{p(z)\overline{p(w_1, 1/\bar{z}_2)} - \tilde{p}(z)\overline{\tilde{p}(w_1, 1/\bar{z}_2)}}{1 - z_1\bar{w}_1}.$$

Then, $L_{w_1} \in \langle p, \tilde{p}\rangle$ for each $w_1 \in \mathbb{C}$.

Proof. Observe

$$L(z; w_1) = p(z)\frac{z_2^m\overline{p(w_1, 1/\bar{z}_2)} - \bar{w}_1^n\tilde{p}(z)}{1 - z_1\bar{w}_1} + \tilde{p}(z)\frac{\bar{w}_1^n p(z) - z_2^m\overline{\tilde{p}(w_1, 1/\bar{z}_2)}}{1 - z_1\bar{w}_1}.$$

The denominator $(1 - z_1\bar{w}_1)$ divides the numerator in both fractions above. ☐

Lemma 8.3. *With L defined as Lemma 8.2, if p is scattering stable then*

$$L(z; w_1) = \Lambda_n(w_1)^* X_n F_1(z_2)^t \vec{E}_1(z) = \Lambda_n(w_1)^* X_n E_1(z_2)^t \vec{F}_1(z).$$

Recall X_n, E_1, F_1 from (5.7) *and* (5.8).

This is just a result of setting $w_2 = 1/\bar{z}_2$ and multiplying through by z_2^m in Theorem 5.6.

Proposition 8.4. *Assume the setup of Theorem 8.1. Then, the entries of $g\vec{E}_1, g\vec{F}_1, h\vec{E}_2, h\vec{F}_2$ belong to $\mathcal{I}_p^\infty$.*

Proof. By Lemmas 8.2, 8.3, the entries of $F_1(z_2)^t\vec{E}_1(z)$ belong to the ideal $\langle p, \tilde{p}\rangle$. After multiplying by the adjugate matrix of F_1^t, we see that the entries of $\det F_1(z_2)\vec{E}_1(z)$ belong to $\langle p, \tilde{p}\rangle$. By (5.8), $\det F_1(z_2)$, $\det E_1(z_2)$ have the same zeros with the same multiplicities on $\mathbb{T}$. Thus, if we divide out the factor of $\det F_1(z_2)$ containing all zeros not on $\mathbb{T}$ we are left with a multiple of g. Therefore, the entries of $g\vec{E}_1(z)$ belong to $\mathcal{I}_p^\infty$. By a similar argument we form the same conclusion for the entries of $g\vec{F}_1 h\vec{E}_2, h\vec{F}_2$. ☐

Proof of Theorem 8.1. Since any $f \in \mathcal{I}_p$ can be written as a combination of $\vec{E}_1, \vec{F}_1, \vec{F}_2$ (Theorem 7.1) it follows that $ghf \in \mathcal{I}_p^\infty$. ☐

9. A commuting pair of contractive matrices

We now begin to study Question/Theorem B which asks for an exact count of the dimension of $\mathcal{P}_{j,k}$. This is accomplished by finding a pair of commuting contractive operators on $\mathcal{G}$ whose joint eigenvalues are directly related to common zeros of p and $\tilde{p}$.

Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable, $\deg p = (n, m)$, and refer to Notation 5.2. Let P be the orthogonal projection onto $\mathcal{G}$ in $L^2(\frac{d\sigma}{|p|^2})$. Define $T_j : \mathcal{G} \to \mathcal{G}$ by

$$T_j f = P z_j f.$$

Our goal in this section is to show T_1 and T_2^* commute and the joint invariant subspaces are directly related to minimal Agler decompositions.

The following is proven in [8] in a more general set-up, but even if we directly applied their theorem here it would still take work to get to this level of specificity. The result is found in the case of p with no zeros on $\overline{\mathbb{D}}^2$ in [20].

Theorem 9.1. *With T_1, T_2 defined as above, the operators T_1 and T_2^* commute.*

Proof. The condition $T_1T_2^* - T_2^*T_1 = 0$ means

$$Pz_1P\bar{z}_2f - P\bar{z}_2Pz_1f = 0$$

for all $f \in \mathcal{G}$. This is equivalent to

$$z_1P\bar{z}_2f - \bar{z}_2Pz_1f \perp \mathcal{G}$$

for all $f \in \mathcal{G}$, which is equivalent to

$$z_1z_2P\bar{z}_2\bar{z}_1f - Pf \perp z_2\mathcal{G}$$

for all $f \in z_1\mathcal{G}$. Let P_1 denote orthogonal projection onto $\mathcal{P}_{n,m}$; let $P_p, P_{\tilde{p}}$ denote orthogonal projection onto $\mathbb{C}p, \mathbb{C}\tilde{p}$ respectively; let $P_{\mathcal{H}}$ denote orthogonal projection onto a subspace $\mathcal{H}$.

By equations (5.2) and (5.3),

$$P_1 = P + P_{\mathcal{F}_1} + P_{\mathcal{F}_2} + P_{\tilde{p}} = z_1z_2P\bar{z}_1\bar{z}_2 + P_{z_1\mathcal{E}_1} + P_{z_2\mathcal{E}_2} + P_p$$

and so

$$z_1z_2P\bar{z}_1\bar{z}_2 - P = P_{\mathcal{F}_1} + P_{\mathcal{F}_2} + P_{\tilde{p}} - (P_p + P_{z_1\mathcal{E}_1} + P_{z_2\mathcal{E}_2}).$$

Then for $g \in z_2\mathcal{G}$ and $f \in z_1\mathcal{G}$ we have

$$\langle (P_{\mathcal{F}_1} + P_{\mathcal{F}_2} + P_{\tilde{p}} - (P_p + P_{z_1\mathcal{E}_1} + P_{z_2\mathcal{E}_2}))f, g\rangle = 0$$

since $z_1\mathcal{E}_1 \perp z_2\mathcal{G}$, $z_2\mathcal{E}_2 \perp z_1\mathcal{G}$, $\mathcal{F}_1 \perp z_1\mathcal{G}$, $\mathcal{F}_2 \perp z_2\mathcal{G}$, and $p, \tilde{p} \perp f, g$. All of this follows from Theorem 5.9. ◻

Lemma 9.2. *Suppose $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ for some subspaces $\mathcal{G}_1, \mathcal{G}_2$. The following are equivalent*

- *$\mathcal{G}_1$ is an invariant subspace of T_1^**
- *$\mathcal{G}_1 \subset z_1\mathcal{G}_1 \oplus \mathcal{E}_2$*
- *$\mathcal{G}_2$ is an invariant subspace of T_1*
- *$z_1\mathcal{G}_2 \subset \mathcal{G}_2 \oplus \mathcal{F}_2$.*

Similarly, the following are equivalent

- *$\mathcal{G}_2$ is an invariant subspace of T_2^**
- *$\mathcal{G}_2 \subset z_2\mathcal{G}_2 \oplus \mathcal{E}_1$*
- *$\mathcal{G}_1$ is an invariant subspace of T_2*
- *$z_2\mathcal{G}_1 \subset \mathcal{G}_1 \oplus \mathcal{F}_1$.*

Thus, if $\mathcal{G}_1$ is an invariant subspace of (T_1^*, T_2) with $\mathcal{G}_2 := \mathcal{G} \ominus \mathcal{G}_1$ the corresponding invariant subspace of (T_1, T_2^*), then the subspaces

$$\mathcal{A}_1 = (z_2\mathcal{G}_2 \oplus \mathcal{E}_1) \ominus \mathcal{G}_2 \text{ and } \mathcal{A}_2 = (z_1\mathcal{G}_1 \oplus \mathcal{E}_2) \ominus \mathcal{G}_1 \tag{9.1}$$

are well-defined and $\dim \mathcal{A}_1 = n, \dim \mathcal{A}_2 = m$.

Proof. Suppose $\mathcal{G}_1$ is invariant under T_1^*. For any $f \in \mathcal{G}$ we can write $f = z_1g + h$ where $g \in \mathcal{G}$ and $h \in \mathcal{E}_2$ since $\mathcal{P}_{n,m-1} = z_1\mathcal{G} \oplus \mathcal{E}_2$. If this f is actually in $\mathcal{G}_1$ then $T_1^*f = P\bar{z}_1f = g + P\bar{z}_1h = g$ since $\mathcal{E}_2 \perp z_1\mathcal{G}$. By invariance, $g \in \mathcal{G}_1$ so that $\mathcal{G}_1 \subset z_1\mathcal{G}_1 \oplus \mathcal{E}_2$. Conversely, if $\mathcal{G}_1 \subset z_1\mathcal{G}_1 \oplus \mathcal{E}_2$, then $\bar{z}_1\mathcal{G}_1 \subset \mathcal{G}_1 \oplus \bar{z}_1\mathcal{E}_2$ and so $T_1^*\mathcal{G}_1 \subset \mathcal{G}_1$ since $\mathcal{E}_2 \perp z_1\mathcal{G}$.

By properties of adjoints, $\mathcal{G}_1$ is invariant for T_1^* iff $\mathcal{G}_2$ is invariant for T_1. If $\mathcal{G}_2$ is invariant for T_1, then for any $f \in \mathcal{G}_2$ we can write $z_1f = g + h$ where $g \in \mathcal{G}_2, h \in \mathcal{F}_2$. Thus, $z_1\mathcal{G}_2 \subset \mathcal{G}_2 \oplus \mathcal{F}_2$. The converse is similar.

The claims about T_2, T_2^* are similar to those for T_1, T_1^*.

Finally, when $\mathcal{G}_1$ is invariant under both T_1^*, T_2, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are well-defined by the inclusions above and the statement about dimensions follows from the fact that $z_j\mathcal{G}_j$ has the same dimension as $\mathcal{G}_j$. $\square$

The following result is closely related to a result in [8], but again they work in higher generality and it takes additional arguments to get to these finite dimensional statements. We emphasize that the point of the theorem is that an Agler pair $(\vec{A}_1, \vec{A}_2)$ corresponds to an invariant subspace of (T_1^*, T_2) exactly when the dimensions of $\vec{A}_1$ and $\vec{A}_2$ match the bidegree of p, namely (n, m).

Theorem 9.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable, $\deg p = (n, m)$ and define T_1, T_2 as above. Let $\mathcal{G}_1$ be an invariant subspace of T_2, T_1^* and let $\mathcal{G}_2 = \mathcal{G} \ominus \mathcal{G}_1$ which will be invariant under T_1, T_2^*. Let $\vec{A}_1 \in \mathbb{C}^n[z_1, z_2], \vec{A}_2 \in \mathbb{C}^m[z_1, z_2]$ be vector polynomials whose entries form an orthonormal basis for $\mathcal{A}_1, \mathcal{A}_2$ from* (9.1). *Then, the pair $(\vec{A}_1, \vec{A}_2)$ is an Agler pair.*

Conversely, suppose $(\vec{A}_1, \vec{A}_2)$ is an Agler pair where either $\vec{A}_1 \in \mathbb{C}^n[z_1, z_2]$ or $\vec{A}_2 \in \mathbb{C}^m[z_1, z_2]$. We assume the entries of $\vec{A}_j$ are linearly independent. Then, there exists an invariant subspace $\mathcal{G}_1$ of (T_1^*, T_2) such that the entries of $\vec{A}_j$ form an orthonormal basis for $\mathcal{A}_j$ as defined in (9.1) *for $j = 1, 2$.*

Proof. ($\Rightarrow$) Let $\vec{G}_j$ be a vector polynomial whose entries form an orthonormal basis for $\mathcal{G}_j$. Then, since $\mathcal{A}_2 \oplus \mathcal{G}_1 = \mathcal{E}_2 \oplus z_1\mathcal{G}_1$, we have $|\vec{A}_2|^2 + |\vec{G}_1|^2 = |\vec{E}_2|^2 + |z_1|^2|\vec{G}_1|^2$ and similarly $|\vec{A}_1|^2 + |\vec{G}_2|^2 = |\vec{E}_1|^2 + |z_2|^2|\vec{G}_2|^2$. But, $|\vec{E}_2|^2 - |\vec{F}_2|^2 = (1-|z_1|^2)|\vec{G}|^2 = (1-|z_1|^2)(|\vec{G}_1|^2 + |\vec{G}_2|^2)$.

Then,

$$\begin{aligned}\sum_{j=1}^{2}(1-|z_j|^2)|\vec{A}_j|^2 =&(1-|z_1|^2)(|\vec{E}_1|^2 - (1-|z_2|^2)|\vec{G}_2|^2) \\ &+ (1-|z_2|^2)(|\vec{E}_2|^2 - (1-|z_1|^2)|\vec{G}_1|^2) \\ =&(1-|z_1|^2)|\vec{E}_1|^2 + (1-|z_2|^2)|\vec{F}_2|^2\end{aligned}$$

which shows $(\vec{A}_1, \vec{A}_2)$ is an Agler pair.

($\Leftarrow$) Suppose $(\vec{A}_1, \vec{A}_2)$ is an Agler pair with $\vec{A}_1 \in \mathbb{C}^n[z_1, z_2]$. By Lemma 6.4, $\Phi_1 = F_1E_1^{-1}, \Phi = A_1E_1^{-1}, \Psi = F_1A_1^{-1}$ are all $n \times n$ matrix inner functions where $\Phi_1 = \Psi\Phi$.

The space $H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1$ therefore possesses the orthogonal decomposition

$$H^2_{1\times n} \ominus H^2_{1\times n}\Phi_1 = (H^2_{1\times n} \ominus H^2_{1\times n}\Phi) \oplus (H^2_{1\times n} \ominus H^2_{1\times n}\Psi)\Phi.$$

By Lemma 6.6, we have $\mathcal{G} = \mathcal{G}_2 \oplus \mathcal{G}_1$ where $\mathcal{G}_2 = (H^2_{1\times n} \ominus H^2_{1\times n}\Phi)\vec{E}_1$ and $\mathcal{G}_1 = (H^2_{1\times n} \ominus H^2_{1\times n}\Psi)\Phi\vec{E}_1 = (H^2_{1\times n} \ominus H^2_{1\times n}\Psi)\vec{A}_1$. The kernel in (6.5), namely

$$k_w(z) \overset{\text{def}}{=} \frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{A}_1(w)^*\vec{A}_1(z)}{1-\bar{w}_2z_2} = \frac{\vec{A}_2(w)^*\vec{A}_2(z) - \vec{F}_2(w)^*\vec{F}_2(z)}{1-\bar{w}_1z_1},$$

is the reproducing kernel for $\mathcal{G}_2$ because

$$k_w(z) = \vec{E}_1(w)^*\underbrace{\frac{I-\Phi(w_2)^*\Phi(z_2)}{1-\bar{w}_2z_2}}_{K_{w_2}(z_2)}\vec{E}_1(z)$$

which means that for $f(z) = \vec{f}(z_2)\vec{E}(z)$ where $\vec{f} \in H^2_{1\times n} \ominus H^2_{1\times n}\Phi$ we have

$$\langle f, k_w\rangle_{\mathcal{G}} = \langle \vec{f}, \vec{E}_1(w)^* K_{w_2}\rangle_{H^2_{1\times n}} = \vec{f}(w_2)\vec{E}_1(w) = f(w)$$

because of Lemma 6.6. By Theorem 5.6, the reproducing kernel for $\mathcal{G}$ is

$$\frac{\vec{E}_1(w)^*\vec{E}_1(z) - \vec{F}_1(w)^*\vec{F}_1(z)}{1 - \bar{w}_2 z_2}$$

and therefore the reproducing kernel for $\mathcal{G}_1 = \mathcal{G} \ominus \mathcal{G}_2$ is the above kernel minus $k_w(z)$ which is just

$$j_w(z) \stackrel{\text{def}}{=} \frac{\vec{A}_1(w)^*\vec{A}_1(z) - \vec{F}_1(w)^*\vec{F}_1(z)}{1 - \bar{w}_2 z_2} = \frac{\vec{E}_2(w)^*\vec{E}_2(z) - \vec{A}_2(w)^*\vec{A}_2(z)}{1 - \bar{w}_1 z_1}.$$

The second equality is (6.6), which holds for Agler pairs in general.

As in Section 4, we can factor the reproducing kernel for $\mathcal{G}_2$ as $\vec{G}_2(w)^*\vec{G}_2(z)$ for some vector polynomial $\vec{G}_2$. So, $k_w(z) = \vec{G}_2(w)^*\vec{G}_2(z)$. The reproducing kernel (on the diagonal $z = w$) for $\mathcal{E}_1 \oplus z_2\mathcal{G}_2$ is thus $|\vec{E}_1|^2 + |z_2|^2|\vec{G}_2|^2$ which equals $|\vec{A}_1|^2 + |\vec{G}_2|^2$ since $|\vec{G}_2|^2 = k_z(z)$. This shows $\mathcal{E}_1 \oplus z_2\mathcal{G}_2$ contains $\mathcal{G}_2$, because we can relate $\begin{pmatrix} \vec{E}_1 \\ z_2\vec{G}_2 \end{pmatrix}$ and $\begin{pmatrix} \vec{A}_1 \\ \vec{G}_2 \end{pmatrix}$ by a unitary matrix, which means $\vec{G}_2$ can be given directly as a combination of $\vec{E}_1, z_2\vec{G}_2$.

Thus, $\mathcal{G}_2$ is invariant under T_2^* by Lemma 9.2 and $\mathcal{A}_1 := (\mathcal{E}_1 \oplus z_2\mathcal{G}_2) \ominus \mathcal{G}_2$ is well-defined and n-dimensional. The vector polynomial $\vec{A}_1$ will be a unitary multiple of a vector polynomial consisting of an orthonormal basis for $\mathcal{A}_1$ (by Section 4), which means the entries of $\vec{A}_1$ also form an orthonormal basis for $\mathcal{A}_1$.

Since $k_z(z) = |\vec{G}_2|^2$, we have $|\vec{F}_2|^2 + |\vec{G}_2|^2 = |\vec{A}_2|^2 + |z_1|^2|\vec{G}_2|^2$ and this shows $z_1\mathcal{G}_2 \subset \mathcal{G}_2 \oplus \mathcal{F}_2$. By Lemma 9.2, $\mathcal{G}_2$ is an invariant subspace of T_1. Thus, $\mathcal{G}_2$ is invariant under T_1, T_2^* and so $\mathcal{G}_1$ is invariant under T_1^*, T_2 and the spaces in (9.1) are well-defined.

The formula $j_z(z) + |\vec{A}_2|^2 = |\vec{E}_2|^2 + |z_1|^2 j_z(z)$ shows that $|\vec{A}_2|^2$ is the reproducing kernel for $\mathcal{A}_2$. Since the entries of $\vec{A}_2$ are assumed to be linearly independent, it follows $\vec{A}_2$ is a unitary multiple of a vector consisting of an orthonormal basis of $\mathcal{A}_2$. □

10. Common zeros of p and $\tilde{p}$ as joint eigenvalues

The pair of commuting contractions from the previous section can be used to count common zeros of p and $\tilde{p}$ in certain regions, since as we show below the joint eigenvalues of T_1, T_2^* are a simple transformation of the common zeros of p and $\tilde{p}$. We also show that T_1, T_2^* *dilate* to multiplication operators $M_{z_1}, M_{\bar{z}_2}$ on $L^2(\frac{d\sigma}{|p|^2})$.

As a side note, the common zeros of p and $\tilde{p}$ are called *intersecting zeros* of p in the paper [21], where they discuss the interesting problem of how to construct a stable polynomial (no zeros in $\overline{\mathbb{D}}^2$) with given intersecting zeros. It would be interesting to pursue their work in the case of scattering stable p.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere and define $\mathbb{D}^{-1} := \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\} \subset \mathbb{C}_\infty$. If $p \in \mathbb{C}[z_1, z_2]$ has bidegree (n, m), we interpret $p(a, \infty) = 0$ to mean $q(z_2) := z_2^m p(a, 1/z_2)$ vanishes at $z_2 = 0$, or equivalently, $p(a, \cdot)$ has degree less than m. We interpret $p(\infty, \infty) = 0$ to mean $z_1^n z_2^m p(1/z_1, 1/z_2)$ vanishes at $(0, 0)$.

Lemma 10.1. *Suppose $p \in \mathbb{C}[z_1, z_2]$ is scattering stable. Then, all common zeros of p and $\tilde{p}$ lie in $(\mathbb{D} \times \mathbb{D}^{-1}) \cup \mathbb{T}^2 \cup (\mathbb{D}^{-1} \times \mathbb{D})$.*

Proof. Evidently, p and $\tilde{p}$ have no common zeros in $\mathbb{D}^2 \cup (\mathbb{D}^{-1})^2$.

Next, p has no zeros on $\mathbb{T} \times \mathbb{D}$ as we now explain. If we define $q_z(w) = p(z, w)$, then q_z has no zeros in $\mathbb{D}$ for each $z \in \mathbb{D}$. If we send $z \in \mathbb{D}$ to a point $a \in \mathbb{T}$, then by Hurwitz's theorem q_a is either identically zero or non-vanishing in $\mathbb{D}$. If q_a is identically zero, then $z_1 - a$ divides $p(z_1, z_2)$. However this would imply $z_1 - a$ divides $\tilde{p}$ which contradicts our assumption that there are no common factors.

We conclude p and $\tilde{p}$ have no common zeros in $\mathbb{T} \times \mathbb{D}$ as well as $\mathbb{T} \times \mathbb{D}^{-1}$, $\mathbb{D} \times \mathbb{T}$, and $\mathbb{D}^{-1} \times \mathbb{T}$. The only place left for common zeros is the set

$$(\mathbb{D} \times \mathbb{D}^{-1}) \cup \mathbb{T}^2 \cup (\mathbb{D}^{-1} \times \mathbb{D}).$$

$\square$

The first key observation is that the common zeros of p and $\tilde{p}$ are closely related to joint eigenvalues of (T_1, T_2^*). When p has no zeros on the closed bidisk, the following can essentially be found in [20] with the minor difference that we deal with joint eigenvalues.

Theorem 10.2. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable. If $(w_1, w_2) \in \mathbb{D} \times \mathbb{D}^{-1}$ is a common zero of p and $\tilde{p}$, then $(\bar{w}_1, 1/\bar{w}_2)$ is a joint eigenvalue of (T_1^*, T_2) and $(w_1, 1/w_2)$ is a joint eigenvalue of (T_1, T_2^*).*

Proof. A point (λ_1, λ_2) is a joint eigenvalue of (T_1^*, T_2) if and only if there is an $f \in \mathcal{G}$ such that

$$\bar{z}_1 f = \lambda_1 f + \bar{z}_1 h \text{ for some } h \in \mathcal{E}_2$$
$$z_2 f = \lambda_2 f + g \text{ for some } g \in \mathcal{F}_1$$

or equivalently, there is $f \in \mathcal{G}$ such that $(1 - z_1\lambda_1)f \in \mathcal{E}_2$ and $(z_2 - \lambda_2)f \in \mathcal{F}_1$.

Let $q(z) := z_2^m p(z_1, 1/z_2)$ and let $\tilde{q}$ be the corresponding reflected polynomial. If we replace w with $(\lambda_1, 1/\lambda_2)$ and multiply through by $\bar{\lambda}_2^m$ in the first formula of Theorem 5.6, we get

$$\begin{aligned}
&\overline{q(\lambda)}p(z) - \overline{\tilde{q}(\lambda)}\tilde{p}(z) \\
&\qquad = (1 - \bar{\lambda}_1 z_1)\bar{\lambda}_2^m \vec{F}_1(\lambda_1, 1/\lambda_2)^* \vec{F}_1(z) + (\bar{\lambda}_2 - z_2)\bar{\lambda}_2^{m-1}\vec{E}_2(\lambda_1, 1/\lambda_2)^*\vec{E}_2(z) \\
&\qquad = (1 - \bar{\lambda}_1 z_1)\Lambda_n(\lambda_1)^* X_n E_1(\bar{\lambda}_2)^t \vec{F}_1(z) + (\bar{\lambda}_2 - z_2)\Lambda_m(\lambda_2)^* X_m E_2(\lambda_1)^* \vec{E}_2(z)
\end{aligned} \tag{10.1}$$

where we use (5.7) and (5.8).

Now, if $(w_1, w_2) \in \mathbb{D} \times \mathbb{D}^{-1}$ is a common zero of p and $\tilde{p}$, then $\lambda = (\lambda_1, \lambda_2) \overset{\text{def}}{=} (w_1, 1/w_2)$ is a common zero of q and $\tilde{q}$. If we substitute this value for λ into (10.1) we get zero.

By Proposition 5.12, we know $\det E_1(\bar{\lambda}_1) \neq 0$, since $\lambda_1 \in \mathbb{D}$. Thus, $\Lambda_n(\lambda_2)^* X_n E_1(\bar{\lambda}_1)^t \neq 0$, and hence

$$f_1(z) := \Lambda_n(\lambda_1)^* X_n E_1(\bar{\lambda}_2)^t \vec{F}_1(z) \in \mathcal{F}_1$$

is nonzero. Since (10.1)$= 0$

$$(1 - \bar{\lambda}_1 z_1) f_1(z) = -(\bar{\lambda}_2 - z_2)\Lambda_m(\lambda_2)^* X_m E_2(\lambda_1)^* \vec{E}_2(z)$$

and so $(\bar{\lambda}_2 - z_2)$ divides f_1. We can then define $f := (\bar{\lambda}_2 - z_2)^{-1} f_1$ which will be an element of $\mathcal{G}$. Note $\lambda_2 \in \mathbb{D}$ so dividing by this factor does not affect whether $f \in L^2(\frac{d\sigma}{|p|^2})$. Then,

$$(1 - \bar{\lambda}_1 z_1) f(z) = -\Lambda_m(\lambda_2)^* X_m E_2(\lambda_1)^* \vec{E}_2(z) \in \mathcal{E}_2.$$

So,

$$f_1 = (\bar{\lambda}_2 - z_2)f \in \mathcal{F}_1 \text{ and } (1 - \bar{\lambda}_1 z_1)f \in \mathcal{E}_2$$

which implies $(\bar{\lambda}_1, \bar{\lambda}_2) = (\bar{w}_1, 1/\bar{w}_2)$ is a joint eigenvalue of (T_1^*, T_2), as desired. $\square$

In operator model theory language, the following theorem says that the commuting contractions T_1, T_2^* have a unitary dilation to $M_{z_1}, M_{\bar{z}_2}$ on $L^2(\frac{d\sigma}{|p|^2})$; see [2]. This is interesting because although Andô's dilation theorem guarantees that some unitary dilation exists, it is surprising that the unitaries are simple and natural. Recall that P is orthogonal projection onto $\mathcal{G}$ in $L^2(\frac{d\sigma}{|p|^2})$.

Theorem 10.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable and define T_1, T_2 as above. For any $s \in \mathbb{C}[z_1, z_2]$, and $g \in \mathcal{G}$*

$$s(T_1, T_2^*)g = Ps(z_1, \bar{z}_2)g.$$

Proof. Let $g \in \mathcal{G}$. Let $j, k \geq 0$ and assume $Pz_1^k \bar{z}_2^j g = T_1^k (T_2^*)^j g$ and we will show $Pz_1^{k+1}\bar{z}_2^j g = T_1^{k+1}(T_2^*)^j g$ and $Pz_1^k \bar{z}_2^{j+1} g = T_1^k (T_2^*)^{j+1} g$. By induction and by linearity the theorem will then follow.

We shall think of $z_1^k g$ as an element of

$$\mathcal{P}_{n-1+k,m-1+j} = z_2^j \mathcal{G} \oplus (\mathcal{P}_{n-1+k,m-1+j} \ominus z_2^j \mathcal{G}).$$

So, using this decomposition we write

$$z_1^k g = z_2^j g_0 + h$$

with $g_0 \in \mathcal{G}, h \in \mathcal{P}_{n-1+k,m-1+j} \ominus z_2^j \mathcal{G}$. Thus,

$$z_1^k \bar{z}_2^j g = g_0 + \bar{z}_2^j h$$

and $Pz_1^k \bar{z}_2^j g = T_1^k (T_2^*)^j g = g_0$ by hypothesis. Since $\mathcal{P}_{n,m-1} = \mathcal{G} \oplus \mathcal{F}_2$ we can write $z_1 g_0 = T_1 g_0 + f$ where $f \in \mathcal{F}_2$. Then,

$$z_1^{k+1} \bar{z}_2^j g = T_1 g_0 + f + z_1 \bar{z}_2^j h$$

and so $P(z_1^{k+1}\bar{z}_2^j g) = T_1^{k+1}(T_2^*)^j g + Pf + P(z_1 \bar{z}_2^j h)$. But, $f \in \mathcal{F}_2 \perp \mathcal{G}$ so $Pf = 0$, and $h \in \mathcal{P}_{n-1+k,m-1+j} \ominus z_2^j \mathcal{G} \perp \bar{z}_1 z_2^j \mathcal{G}$ by Remark 5.11 since

$$\bar{z}_1 z_2^j \mathcal{G} \subset \{f \in L^2(\frac{d\sigma}{|p|^2}) : \text{supp} \hat{f} \subset \{(a, b) \in \mathbb{Z}^2 : a < n \text{ and } b \geq j\}\}.$$

Thus, $Pz_1 \bar{z}_2^j h = 0$ and we conclude that $T_1^{k+1}(T_2^*)^j g = Pz_1^{k+1}\bar{z}_2^j g$.

The claim involving T_2^* is similar with the key ingredient found in Remark 5.11. $\square$

For $Q \in \mathbb{C}[z_1, z_2]$, we let $Z_Q = \{z \in \mathbb{C}^2 : Q(z) = 0\}$.

Theorem 10.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable and $\deg p = (n, m)$. Let $q(z) = z_2^m p(z_1, 1/z_2), \tilde{q}(z) = z_2^m \tilde{p}(z_1, 1/z_2)$. Then, $q(T_1, T_2^*) = \tilde{q}(T_1, T_2^*) = 0$, and the joint spectrum $\sigma(T_1, T_2^*)$ is $Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2$.*

Proof. Notice that on $\mathbb{T}^2$, $q(z_1, \bar{z}_2) = \bar{z}_2^m p(z)$. Let $f, g \in \mathcal{G}$. By Theorem 10.3

$$q(T_1, T_2^*)f = Pq(z_1, \bar{z}_2)f = P\bar{z}_2^m p(z)f.$$

The inner product of this with g is

$$\langle p, z_2^m \bar{f} g \rangle_{L^2(\frac{d\sigma}{|p|^2})}.$$

The frequency support of $z_2^m \bar{f} g$ is in $\{(j,k) : j < n, k > 0\}$, and such a function is orthogonal to p and $\tilde{p}$ by Theorem 5.9. This shows $q(T_1, T_2^*)f = 0$. A similar argument applies to $\tilde{q}$.

Thus, if we have a joint eigenvalue λ with joint eigenvector f, then $q(T_1, T_2^*)f = q(\lambda)f = 0$ and $\tilde{q}(T_1, T_2^*)f = \tilde{q}(\lambda)f = 0$. So, $q(\lambda) = \tilde{q}(\lambda) = 0$. This shows $\sigma(T_1, T_2^*) \subset Z_q \cap Z_{\tilde{q}} \cap \overline{\mathbb{D}}^2$. Neither T_1 nor T_2^* can have unimodular eigenvalues because we would get $(z_1 - \lambda_1)f = g \in \mathcal{F}_2$ and this implies $\|f\|^2 = |\lambda_1|^2 \|f\|^2 + \|g\|^2$ and so $|\lambda_1| < 1$; and similarly for T_2^*.

By Theorem 10.2, common zeros of $q, \tilde{q}$ inside $\mathbb{D}^2$ are joint eigenvalues of (T_1, T_2^*). Thus,

$$\sigma(T_1, T_2^*) = Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2.$$

□

We see that $\dim \mathcal{G}$ is at least the number of elements of $(Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2)$. Our goal is to show $\dim \mathcal{G} = \#(Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2)$ if we count roots with appropriate multiplicities.

11. Switching from $\mathbb{D}^2$ to $\mathbb{D} \times \mathbb{D}^{-1}$

As the previous section indicates, the polynomial $q(z) = z_2^m p(z_1, 1/z_2)$, which has no zeros in $\mathbb{D} \times \mathbb{D}^{-1}$, is in some ways more natural than p. One approach to counting $\dim \mathcal{G}$ is to study $L^2(\frac{d\sigma}{|q|^2})$ instead and write out formulas and orthogonality relations analogous to Theorems 5.6 and 5.9.

Rather than go through all of that, we shall do some simple conversions between p and q that we will need later. The main technical fact we need is as follows.

Proposition 11.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable and define q as above. There exist one variable polynomials $h_1 \in \mathbb{C}[z_2], h_2 \in \mathbb{C}[z_1]$ with no zeros in $\mathbb{D}$ such that the entries of*

$$h_1(z_2)\bar{F}_1(z_2)X_n\Lambda_n(z_1) \text{ and } h_2(z_1)F_2(z_1)X_m\Lambda_m(z_2)$$

belong to the ideal $\langle q, \tilde{q} \rangle$. Here $\bar{F}_1(z_2) = \overline{F_1(\bar{z}_2)}$.

Proof. Recall from Lemma 8.2 and Lemma 8.3

$$\begin{aligned} L_{w_1}(z) = L(z_1, z_2; w_1) &= z_2^m \frac{p(z)\overline{p(w_1, 1/\bar{z}_2)} - \tilde{p}(z)\overline{\tilde{p}(w_1, 1/\bar{z}_2)}}{1 - z_1\bar{w}_1} \\ &= \Lambda_n(w_1)^* X_n F_1(z_2)^t E_1(z_2)\Lambda_n(z_1). \end{aligned}$$

Now let $H_{w_1}(z) = H(z_1, z_2; w_1) = z_2^{2m} L(z_1, 1/z_2; w_1)$. Then, $H_{w_1} \in \langle q, \tilde{q} \rangle$ for each w_1 by Lemma 8.2 since

$$H(z_1, z_2; w_1) = z_2^m \frac{q(z)\overline{q(w_1, 1/\bar{z}_2)} - \tilde{q}(z)\overline{\tilde{q}(w_1, 1/\bar{z}_2)}}{1 - z_1\bar{w}_1}.$$

On the other hand,

$$\begin{aligned} H(z_1, z_2; w_1) &= \Lambda_n(w_1)^* X_n z_2^m F_1(1/z_2)^t z_2^m E_1(1/z_2)\Lambda_n(z_1) \\ &= \Lambda_n(w_1)^* \overline{E_1(\bar{z}_2)F_1(\bar{z}_2)} X_n \Lambda_n(z_1). \end{aligned}$$

Therefore, the entries of $\bar{E}_1(z_2)\bar{F}_1(z_2)X_n\Lambda_n(z_1)$ belong to $\langle q, \tilde{q} \rangle$. If we multiply by the adjugate of $\bar{E}_1$ we see that the entries of $\det(\bar{E}_1(z_2))\bar{F}_1(z_2)X_n\Lambda_n(z_1)$ belong to $\langle q, \tilde{q} \rangle$. Since $\det(\bar{E}_1(z_2))$ has no zeros in $\mathbb{D}$, the first part of the proposition is proved.

The second part is similar. Let

$$K_{w_2}(z) = K(z_1, z_2; w_2) = z_1^n \frac{p(z)\overline{p(1/\bar{z}_1, w_2)} - \tilde{p}(z)\overline{\tilde{p}(1/\bar{z}_1, w_2)}}{1 - z_2\bar{w}_2}.$$

Then, by Theorem 5.6

$$K(z_1, z_2; w_2) = \Lambda_m(w_2)^* X_m E_2(z_1)^t F_2(z_1)\Lambda_m(z_2)$$

and if we define $J(z_1, z_2; w_2) = (z_2\bar{w}_2)^{m-1}K(z_1, 1/z_2; 1/w_2)$ then

$$\begin{aligned} J(z_1, z_2; w_2) &= z_1^n \frac{q(z)\overline{q(1/\bar{z}_1, w_2)} - \tilde{q}(z)\overline{\tilde{q}(1/\bar{z}_1, w_2)}}{1 - z_2\bar{w}_2} \\ &= \Lambda_m(w_2)^* E_2(z_1)^t F_2(z_1) X_m \Lambda_m(z_2). \end{aligned}$$

As before, the entries of $\det E_2(z_1)F_2(z_1)X_m\Lambda_m(z_2)$ belong to $\langle q, \tilde{q}\rangle$. Since $\det E_2$ has no zeros in $\mathbb{D}$, this proves the second claim of the proposition. $\square$

12. Background: intersection multiplicities

This section discusses intersection multiplicities for plane curves. We also discuss Bézout's theorem for $\mathbb{C}_\infty \times \mathbb{C}_\infty$.

Historically, there are at least 3 equivalent ways to compute the intersection multiplicity of a common zero of two plane curves. One can use resultants (see [17] section 2.7), however this method requires putting the polynomials into general position through linear change of variables. This simple approach seems to be fraught since the polynomials we are interested in do not behave well under linear transformations. The other ways to compute intersection multiplicity, outlined below, are dimension counts of quotients of local rings, dimension counts of generalized eigenspaces, and order of vanishing of resultants of Puiseux expansions.

Let $I \subset \mathbb{C}[z_1, z_2]$ be a zero-dimensional ideal; meaning $V(I) \overset{\text{def}}{=} \{z : \forall f \in I, f(z) = 0\}$ is a finite set. For $\lambda \in V(I)$ we let $\mathcal{O}_\lambda$ denote the localization of $\mathbb{C}[z_1, z_2]$ at λ, or in concrete terms the ring of rational functions whose denominators do not vanish at λ. The intersection multiplicity $N_\lambda(I)$ is defined by

$$N_\lambda(I) \overset{\text{def}}{=} \dim(\mathcal{O}_\lambda / I\mathcal{O}_\lambda).$$

See [13, 18]. Here the ideal $I\mathcal{O}_\lambda$ is the ideal generated by I in $\mathcal{O}_\lambda$. If $I = \langle p, q\rangle$, the ideal generated by $p, q \in \mathbb{C}[z_1, z_2]$, we may write $N_\lambda(p, q)$ for $N_\lambda(I)$.

The intersection multiplicities can be computed as the dimensions of certain generalized eigenspaces as well. Let $[f]$ denote the equivalence class of $f \in \mathbb{C}[z_1, z_2]$ in $Q = \mathbb{C}[z_1, z_2]/I$. Then, the maps

$$M_1[f] := [z_1 f] \qquad M_2[f] := [z_2 f]$$

are well-defined linear maps on Q such that M_1 and M_2 commute. A point λ is in $V(I)$ if and only if $\lambda = (\lambda_1, \lambda_2)$ is a joint eigenvalue of M_1, M_2 and the corresponding joint generalized eigenspace is isomorphic to $\mathcal{O}_\lambda / I\mathcal{O}_\lambda$. The book [13] proves something that amounts to the same thing, namely if $g \in \mathbb{C}[z_1, z_2]$, then the eigenvalues of the map $[f] \mapsto [gf]$ are the values of g on $V(I)$. Furthermore, if g takes on distinct values $g(\lambda_1), \dots, g(\lambda_M)$ on the points of $V(I)$ then the generalized eigenspaces are isomorphic to $\mathcal{O}_{\lambda_j}/I\mathcal{O}_{\lambda_j}$ for $j = 1, \dots, M$. See exercise 12 chapter 4.2 of [13]. Thus, the dimensions of generalized eigenspaces can be used to compute intersection multiplicities. In other words, if

$$G_\lambda = \{[f] \in Q : \exists N, M \text{ such that } (z_1-\lambda_1)^N f, (z_2-\lambda_2)^M f \in I\} \tag{12.1}$$

then $\dim G_\lambda = N_\lambda(I)$.

Remark 12.1. The book [18] also gives a list of properties of the intersection multiplicity of two polynomials p, q with no common factor that yields an algorithm for its computation. Among these are

(1) $N_\lambda(p,q) = 0$ if and only if λ is not a common zero of p, q.
(2) $N_\lambda(p,q) \geq m_\lambda(p) m_\lambda(q)$ where m_λ denotes the order of vanishing at λ (the degree of the lowest non-zero term in the homogeneous expansion at λ) of the given polynomial.
(3) $N_\lambda(p,q) = N_\lambda(p, q+rp)$ for any $r \in \mathbb{C}[z_1, z_2]$.
(4) If $p = \prod p_j^{s_j}$ and $q = \prod q_j^{t_j}$ is the decomposition into irreducible components, then $N_\lambda(p,q) = \sum s_j t_k N_\lambda(p_j, q_k)$.

In Appendix C, we make use of an older method for computing intersection multiplicity using Puiseux series if $I = \langle p, q\rangle$; see [17] section 8.7. For simplicity we assume $\lambda = (0,0) \in V(I)$. We may write $p = u_1 \prod p_j^{s_j}$ and $q = u_2 \prod q_j^{t_k}$ where now u_1, u_2 are functions analytic and non-vanishing in a neighborhood of $(0,0)$ and the p_j's and q_k's are irreducible Weierstrass polynomials. The intersection multiplicity can be computed via

$$N_0(p,q) = \sum s_j t_k N_0(p_j, q_k)$$

where it is shown separately in [17] how to compute $N_0(p_j, q_k)$ for Weierstrass polynomials. We may as well assume for simplicity of notation that $p = p_j, q = q_k$. By Puiseux's theorem (see Chapter 7 of [17]), there exist univariate functions ϕ, ψ which are analytic in a neighborhood of 0 such that

$$p(t^N, \phi(t)) = 0 \text{ and } q(t^M, \psi(t)) = 0$$

for some positive integers N, M. The intersection multiplicity of p and q at 0 can now be computed as the order of vanishing of the following formal power series in fractional powers of t

$$f(t) = \prod_{j=1}^{N} \prod_{k=1}^{M} (\phi(\mu^j t^{1/N}) - \psi(\nu^k t^{1/M})).$$

Here $\mu = e^{2\pi i/N}, \nu = e^{2\pi i/M}$. One can show f is actually a power series in t (and does not involve fractional powers in the end) and the order of vanishing of f at 0 equals $N_0(p,q)$.

A few words about Bézout's theorem for $\mathbb{P} \times \mathbb{P}$ will be helpful for later. Although this is a standard result in algebraic geometry it is difficult to find an elementary discussion of it in the literature, in contrast to the setting of two-dimensional projective space $\mathbb{P}^2 \neq \mathbb{P} \times \mathbb{P}$.

Let $F, G \in \mathbb{C}[z_0, z_1, w_0, w_1]$ be bihomogeneous, meaning homogeneous in (z_0, z_1) and (w_0, w_1) separately. We can then associate bidegrees $(n_1, n_2), (m_1, m_2)$ to F and G respectively; e.g. n_1 is the degree of F with respect to (z_0, z_1). Assuming F, G have no common factors, Bézout's theorem for $\mathbb{P} \times \mathbb{P}$ says that F, G have

$$n_1 m_2 + n_2 m_1$$

common zeros in $\mathbb{P}\times\mathbb{P}$ with multiplicities counted using the local ring definition as presented in Section 12. This is found in [38] (Chapter 4, Section 2.1, Example 4.9), however we caution that it is stated for "divisors in general position," which if one tracks through the definitions in [38] gives the result above.

In this paper we deal with the related situation of $p, q \in \mathbb{C}[z, w]$ with bidegrees $(n_1, n_2), (m_1, m_2)$ and no common factors which we can "bihomogenize" via

$$F(z_0 : z_1, w_0 : w_1) = z_0^{n_1} w_0^{n_2} p(z_1/z_0, w_1/w_0)$$

$$G(z_0 : z_1, w_0 : w_1) = z_0^{m_1} w_0^{m_2} q(z_1/z_0, w_1/w_0).$$

Then for instance a common zero of p, q at (a, ∞) is just a common zero of F, G at $(z_0, z_1, w_0, w_1) = (1, a, 0, 1)$; we are simply using the Riemann sphere $\mathbb{C}_\infty$ model instead of projective space $\mathbb{P}$. Thus, p, q will have $n_1m_2+n_2m_1$ common zeros in $\mathbb{C}_\infty\times\mathbb{C}_\infty$ as before.

13. The dimension theorem: Theorem B

Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable and define $q(z) = z_2^m p(z_1, 1/z_2)$ and $\tilde{q}$ as in Theorem 10.4.

For $\lambda \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2$ we define the joint generalized eigenspace of (T_1, T_2^*) for eigenvalue λ to be

$$\mathcal{G}_\lambda := \{f \in \mathcal{G} : \exists N, M \text{ such that } (T_1 - \lambda_1)^N f = (T_2^* - \lambda_2)^M f = 0\}$$

The goal of this section is to prove the following theorem.

Theorem 13.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable with $\deg p = (n, m)$. Let $q(z) = z_2^m p(z_1, 1/z_2)$. Then, for each $\lambda \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2$*

$$N_\lambda(q, \tilde{q}) = \dim \mathcal{G}_\lambda.$$

Therefore,

$$\dim \mathcal{G} = \sum_{\lambda \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2} N_\lambda(q, \tilde{q})$$

Let $\mathcal{I} = \langle q, \tilde{q}\rangle$, the ideal generated by $q, \tilde{q}$, and $Q = \mathbb{C}[z_1, z_2]/\mathcal{I}$ which is necessarily finite dimensional. We will use $[f]$ to denote the equivalence class of $f \in \mathbb{C}[z_1, z_2]$ in Q. Recall (12.1)

$$G_\lambda = \{[f] \in Q : \exists N, M \text{ such that } (z_1 - \lambda_1)^N f, (z_2 - \lambda_2)^M f \in \mathcal{I}\}.$$

and $\dim G_\lambda = N_\lambda(q, \tilde{q})$.

For $f \in \mathbb{C}[z_1, z_2]$ we define

$$f^\#(z_1, z_2) \overset{\text{def}}{=} z_2^{m-1} f(z_1, 1/z_2) \in \mathbb{C}[z_1, z_2, z_2^{-1}]$$

a Laurent polynomial in z_2. Notice that on $\mathbb{T}^2$, $f^\#(z) = z_2^{m-1} f(z_1, \bar{z}_2)$ and also $q^\#(z) = \bar{z}_2 p(z)$. Recall that P is the operator of orthogonal projection onto $\mathcal{G}$ in $L^2(\frac{d\sigma}{|p|^2})$.

Lemma 13.2. *Suppose $\lambda \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2$. If $[f] \in G_\lambda$, then $f^\# \in L^2(\frac{d\sigma}{|p|^2})$, $Pf^\# \in \mathcal{G}_\lambda$, and if $[f] = 0$, then $f^\# \perp \mathcal{G}$. Thus, the linear map $V[f] \overset{def}{=} Pf^\#$ from G_λ to $\mathcal{G}_\lambda$ is well-defined.*

Proof. If $[f] \in G_\lambda$, then there exist N, M such that $(z_1 - \lambda_1)^N f, (z_2 - \lambda_2)^M f \in \mathcal{I}$. So, we may write

$$(z_1 - \lambda_1)^N f = Aq + B\tilde{q} \text{ and } (z_2 - \lambda_2)^M f = Cq + D\tilde{q}$$

for some $A, B, C, D \in \mathbb{C}[z_1, z_2]$. Then, applying the $\#$ operation and restricting $z \in \mathbb{T}^2$

$$(z_1 - \lambda_1)^N f^\# = A(z_1, \bar{z}_2)\bar{z}_2 p + B(z_1, \bar{z}_2)\bar{z}_2\tilde{p}$$
$$(\bar{z}_2 - \lambda_2)^N f^\# = C(z_1, \bar{z}_2)\bar{z}_2 p + D(z_1, \bar{z}_2)\bar{z}_2\tilde{p}$$

Since $\lambda_1 \in \mathbb{D}$, we see that $f^\# \in L^2(\frac{d\sigma}{|p|^2})$. By (5.1), $z_1^j \bar{z}_2^k p$ is orthogonal to $\mathcal{G}$ for $j \geq 0$ and $k > 0$ and the same holds for $\tilde{p}$. So, $A(z_1, \bar{z}_2)\bar{z}_2 p \perp \mathcal{G}, B(z_1, \bar{z}_2)\bar{z}_2\tilde{p} \perp \mathcal{G}$ and similarly for C, D. Therefore, $(z_1 - \lambda_1)^N f^\#, (\bar{z}_2 - \lambda_2)^M f^\# \perp \mathcal{G}$. We may write $f^\# = Pf^\# + (I - P)f^\#$.

By Remark 5.11, since $(I - P)f^\# \perp \mathcal{G}$ we see that $(I - P)f^\#$ is in fact orthogonal to $\bar{z}_1^j z_2^k \mathcal{G}$ for $j, k \geq 0$. Therefore, $(z_1 - \lambda_1)^N (I - P)f^\#, (\bar{z}_2 - \lambda_2)^M (I - P)f^\#$ are both orthogonal to $\mathcal{G}$, whence we conclude $(z_1 - \lambda_1)^N Pf^\#, (\bar{z}_2 - \lambda_2)^M Pf^\#$ are orthogonal to $\mathcal{G}$. By Theorem 10.3,

$$0 = P(z_1 - \lambda_1)^N Pf^\# = (T_1 - \lambda_1)^N Pf^\#$$
$$0 = P(\bar{z}_2 - \lambda_1)^M Pf^\# = (T_2^* - \lambda_2)^M Pf^\#$$

and this implies $Pf^\# \in \mathcal{G}_\lambda$.

Next, if $[f] = 0$, we know $f = Aq + B\tilde{q}$ for some $A, B \in \mathbb{C}[z_1, z_2]$. Again, $f^\# = \bar{z}_2 A(z_1, \bar{z}_2)p + \bar{z}_2 B(z_1, \bar{z}_2)\tilde{p}$. Again by (5.1), $f^\# \perp \mathcal{G}$ so that $V[f] = 0$. $\square$

Lemma 13.3. *Assume the setup of the previous lemma. The map $V : G_\lambda \to \mathcal{G}_\lambda$ is injective.*

Proof. Suppose $[f] \in G_\lambda$ and $f^\# \perp \mathcal{G}$. Letting $(J, K) = \deg f$, $g := z_2^{K-m+1} f^\# \in \mathcal{P}_{J,K} \ominus z_2^{K-m+1}\mathcal{G}$. By (5.5)

$$g(z) = g_0(z_1, z_2)p(z) + z_1^{L-n+1}\vec{g}_1(z_2)\vec{E}_1(z) + z_2^{K-m+1}\vec{g}_2(z_1)\vec{F}_2(z)$$

for $g_0 \in \mathbb{C}[z_1, z_2]$ of degree at most $(L - n, K - m)$, $\vec{g}_2 \in \mathbb{C}^m[z_1]$ of degree at most $L - n$, $\vec{g}_1 \in \mathbb{C}^n[z_2]$ of degree at most $K - m$. We convert back to f by replacing z_2 with $1/z_2$ and multiplying through by z_2^K to get

$$f(z) = f_0(z_1, z_2)q(z) + z_1^{L-n+1}\vec{f}_1(z_2)\bar{F}_1(z_2)X_n\Lambda_n(z_1) + \vec{f}_2(z_1)F_2(z_1)X_m\Lambda_m(z_2)$$

for appropriate polynomials $f_0, \vec{f}_1, \vec{f}_2$; here $\bar{F}_1(z_2) = \overline{F_1(\bar{z}_2)}$, where we are taking an entrywise conjugate. By Proposition 11.1, there exist one variable polynomials $h_1 \in \mathbb{C}[z_2], h_2 \in \mathbb{C}[z_1]$ with no zeros in $\mathbb{D}$ such that $h_1(z_2)h_2(z_1)f(z) \in \langle q, \tilde{q}\rangle$. Since $[f] \in G_\lambda$, we know $(z_1 - \lambda_1)^N f, (z_2 - \lambda_2)^M f \in \langle q, \tilde{q}\rangle$. There exist $a_0, a_1, a_2 \in \mathbb{C}[z_1, z_2]$ such that

$$1 = a_0(z)h_1(z_1)h_2(z_2) + a_1(z)(z_1 - \lambda_1)^N + a_2(z)(z_2 - \lambda_2)^M$$

since the ideal $\langle h_1 h_2, (z_1 - \lambda_1)^N, (z_2 - \lambda_2)^M\rangle$ is all of $\mathbb{C}[z_1, z_2]$ (by Nullstellensatz). Hence, $f \in \langle q, \tilde{q}\rangle$, which shows $[f] = 0$ and the map V is injective. $\square$

Lemma 13.4. *Assume the setup of the previous lemma. The map $V : G_\lambda \to \mathcal{G}_\lambda$ is surjective.*

Proof. Let $g \in \mathcal{G}_\lambda$. Then, $(z_1 - \lambda_1)^N g, (\bar{z}_2 - \lambda_2)^M g \perp \mathcal{G}$ for some N, M. So, $(z_1 - \lambda_1)^N g \in \mathcal{P}_{N+n-1,m-1} \ominus \mathcal{G}$ and by Corollary 5.10

$$(z_1 - \lambda_1)^N g = \vec{g}_2(z_1)\vec{F}_2(z)$$

where $\vec{g}_2$ has degree at most $N-1$. Also, $(1-\lambda_2 z_2)^M g \in \mathcal{P}_{n-1,M+m-1} \ominus z_2^M \mathcal{G}$ and by (5.4)

$$(1-\lambda_2 z_2)^M g = \vec{g}_1(z_2)\vec{E}_1(z)$$

where $\vec{g}_1$ has degree at most $M-1$. Applying the # operation yields

$$(z_1-\lambda_1)^N g^{\#} = \vec{g}_2(z_1)F_2(z_1)X_m\Lambda_m(z_2)$$

$$(z_2-\lambda_2)^M g^{\#} = z_2^{M-1}\vec{g}_1(1/z_2)\bar{F}_1(z_2)X_n\Lambda_n(z_1).$$

By Proposition 11.1, there exist $h_1 \in \mathbb{C}[z_2], h_2 \in \mathbb{C}[z_1]$ with no zeros in $\mathbb{D}$ such that

$$h_2(z_1)(z_1-\lambda_1)^N g^{\#}, h_1(z_2)(z_2-\lambda_2)^M g^{\#} \in \langle q, \tilde{q}\rangle,$$

so that $h_1(z_2)h_2(z_1)g^{\#} \in G_\lambda$. Following [13, 18], we define

$$h = 1 - \left(1 - \frac{h_1(z_2)h_2(z_1)}{h_1(\lambda_2)h_2(\lambda_1)}\right)^{N+M}.$$

Now $1-h = (1-\frac{h_1(z_2)h_2(z_1)}{h_1(\lambda_2)h_2(\lambda_1)})^{N+M}$ can be expanded as a combination of terms $(z_1-\lambda_1)^j(z_2-\lambda_2)^k$ where $j \geq N, k \geq M$. Thus, $H := 1-h(z_1,\bar{z}_2)$ is a combination of terms $(z_1-\lambda_1)^j(\bar{z}_2-\lambda_2)^k$ where $j \geq N, k \geq M$ and therefore Hg is orthogonal to $\mathcal{G}$ by Remark 5.11 (for instance the remark could be applied to $(z_1-\lambda_1)^N g$). But, h is a combination of powers of h_1h_2 so that $hg^{\#} \in G_\lambda$. Therefore,

$$V[hg^{\#}] = P(hg^{\#})^{\#} = P((1-H)g) = Pg = g$$

which shows V is surjective. □

Proof of Theorem 13.1. We conclude from these lemmas that $N_\lambda(q,\tilde{q}) = \dim G_\lambda = \dim \mathcal{G}_\lambda$ for $\lambda \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{D}^2$. The theorem follows immediately because the sum of the dimensions of the generalized eigenspaces equals the dimension of the underlying space. □

If $\lambda = (\lambda_1, \lambda_2) \in Z_q \cap Z_{\tilde{q}} \cap \mathbb{T}^2$, then $\tilde{\lambda} = (\lambda_1, \bar{\lambda}_2) \in Z_p \cap Z_{\tilde{p}} \cap \mathbb{T}^2$ and the multiplicities match $N_\lambda(q,\tilde{q}) = N_{\tilde{\lambda}}(p,\tilde{p})$. This follows from the isomorphism between the localizations

$$\mathcal{O}_\lambda/(\langle q,\tilde{q}\rangle \mathcal{O}_\lambda) \to \mathcal{O}_{\tilde{\lambda}}/(\langle p,\tilde{p}\rangle \mathcal{O}_{\tilde{\lambda}})$$

given by $f(z) \mapsto f(z_1, 1/z_2)$.

Let $N_{\mathbb{T}^2}(p,\tilde{p})$ denote the sum of the multiplicities of the common roots of p and $\tilde{p}$ on $\mathbb{T}^2$. By the above remarks, $N_{\mathbb{T}^2}(p,\tilde{p}) = N_{\mathbb{T}^2}(q,\tilde{q})$. Theorem B from the introduction is given by the following corollary.

Corollary 13.5. *Let $p \in \mathbb{C}[z_1,z_2]$ be scattering stable and $\deg p = (n,m)$. For $j \geq n-1, k \geq m-1$*

$$\dim \mathcal{P}_{j,k} = (j+1)(k+1) - \frac{1}{2}N_{\mathbb{T}^2}(p,\tilde{p}).$$

Proof. By the Bézout theorem for $\mathbb{C}_\infty \times \mathbb{C}_\infty$ (see Section 12), p and $\tilde{p}$ have $2nm$ common zeros in $\mathbb{C}_\infty \times \mathbb{C}_\infty$, where we count zeros with appropriate multiplicities. Let $N_{\mathbb{D}^2}(q,\tilde{q})$ be the sum of the intersection multiplicities of the common roots of q and $\tilde{q}$ in $\mathbb{D}^2$. By reflective symmetry of the common roots of $q,\tilde{q}$ we have

$$2nm = 2N_{\mathbb{D}^2}(q,\tilde{q}) + N_{\mathbb{T}^2}(q,\tilde{q}) = 2\dim \mathcal{G} + N_{\mathbb{T}^2}(p,\tilde{p})$$

by Theorem 13.1 and since $Z_q \cap Z_{\tilde{q}} \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup (\mathbb{D}^{-1})^2$; this follows from Lemma 10.1.

This proves the corollary for $j = n-1, k = m-1$. In general we use the orthogonal decomposition of Corollary 5.10 to see that

$$\begin{aligned}\dim \mathcal{P}_{j,k} =& \dim \mathcal{G} + n(k-m+1) \\ &+ m(j-n+1) + (k-m+1)(j-n+1) \\ =& nm + n(k-m+1) + m(j-n+1) \\ &+ (k-m+1)(j-n+1) - \frac{1}{2}N_{\mathbb{T}^2}(p,\tilde{p}) \\ =& (j+1)(k+1) - \frac{1}{2}N_{\mathbb{T}^2}(p,\tilde{p}).\end{aligned}$$

□

Corollary 13.6. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable. Then, p has a unique Agler pair (up to unitary multiplication) iff p and $\tilde{p}$ have $2nm$ common zeros on $\mathbb{T}^2$, counting multiplicities; i.e. all common roots in $\mathbb{C}_\infty \times \mathbb{C}_\infty$ must be on $\mathbb{T}^2$.*

Proof. By Corollary 6.5, uniqueness of Agler pairs is equivalent to $\mathcal{G} = \{0\}$. By the previous corollary, $\mathcal{G}$ is trivial iff $N_{\mathbb{T}^2}(p,\tilde{p}) = 2nm$. □

We show in Appendix C that the multiplicity at every common zero on $\mathbb{T}^2$ is even. This is obvious given a local Bézout theorem which would say that if two polynomials have k common zeros counting multiplicities in an open set, then small perturbations of the polynomials have this property. We give a direct proof using Puiseux series.

14. Non-tangential boundary behavior and Theorem C

Next we examine the non-tangential boundary behavior of rational functions holomorphic in $\mathbb{D}^2$. Some of our results hold naturally in d variables, so we keep this level of generality until we need machinery that is only valid in two dimensions.

Let $u = (1, 1, \ldots, 1) \in \mathbb{T}^d$. For $z \in \mathbb{C}^d$ we write $1/z \stackrel{\text{def}}{=} (1/z_1, \ldots, 1/z_d)$. This section is entirely about local behavior at a point of $\mathbb{T}^d$ so we can without loss of generality focus on u. Let $RHP = \{z \in \mathbb{C} : \text{Re}z > 0\}$. Everything in this section hinges on the following fact.

Theorem 14.1. *Let $p \in \mathbb{C}[z_1, \ldots, z_d]$ have total degree n, no zeros in $\mathbb{D}^d$, and assume $p(u) = 0$ with order M, meaning*

$$p(u - \zeta) = \sum_{j=M}^{n} P_j(\zeta)$$

where the P_j are homogeneous polynomials of degree j. Then, P_M has no zeros in RHP^d.

Proof. Observe that

$$P_M(\zeta) = \lim_{r \searrow 0} \frac{1}{r^M} p(u - r\zeta).$$

Now, $\zeta \mapsto p(u - r\zeta)$ has no zeros in the region

$$R_r = \{\zeta \in \mathbb{C}^d : \text{Re}\zeta_j > r|\zeta_j|^2 \text{ for } j = 1, \ldots, d\}.$$

These regions increase as $r > 0$ decreases to 0. By Hurwitz's theorem P_M has no zeros in R_r for every $r > 0$ (P_M is not identically zero by construction). Since $\bigcup_{r>0} R_r = RHP^d$, P_M has no zeros in RHP^d. □

When studying $\zeta \in RHP^d$ approaching 0 non-tangentially we will think of the elements of

$$D_\zeta = \{|\zeta_1|, \dots, |\zeta_d|, \mathrm{Re}\zeta_1, \dots, \mathrm{Re}\zeta_d\}$$

all comparable to a quantity r which is going to 0. To be specific we can arbitrarily say $r = |\zeta_1|$.

A non-tangential approach region to 0 in RHP^d will be a region of the form

$$AR_c = \{\zeta \in RHP^d : c \geq x/y \geq 1/c \text{ for any } x, y \in D_\zeta\}$$

for $c > 1$. (AR = "approach region.") Notice that $AR_c \cap \{|\zeta_1| = r\}$ is a compact set in RHP^d where every element of D_ζ is between cr and r/c. It is useful to point out that if P is homogeneous of degree M and non-vanishing in RHP^d then

$$|P(\zeta)| \geq Cr^M$$

where $r = |\zeta_1|$ and $C = \inf\{|P(\zeta)| : |\zeta_1| = 1, \zeta \in AR_c\} > 0$.

We say f is non-tangentially bounded at u if f is bounded on non-tangential approach regions to u. We say $f = q/p$ has a non-tangential limit at u if the limit

$$\lim_{\substack{\zeta \to 0 \\ \zeta \in AR_c}} f(u - \zeta)$$

exists. We say f is non-tangentially C^k at u if there exists a polynomial L of degree at most k such that

$$f(u - \zeta) - L(\zeta) = o(r^k)$$

for $\zeta \to 0$ in AR_c where $|\zeta_1| = r$.

Proposition 14.2. *Let $p \in \mathbb{C}[z_1, \dots, z_d]$ have no zeros in $\mathbb{D}^d$ and assume p vanishes to order M at u. Let $q \in \mathbb{C}[z_1, \dots, z_d]$. If $f = q/p$ then f is bounded along non-tangential approach regions to u iff q vanishes to order at least M at u.*

Proof. Write

$$q(u - \zeta) = \sum_{j \geq 0} Q_j(\zeta) \tag{14.1}$$

where each Q_j is homogeneous of degree j. Then, let

$$g(\zeta) = f(u - \zeta) = \frac{\sum_{j \geq 0} Q_j(\zeta)}{\sum_{j=M}^{n} P_j(\zeta)}.$$

If f is bounded along non-tangential approach regions then certainly $Q_0 = 0$. If q vanishes to order K, then $Q_1 = \cdots = Q_{K-1} = 0$ and $Q_K \neq 0$. Choose $a_1, \dots, a_d > 0$ such that for $a = (a_1, \dots, a_d)$, $Q_K(a) \neq 0$. Then, as $r \searrow 0$

$$g(ra) = \frac{r^K Q_K(a) + O(r^{K+1})}{r^M P_M(a) + O(r^{M+1})} = r^{K-M} \frac{Q_K(a) + O(r)}{P_M(a) + O(r)}$$

which can only be bounded if $K \geq M$ since $Q_K(a), P_M(a) \neq 0$.

Conversely, if q vanishes to order at least M, then $Q_j = 0$ for $j < M$, and since $|P_M(\zeta)| \geq cr^M$ for ζ in a non-tangential approach region and $r = |\zeta_1|$ (or any other comparable quantity) we have

$$|g(\zeta)| \leq \frac{O(r^M)}{cr^M + O(r^{M+1})} = O(1).$$

□

Proposition 14.3. *Let $p \in \mathbb{C}[z_1, \dots, z_d]$ have no zeros in $\mathbb{D}^d$ and assume p vanishes to order M at u. Let $q \in \mathbb{C}[z_1, \dots, z_d]$ vanish to order at least M at u. If $f = q/p$ then f has a limit along non-tangential approach regions to u iff $Q_M = bP_M$ for some constant b, with Q_M defined as in* (14.1). *In this case, the non-tangential limit will equal the constant b.*

Proof. If f has a limit along non-tangential approach regions to u then, employing g as in the previous proof, there exists b such that

$$o(1) = g(\zeta) - b = \frac{\sum_{j\geq M} Q_j(\zeta) - bP_j(\zeta)}{\sum_{j\geq M} P_j(\zeta)} = \frac{Q_M(\zeta) - bP_M(\zeta)}{P_M(\zeta)} \frac{1}{1 + O(r)} + O(r).$$

Thus, $\frac{Q_M(\zeta)-bP_M(\zeta)}{P_M(\zeta)}$ goes to 0 as $r \searrow 0$. This is not possible unless $Q_M = bP_M$ by homogeneity. Indeed, if $Q_M(a) - bP_M(a) \neq 0$ for some $a \in (0, \infty)^d$, then $\frac{r^M(Q_M(a)-bP_M(a))}{r^M P_M(a)}$ is a nonzero constant. Thus, $Q_M - bP_M$ vanishes identically.

If $Q_M = bP_M$, the above computation shows $g(\zeta) - b = O(r)$, so $f(u - \zeta)$ goes to b as $\zeta \to 0$ non-tangentially. □

The next fact is included for convenience.

Lemma 14.4. *Let $p \in \mathbb{C}[z_1, \dots, z_d]$ have no zeros in $\mathbb{D}^d$, multidegree $n = (n_1, n_2, \dots, n_d)$, and set*

$$\tilde{p}(z) \overset{def}{=} z^n \overline{p(1/\bar{z})}.$$

Then, $|\tilde{p}(z)| \leq |p(z)|$ for $z \in \mathbb{D}^d$.

Proof. If p has no zeros in $\overline{\mathbb{D}}^d$ then $\tilde{p}/p$ is analytic in a neighborhood of $\overline{\mathbb{D}}^d$ and unimodular on $\mathbb{T}^d$. Therefore, by the maximum principle $|\tilde{p}/p| \leq 1$ on $\mathbb{D}^d$. If there are zeros on the boundary we look at $p_t(z) \overset{\text{def}}{=} p(tz)$ for $t \in (0, 1)$ and

$$\tilde{p}_t(z) = t^{|n|}\tilde{p}(z/t).$$

We have $|t^{|n|}\tilde{p}(z/t)| \leq |p(tz)|$ for $z \in \mathbb{D}^d$ and if we let $t \nearrow 1$ we get $|\tilde{p}(z)| \leq |p(z)|$. □

Proposition 14.5. *Assume the setup of Lemma 14.4. Suppose p vanishes to order M at u so that we can write*

$$p(u - \zeta) = \sum_{j=M}^{|n|} P_j(\zeta) \qquad \tilde{p}(u - \zeta) = \sum_{j=M}^{|n|} Q_j(\zeta)$$

where $P_j, Q_j \in \mathbb{C}[\zeta_1, \dots, \zeta_d]$ are homogeneous of degree j. Then, νP_M has real coefficients for some $\nu \in \mathbb{T}$ and Q_M is a unimodular multiple of P_M.

Proof. If we perform the reflection operation $f \mapsto \tilde{f}$ at degree n to $(z - u)^\alpha = (z_1 - 1)^{\alpha_1} \cdots (z_d - 1)^{\alpha_d}$ we get

$$\begin{aligned} z^{n-\alpha}(1 - z_1)^{\alpha_1} \cdots (1 - z_d)^{\alpha_d} &= z^{n-\alpha}(-1)^{|\alpha|}(z - u)^\alpha \\ &= (-1)^{|\alpha|}(z - u)^\alpha \pm (z^{n-\alpha} - 1)(z - u)^\alpha \end{aligned}$$

which shows that reflecting $(z - u)^\alpha$ yields $(-1)^{|\alpha|}(z - u)^\alpha$ plus terms of higher total degree. This implies that in the homogeneous expansion of $\tilde{p}$ we have $Q_M = (-1)^M \bar{P}_M$ where $\bar{P}_M$ denotes taking conjugates of the coefficients of P_M.

By Lemma 14.4, we have that for $\zeta \in RHP^d$ and $t > 0$ sufficiently small

$$|p(u - t\zeta)|^2 - |\tilde{p}(u - t\zeta)|^2 \geq 0$$

whereas when $t < 0$ we have the opposite inequality. In terms of homogeneous expansions this expression on the left is

$$t^{2M}|P_M(\zeta)|^2 - t^{2M}|Q_M(\zeta)|^2 + O(t^{2M+1}).$$

Dividing by t^{2M} and sending t to 0 from the left and right we see that $|P_M(\zeta)|^2 - |Q_M(\zeta)|^2$ is both $\leq$ and ≥ 0. Thus $|P_M(\zeta)|^2 = |Q_M(\zeta)|^2$ for $\zeta \in RHP^d$ which implies $P_M = \mu Q_M = \mu(-1)^M \bar{P}_M$ for some $\mu \in \mathbb{T}$. In turn, it follows that for $\nu = \sqrt{\bar{\mu}(-1)^M}$, $\nu P_M = \bar{\nu}\bar{P}_M$ has real coefficients.

□

As mentioned in the introduction, rational inner functions on $\mathbb{D}^d$ are of the form

$$\mu z^\alpha \frac{\tilde{p}(z)}{p(z)}$$

where $\mu \in \mathbb{T}$ and α is a multi-index (see [36]). Therefore, Theorem C from the introduction follows from the next corollary, which is a direct consequence of Propositions 14.3 and 14.5.

Corollary 14.6. *If $p \in \mathbb{C}[z_1, \ldots, z_d]$ has no zeros in $\mathbb{D}^d$ then for any $\zeta \in \mathbb{T}^d$*

$$\lim_{z \to \zeta} \frac{\tilde{p}(z)}{p(z)}$$

exists as $z \to \zeta$ non-tangentially. Moreover, this limit will be an element of $\mathbb{T}$.

Proposition 14.5 has the following corollary in two dimensions.

Corollary 14.7. *Suppose $p \in \mathbb{C}[z_1, z_2]$ is scattering stable and vanishes to order M at $\lambda \in \mathbb{T}^2$. Then, $N_\lambda(p, \tilde{p}) \geq M(M+1)$.*

Proof. By Proposition 14.5 there is a $\nu \in \mathbb{T}$, such that $p - \nu\tilde{p}$ vanishes to order at least $M+1$ at λ. By Remark 12.1

$$N_\lambda(p, \tilde{p}) = N_\lambda(p, p - \nu\tilde{p}) \geq M(M+1).$$

□

We now study higher regularity for rational inner functions.

Theorem 14.8. *Suppose $f = \tilde{p}/p$ has non-tangential value ν at u. Then, f is non-tangentially C^1 at u iff P_M divides $Q_{M+1} - \nu P_{M+1}$. More generally, f is non-tangentially C^k at u iff*

$$F_1 \stackrel{def}{=} \frac{Q_{M+1} - \nu P_{M+1}}{P_M} \in \mathbb{C}[\zeta_1, \ldots, \zeta_d]$$

$$F_2 \stackrel{def}{=} \frac{Q_{M+2} - \nu P_{M+2} - F_1 P_{M+1}}{P_M} \in \mathbb{C}[\zeta_1, \ldots, \zeta_d]$$

and so on up to the last condition

$$F_k \stackrel{def}{=} \frac{Q_{M+k} - \nu P_{M+k} - \sum_{j=1}^{k-1} F_j P_{M+k-j}}{P_M} \in \mathbb{C}[\zeta_1, \ldots, \zeta_d].$$

In this case the non-tangential Taylor expansion is given by $\sum_{j=1}^{k} F_j$ *in the sense that*

$$f(u-\zeta) - (\nu + \sum_{j=1}^{k} F_j(\zeta)) = o(r^k)$$

Proof. We can multiply by a unimodular constant to put p in the form

$$p(u-\zeta) = \sum_{j=M}^{|n|} P_j(\zeta)$$

$$\tilde{p}(u-\zeta) = \nu P_M(\zeta) + \sum_{j=M+1}^{|n|} Q_j(\zeta)$$

where P_M has real coefficients and no zeros in RHP^d.

Observe that

$$\frac{\tilde{p}(z)}{p(z)} - \nu = \frac{\sum_{j\geq 1} Q_{M+j}(\zeta) - \nu P_{M+j}(\zeta)}{\sum_{j\geq 0} P_{M+j}(\zeta)}$$

$$= \left(\sum_{j\geq 1} \frac{Q_{M+j}(\zeta) - \nu P_{M+j}(\zeta)}{P_M(\zeta)}\right)\left(\frac{1}{1+\sum_{j\geq 1}\frac{P_{M+j}(\zeta)}{P_M(\zeta)}}\right). \tag{14.2}$$

Next, if $F_1 = (Q_{M+1} - \nu P_{M+1})/P_M$, then

$$f(u-\zeta) - (\nu + F_1(\zeta))$$

$$= \left(\sum_{j\geq 1} \frac{Q_{M+j+1}(\zeta) - \nu P_{M+j+1}(\zeta) - F_1(\zeta)P_{M+j}(\zeta)}{P_M(\zeta)}\right)\left(\frac{1}{1+\sum_{j\geq 1}\frac{P_{M+j}(\zeta)}{P_M(\zeta)}}\right)$$

$$= \frac{O(r^2)}{1+O(r)} = O(r^2)$$

which shows f is non-tangentially C^1 at u assuming $F_1 \in \mathbb{C}[\zeta_1, \dots, \zeta_d]$. On the other hand, if f is non-tangentially C^1 at u, then there is a degree 1 homogeneous polynomial Q such that

$$f(u-\zeta) - (\nu + Q(\zeta)) = o(r)$$

so

$$F_1(\zeta) - Q(\zeta) = o(r) \text{ by (14.2)}$$

which means $F_1 = Q$ by homogeneity.

The general case is proved similarly by induction using the formula

$$f(u-\zeta) - (\nu + \sum_{j=1}^{k} F_j(\zeta))$$

$$= \left(\sum_{j\geq 1} \frac{Q_{M+j+k} - \nu P_{M+j+k} - \sum_{m=1}^{k} F_m P_{M+k+j-m}}{P_M}\right)\left(\frac{1}{1+\sum_{j\geq 1}\frac{P_{M+j}}{P_M}}\right)$$

$$= O(r^{k+1}).$$

$\square$

We get from the above proof the existence of a non-tangential directional derivative function

$$F_1(\zeta) = \frac{Q_{M+1}(\zeta) - \nu P_{M+1}(\zeta)}{P_M(\zeta)}$$

for $f = \tilde{p}/p$ even when f is not non-tangentially C^1. This is closely related to a main result of [4], which holds for bounded analytic functions on $\mathbb{D}^2$ (i.e. not just rational inner functions). The paper [6] goes further and characterizes the possible "slope functions" in two variables.

Restricting to two variables, we see that if f is non-tangentially C^k at u, then $N_u(p, \tilde{p}) \geq M(M+k+1)$ because of the following observation:

$$\begin{aligned}
N_u(p,\tilde{p}) &= N_0(P_M + \sum_{j\geq 1} P_{M+j}, \nu P_M + \sum_{j\geq 1} Q_{M+j}) \\
&= N_0(P_M + \sum_{j\geq 1} P_{M+j}, \sum_{j\geq 1}(Q_{M+j} - \nu P_{M+j})) \\
&= N_0(P_M + \sum_{j\geq 1} P_{M+j}, \sum_{j\geq 1}(Q_{M+j+1} - \nu P_{M+j+1} - F_1 P_{M+j})) \\
&= \cdots \\
&= N_0(P_M + \sum_{j\geq 1} P_{M+j}, \sum_{j\geq 1}(Q_{M+j+k} - \nu P_{M+j+k} - \sum_{m=1}^{k} F_m P_{M+k+j-m})) \\
&\geq M(M+k+1)
\end{aligned}$$

This computation is based on the rules from Remark 12.1.

Corollary 14.9. *Suppose $p \in \mathbb{C}[z_1, z_2]$ is scattering stable and $\tilde{p}/p$ is non-tangentially C^k at a point $\lambda \in \mathbb{T}^2$. If p vanishes to order M at λ, then*

$$N_\lambda(p, \tilde{p}) \geq M(M+k+1).$$

For example, if $\tilde{p}/p$ is C^1 at λ, then $N_\lambda(p,\tilde{p}) \geq 4$, since the intersection multiplicity must be even. An interesting consequence is that the number of C^1 points which are not C^2 is finite (i.e. at most $nm/4$).

Finally, we point out that at least in two variables, if $f = q/p \in L^2(\mathbb{T}^2)$ then f is non-tangentially bounded at every point in $\mathbb{T}^2$. By Proposition 14.2, this is equivalent to showing that q vanishes at least to the same order as p at every zero of p on $\mathbb{T}^2$.

Theorem 14.10. *Assume $p \in \mathbb{C}[z_1, z_2]$ is scattering stable and $q \in \mathcal{I}_p$. Then, $f := q/p$ is non-tangentially bounded at every point of $\mathbb{T}^2$; equivalently, if p vanishes to order M at some point of $\mathbb{T}^2$ then every element of $\mathcal{I}_p$ vanishes to at least order M.*

Proof. We may assume p vanishes to order M at $u = (1,1)$. Let $\phi(\zeta) = \frac{\tilde{p}}{p}(u - \zeta)$. By Corollary 14.6, in a non-tangential approach region to $(0,0)$ in RHP^2, $\phi(\zeta) = \nu + O(r)$ for some $\nu \in \mathbb{T}$—actually this is the last line of the *proof* of Proposition 14.3.

For any Agler pair $(\vec{A}_1, \vec{A}_2)$, we see that

$$1 - |\phi(\zeta)|^2 \geq (1 - |1-\zeta_1|^2)\frac{|\vec{A}_1(u-\zeta)|^2}{|p(u-\zeta)|^2}.$$

A similar inequality could be written for $\vec{A}_2$. Now, $1 - |\phi(\zeta)|^2 = 1 - |\nu + O(r)|^2 = O(r)$ and $1 - |1-\zeta_1|^2 = 2\text{Re}\zeta_1 - |\zeta_1|^2 \geq cr$ for $|\zeta_1|$ small enough (because we are in a non-tangential approach region). Thus, $O(1) \geq \frac{|\vec{A}_1(u-\zeta)|^2}{|p(u-\zeta)|^2}$ is bounded along every non-tangential approach region to $(0,0)$. Similarly, $\frac{|\vec{A}_2|^2}{|p|^2}$ is bounded along non-tangential approach regions to u.

This allows us to conclude that $1/p(z)$ times any of $\vec{E}_1, \vec{E}_2, \vec{F}_1, \vec{F}_2$ gives a rational function bounded along non-tangential approach regions to u.

By Theorem 7.1, every element of $\mathcal{I}_p$ can be written in terms of polynomial multiples of $\mathcal{E}_1, \mathcal{F}_1, \mathcal{F}_2$. Therefore, every element q of $\mathcal{I}_p$ will vanish to at least order M at u, or equivalently q/p will be non-tangentially bounded at u. $\square$

15. Examples

This section contains three examples to illustrate Theorems A,B,and C. See [10] for a construction of more examples.

Example 15.1. The following example is taken from [4]. Let

$$p(z_1, z_2) = 4 - z_1 - 3z_2 - z_1 z_2 + z_2^2 \qquad \tilde{p}(z_1, z_2) = 4z_1 z_2^2 - z_2^2 - 3z_1 z_2 - z_2 + z_1.$$

The special Agler pairs for p can be constructed as described in Appendix B. Namely, set $|z_2| = 1$ and consider

$$\frac{|p(z)|^2 - |\tilde{p}(z)|^2}{1 - |z_1|^2} = 4|(1-z_2)^2|^2.$$

Since $(1-z_2)^2$ has no zeros in $\mathbb{D}$ it follows that $\vec{E}_1(z) = 2(1-z_2)^2$. Since the reflection of this equals itself, we see that $\vec{F}_1 = \vec{E}_1$. This automatically implies that p has unique Agler decomposition (up to unitary multiples of Agler pairs).

The vector polynomial $\vec{E}_2 = \vec{F}_2$ can be constructed as in Remark 7.2. We get

$$\vec{F}_2(z) = \vec{E}_2(z) = 2\begin{pmatrix} (1-z_1)(1-z_2) \\ \sqrt{2}(1-z_1 z_2) \end{pmatrix}.$$

The Agler decomposition for p is given by

$$|p|^2 - |\tilde{p}|^2 = 4(1-|z_1|^2)|(1-z_2)^2|^2 + 4(1-|z_2|^2)(|(1-z_1)(1-z_2)|^2 + 2|1-z_1 z_2|^2).$$

Because this is unique we know $\mathcal{P}_{0,1} = \{0\}$. We can also see this by computing the intersection multiplicity at $(1,1)$.

The expansion of p at $(1,1)$ is given by

$$p(1-\zeta, 1-\eta) = 2(\zeta+\eta) + \eta^2 - \zeta\eta \qquad \tilde{p}(1-\zeta, 1-\eta) = -2(\zeta+\eta) + 5\zeta\eta + 3\eta^2 - 4\zeta\eta^2$$

and so by Remark 12.1

$$\begin{aligned} N_{(1,1)}(p, \tilde{p}) &= N_{(1,1)}(p, \tilde{p} + p) \\ &= N_0(2(\zeta+\eta) + \eta^2 - \zeta\eta, \eta(\zeta+\eta) - \zeta\eta^2) \\ &= N_0(2(\zeta+\eta) + \eta^2 - \zeta\eta, \eta) + N_0(2(\zeta+\eta) + \eta^2 - \zeta\eta, (\zeta+\eta) - \zeta\eta) \\ &= 1 + N_0(\eta^2 + \zeta\eta, \zeta + \eta - \zeta\eta) \\ &= 1 + N_0(\eta, \zeta+\eta-\zeta\eta) + N_0(\zeta+\eta, \zeta+\eta-\zeta\eta) \\ &= 1 + 1 + N_0(\zeta+\eta, \zeta\eta) = 4. \end{aligned}$$

Thus, $\dim \mathcal{P}_{0,1} = 2 - (1/2)(4) = 0$.

More generally, $\dim \mathcal{P}_{j,k} = (j+1)(k+1) - 2$. This suggests 2 conditions force $q \in \mathcal{I}_p$. They are $q(1,1) = 0$ and $\partial_1 q(1,1) = \partial_2 q(1,1)$. To see this, note that $\{(1-z_2)^2, (1-z_1)(1-z_2), (1-z_1z_2)\}$ generates $\mathcal{I}_p$. These generators satisfy the two conditions $q(1,1) = 0, \partial_1 q(1,1) = \partial_2 q(1,1)$ and it can be shown that these conditions determine an ideal in $\mathbb{C}[z,w]$ with codimension 2. Therefore, $q/p \in L^2(\mathbb{T}^2)$ iff $q(1,1) = 0$ and $\partial_1 q(1,1) = \partial_2 q(1,1)$.

The rational inner function $f = \tilde{p}/p$ is non-tangentially C^1 because $\zeta + \eta$ divides the second order term of $p + \tilde{p}$, which is $4\eta^2 + 4\zeta\eta$. In fact, f is non-tangentially C^2 because $\nu = -1$, $F_1 = 2\eta$ and $F_2 = -\eta^2$:

$$\frac{\tilde{p}}{p} - (-1 + 2\eta - \eta^2) = \eta^3 \frac{\eta - \zeta}{2(\zeta + \eta) + \eta^2 - \zeta\eta}.$$

No higher regularity is possible because this would force an intersection multiplicity at least 6. This can also be seen directly. ♦

Example 15.2. The next example is taken from [23]. Let

$$\begin{aligned} p(z) =& \frac{1}{18}\left(3\sqrt{5}z_2^2 - 2z_2^2 - 6\sqrt{5}z_2 - 9z_2 + 18\right) \\ &+ \frac{1}{18}\left(9z_2^2 - 14z_2 + 6\sqrt{5} - 9\right) z_1 \\ &+ \frac{1}{18}\left(9z_2 - 3\sqrt{5} - 2\right) z_1^2. \end{aligned}$$

This example was designed to have the feature that for $f = \tilde{p}/p$

$$f(z_1, z_1) = z_1 \text{ and } f\left(\frac{z_1 - \sqrt{5}/3}{1 - (\sqrt{5}/3)z_1}, \frac{z_1 + \sqrt{5}/3}{1 + (\sqrt{5}/3)z_1}\right) = z_1$$

which means f acts as an automorphism of the disk when restricted to certain embedded disks. One can check (with simple computer algebra) that $N_{(1,1)}(p, \tilde{p}) = 6$ and $N_{(-1,-1)}(p, \tilde{p}) = 2$. Therefore,

$$\dim \mathcal{P}_{1,1} = 2 \cdot 2 - \frac{1}{2}(6 + 2) = 0$$

by Theorem B. So, p has a unique Agler pair.

Now, p vanishes to order 2 at $(1,1)$, and elements of $\mathcal{I}_p$ must have the same property. This puts 3 conditions on elements of $\mathbb{C}[z_1, z_2]$. Also, p vanishes to order 1 at $(-1,-1)$ and this puts one additional condition on elements of $\mathbb{C}[z_1, z_2]$. Thus, $q \in \mathcal{I}_p$ iff

$$q(1,1) = \partial_1 q(1,1) = \partial_2 q(1,1) = 0 = q(-1,-1).$$

These conditions are enough to show $\mathcal{P}_{2,1}$ is spanned by $\{z_1^2 - z_1 - z_1z_2 + z_2, z_1^2 z_2 - z_1 - z_1z_2 + 1\}$ and $\mathcal{P}_{1,2}$ is spanned by $\{z_2^2 - z_2 - z_1z_2 + z_1, z_1z_2^2 - z_2 - z_1z_2 + 1\}$. Since $\mathcal{G}$ is trivial, this implies $\mathcal{E}_1 = \mathcal{F}_1, \mathcal{E}_2 = \mathcal{F}_2$ and therefore we can use spanning sets for $\mathcal{P}_{2,1} = \mathcal{E}_2, \mathcal{P}_{1,2} = \mathcal{E}_1$ to generate $\mathcal{I}_p$. Therefore,

$$\mathcal{I}_p = \langle z_1^2 - z_1 - z_1z_2 + z_2, z_1^2z_2 - z_1 - z_1z_2 + 1, z_2^2 - z_2 - z_1z_2 + z_1, z_1z_2^2 - z_2 - z_1z_2 + 1\rangle$$

which illustrates Theorem 7.1.

Next, we discuss non-tangential regularity. The bottom homogeneous term of $p(1-\zeta, 1-\eta)$ is

$$(7/18 - \sqrt{5}/6)\zeta^2 + (11/9)\zeta\eta + (7/18 + \sqrt{5}/6)\eta^2$$

and the bottom homogeneous term of $p(1-\zeta, 1-\eta) - \tilde{p}(1-\zeta, 1-\eta)$ is

$$(1 - \sqrt{5}/3)\zeta^2\eta + (1 + \sqrt{5}/3)\zeta\eta^2.$$

Since the former does not divide the latter, $f = \tilde{p}/p$ is not non-tangentially C^1 at $(1, 1)$; notice that order of vanishing at $(1, 1)$ alone does not reveal this. Since $N_{(-1,-1)}(p, \tilde{p}) = 2$, f is also not non-tangentially C^1 at $(-1, -1)$.

♦

Example 15.3. J. Pascoe has a method to construct rational inner functions which are non-tangentially C^k but not C^{k+1} at a point of $\mathbb{T}^2$—his construction will appear in forthcoming work [33]. He has generously allowed us to include the following example which comes from his construction.

Let

$$p(z) = 4 - 5z_1 - 2z_2 + 2z_1z_2 + 3z_1^2 - z_1^2z_2 - z_1^3z_2$$
$$\tilde{p}(z) = 4z_2z_1^3 - 5z_2z_1^2 + 3z_2z_1 - 2z_1^3 + 2z_1^2 - z_1 - 1.$$

Note p has degree $(3, 1)$. One can compute that

$$N_{(1,1)}(p, \tilde{p}) = 6$$

which again means $\mathcal{G} = \mathcal{P}_{2,0} = \{0\}$ and p has a unique Agler pair. Thus, $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{P}_{2,1}, \mathcal{E}_2 = \mathcal{F}_2 = \mathcal{P}_{3,0}$.

By the dimension theorem $\mathcal{P}_{j,k} = (j+1)(k+1)-3$ for $j \geq 2, k \geq 0$. So, $\mathcal{I}_p$ has codimension 3 in $\mathbb{C}[z_1, z_2]$ and it is of interest to determine the 3 conditions imposed on elements of $\mathcal{I}_p$. Necessarily, $q(1, 1) = 0$ for all $q \in \mathcal{I}_p$.

Using the method of Appendix B, one can compute that $\vec{E}_2(z) = \sqrt{2}(1 - z_1)^3$. Once $\vec{E}_2$ is known, we can use the method outlined after the proof of Theorem 7.1 to find $\vec{E}_1 = \vec{F}_1$. In this case, we get the following orthonormal basis for $\mathcal{F}_1 = \mathcal{E}_1$

$$\{2\sqrt{\frac{2}{7}}(1 - z_2z_1), \sqrt{\frac{2}{133}}(-21z_2z_1^2 + 20z_2z_1 - 7z_2 + 7z_1^2 + 1),$$
$$\frac{1}{\sqrt{19}}(11z_2z_1^2 - 15z_2z_1 + 10z_2 - 10z_1^2 + 19z_1 - 15)\}.$$

Thus, if we put these polynomials into a vector we get $\vec{F}_1$ and hence we have computed the unique Agler pair $(\vec{F}_1, \vec{E}_2)$ (up to unitary multiplication).

Since this is messy, we use a slightly different approach to get manageable numbers. The coefficients of powers of w in $\vec{E}_1(w)^*\vec{E}_1(z)$ will span $\mathcal{E}_1$ and these can be found directly in terms of $p, \tilde{p}, \vec{E}_2$ by Theorem 5.6. This makes it possible to find the following non-orthonormal basis for $\mathcal{E}_1$

$$\{(1 - z_1)^2(1 - z_2), (1 - z_1)(2 - z_1 - z_2), (1 - z_1z_2)\}.$$

Thus, the ideal $\mathcal{I}_p$ is generated by

$$\{(1 - z_1)(2 - z_1 - z_2), (1 - z_1z_2), (1 - z_1)^3\}.$$

The polynomial $(1-z_1)^2(1-z_2)$ in the basis for $\mathcal{E}_1$ can be written in terms of these so we can safely remove it. Using this we can find defining relations for $\mathcal{I}_p$; namely, $q \in \mathcal{I}_p$ iff $q(1,1) = \partial_1 q(1,1) - \partial_2 q(1,1) = 0$ and

$$\partial_{11} q(1,1) - 2\partial_{12} q(1,1) + \partial_{22} q(2,2) + 2\partial_1 q(1,1) = 0.$$

One can check that these conditions actually define an ideal which has codimension 3 in $\mathbb{C}[z_1, z_1]$ which must then coincide with $\mathcal{I}_p$.

We omit the details, but using Theorem 14.8 we can show $f = \tilde{p}/p$ is non-tangentially C^4 but not C^5 at $(1,1)$.

♦

16. Appendix A: Theorem 5.9 and Lemma 6.1

In this appendix we explain how Theorem 5.9 in Section 5 follows from the work in [11,25].

The orthogonality relations for p follow from Proposition 7.1 of [25]. Since reflection $f \mapsto \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$ is an anti-unitary, the orthogonality relations for $\tilde{p}$ follow from those for p.

We only need to establish the orthogonality relations for $\mathcal{F}_1$ since the relations for $\mathcal{E}_1$ follow by applying the anti-unitary reflection, and the relations for $\mathcal{E}_2, \mathcal{F}_2$ follow by symmetry.

The orthogonality relation for $\mathcal{F}_1$ is not easily quotable from [25], so we shall carefully explain how it follows from work in [11].

The setup of [11] is slightly different. Instead of working in $L^2(\frac{d\sigma}{|p|^2})$ we work in $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ with Lebesgue measure but the spaces of interest end up containing functions of the form f/p so that there is a direct comparison between this paper and [11]. Let $\phi = \frac{\tilde{p}}{p}$; notice that $\phi \in H^\infty$ and multiplication by ϕ is a unitary on $L^2(\mathbb{T}^2)$ since $|\phi| = 1$ a.e. on $\mathbb{T}^2$. Define

$$\begin{aligned}
\mathcal{H}_\phi &= H^2 \ominus \phi H^2 \\
\mathcal{H}^1_\phi &= H^2 \cap \phi L^2_{\bullet -} \\
\mathcal{H}^2_\phi &= H^2 \cap \phi L^2_{- \bullet} \\
\mathcal{K}_\phi &= H^2 \cap \phi L^2_{--} \\
\mathcal{K}^1_\phi &= H^2 \cap z_1 \phi L^2_{--} \\
\mathcal{K}^2_\phi &= H^2 \cap z_2 \phi L^2_{--}
\end{aligned}$$

where

$$\begin{aligned}
L^2_{\bullet -} &= \{f \in L^2(\mathbb{T}^2) : \text{supp}\hat{f} \subset \{(j,k) : k < 0\}\} \\
L^2_{- \bullet} &= \{f \in L^2(\mathbb{T}^2) : \text{supp}\hat{f} \subset \{(j,k) : j < 0\}\} \\
L^2_{--} &= \{f \in L^2(\mathbb{T}^2) : \text{supp}\hat{f} \subset \{(j,k) : j, k < 0\}\}.
\end{aligned}$$

We emphasize we are taking orthogonal complements in $L^2(\mathbb{T}^2)$. We will also use $L^2_{+\bullet}, L^2_{\bullet +}$ which are the functions in $L^2(\mathbb{T}^2)$ with Fourier support in $\{(j,k) : j \geq 0\}, \{(j,k) : k \geq 0\}$ respectively. Warning: "+" refers to a non-strict inequality in this notation and "-" refers to a strict inequality.

The following Proposition is similar to Proposition 5.1 of [11].

Proposition 16.1.
$$H^2 \ominus \mathcal{H}^1_\phi = L^2_{+\bullet} \ominus (L^2_{+\bullet} \cap \phi L^2_{\bullet -})$$

Proof. Let $P_{L^2_{+\bullet}}$ denote orthogonal projection onto $L^2_{+\bullet}$. Observe
$$\begin{aligned} L^2_{+\bullet} \ominus (L^2_{+\bullet} \cap \phi L^2_{\bullet -}) &= P_{L^2_{+\bullet}}((L^2_{+\bullet} \cap \phi L^2_{\bullet -})^\perp) \\ &= P_{L^2_{+\bullet}}(L^2_{-\bullet} \vee \phi L^2_{\bullet +}) \\ &= \text{closure}(P_{L^2_{+\bullet}}(L^2_{-\bullet} + \phi L^2_{\bullet +})) \\ &= \text{closure}(P_{L^2_{+\bullet}}(\phi L^2_{\bullet +})) \\ &\subset \text{closure}(P_{L^2_{+\bullet}}(L^2_{\bullet +})) \\ &\subset H^2 \end{aligned}$$
The main facts we are using are $(L^2_{+\bullet})^\perp = L^2_{-\bullet}$ and $(\phi L^2_{\bullet -})^\perp = \phi L^2_{\bullet +}$ since multiplication by ϕ is a unitary. Also, $\phi L^2_{\bullet +} \subset L^2_{\bullet +}$ since $\phi \in H^\infty$.

Then, we can conclude that
$$H^2 \ominus (H^2 \cap \phi L^2_{\bullet -}) = L^2_{+\bullet} \ominus (L^2_{+\bullet} \cap \phi L^2_{\bullet -})$$
by a simple Hilbert space lemma from [11]. Lemma 2.6 of [11] says that if $\mathcal{K}_1, \mathcal{K}_2$ are closed subspaces of a Hilbert space $\mathcal{H}$ satisfying $\mathcal{H} \ominus \mathcal{K}_1 \subset \mathcal{K}_2$, then $\mathcal{H} \ominus \mathcal{K}_1 = \mathcal{K}_2 \ominus (\mathcal{K}_1 \cap \mathcal{K}_2)$. $\square$

By Proposition 2.3, 5.1 and Corollary 9.2 of [11] we have
$$\begin{gathered} \mathcal{H}_\phi \ominus \mathcal{H}^1_\phi = \mathcal{H}^2_\phi \ominus \mathcal{K}_\phi \\ (\mathcal{H}^2_\phi \ominus \mathcal{K}_\phi) \ominus z_2(\mathcal{H}^2_\phi \ominus \mathcal{K}_\phi) = \mathcal{K}^2_\phi \ominus \mathcal{K}_\phi \\ \mathcal{K}_\phi = \{q/p \in H^2 : q \in \mathbb{C}[z_1, z_2], \deg q \le (n-1, m-1)\} \\ \mathcal{K}^2_\phi = \{q/p \in H^2 : q \in \mathbb{C}[z_1, z_2], \deg q \le (n-1, m).\} \end{gathered}$$

Proposition 16.2. *If $f \in L^2_{+\bullet}$ and $f/p \in L^2$, then $f/p \in L^2_{+\bullet}$.*

This is Proposition 8.1 of [11] except [11] has the hypothesis $f \in H^2$ and conclusion $f/p \in H^2$. However, the proof of Proposition 8.1 of [11] is structured so that it proves Proposition 16.2 and then by symmetry $f \in L^2_{\bullet +}$ and $f/p \in L^2$ implies $f/p \in L^2_{\bullet +}$, which yields the result for H^2.

This shows
$$\mathcal{K}_\phi = p^{-1}\mathcal{G} \text{ and } \mathcal{K}^2_\phi = p^{-1}\mathcal{P}_{n-1,m}$$
which shows the connection to this paper. Thus, $\mathcal{K}^2_\phi \ominus \mathcal{K}_\phi = p^{-1}\mathcal{F}_1 \subset \mathcal{H}^2_\phi \ominus \mathcal{K}_\phi \subset H^2 \ominus \mathcal{H}^1_\phi \perp L^2_{+\bullet} \cap \phi L^2_{\bullet -}$ by Proposition 16.1 above. Therefore, $p^{-1}\mathcal{F}_1$ is orthogonal (in $L^2(\mathbb{T}^2)$) to $\mathcal{Q} \overset{\text{def}}{=} L^2_{+\bullet} \cap \phi L^2_{\bullet -}$. If we can show
$$\{f/p \in L^2 : \text{supp}\hat{f} \subset \{(j,k) : j \ge 0 \text{ and } k < m\}\} \subset \mathcal{Q}$$
then we will be finished.

Now, if $f/p \in L^2$ and $\text{supp}\hat{f} \subset \{(j,k) : j \ge 0, k < m\}$ then $f/p \in L^2_{+\bullet}$ by Proposition 16.2. We need to show $(f/p)\bar{\phi} \in L^2_{\bullet -}$ or $\overline{(f/p)}\phi \in z_2 L^2_{\bullet +}$ so we compute
$$\frac{\bar{f}}{\bar{p}}\frac{\tilde{p}}{p} = \frac{z_1^n z_2^m \bar{f}}{p} \text{ on } \mathbb{T}^2.$$

Since $z_2^{m-1}\bar{f} \in L^2_{\bullet+}$ we see that the above is in $z_2 L^2_{\bullet+}$ by Proposition 16.2. This proves $f/p \in \mathcal{Q}$.

Thus, $p^{-1}\mathcal{F}_1 \perp f/p$ for any f with $f/p \in L^2$ and $\operatorname{supp}\hat{f} \subset \{(j,k) : j \geq 0 \text{ and } k < m\}$, where the orthogonality "$\perp$" is in $L^2(\mathbb{T}^2)$. But, this exactly means

$$\mathcal{F}_1 \perp \{f \in L^2(\frac{d\sigma}{|p|^2}) : \operatorname{supp}\hat{f} \subset \{(j,k) : j \geq 0 \text{ and } k < m\}\}$$

using the inner product of $L^2(\frac{d\sigma}{|p|^2})$. This concludes our explanation of Theorem 5.9. We now prove Lemma 6.1.

Proof of Lemma 6.1. By definition of "Agler pair" we have,

$$|p|^2 + \sum_{j=1}^{2} |z_j \vec{A}_j|^2 = |\tilde{p}|^2 + \sum_{j=1}^{2} |\vec{A}_j|^2$$

and by Proposition 4.2 there exists a $(1+N+M)\times(1+N+M)$ isometric matrix U, which is necessarily a unitary because it is square, such that

$$U \begin{pmatrix} p \\ z_1\vec{A}_1 \\ z_2\vec{A}_2 \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \vec{A}_1 \\ \vec{A}_2 \end{pmatrix}.$$

Write U in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where the blocks correspond to the direct sum $\mathbb{C}^{1+N+M} = \mathbb{C} \oplus \mathbb{C}^{N+M}$. Recall $\Delta(z) = \begin{pmatrix} z_1 I_N & 0 \\ 0 & z_2 I_M \end{pmatrix}$. Then, for $\vec{A} = \begin{pmatrix} \vec{A}_1 \\ \vec{A}_2 \end{pmatrix}$ we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p \\ \Delta\vec{A} \end{pmatrix} = \begin{pmatrix} Ap + B\Delta\vec{A} \\ Cp + D\Delta\vec{A} \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \vec{A} \end{pmatrix}.$$

Then, $\vec{A} = p(I - D\Delta)^{-1}C$ and consequently $p(A + B\Delta(I - D\Delta)^{-1}C) = \tilde{p}$. □

17. Appendix B: Constructing Agler pairs

Theorem 5.6 and Proposition 5.12 make it possible to construct $\vec{E}_j, \vec{F}_j$ using the one variable matrix Fejér-Riesz lemma (see [16,32,34,35]). This approach can actually be pushed further to prove the main formula in Theorem 5.6 using the method of Kummert [29], but we will not do this here. The construction goes as follows. For $z_2 \in \mathbb{T}$ we write

$$\frac{\overline{p(w_1,z_2)}p(z) - \overline{\tilde{p}(w_1,z_2)}\tilde{p}(z)}{1 - \bar{w}_1 z_1} = \Lambda_n(w_1)^* T_1(z_2) \Lambda_n(z_1)$$

for some matrix Laurent polynomial $T_1(z_2) \in \mathbb{C}^{n\times n}[z_2, z_2^{-1}]$. In fact, if we write $p(z) = \sum_{j=0}^n p_j(z_2) z_1^j$ and define $\tilde{p}_j(z_2) = z_2^m \overline{p_j(1/\bar{z}_2)}$ as well as

$$R(z_2) = \begin{pmatrix} p_0(z_2) & p_1(z_2) & \cdots & p_{n-1}(z_2) \\ 0 & p_0(z_2) & \cdots & p_{n-2}(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_0(z_2) \end{pmatrix}$$

$$S(z_2) = \begin{pmatrix} \tilde{p}_n(z_2) & \tilde{p}_{n-1}(z_2) & \cdots & \tilde{p}_1(z_2) \\ 0 & \tilde{p}_n(z_2) & \cdots & \tilde{p}_2(z_2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{p}_n(z_2) \end{pmatrix}$$

then by direct calculation $T_1(z_2) = R(z_2)^* R(z_2) - S(z_2)^* S(z_2)$ for $z_2 \in \mathbb{T}$.

By Theorem 5.6, for $z_1, w_1 \in \mathbb{C}, z_2 \in \mathbb{T}$

$$\Lambda_n(w_1)^* T_1(z_2) \Lambda_n(z_1) = \vec{E}_1(w_1, z_2)^* \vec{E}_1(z_1, z_2) = \Lambda_n(w_1)^* E_1(z_2)^* E_1(z_2) \Lambda_n(z_1)$$

where we have used (5.7). As this formula holds for all $z_1, w_1 \in \mathbb{C}$, we get

$$T_1(z_2) = E_1(z_2)^* E_1(z_2)$$

for $z_2 \in \mathbb{T}$. By the matrix Fejér-Riesz lemma, there exists a matrix polynomial $A(z_2)$ with $\det A(z_2)$ non-vanishing for $z_2 \in \mathbb{D}$ such that $A(z_2)^* A(z_2) = E_1(z_2)^* E_1(z_2)$ for $z_2 \in \mathbb{T}$. The matrix function $\Psi := A E_1^{-1}$ will be analytic on $\mathbb{D}$ by Proposition 5.12 and unitary valued on $\mathbb{T}$, hence an inner function. The matrix function $\Psi^{-1} = E_1 A^{-1}$ is also inner since $(A^{-1})^* E_1^* E_1 A^{-1} = I$ on $\mathbb{T}$ and A is invertible in $\mathbb{D}$ by the matrix Fejér-Riesz lemma. By the maximum principle, the only way Ψ and Ψ^{-1} can both be inner is if Ψ is a constant unitary matrix (we omit some details). Thus, E_1 can be constructed via the Fejér-Riesz lemma, which yields a construction for F_1 via (5.8). The construction for E_2, F_2 is analogous.

If p has no zeros on $\overline{\mathbb{D}}^2$ this construction can be done numerically since there are algorithms for performing Fejér-Riesz factorizations for univariate matrix Laurent polynomials which are positive on $\mathbb{T}$; see Theorem 3.1 of [19] as well as [16, 32]. In our case, T_1 is positive definite on $\mathbb{T}$ except at finitely many points. The paper [16] describes how to factor out these singularities (see Section III of [16]).

18. Appendix C: Multiplicities on $\mathbb{T}^2$ are even

This section is technically not necessary for the main theorems of this paper, but it is perhaps reassuring to know that our formula for the dimension of $\mathcal{P}_{j,k}$ in Theorem B does not actually involve any fractions. As mentioned in Section 12, the method used in this Appendix for computing intersection multiplicities is an older method based on Puiseux series and is discussed in detail in [17, Section 8.7]. Lemma 18.3, describing the initial power series development of a Puiseux series associated to the zero set of a scattering stable polynomial around a zero in the boundary, may be of some independent interest.

Proposition 18.1. *Let $p \in \mathbb{C}[z_1, z_2]$ be scattering stable. If $p(t) = 0$ for $t \in \mathbb{T}^2$, then $N_t(p, \tilde{p})$ is even.*

To prove this, we switch to the product upper half plane $\mathbb{C}^2_+ = \{(z_1, z_2) \in \mathbb{C}^2 : \text{Im} z_1, \text{Im} z_2 > 0\}$ using a Cayley transform. Thus, we assume $p \in \mathbb{C}[z_1, z_2]$ has no zeros in $\mathbb{C}^2_+$ and no common factors with $\bar{p}(z_1, z_2) \stackrel{\text{def}}{=} \overline{p(\bar{z}_1, \bar{z}_2)}$. We assume $p(0, 0) = 0$ and we will show $N_0(p, \bar{p})$ is even. Using the local ring definition of multiplicity and the Cayley transform, this yields our original proposition, but we will not go through all of the details of this conversion.

First, note that $p(0, z_2)$ is not identically zero, and by the Weierstrass preparation theorem (see [17]) we can factor

$$p = q p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

where $p_1, \dots, p_m$ are irreducible Weierstrass polynomials in z_2 and q is analytic and non-vanishing in a neighborhood of $(0,0)$. We can reflect this formula to obtain a factorization of $\bar{p}$

$$\bar{p} = \bar{q}\bar{p}_1^{n_1} \cdots \bar{p}_m^{n_m}.$$

Section 12 explains how to compute the intersection multiplicity at 0 of p and $\bar{p}$. First,

$$N_0(p, \bar{p}) = \sum_{j,k} n_j n_k N_0(p_j, \bar{p}_k) = 2\sum_{j<k} n_j n_k N_0(p_j, \bar{p}_k) + \sum_j n_j^2 N_0(p_j, \bar{p}_j)$$

and therefore it suffices to show $N_0(p_j, \bar{p}_j)$ is even for each j. We may as well drop the j and prove the following lemma.

Lemma 18.2. *Let $p \in \mathbb{C}\langle z_1\rangle[z_2]$ be an irreducible Weierstrass polynomial, analytic in a neighborhood Ω of $(0,0) \in \mathbb{C}^2$. Suppose p is non-vanishing in $\Omega \cap \mathbb{C}^2_+$. Then, $N_0(p, \bar{p})$ is even.*

Proof. By Puiseux's theorem (see [17]), there is a function $\phi(t) = a_r t^r + a_{r+1}t^{r+1} + \dots$, analytic in a neighborhood of $0 \in \mathbb{C}$, and a positive integer k such that

$$p(X, Y) = \prod_{j=1}^{k} (Y - \phi(\mu^j X^{1/k}))$$

where μ is a primitive k-th root of unity. The expression $X^{1/k}$ can be interpreted as a formal symbol whose k-th power is X and the above can be viewed as a formal power series computation. Alternatively, the map $t \mapsto (t^k, \phi(t))$ gives a local parametrization of the zero set of p. By performing the reflection operation,

$$\bar{p}(X, Y) = \prod_{j=1}^{k} (Y - \bar{\phi}(\mu^j X^{1/k}))$$

and $t \mapsto (t^k, \bar{\phi}(t))$ gives a local parametrization of the zero set of $\bar{p}$.

Now, $N_0(p, \bar{p})$ is given as the order of vanishing of the resultant of p and $\bar{p}$ which is given by

$$\prod_{i=1}^{k} \prod_{j=1}^{k} (\phi(\mu^i X^{1/k}) - \bar{\phi}(\mu^j X^{1/k})).$$

We will show the order of vanishing is even by examining ϕ separately. We will use the following lemma which is proved at the end of the section.

Lemma 18.3. *Let ϕ be analytic in a neighborhood of $0 \in \mathbb{C}$, $\phi(0) = 0$ and assume $t \mapsto (t^k, \phi(t))$ is injective into $\mathbb{C}^2 \setminus \mathbb{C}^2_+$. Then, ϕ has power series expansion*

$$\phi(t) = \sum_{j=1}^{M} a_j t^{jk} + bt^{2kL} + \sum_{j=2kL+1}^{\infty} b_j t^j$$

where $a_1 < 0$, $a_2, \dots, a_M \in \mathbb{R}$, $\arg b \in (\pi, 2\pi)$.

Thus, the initial terms of ϕ must be powers of t^k taken with real coefficients until an even power of t^k is attained with coefficient in the lower half plane, and after that point all we say is that there must be terms that are not powers of t^k, else $(t^k, \phi(t))$ would not be injective.

We proceed to look at the resultant computation

$$\prod_{i=1}^{k}\prod_{j=1}^{k}(\phi(\mu^i X^{1/k}) - \bar{\phi}(\mu^j X^{1/k})) = \prod_{i=1}^{k}\prod_{j=1}^{k}(b-\bar{b})X^{2L} + \text{ higher order}$$

which vanishes to order $2Lk^2$ since $b \neq \bar{b}$. Therefore the intersection multiplicity is even and we are finished aside from the proof of Lemma 18.3 □

Proof of Lemma 18.3. To begin we analyze the first term of $\phi(t) = at^r + \cdots$, $a \neq 0$. Our assumption is that $t \mapsto (t^k, \phi(t))$ does not map into the upper half plane. So, if $t = |t|e^{i\theta}$, then $\sin k\theta > 0$ implies $\text{Im}\phi(t) \leq 0$. So, writing $a = |a|e^{i\alpha}$ and letting θ be fixed and satisfy $\sin k\theta > 0$ we have

$$0 \geq \lim_{|t|\to 0} \frac{1}{|t|^r}\text{Im}\phi(t) = |a|\text{Im}e^{i(\alpha+r\theta)} = |a|\sin(\alpha + r\theta).$$

The fact that $\sin(\alpha+r\theta)$ has constant sign on an interval of length π/k means $r \leq k$. On the other hand, if $\sin(\alpha + r\theta) > 0$, then the above limit calculation shows that $\text{Im}\phi(t) > 0$ for $|t|$ sufficiently small in which case we must have $\sin(k\theta) \leq 0$. Thus, $\sin(k\theta)$ has constant sign on an interval of length π/r which means $k \leq r$. Therefore, $k = r$. As a result, $\sin(\theta) > 0$ implies $\sin(\alpha + \theta) \leq 0$ which is only possible if $\alpha = \pi$ (modulo multiples of 2π). Thus, a is a negative real number.

Next, we may suppose

$$\phi(t) = a_1 t^k + a_2 t^{2k} + \cdots a_M t^{Mk} + bt^L + \cdots$$

where $a_1, \ldots, a_M \in \mathbb{R}$ and bt^L is the first term not of this type (either b is not real or L is not a multiple of k). Note that we allow $M = 1$. Choose θ so $\sin(k\theta) = 0$; namely θ is an integer multiple of π/k. Then, writing $b = |b|e^{i\beta}$

$$\lim_{|t|\to 0} \frac{1}{|t|^L}\text{Im}\phi(t) = |b|\sin(\beta + L\theta).$$

This must be non-positive. Otherwise, $\text{Im}\phi(t)$ would be positive for $|t|$ small enough and then we could perturb θ to get a point where $(t^k, \phi(t))$ is in the upper half plane. So, $\beta + L\pi j/k \in [\pi, 2\pi] + 2\pi\mathbb{Z}$ for $j = 0, 1, 2, \ldots$. This can only happen if L is a multiple of k—by our assumption this means b is not real. If L is an odd multiple of k then $\beta, \beta+\pi \in [\pi, 2\pi]$ which can only happen if b is real which is not true by assumption. Thus, L must be an even multiple of k in which case $\beta \in (\pi, 2\pi)$, again since b is not real. □

Notation

We collect the notation of the paper in one place and refer to where it was defined if possible.

$$\begin{aligned}
\mathbb{C} &= \text{ the set of complex numbers} \\
\mathbb{D} &= \{z \in \mathbb{C} : |z| < 1\} = \text{ the unit disk} \\
\mathbb{D}^2 &= \mathbb{D} \times \mathbb{D} = \text{ the bidisk} \\
\mathbb{T} &= \{z \in \mathbb{C} : |z| = 1\} = \text{ the unit circle} \\
\mathbb{T}^2 &= \mathbb{T} \times \mathbb{T} = \text{ the two-torus}
\end{aligned}$$

$$\begin{aligned}
\mathbb{C}^n &= \text{ n-dimensional column vectors with entries in } \mathbb{C}\\
\mathbb{C}^{1\times n} &= \text{ n-dimensional row vectors with entries in } \mathbb{C}\\
\mathbb{C}^{m\times n} &= \text{ $n\times m$ matrices with entries in } \mathbb{C}\\
\mathbb{C}[z_1,z_2] &= \text{ polynomials in z_1,z_2 with coefficients in } \mathbb{C}\\
\deg p &= \text{ the bidegree of } p\in\mathbb{C}[z_1,z_2]\\
V[z_1,z_2] &= \text{ polynomials in z_1,z_2 with coefficients in } V\\
L^2(\mathbb{T}^2) &= L^2 \text{ with respect to Lebesgue measure on } \mathbb{T}^2\\
\mathcal{I}_p &= \{q\in\mathbb{C}[z_1,z_2] : q/p\in L^2(\mathbb{T}^2)\}\\
\mathcal{P}_{j,k} &= \{q\in\mathcal{I}_p : \deg q\le (j,k)\} \qquad \text{Note p is taken from context}\\
\tilde{p}(z_1,z_2) &= z_1^n z_2^m\overline{p(1/\bar{z}_1,1/\bar{z}_2)} \text{ for } p\in\mathbb{C}[z_1,z_2] \text{ with } \deg p=(n,m)\\
\mathcal{I}_p^\infty &= \{q\in\mathbb{C}[z_1,z_2] : q/p\in L^\infty(\mathbb{T}^2)\}\\
N_{\mathbb{T}^2}(p,q) &= \text{ the number of common zeros of p,q in } \mathbb{T}^2,\\
&\qquad \text{counted with multiplicity as in Section 12}\\
L^2\Big(\frac{d\sigma}{|p|^2}\Big) &= L^2 \text{ space with respect to Lebesgue measure on $\mathbb{T}^2$ times } \frac{1}{|p|^2}\\
RHP &= \{z\in\mathbb{C} : \mathrm{Re} z>0\} = \text{ right half plane}\\
H^2_{1\times n} &= \text{ row-vector valued Hardy space on } \mathbb{T}\\
\mathcal{E}_j,\mathcal{F}_j,\mathcal{G} & \quad \text{See Notation 5.2}\\
\vec{E}_j,\vec{F}_j,\vec{G} & \quad \text{See Notation 5.3}\\
\oplus &= \text{ orthogonal direct sum in Hilbert space}\\
\Lambda_n &= \text{ See (4.1)}\\
E_j,F_j &= \text{ See (5.7)}\\
X_n &= \text{ See (5.9)}\\
P &= \text{ projection onto $\mathcal{G}$ in Section 9}\\
T_j &= \text{ the linear map on $\mathcal{G}$ given by } f\mapsto Pz_jf\\
\mathbb{C}_\infty &= \mathbb{C}\cup\{\infty\} = \text{ the Riemann sphere}\\
\mathbb{D}^{-1} &= \{z\in\mathbb{C} : |z|>1\}\cup\{\infty\}\subset\mathbb{C}_\infty\\
Z_Q &= \{z\in\mathbb{C}^2 : Q(z)=0\} \text{ for } Q\in\mathbb{C}[z_1,z_2]\\
N_\lambda(I) &= \text{ Intersection multiplicity of λ in the ideal I; see Section 12}\\
\langle p,q\rangle &= \text{ the ideal generated by } p,q\in\mathbb{C}[z_1,z_2]\\
\langle f,g\rangle_{\mathcal{H}} &= \text{ Inner product in Hilbert space } \mathcal{H}\\
N_\lambda(p,q) &= N_\lambda(\langle p,q\rangle)\\
&= \text{ intersection multiplicity of the common zero λ of } p,q\\
f^\#(z_1,z_2) &= z_2^{m-1}f(z_1,1/z_2)
\end{aligned}$$

$$u = (1, 1, \ldots, 1) \in \mathbb{C}^d \text{ in Section 14}$$
$$\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im} z > 0\} = \text{ upper half plane}$$

References

[1] Jim Agler and John E. McCarthy, *The three point Pick problem on the bidisk*, New York J. Math. **6** (2000), 227–236 (electronic). MR1781508 (2001j:32007)

[2] ———, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. MR1882259 (2003b:47001)

[3] ———, *Hankel vector moment sequences and the non-tangential regularity at infinity of two variable Pick functions*, Trans. Amer. Math. Soc. **366** (2014), no. 3, 1379–1411, DOI 10.1090/S0002-9947-2013-05952-1. MR3145735

[4] Jim Agler, John E. McCarthy, and N. J. Young, *A Carathéodory theorem for the bidisk via Hilbert space methods*, Math. Ann. **352** (2012), no. 3, 581–624, DOI 10.1007/s00208-011-0650-7. MR2885589

[5] ———, *Operator monotone functions and Löwner functions of several variables*, Ann. of Math. (2) **176** (2012), no. 3, 1783–1826, DOI 10.4007/annals.2012.176.3.7. MR2979860

[6] J. Agler, R. Tully-Doyle, and N. J. Young, *Boundary behavior of analytic functions of two variables via generalized models*, Indag. Math. (N.S.) **23** (2012), no. 4, 995–1027, DOI 10.1016/j.indag.2012.07.003. MR2991931

[7] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. MR0051437 (14,479c)

[8] Joseph A. Ball, Cora Sadosky, and Victor Vinnikov, *Scattering systems with several evolutions and multidimensional input/state/output systems*, Integral Equations Operator Theory **52** (2005), no. 3, 323–393, DOI 10.1007/s00020-005-1351-y. MR2184571 (2006h:47013)

[9] Catherine Bénéteau, Greg Knese, Łukasz Kosiński, Constanze Liaw, Daniel Seco, and Alan Sola, *Cyclic polynomials in two variables*, 2014, to appear in Trans. Amer. Math. Soc.

[10] Kelly Bickel, *Fundamental Agler decompositions*, Integral Equations Operator Theory **74** (2012), no. 2, 233–257, DOI 10.1007/s00020-012-1992-6. MR2983064

[11] Kelly Bickel and Greg Knese, *Inner functions on the bidisk and associated Hilbert spaces*, J. Funct. Anal. **265** (2013), no. 11, 2753–2790, DOI 10.1016/j.jfa.2013.08.002. MR3096989

[12] Brian J. Cole and John Wermer, *Ando's theorem and sums of squares*, Indiana Univ. Math. J. **48** (1999), no. 3, 767–791, DOI 10.1512/iumj.1999.48.1716. MR1736979 (2000m:47014)

[13] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005. MR2122859 (2005i:13037)

[14] John P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993. MR1224231 (94i:32022)

[15] ———, *Inequalities from complex analysis*, Carus Mathematical Monographs, vol. 28, Mathematical Association of America, Washington, DC, 2002. MR1899123 (2003e:32001)

[16] P. Delsarte, Y. Genin, and Y. Kamp, *A simple approach to spectral factorization*, IEEE Trans. Circuits and Systems **25** (1978), no. 11, 943–946, DOI 10.1109/TCS.1978.1084403. MR508983 (82g:93021)

[17] Gerd Fischer, *Plane algebraic curves*, Student Mathematical Library, vol. 15, American Mathematical Society, Providence, RI, 2001. Translated from the 1994 German original by Leslie Kay. MR1836037 (2002g:14042)

[18] William Fulton, *Algebraic curves*, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989. An introduction to algebraic geometry; Notes written with the collaboration of Richard Weiss; Reprint of 1969 original. MR1042981 (90k:14023)

[19] Jeffrey S. Geronimo and Ming-Jun Lai, *Factorization of multivariate positive Laurent polynomials*, J. Approx. Theory **139** (2006), no. 1-2, 327–345, DOI 10.1016/j.jat.2005.09.010. MR2220044 (2007a:47023)

[20] Jeffrey S. Geronimo and Hugo J. Woerdeman, *Positive extensions, Fejér-Riesz factorization and autoregressive filters in two variables*, Ann. of Math. (2) **160** (2004), no. 3, 839–906, DOI 10.4007/annals.2004.160.839. MR2144970 (2006b:42036)

[21] ———, *Two-variable polynomials: intersecting zeros and stability*, IEEE Trans. Circuits Syst. I Regul. Pap. **53** (2006), no. 5, 1130–1139, DOI 10.1109/TCSI.2005.862180. MR2235187 (2007c:93050)

[22] Dennis Goodman, *Some stability properties of two-dimensional linear shift-invariant digital filters*, IEEE Trans. Circuits and Systems **CAS-24** (1977), no. 4, 201–208. MR0444266 (56 #2624)
[23] Greg Knese, *A Schwarz lemma on the polydisk*, Proc. Amer. Math. Soc. **135** (2007), no. 9, 2759–2768 (electronic), DOI 10.1090/S0002-9939-07-08766-7. MR2317950 (2008j:32004)
[24] ———, *Polynomials defining distinguished varieties*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 5635–5655, DOI 10.1090/S0002-9947-2010-05275-4. MR2661491 (2011f:47022)
[25] ———, *Polynomials with no zeros on the bidisk*, Anal. PDE **3** (2010), no. 2, 109–149, DOI 10.2140/apde.2010.3.109. MR2657451 (2011i:42051)
[26] ———, *Rational inner functions in the Schur-Agler class of the polydisk*, Publ. Mat. **55** (2011), no. 2, 343–357, DOI 10.5565/PUBLMAT_55211_04. MR2839446 (2012k:47033)
[27] ———, *Determinantal representations for semi-hyperbolic polynomials*, 2013, submitted.
[28] Steffi Knorn and Richard H. Middleton, *Stability of two-dimensional linear systems with singularities on the stability boundary using LMIs*, IEEE Trans. Automat. Control **58** (2013), no. 10, 2579–2590, DOI 10.1109/TAC.2013.2264852. MR3106063
[29] Anton Kummert, *Synthesis of two-dimensional lossless m-ports with prescribed scattering matrix*, Circuits Systems Signal Process. **8** (1989), no. 1, 97–119, DOI 10.1007/BF01598747. MR998029 (90e:94048)
[30] Douglas Lind and Klaus Schmidt, *Homoclinic points of algebraic $\mathbf{Z}^d$-actions*, J. Amer. Math. Soc. **12** (1999), no. 4, 953–980, DOI 10.1090/S0894-0347-99-00306-9. MR1678035 (2000d:37002)
[31] Douglas Lind, Klaus Schmidt, and Evgeny Verbitskiy, *Homoclinic points, atoral polynomials, and periodic points of algebraic $\mathbb{Z}^d$-actions*, Ergodic Theory Dynam. Systems **33** (2013), no. 4, 1060–1081, DOI 10.1017/S014338571200017X. MR3082539
[32] D. N. Prabhakar Murthy, *Factorization of discrete-process spectral matrices*, IEEE Trans. Information Theory **IT-19** (1973), 693–696. MR0376712 (51 #12887)
[33] J.E. Pascoe, *An inductive Julia-Carathéodory theorem for Pick functions in two variables*, preprint.
[34] Murray Rosenblatt, *A multi-dimensional prediction problem*, Ark. Mat. **3** (1958), 407–424. MR0092332 (19,1098c)
[35] Marvin Rosenblum, *Vectorial Toeplitz operators and the Fejér-Riesz theorem*, J. Math. Anal. Appl. **23** (1968), 139–147. MR0227794 (37 #3378)
[36] Walter Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0255841 (41 #501)
[37] David Scheinker, *Hilbert function spaces and the Nevanlinna-Pick problem on the polydisc II*, J. Funct. Anal. **266** (2014), no. 1, 355–367, DOI 10.1016/j.jfa.2013.07.027. MR3121734
[38] Igor R. Shafarevich, *Basic algebraic geometry. 1*, Translated from the 2007 third Russian edition, Springer, Heidelberg, 2013. Varieties in projective space. MR3100243
[39] Barry Simon, *Orthogonal polynomials on the unit circle. Part 1*, American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2005. Classical theory. MR2105088 (2006a:42002a)

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