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Optical Interfference II

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Two Astronomical Observations Based on Interference

I. A. A. Michelson's Observations of the Diameter of Stars

The Problem

Telescopes built before the twentieth century were built to compensate for the limited sensitivity of the human eye. They simply gathered *more light* coming from some direction in space. With technological development we could see more stars in the sky.

Terrestrial observations taken six months apart in time showed a slight shift in position of many stars (their "parallax") in the sky from which triangulation can be used to determine how far these stars are from us.‡

Human observers always *saw* stars as bright disks in the sky. Projecting the observed disks to the known distances from the sun led to the conclusion that sun was the smallest star in the universe. With technological development, however, came the realization that the observed size of the disk was an artifact: The aberration of lenses limited the resolution of images produced by lenses and fluctuations in the earth's atmosphere (the stars "twinkle") leads to the existence of multiple paths for light to travel to humans. If the stars were comparable to the sun in size, if telescopes had no abberation, and if there was no atmosphere, stars should appear to terrestrials as point light sources ... It is not possible to *see* the diameters of stars in the usual sense of the word.

How do optical instruments work?

Maxwell provided a formal approach to determing how light moved through an apparatus: Just solve a set of partial differential equations for the $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ fields associated with the appropriate boundary conditions.

At the time Maxwell deduced the equations of motion for the $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ fields, however, the problem of understanding how light moves through optical instruments already had a long history. I. Newton had argued that a light beam was simply a stream of particles. C. Huygens* and Robert Hooke had argued that light was a wave phenomena. Huygens' understanding is more intuitively attractive than Maxwell's formal

‡ F. W. Bessel used this in 1838 to determine that the closest (pair of) stars to us *61 Cygni* was about 10.3 light years from us.

* Christiaan Huygens (1629-1695) was a prominent Dutch mathematician and scientist.

analysis and in most cases agrees essentially with experiment. So I reproduce here a description of A. A. Michelson's measurement of the diameter of stars using Huygens' argument.

Huygens' Principle

A monochromatic wave will have an amplitude and a phase at all points in space. We identify the locus of points where the phase is the same as a *wave front* of the wave. This will be a point in a 1D space, a curve in a 2D space, and a surface in a 3D space. Huygens argued that we could deduce where a wave would go if we knew the wave's amplitude on some wave front at some particular time: Simply assume that spherical waves are emitted from all the points on a wave front with amplitudes that are proportional to the amplitude of the wave on the wavefront and a phase that is the same as the phase on the wavefront.

If all these waves are superimposed at some observation point, the fact that the distances between the observation point and the different points on the wave front differ means that there is interference between the various waves. When this interference is resolved, we learn the amplitude and phase of the wave at the new observation point.

Huygens used this argument to deduce Snell's law, to design lenses for telescopes, etc. Fresnel used the same argument to explain interference phenomena that he had observed. We use essentially the same argument to explain to freshman the interference seen in T. Young's double slit experiment.

Application to Star Light

Let me model a star as a spherical object that has a surface (of radius R_S) that emits light. To apply Huygens's analysis, I assume that the radiation is monchromatic. This is not the case, but the radiation actually has a black-body distribution that is sharply peaked and the earth's atmosphere and the lenses in telescopes partially filter the spectrum of the star light seen. More modern versions (see below) of the experiment select the spectrum extremely tightly, but this was not done originally by Michelson. In any case, the monchromatic description can be applied to the real experiment by simply superimposing solutions with the correct spectral distribution.

Let me use a spherical coordinate system centered on the star. Then, as a first approximation the $r = R_S$ surface will be a wave-front. This is only a rough approximation, of course. But a more realistic model is that $r = R_S$ can be taken as a slight displacement of the real wave-front. Because of this displacement, I assume that I can treat $r = R_S$ as a surface on which radiation is emitted but that the *phase* of the emitted light is different at different points on the suface. That is, the actual phase is $\chi(\theta,\phi)$ at (R_S,θ,ϕ), rather than 0. I expect $\chi(\theta,\phi)$ to be a stochastic variable with

$$\langle\chi(\theta,\phi)\rangle = 0$$

at all θ, ϕ.

Complex Waves

Working out the interference between real waves involves a lot of trigonometry. It is easier to execute the analysis if the waves are described in term of complex waves.* The *real* part of

$$A\, e^{i[\boldsymbol{k}\cdot\boldsymbol{r}-\omega\, t]}$$

for example, describes a monochromatic wave of amplitude A that oscillates with a frequency ω and propagates in the direction of $\boldsymbol{k}$ with a wavelength $\lambda = 2\pi/k$ and a speed $c = k/c$. Any superposition of such waves

$$\sum_j A_j\, e^{i[\boldsymbol{k}_j\cdot\boldsymbol{r}-\omega_j\, t]}$$

at a point $\boldsymbol{r}$ can be resolved into an amplitude and a phase at $\boldsymbol{r}$.

Star Light

We now use Huygens principle to determine the light of a star observed at a distance $D \gg R_S$ from the star. That is, at the point $x = 0,\ y = 0,\ z = D$ we want to know the sum of the fields due to wavelets emitted at $r = R_S,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$'

In *Appendix A* I assume that light with a known speed c and deduce how the wave properties propagate from some source. Applying Eq.(A6) we predict that the electric field at the observation point will be

$$\boldsymbol{E}(D\,\hat{k},t) = -\frac{1}{4\pi}\int_0^{\pi/2} R_S\, d\theta \int_0^{2\pi} d\phi\, \frac{\boldsymbol{e}(t-\frac{|\boldsymbol{r}-D\,\hat{k}|}{c})}{|\boldsymbol{r}-D\,\hat{k}|}$$

with

$$|\boldsymbol{r}-D\,\hat{k}| = \sqrt{R_S^2\,\sin^2\theta + (R_S\cos\theta - D)^2}$$

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* The concept of ***analytic signal*** was introduced by D. Gabor (1946).

Appendix A

While the fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ are vector fields*, Maxwell found that all components had the same equation of motion in a vacuum. If A($\boldsymbol{r}$,t) is one of the components,

$$\nabla^2 A(\boldsymbol{r},t) = \frac{1}{c^2}\frac{\partial^2 A(\boldsymbol{r},t)}{\partial t^2}, \tag{A1}$$

the *wave equation* for a 3D wave with speed c. Equation(A1) becomes a well posed problem when boundary conditions for some real situation are provided.

Here I provide *a generic* set of boundary conditions that we can use in a large class of applications: Suppose that the field is generated at $\boldsymbol{r} = 0$ with a source $S(t)$ and that the produced wave is *out-going* when observed at $r \gg 0$. Then the differential equation for $A(\boldsymbol{r},t)$ is

$$\nabla^2 A(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 A(\boldsymbol{r},t)}{\partial t^2} = \delta(\boldsymbol{r})\, S(t) \tag{A2}$$

at all $\boldsymbol{r}, t$ and an out-going solution is sought.

The problem has spherical symmetry around $\boldsymbol{r} = 0$, so we suppose that the solution has the form

$$A(\boldsymbol{r},t) = \frac{a(r,t)}{r} \tag{A3}$$

Inserting this into (A2) gives

$$\frac{\partial^2 a(r,t)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 a(r,t)}{\partial t^2} = r\,\delta(\boldsymbol{r})\, S(t) \tag{A4}$$

The right hand side of Eq(A4) is zero so the equation for $a(r,t)$ is just the wave equation in one dimension. The general solution to Eq.(A4) is

$$a(r,t) = f(r - c\,t) + g(r + c\,t)$$

with $f(x)$ and g(x) *any* functions possessing a second derivative at all x.

$f(r - c\,t)$ represents an out-going wave. Since $g(r + c\,t)$ represents an in-coming wave, we drop the $g(x)$ term and conclude

$$A(\boldsymbol{r},t) = \frac{f(r - c\,t)}{r} \tag{A5}$$

* These fields are often combined into a single (second rank) electromagnetic field tensor.

To satisfy the boundary condition at $r = 0$, we integrate Eq.(A2) over a sphere centered at $r = 0$ of (small) radius R. The result is that

$$S(t) = 4\pi \left[R f'(R - ct) - f(R - ct)\right]$$

at all $R > 0$. The $R \to 0$ limit gives

$$-4\pi f(-ct) = S(t)$$

at all t,whence

$$f(r - ct) = -\frac{S(t - \frac{r}{c})}{4\pi}$$

and the solution to (A2) is

$$A(r,t) = -\frac{S(t - \frac{r}{c})}{4\pi r} \qquad (A6)$$

In words, the amplitude of the fields decay as $\sim 1/r$ with distance r from the source and the amplitude at a distance r from the source depends on the amplitude of the source at the earlier time $t - \frac{r}{c}$.