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Inter-Sampling Safety of Stochastic Systems with Control Barrier Functions

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Purpose

Many engineered systems have a requirement that they be designed to be safe. In a more rigid sense, there are situations and conditions that are unacceptable and must be avoided. For example, if designing a self-driving car, a collision would be an unacceptable condition and safe control would be the aspects of the control system that prevent the collision from occurring. Typically, Control Barrier Functions are used to ensure safety.

When applying this control to real-life systems, often sensing only occurs and control can only be applied during discrete sampling times. Special concern must be taken so that safety is ensured through a sampling period. Also, there are often stochastic dynamics that a system might experience in the form of sensor noise or external disturbances. The system must be robust to these random dynamics.

Work has been done to ensure safety of a deterministic system with discrete sampling periods [1]. Work has also been done to ensure safety of a stochastic system when there is access to an unbounded continuous control [2], [3], [4]. However, there is currently a gap in the literature for the combination of those topics. The purpose of this project is to explore the topic of Safe Control when dealing with systems that both have stochastic dynamics and utilize discrete time sample-and-hold control.

Control Barrier Functions (CBFs)

- Safe control is accomplished using Control Barrier Functions (CBFs) [5]. These provide the conditions that the control inputs must satisfy to ensure safety.
- The safe set, C, is defined by when the CBF, h(x), is non-negative
$$C = \{x \in D \subset \mathbb{R}^n : h(x) \geq 0\}$$
- Safety is ensured so long as when on the boundary of the set, h(x) is non-decreasing. This can be accomplished with class-K function α(h(x)) so long as an input is selected such that
$$\dot{h}(x) \geq -\alpha(h(x))$$

Problem Statement

- The problem considers a small robotic car driving a track on a table. The velocity of the car is input directly. Therefore, the system is

Deterministic Case: $\dot{x} = u$

Stochastic Case: $dx_t = u dt + \sigma dW_t$

$x, u, Wt \in \mathbb{R}^2$

- Safety is defined by the car remaining on the table. The table is circular with radius of 3 feet. So the CBF is
$$h(x) = R^2 - (x_1^2 + x_2^2)$$
- Therefore, h(x) < 0 refers to a condition when the car is no longer on the table. The function of the car is to drive around the track on the table, so the nominal controller tends to drive in a circle about the center of the table.

3 different systems were simulated

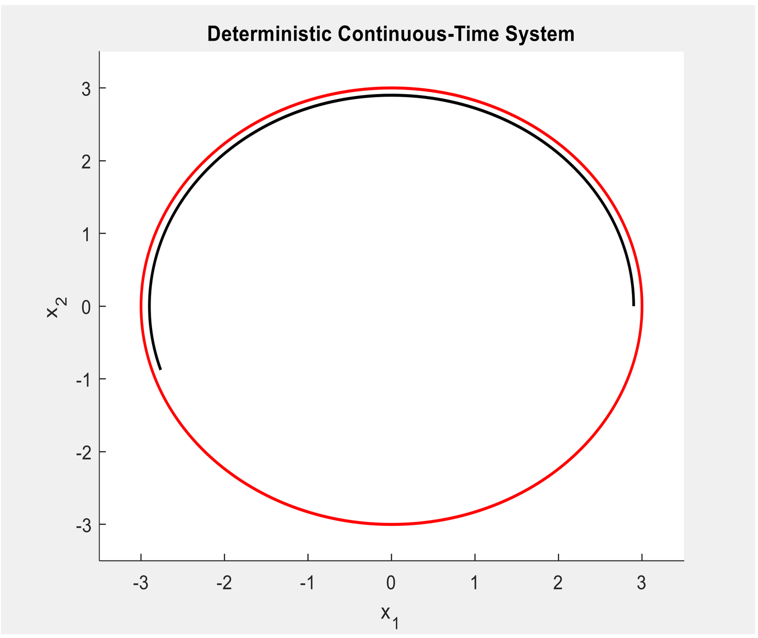
1. Continuous Time Deterministic System
2. Sample-and-Hold Deterministic System
3. Sample-and-Hold Stochastic System

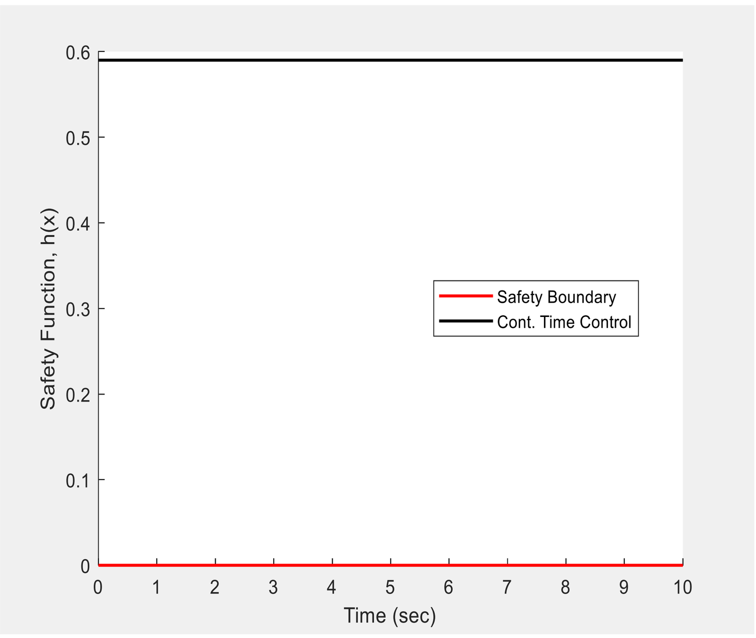
3 different control systems were used such that

1. $\dot{h}(x) \geq -\alpha(h(x))$: Cont. Time Control
2. 1. $\dot{h}(x) \geq -\alpha(h(x)) + d$: Disc. Time Control
3. 1. $\dot{h}(x) \geq -\alpha(h(x)) + d + B_\sigma$: Stochastic Control

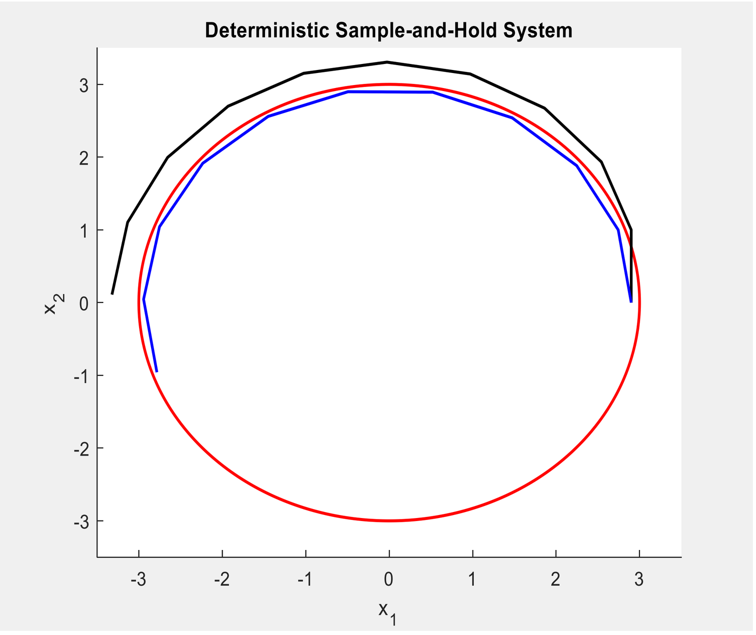
Simulation

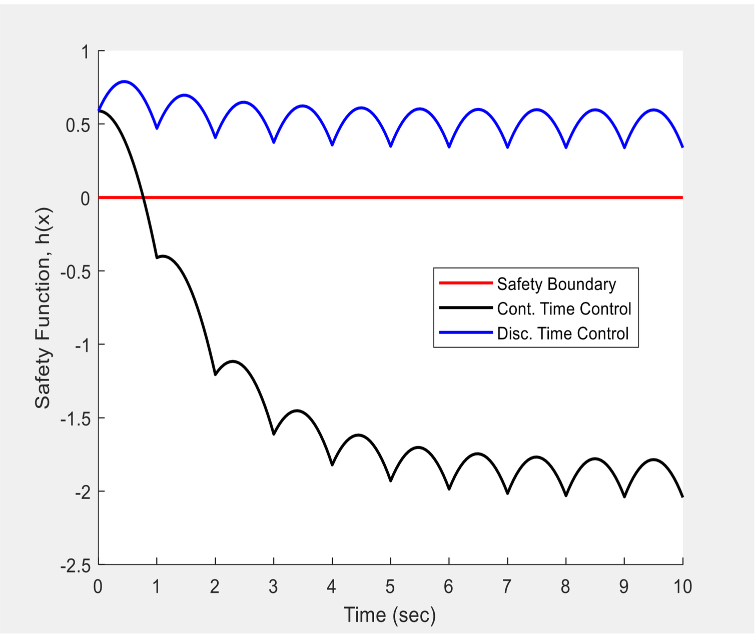
For the Deterministic Continuous time case, the system remains safe with use of the typical CBF



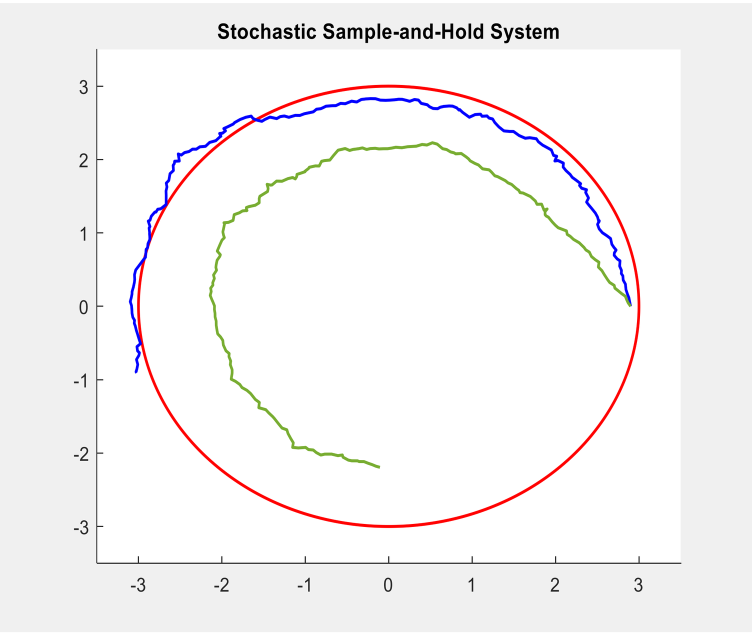


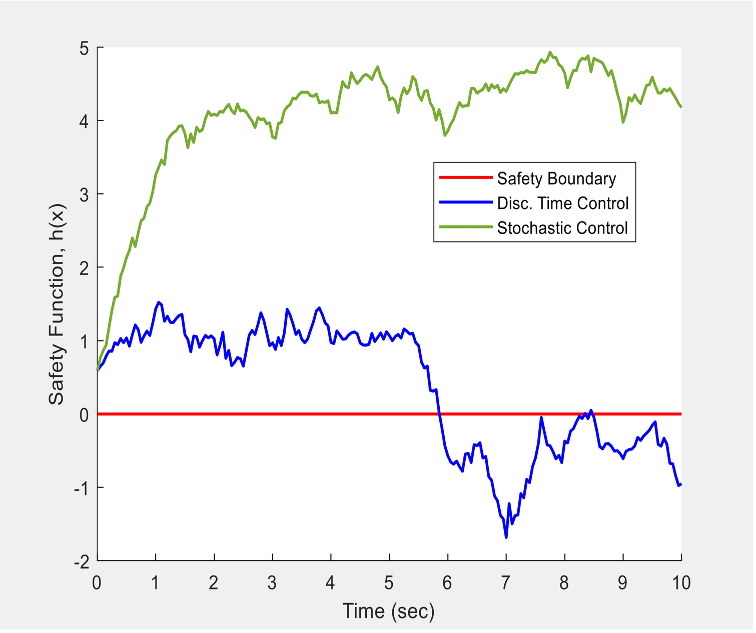
For the Deterministic Sample-and-Hold System, the typical CBF experiences safety violations. The car drives forward for a time sample, without turning. The Discrete Time control system anticipates the edge and avoids the safety violation.





For the Stochastic Sample-and-Hold System, the Discrete time control system drifts off the table due to the random process. The stochastic control system has stricter conditions and avoids the edge, remaining safe.





Sample-and-Hold Systems

- Systems that only sample and update their inputs after discrete sampling times are Sample-and-Hold Systems [1]. These systems will have a constant input through the sampling time.
- From a safety perspective, if the safety of the system changes as the state changes during a sampling period, there will be no opportunity to update the input until the next sampling time. These can lead to safety violations.
- To avoid these violations and ensure safety throughout, we add a measure of strength to the CBF condition that is large enough to account for any departures
$$\dot{h}(x) \geq -\alpha(h(x)) + d$$

Bounded Error for Stochastic Systems

- When Brownian Motion is added to the system, the random process will lead to departures from a typical trajectory, which may lead to safety violations.
- We can probabilistically bound the error experienced by the random process in a sampling period [6]. This error bound, B_σ, is dependent on the variance of the process, the sampling time, and a desired level of confidence that the system will remain safe.
- We can not ensure safety with probability 1. We can however, improve the probability of remaining safe by accounting for the error bound and strengthening our CBF condition further.
$$\dot{h}(x) \geq -\alpha(h(x)) + d + B_\sigma$$

Possible Future Work

- Determining safety confidence for multiple time samples as time advances towards infinity.
- Finding conditions for the necessary sampling time to have confidence in safety with a given control system
- Investigate altering sample time as we approach the bound to only apply strict constraints in most dire conditions

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Acknowledgements

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