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THIN SEQUENCES AND THE GRAM MATRIX

PAMELA GORKIN, JOHN E. MCCARTHY, SANDRA POTT, AND BRETT D. WICK

ABSTRACT. We provide a new proof of Volberg's Theorem characterizing thin interpolating sequences as those for which the Gram matrix associated to the normalized reproducing kernels is a compact perturbation of the identity. In the same paper, Volberg characterized sequences for which the Gram matrix is a compact perturbation of a unitary as well as those for which the Gram matrix is a Schatten-2 class perturbation of a unitary operator. We extend this characterization from 2 to p , where $2 \leq p \leq \infty$.

1. INTRODUCTION

Let \mathbb{D} denote the open unit disk and \mathbb{T} the unit circle. Given $\{\alpha_j\}$, a Blaschke sequence of points in \mathbb{D} , we let B denote the corresponding Blaschke product and B_n denote the Blaschke product with the zero α_n removed. Further, we let $\delta_j = |B_j(\alpha_j)|$, $k_j = \frac{1}{1-\alpha_j z}$ denote the Szegő kernel (the reproducing kernel for H^2) at α_j , $g_j = k_j/\|k_j\|$ the H^2 -normalized kernel, and G the Gram matrix with entries $G_{ij} = \langle g_j, g_i \rangle$. In the second part of [10, Theorem 2], Volberg's goal was to develop a condition ensuring that $\{g_n\}$ is near an orthogonal basis; by this, one means that there exist U unitary and K compact such that

$$g_n = (U + K)e_n,$$

where $\{e_n\}$ is the standard orthogonal basis for ℓ^2 . By [7, Section 3] or [4, Proposition 3.2], this is equivalent to the Gram matrix defining a bounded operator of the form $I + K$ with K compact. Following Volberg and anticipating the connection to the Schatten- p classes, we call such bases $\mathcal{U} + \mathcal{S}_\infty$ bases. Volberg showed that $\{g_n\}$ is a $\mathcal{U} + \mathcal{S}_\infty$ basis if and only if $\lim_n \delta_n = 1$; in other words, if and only if $\{\alpha_n\}$ is a thin sequence. Assuming $\{g_n\}$ is a $\mathcal{U} + \mathcal{S}_\infty$ basis, it is not difficult to show that the sequence $\{\alpha_n\}$ must be thin. But Volberg's proof of the converse is more difficult and depends on the main lemma of a paper of Axler, Chang and Sarason [2, Lemma 5], estimating the norm of a certain product of Hankel operators as well as a factorization theorem for Blaschke products. The lemma in [2] uses maximal functions and a certain distribution function inequality. A more direct proof of Volberg's result is desirable, and we provide a simpler proof of this result in Theorem 3.5 of this paper.

In a second theorem, letting \mathcal{S}_2 denote the class of Hilbert-Schmidt operators, Volberg showed (see [10, Theorem 3]) that $\{g_n\}$ is a $\mathcal{U} + \mathcal{S}_2$ basis if and only if $\prod_{n=1}^\infty \delta_n$ converges. We are interested in estimates for the "in-between" cases. We provide a new proof of Volberg's theorem for $p = \infty$ and prove the following theorem.

Theorem 1.1. *For $2 \leq p < \infty$, the operator $G - I \in \mathcal{S}_p$ if and only if $\sum_n (1 - \delta_n^2)^{p/2} < \infty$.*

Volberg's theorem covered the cases $p = 2$ and $p = \infty$, but our proofs differ in the following way: Instead of using the results of [2] and theorems about Hankel operators, we use the

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relationship between growth estimates of functions that do interpolation on thin sequences (see [5], [6]) and the norm of the Gram matrix. This simplifies previous proofs and provides the best estimates available.

2. PRELIMINARIES AND NOTATION

Let $\{\alpha_j\}$ be a sequence in \mathbb{D} with corresponding Blaschke product B , and B_j be the Blaschke product with zeroes at every point in the sequence except α_j and $\delta_j = |B_j(\alpha_j)|$. The separation constant δ is defined to be $\delta := \inf_j \delta_j$. Carleson's interpolation theorem says that the sequence $\{\alpha_j\}$ is interpolating if and only if $\delta > 0$, [3]. The sequence $\{\alpha_j\}$ is said to be *thin* if $\lim_{j \rightarrow \infty} \delta_j = 1$. Given a thin sequence we may arrange the δ_j in increasing order and rearrange the zeros of the Blaschke product accordingly.

Recall that if T is an operator on a Hilbert space \mathcal{H} and λ_n is the n th singular value of T , then given p with $1 \leq p < \infty$ the Schatten- p class, \mathcal{S}_p , is defined to be the space of all compact operators with corresponding singular sequence in ℓ^p , the space of p -summable sequences. Then \mathcal{S}_p is a Banach space with norm

$$\|T\|_p = \left(\sum |\lambda_n|^p \right)^{1/p}.$$

For $p = \infty$, we let \mathcal{S}_∞ denote the space of compact operators.

Recall that k_j denotes the Szegő kernel, $g_j = k_j/\|k_j\|$, and G the Gram matrix with entries $G_{ij} = \langle g_j, g_i \rangle$. (The Gram matrix depends of course on the sequence $\{\alpha_j\}$, but we suppress this in the notation). For $\{\alpha_j\}$ interpolating, we let D be the diagonal matrix with entries $1/B_j(\alpha_j)$. It is known (see, for example, formula (26) of [8]) that

$$(2.1) \quad G^{-1} = D^* G^t D.$$

For a given sequence $\{\alpha_j\}$, the interpolation constant is the infimum of those M such that for any sequence $\{a_j\}$ in ℓ^∞ , one can find a function f in H^∞ with $f(\alpha_j) = a_j$ and $\|f\|_\infty \leq M \|a\|_{\ell^\infty}$. We shall let $M(\delta)$ denote the supremum of the interpolation constants over all sequences $\{\alpha_j\}$ with separation constant δ .

The following result is due essentially to A. Shields and H. Shapiro [9]. See [1, Proposition 9.5] for a proof of this version.

Proposition 2.1. *Let $\{\alpha_j\}$ be an interpolating sequence in \mathbb{D} .*

- (i) *If the interpolation constant is M , then both $\|G\|$ and $\|G^{-1}\|$ are bounded by M^2 .*
- (ii) *If $\|G\| = C_1$ and $\|G^{-1}\| = C_2$, then the interpolation constant is bounded by $\sqrt{C_1 C_2}$.*

We shall use the following estimate of J.P. Earl (see [5] or [6]) to obtain our results.

Theorem 2.2 (Earl's Theorem). *The interpolation constant $M(\delta)$ satisfies*

$$M(\delta) \leq \left(\frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^2.$$

3. SCHATTEN- p CLASSES

In this section we provide estimates on the Schatten- p norm of $G - I$. We will need the theorem and lemma below.

Theorem 3.1. (see e.g. [11, Theorem 1.33]) *Let T be an operator on a separable Hilbert space, \mathcal{H} .*

If $0 < p \leq 2$ then

$$\|T\|_{\mathcal{S}_p}^p = \inf \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}$$

and if $2 \leq p < \infty$

$$\|T\|_{\mathcal{S}_p}^p = \sup \left\{ \sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H} \right\}.$$

We say that two sequences $\{x_n\}$ and $\{y_n\}$ of positive numbers are equivalent if there exist constants c and C , independent of n , such that $cy_n \leq x_n \leq Cy_n$ for all n . We write $x_n \asymp y_n$. We will also write $A \lesssim B$ to indicate that there exists a constant C such that $A \leq CB$.

Lemma 3.2. *Let $\{e_n\}$ denote the standard orthonormal basis for ℓ^2 and $\{\alpha_j\}$ be an interpolating sequence in \mathbb{D} with corresponding δ_j . Then*

$$\|(G - I)e_n\| \asymp \sqrt{1 - \delta_n^2}.$$

Proof. We have

$$\begin{aligned} \|(G - I)e_n\|^2 &= \langle (G^* - I)(G - I)e_n, e_n \rangle \\ &= \left\langle \begin{pmatrix} \langle g_n, g_1 \rangle \\ \dots \\ \langle g_n, g_{n-1} \rangle \\ 0 \\ \langle g_n, g_{n+1} \rangle \\ \dots \end{pmatrix}, \begin{pmatrix} \langle g_n, g_1 \rangle \\ \dots \\ \langle g_n, g_{n-1} \rangle \\ 0 \\ \langle g_n, g_{n+1} \rangle \\ \dots \end{pmatrix} \right\rangle \\ &= \sum_{j \neq n} \langle g_n, g_j \rangle \overline{\langle g_n, g_j \rangle} \\ &= \sum_{j \neq n} \left| \frac{\sqrt{1 - |\alpha_j|^2} \sqrt{1 - |\alpha_n|^2}}{1 - \overline{\alpha_n} \alpha_j} \right|^2 \\ &= \sum_{j \neq n} 1 - \left| \frac{\alpha_j - \alpha_n}{1 - \overline{\alpha_n} \alpha_j} \right|^2. \end{aligned}$$

But $-\log x \geq 1 - x$ for $x > 0$ and $-\log x < c(1 - x)$ for $x < 1$ bounded away from 0 and some constant c independent of x , so $-\log x \asymp 1 - x$ for x bounded away from 0. Consequently,

$$\begin{aligned} \|(G - I)e_n\|^2 &\asymp \sum_{j \neq n} -\log \left| \frac{\alpha_j - \alpha_n}{1 - \overline{\alpha_n} \alpha_j} \right|^2 \\ &= -\log \prod_{j \neq n} \left| \frac{\alpha_j - \alpha_n}{1 - \overline{\alpha_n} \alpha_j} \right|^2 \\ &= -\log \delta_n^2 \\ &\asymp 1 - \delta_n^2. \end{aligned}$$

Note that the constants involved do not depend on n . □

Combining the lemma with Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *The following estimates hold:*

- If $2 \leq p < \infty$ then

$$\sum_n (1 - \delta_n)^{\frac{p}{2}} \lesssim \|G - I\|_{\mathcal{S}_p}^p;$$

- If $0 < p \leq 2$ then

$$\|G - I\|_{\mathcal{S}_p}^p \lesssim \sum_n (1 - \delta_n)^{\frac{p}{2}};$$

- If $p = 2$ then

$$\sum_n (1 - \delta_n) \asymp \|G - I\|_{\mathcal{S}_2}^2.$$

Lemma 3.4. *Let $\{\alpha_j\}$ be an interpolating sequence and G the corresponding Gram matrix. Let $C = \|G^{-1}\|$. Then $\|G - I\| \leq C - 1$.*

Proof. By (2.1), we have $G \leq CI$, and as G is a positive operator, we have $G \geq (1/C)I$. Therefore

$$\left(\frac{1}{C} - 1\right)I \leq G - I \leq (C - 1)I,$$

and as $C > 1$, we get $\|G - I\| \leq C - 1$. □

In what follows, for a positive integer N , we let G_N denote the lower right-hand corner of the Gram matrix obtained by deleting the first N rows and columns of G . Thus,

$$G_N = \begin{pmatrix} 1 & \langle g_{N+1}, g_N \rangle & \cdots & \langle g_{N+j}, g_N \rangle & \cdots \\ \langle g_N, g_{N+1} \rangle & 1 & \cdots & \langle g_{N+j}, g_{N+1} \rangle & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and $\lambda_n \geq 0$ denotes the n -th singular value of $G - I$, where the singular values are arranged in decreasing order.

We are now ready to provide our simpler proof of Volberg's result [10, Theorem 2, p. 215].

Theorem 3.5. *The sequence $\{\alpha_n\}$ is a thin sequence if and only if the Gram matrix G is the identity plus a compact operator.*

Proof. (\Rightarrow) Suppose $\{\alpha_n\}$ is thin. By discarding finitely many points in the sequence (α_n) , we can assume that the sequence has a positive separation constant, and hence is interpolating.

Let G_N be the Gram matrix of $\{g_j\}$ for $j \geq N$. We shall let $\delta_{N,j}$ denote the δ_j defined for G_N (that is, corresponding to the Blaschke sequence $\{\alpha_j\}_{j \geq N}$). Note that $\delta_{N,j} \geq \delta_{N+j}$ for $j = 0, 1, 2, \dots$, and so we have that $\delta(N) := \inf_j \delta_{N,j} \geq \inf_j \delta_{N+j} = \delta_N$. By Theorem 2.2 and Proposition 2.1,

$$\|G_N^{-1}\| \leq (M(\delta(N)))^2 \leq \frac{\left(1 + \sqrt{1 - \delta(N)}\right)^4}{\delta(N)^4} \leq \frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4}.$$

Applying Lemma 3.4

$$\|G_N - I_N\| \leq \left(\frac{\left(1 + \sqrt{1 - \delta_N^2}\right)^4}{\delta_N^4} - 1 \right) \leq C\sqrt{1 - \delta_N},$$

where C is a constant independent of N . Since $\sqrt{1 - \delta_N} \rightarrow 0$ as $N \rightarrow \infty$, we conclude that $G - I$ is compact.

(\Leftarrow) From (2.1), we have

$$(3.1) \quad G^{-1} - I = D^*(G^t - I)D + [D^*D - I].$$

If $G - I$ is compact, then so are $G^t - I$ and $G^{-1} - I = G^{-1}(I - G)$. Therefore from (3.1), we have $D^*D - I$ is compact, which means $\lim_{j \rightarrow \infty} \delta_j^2 = 1$. Consequently, the sequence is thin. \square

Theorem 3.6. *For $2 \leq p < \infty$, the operator $G - I \in \mathcal{S}_p$ if and only if $\sum_n (1 - \delta_n^2)^{p/2} < \infty$.*

Proof. By Theorem 3.3, if $G - I \in \mathcal{S}_p$, then the sum is finite.

Now suppose the sum is finite. Using Lemma 3.4 as in Theorem 3.5, we have

$$\|G_N - I_N\| \leq C\sqrt{1 - \delta_N},$$

where C is independent of N .

By [11, Theorem 1.4.11],

$$|\lambda_{N+1}| \leq \inf\{\|(G - I) - F\| : F \in \mathcal{F}_N\},$$

where \mathcal{F}_N is the set of all operators of rank less than or equal to N . Therefore, taking F to be the matrix with the same first N rows and columns as $G - I$, which is of rank at most $2N$, we have

$$|\lambda_{2N+1}| \leq \|G_{N+1} - I_{N+1}\| \leq C\sqrt{1 - \delta_{N+1}},$$

by our computation above. Therefore

$$|\lambda_{2N+1}|^p \leq C^p(1 - \delta_{N+1})^{p/2}.$$

Since the singular values are arranged in decreasing order, $|\lambda_{2n+1}| > |\lambda_{2n}|$ for each n . Thus, if $\sum_N (1 - \delta_N)^{p/2} < \infty$, then $\sum_n |\lambda_{2n}|^p \leq 2 \sum_n |\lambda_{2n+1}|^p < \infty$ and we conclude that $G - I \in \mathcal{S}_p$. \square

We conclude by remarking that it is possible to trace through the proofs above to determine constants c and C , which depend only on $\delta = \inf_n \delta_n$, such that for $2 \leq p \leq \infty$,

$$(3.2) \quad c\|\sqrt{1 - \delta_n}\|_{\ell^p} \leq \|G - I\|_{\mathcal{S}_p} \leq C\|\sqrt{1 - \delta_n}\|_{\ell^p}.$$

In particular, by choosing δ close enough to 1, one can choose c and C in (3.2) arbitrarily close to $\sqrt{2}$ and $4\sqrt{2}(2^{1/p})$, respectively.

Question 3.7. *Is Theorem 3.6 true for $p < 2$?*

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