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Hankel vector moment sequences and the non-tangential regularity at infinity of two variable Pick functions

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HANKEL VECTOR MOMENT SEQUENCES AND THE NON-TANGENTIAL REGULARITY AT INFINITY OF TWO VARIABLE PICK FUNCTIONS

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ABSTRACT. A Pick function of d variables is a holomorphic map from Π^d to Π, where Π is the upper halfplane. Some Pick functions of one variable have an asymptotic expansion at infinity, a power series $\sum_{n=1}^{\infty} \rho_n z^{-n}$ with real numbers ρ_n that gives an asymptotic expansion on non-tangential approach regions to infinity. In 1921 H. Hamburger characterized which sequences $\{\rho_n\}$ can occur. We give an extension of Hamburger's results to Pick functions of two variables.

1. INTRODUCTION

A Pick function of one variable is a holomorphic map from the upper half-plane, which we shall denote by Π, into $\overline{\Pi}$. A Pick function of two variables is a holomorphic map from Π^2 to $\overline{\Pi}$. The purpose of this note is to extend to two variables certain well-known results about the asymptotic analysis of Pick functions in one variable.

1.1. **One variable results.** In 1922, R. Nevanlinna showed that a Pick class function of one variable that decays at infinity is the Cauchy transform of a finite measure on $\mathbb{R}$.

Theorem 1.1 ([13]). *If $F : \Pi \to \Pi$ is analytic and satisfies*

$$(1.2) \qquad \limsup_{y \to \infty} |yF(iy)| < \infty,$$

then there exists a unique finite positive Borel measure μ on $\mathbb{R}$ so that

$$(1.3) \qquad F(z) = \int \frac{d\mu(t)}{t - z}.$$

We shall say that a set S in Π approaches ∞ non-tangentially, $S \overset{nt}{\to} \infty$, if ∞ is in the closure, and there is a constant c such that $|z| \leq c \operatorname{Im}(z)$ for all $z \in S$. If F has a representation as in (1.3), then

$$F(z) = \frac{\rho}{z} + o(1/|z|)$$

as $z \overset{nt}{\to} \infty$, where $\rho = -\|\mu\|$. If μ has more moments, then there is a higher order asymptotic expansion at ∞. H. Hamburger proved the following two theorems [8, 9]. For a proof of Theorem 1.4 as stated, see [17, Thm. 2.2] or [5, Thm 3.2.1].

Theorem 1.4. *Let real constants $\rho_1, \ldots, \rho_{2N-1}$ be given. There exists a Pick function F satisfying*

$$(1.5) \qquad F(z) = \frac{\rho_1}{z} + \frac{\rho_2}{z^2} + \cdots + \frac{\rho_{2N-1}}{z^{2N-1}} + o(|z|^{-(2N-1)})$$

as $z \overset{nt}{\to} \infty$ if and only if there is a measure μ on $\mathbb{R}$ whose first $2(N-1)$ moments are finite and satisfy

$$(1.6) \qquad \int t^k d\mu(t) = -\rho_{k+1}, \qquad 0 \le k \le 2(N-1).$$

Moreover, in this case F has a representation as in (1.3) for some measure μ satisfying (1.6).

Hamburger gave an alternate equivalent condition. There is also a proof in [17, Thm. 1.2], and see [12, Thm. 3.3] for an alternative formulation (but without a proof).

Theorem 1.7. *Let $\rho_1, \ldots, \rho_{2N-1}$ be given real numbers. There exists a Pick function F satisfying (1.5) if and only if the N-by-N Hankel matrix*

$$H = - \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_N \\ \rho_2 & \rho_3 & \cdots & \rho_{N+1} \\ \vdots & \vdots & & \vdots \\ \rho_N & \rho_{N+1} & \cdots & \rho_{2N-1} \end{pmatrix}$$

is positive semi-definite and has the property that whenever $(c_1, c_2, \ldots, c_{N-1}, 0)^t$ is in the kernel of H, then $(0, c_1, c_2, \ldots, c_{N-1})^t$ is also in the kernel.

In 1881, L. Kronecker proved the following theorem [11] (see [14, Thm. I.3.1] for a modern treatment).

Theorem 1.8. *The infinite Hankel form*

$$\begin{pmatrix} \rho_1 & \rho_2 & \rho_3 & \cdots \\ \rho_2 & \rho_3 & \rho_4 & \cdots \\ \rho_3 & \rho_4 & \rho_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

is of finite rank if and only if

$$F(z) = \sum_{n=1}^{\infty} \frac{\rho_n}{z^n}$$

is a rational function.

1.2. **Two variable results.** A two variable version of Theorem 1.1 was proved in [4]; see also Theorem 8.4 below. Before stating it, let us introduce some notation. If Y is an operator on a Hilbert space, and $z = (z_1, z_2)$ is a point in $\mathbb{C}^2$, we shall use z_Y to denote the operator

$$z_Y = z_1 Y + z_2(I - Y).$$

Theorem 1.9 ([4]). *Let* $h : \Pi^2 \to \Pi$ *be a Pick function of two variables. Then*

$$\limsup_{s \to \infty} |sF(is, is)| \ < \ \infty$$

if and only if there is a Hilbert space $\mathcal{H}$, *a self-adjoint densely defined operator* A *on* $\mathcal{H}$, *a positive contraction* Y *on* $\mathcal{H}$, *and a vector* α *in* $\mathcal{H}$, *such that*

$$(1.10) \qquad h(z) \ = \ \langle (A - z_Y)^{-1} \alpha, \alpha \rangle, \quad z \in \Pi^2.$$

We shall say that h has a *type I Nevanlinna representation* if it has a representation as in (1.10).

In one variable, the Poisson integral of any finite positive measure on $\mathbb{R}$ is the real part of a Pick function that decays like (1.2), so the study of asymptotic expansions (1.5) and solutions to the moment problem (1.6) for arbitrary measures are tightly bound. In two variables, their study diverges. The infinite Hamburger moment problem in several variables is studied in [16] and [18]; for an algorithm for solving the problem in two variables, see [19]. For the truncated problem, see for example the memoir [7] and subsequent papers. Our objective is to study the two variable analogue of (1.5).

If one restricts z to the diagonal $\{z_1 = z_2\}$, then (1.10) becomes (1.3), where μ is the scalar spectral measure of A for the vector α. Saying that an even moment γ_{2k} exists in this case is the assertion that t^{k-1} is in the domain of A. We shall generalize this idea to two variables.

We shall let m and n denote ordered pairs of non-negative integers. We set $e_1 = (1,0)$ and $e_2 = (0,1)$. If $n = (n_1, n_2)$, we set $|n| = n_1 + n_2$, and for a pair $z = (z_1, z_2)$ we follow the usual convention of letting $z^n = z_1^{n_1} z_2^{n_2}$. For N a positive integer we set $I_N = \{n \mid 1 \leq |n| \leq N\}$.

We now define an object that we shall call a *finite Hankel vector moment sequence*, or for short, a *finite HVMS*. For simplicity, we take $N \geq 2$; see (2.1) for general N.

Definition 1.11. For a fixed positive integer $N \geq 2$, a finite Hankel vector moment sequence is a 3-tuple, $(\{\alpha_n\}_{n \in I_N}, Y, A)$, where: $\{\alpha_n\}_{n \in I_N}$ is a sequence of vectors in some Hilbert space $\mathcal{H}$; Y is a positive contraction acting on $\mathcal{H}$, satisfying for each $l = 1, \dots, N$

$$(1.12) \qquad Y\alpha_{(0,l)} = 0 = (1 - Y)\alpha_{(l,0)} = 0;$$

A is a partially defined symmetric operator on $\mathcal{H}$ with the property that

$$(1.13) \qquad \{\alpha_n \mid 1 \leq |n| \leq N - 1\} \subset \mathrm{Dom}(A);$$

and for each $n \in I_{N-1}$,

$$(1.14) \qquad A\alpha_n = Y\alpha_{n+e_1} + (1 - Y)\alpha_{n+e_2}.$$

Here is the main result of this paper.

Theorem 1.15. *A Pick function* h *of two variables satisfies*

$$(1.16) \qquad h(z) \ = \ \sum_{n \in I_{2N-1}} \frac{\rho_n}{z^n} \ + \ o(\|z\|^{-(2N-1)})$$

as $z \xrightarrow{nt} \infty$, for some real numbers ρ_n, if and only if it has a representation as in (1.10) and there is a finite HVMS $(\{\alpha_n\}_{n \in I_N}, Y, A)$ with $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$. Moreover, ρ_k is given by the formula

$$\rho_k = -\sum \{\langle \alpha_n, A\alpha_m \rangle \; : \; m_1 + n_1 = k_1, \; m_2 + n_2 = k_2, \; m_1 + m_2 = \lfloor |k|/2 \rfloor \}.$$

When $k = 1$, one interprets the right-hand side of the inner product as α (so $\rho_{(1,0)} = -\langle \alpha_{(1,0)}, \alpha \rangle$ and $\rho_{(0,1)} = -\langle \alpha_{(0,1)}, \alpha \rangle$). By $z \xrightarrow{nt} \infty$ we mean that $\|z\| \to \infty$, while z stays in an approach region

$$\{z \in \Pi^2 \; : \; \|z\| \le c \min\{\mathrm{Im}z_1, \; \mathrm{Im}z_2\}\}$$

for some c. The notation $\lfloor M/2 \rfloor$ stands for the greatest integer less than or equal to $M/2$.

The forward implication of (1.15) is Theorem 4.2; the converse is Theorem 3.10. To relate Theorem 1.15 to Theorems 1.4 and 1.7, think in one variable of α_n as t^{n-1} in $L^2(\mu)$, and A as multiplication by t on $L^2(\mu)$. Then ρ_k is given by a single term, $-\langle t^{\lceil k/2 \rceil - 1}, t^{\lfloor k/2 \rfloor} \rangle$.

Theorem 1.7 also has a two variable analogue, which we give in Theorem 5.7. This justifies our nomenclature of Hankel vector moment sequence. The last condition in Theorem 5.7 is an analogue of the last condition in Theorem 1.7; for an explanation of it, see Section 5.

Theorem 5.7. *Let $a = (a^1, a^2)$ be a pair of matrices on I_N. Then there is a finite HVMS $(\{\alpha_n\}_{n \in I_N}, Y, A)$ such that*

$$\begin{aligned} a^1_{mn} &= \langle Y\alpha_n, \alpha_m \rangle, \\ a^2_{mn} &= \langle (1 - Y)\alpha_n, \alpha_m \rangle \end{aligned}$$

if and only if the following four conditions obtain:

$$a^1 \text{ and } a^2 \text{ are positive semi-definite,}$$

$$a^1_{m+e_1,n} + a^2_{m+e_2,n} = a^1_{m,n+e_1} + a^2_{m,n+e_2} \quad \text{whenever} \quad m, n \in I_{N-1},$$

$$a^1_{(0,l),(0,l)} = a^2_{(l,0),(l,0)} = 0 \quad \text{for} \quad l = 1, \dots, N,$$

$$\mathrm{supp}(f) \in I_{N-1} \text{ and } (a^1 + a^2)f = 0 \Rightarrow (a^1 S_1 + a^2 S_2)f = 0.$$

In Section 6, we discuss infinite sequences. One multi-variable generalization of Kronecker's Theorem 1.8 was proved by S. C. Power [15]. In Theorem 6.6, we prove another.

Theorem 6.6. *Let h have non-tangential asymptotic expansions of all orders at infinity. Then there is an infinite HVMS $(\{\alpha_n\}, Y, A)$ with $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, and $h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle$. The sequence can be chosen with $\mathrm{rank}\langle \alpha_n, \alpha_m \rangle < \infty$ if and only if h is a rational function.*

In Section 7, we give an example of a construction of functions in the Pick class that have asymptotic expansions. In Section 8, we give some technical results on models.

2. Finite Hankel vector moment sequences

Definition 2.1. For a fixed positive integer N, a finite Hankel vector moment sequence is a 3-tuple, $(\{\alpha_n\}_{n \in I_N}, Y, A)$, where: $\{\alpha_n\}_{n \in I_N}$ is a sequence of vectors in some Hilbert space $\mathcal{H}$; Y is a positive contraction acting on $\mathcal{H}$, satisfying for each $l = 1, \ldots, N$

$$(2.2) \qquad Y\alpha_{(0,l)} = 0 = (1-Y)\alpha_{(l,0)} = 0;$$

A is a partially defined symmetric operator on $\mathcal{H}$ with the properties that, if $N \geq 2$,

$$(2.3) \qquad \{\alpha_n \mid 1 \leq |n| \leq N-1\} \subset \mathrm{Dom}(A);$$

and, for each $n \in I_{N-1}$,

$$(2.4) \qquad A\alpha_n = Y\alpha_{n+e_1} + (1-Y)\alpha_{n+e_2}.$$

When $N = 1$ conditions (2.3) and (2.4) are dropped.

Every symmetric operator has a self-adjoint extension on a possibly larger Hilbert space, so there is no loss in generality in assuming A is self-adjoint.

If $(\{\alpha_n\}_{n \in I_N}, Y, A)$ is a finite HVMS, we shall frequently abuse the notation somewhat and refer to the entire tuple by simply $\{\alpha_n\}$. If $\{\alpha_n\}$ is an HVMS as above, we refer to N as the *size* of $\{\alpha_n\}$, Y as the *Hankel weight* of $\{\alpha_n\}$, A as the *Hankel shift* of $\{\alpha_n\}$, and finally the vectors, α_n, are called the *vector moments* of $\{\alpha_n\}$.

Our first proposition gives a simple yet fundamental property of HVMS's. If $z \in \mathbb{C}^2$ and Y is a positive contraction on a Hilbert space $\mathcal{H}$, we define $z_Y = z_1 Y + z_2(1-Y)$. As Y is a positive contraction, the spectral theorem implies that z_Y^{-1} is a well-defined analytic operator valued function on the set $\{z \in \mathbb{C}^2 \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$. If $\{\alpha_n\}$ is an HVMS with shift A and weight Y, and l is a positive integer, we shall adopt the notation

$$R_l(z) = z_Y^{-1}(A z_Y^{-1})^{l-1}.$$

Note that if $z \in \{z \in \mathbb{C}^2 \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$, then the domain of $R_l(z)$ is all of $\mathcal{H}$ if $l = 1$ and for $l \geq 2$ it is inductively defined by

$$\mathrm{Dom}((R_l(z))) = \{\alpha \in \mathcal{H} \mid (z_Y^{-1}A)^i z_Y^{-1}\alpha \in \mathrm{Dom}(A), \ i = 0, \ldots, l-2\}.$$

Note also that

$$R_l(\bar{z}) \ \subseteq \ R_l(z)^*.$$

Proposition 2.5. *Let* $\{\alpha_n\}$ *be an HVMS of size* N *and let*

$$(2.6) \qquad \alpha = \alpha_{(1,0)} + \alpha_{(0,1)}.$$

If $1 \leq l \leq N$, *then*

$$(2.7) \qquad \alpha \in \mathrm{Dom}(R_l(z))$$

and

$$(2.8) \qquad R_l(z)\alpha = \sum_{|n|=l} \frac{1}{z^n}\alpha_n$$

for all z in $\{z \in \mathbb{C}^2 \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$.

Proof. We induct on N. If $N = 1$ and $l = 1$, then trivially (2.7) holds. Also, by (2.2), $Y\alpha_{(0,1)} = 0 = (1 - Y)\alpha_{(1,0)} = 0$. Hence,

$$\begin{aligned} R_1(z)\alpha &= z_Y^{-1}\alpha \\ &= z_Y^{-1}\alpha_{(1,0)} + z_Y^{-1}\alpha_{(0,1)} \\ &= \frac{1}{z_1}\alpha_{(1,0)} + \frac{1}{z_2}\alpha_{(1,0)} \\ &= \sum_{|n|=1} \frac{1}{z^n}\alpha_n. \end{aligned}$$

Now assume that the proposition holds for HVMS's of size N. Fix an HVMS, $\{\alpha_n\}_{n \in I_{N+1}}$, of size $N + 1$. The case when $l = 1$ is handled as in the previous paragraph. If $2 \leq l \leq N + 1$, as $\{\alpha_n\}_{n \in I_N}$ is an HVMS of size N, the inductive hypothesis implies that

$$(2.9) \qquad R_l(z)\alpha = z_Y^{-1} A R_{l-1}(z)\alpha = z_Y^{-1} A \sum_{|n|=l-1} \frac{1}{z^n}\alpha_n.$$

As $\{\alpha_n\}_{n \in I_{N+1}}$ is of size $N + 1$ and $l - 1 \leq N$, (2.3) implies that $\alpha_n \in \text{Dom}(A)$ whenever $|n| = l - 1$. Hence, (2.9) implies that $\alpha \in \text{Dom}(R_l(z))$. Also, using (2.2) and (2.4) we see via (2.9) that

$$\begin{aligned} R_l(z)\,\alpha &= z_Y^{-1} A \sum_{|n|=l-1} \frac{1}{z^n}\alpha_n \\ &= z_Y^{-1} \sum_{|n|=l-1} \frac{1}{z^n} Y\alpha_{n+e_1} + z_Y^{-1} \sum_{|n|=l-1} \frac{1}{z^n}(1-Y)\alpha_{n+e_2} \\ &= z_Y^{-1} \frac{1}{z_1^{l-1}} Y\alpha_{(l,0)} + z_Y^{-1} \sum_{\substack{|m|=l \\ m \neq (l,0),(0,l)}} \frac{z_1}{z^m} Y\alpha_m \\ &\quad + z_Y^{-1} \sum_{\substack{|m|=l \\ m \neq (l,0),(0,l)}} \frac{z_2}{z^m}(1-Y)\alpha_m + z_Y^{-1}\frac{1}{z_2^{l-1}}(1-Y)\alpha_{(0,l)} \\ &= \frac{1}{z_1^l}\alpha_{(l,0)} + z_Y^{-1} \sum_{\substack{|m|=l \\ m \neq (l,0),(0,l)}} \frac{z_1}{z^m} Y\alpha_m \\ &\quad + z_Y^{-1} \sum_{\substack{|m|=l \\ m \neq (l,0)(0,l)}} \frac{z_2}{z^m}(1-Y)\alpha_m + \frac{1}{z_2^l}\alpha_{(0,l)} \\ &= \frac{1}{z_1^l}\alpha_{(l,0)} + z_Y^{-1} \sum_{\substack{|m|=l \\ m \neq (l,0)(0,l)}} (z_1 Y + z_2(1-Y))\frac{1}{z^m}\alpha_m + \frac{1}{z_2^l}\alpha_{(0,l)} \end{aligned}$$

$$= \frac{1}{z_1^l}\alpha_{(l,0)} + \sum_{\substack{|m|=l \\ m \neq (l,0)(0,l)}} \frac{1}{z^m}\alpha_m + \frac{1}{z_2^l}\alpha_{(0,l)}$$

$$= \sum_{|n|=l} \frac{1}{z^n}\alpha_n.$$

$\square$

The property described by (2.7) in Proposition 2.5 arises as an issue in many of the applications of HVMS's that we have in mind. Accordingly, we introduce the following definition.

Definition 2.10. Let $\mathcal{H}$ be a Hilbert space, $\alpha \in \mathcal{H}$, and assume that Y is a positive contraction on $\mathcal{H}$. If A is a symmetric operator on H, we say that A has finite complex vector (Y,α)-moments to order N if for each $z \in \{z \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$, $\alpha \in \mathrm{Dom}((Az_Y^{-1})^l)$ for $l = 1, \ldots, N$. We say that A has finite real vector (Y,α)-moments to order N if for each $b \in \mathbb{R}^{+^2}$, $\alpha \in \mathrm{Dom}((Ab_Y^{-1})^l)$ for $l = 1, \ldots, N$.

The following converse to Proposition 2.5 provides a useful criterion to verify that a given symmetric operator and positive operator are associated with an HVMS.

Proposition 2.11. Let $\mathcal{H}$ be a Hilbert space, let $\alpha \in \mathcal{H}$ and assume that A and Y are operators acting on $\mathcal{H}$, with A symmetric and Y a positive contraction. The following conditions are equivalent.

  (i) There exists a sequence $\{\alpha_n\}_{n \in I_N}$ in $\mathcal{H}$ such that $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$ and $(\{\alpha_n\}, A, Y)$ is an HVMS.

  (ii) A has finite complex vector (Y,α)-moments to order $N - 1$, and for each $l = 1, \ldots, N$ there exist vectors α_n, $|n| = l$, such that

$$R_l(z)\alpha = \sum_{|n|=l} \frac{1}{z^n}\alpha_n$$

  whenever $z \in \{z \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$.

  (iii) A has finite real vector (Y,α)-moments to order $N - 1$, and for each $l = 1, \ldots, N$ there exist vectors α_n, $|n| = l$, such that

(2.12) $$R_l(b)\alpha = \sum_{|n|=l} \frac{1}{b^n}\alpha_n$$

  whenever $b \in \mathbb{R}^{+^2}$.

Proof. That (i) implies (ii) follows from Proposition 2.5. Obviously, (ii) implies (iii).

Assume that (iii) holds. To show that (2.2) holds when $l = 1$ and that $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, equate coefficients in the following equation obtained from (2.12) when $l = 1$:

$$\alpha = b_Y b_Y^{-1}\alpha$$
$$= b_Y R_1(b)\alpha$$
$$= (b_1 Y + b_2(1 - Y))(\frac{1}{b_1}\alpha_{(1,0)} + \frac{1}{b_2}\alpha_{(0,1)}).$$

Now assume $N \geq 2$. Note that the moment condition implies that for $1 \leq l \leq N-1$, $R_l(b)\alpha \in \text{Dom}(A)$. Hence by (2.12),

$$\sum_{|n|=l} \frac{1}{b^n}\alpha_n \in \text{Dom}(A),$$

for all $b \in \mathbb{R}^{+2}$. As

$$\text{span}\{\sum_{|n|=l} \frac{1}{b^n}\alpha_n \mid b \in \mathbb{R}^{+2}\} = \text{span}\{\alpha_n \mid |n|=l\},$$

it follows that (2.3) holds.

Now fix l with $1 \leq l \leq N-1$. Noting that $b_Y R_{l+1}(b) = AR_l(b)$, we compute using (2.12) that

$$\sum_{|m|=l} \frac{1}{b^m} A\alpha_m = A \sum_{|m|=l} \frac{1}{b^m}\alpha_m$$

$$= b_Y \sum_{|n|=l+1} \frac{1}{b^n}\alpha_n$$

$$= \frac{b_1}{b_2^{l+1}} Y\alpha_{(0,l+1)} + \sum_{|m|=l} \frac{1}{b^m}(Y\alpha_{m+e_1} + (1-Y)\alpha_{m+e_2})$$

$$+ \frac{b_2}{b_1^{l+1}}(1-Y)\alpha_{(l+1,0)}.$$

Equating terms in this formula yields that (2.4) holds for $2 \leq l \leq N-1$ and that (2.2) holds for $2 \leq l \leq N$. $\qquad\square$

We now turn to a much more subtle characterization of the HVMS's given in Theorem 2.21 below. Suppose that $\{\alpha_n\}$ is an HVMS of size N with weight Y and shift A and let α be as in (2.6). Let $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t > 0\}$. For $1 \leq k \leq 2N-1$ define functions $r_k : \mathbb{R}^{+2} \to \mathbb{R}$ by the formulas

$$(2.13) \qquad r_1(b) \;=\; \langle R_1(b)\alpha, \alpha\rangle \;=\; \langle b_Y^{-1}\alpha, \alpha\rangle \qquad\qquad \text{if } k=1,$$

$$(2.14) \qquad r_k(b) \;=\; \langle R_l(b)\alpha, AR_{l-1}(b)\alpha\rangle \qquad\quad \text{if } 3 \leq k = 2l-1,$$

$$(2.15) \qquad r_k(b) \;=\; \langle R_l(b)\alpha, AR_l(b)\alpha\rangle \qquad\qquad \text{if } 2 \leq k = 2l,$$

where the expressions $R_l(b)\alpha$ make sense by Proposition 2.5. Computing $r_k(b)$ using (2.8) yields the qualitative information that for each k with $1 \leq k \leq 2N-1$, $r_k(b)$ is a homogenous polynomial in $\frac{1}{b} = (\frac{1}{b_1}, \frac{1}{b_2})$ of degree k. To formalize these properties of α, Y, and A we introduce the following definition.

Definition 2.16. Let $\mathcal{H}$ be a Hilbert space, $\alpha \in \mathcal{H}$, and assume that Y is a positive contraction on $\mathcal{H}$. Assume that A is a symmetric operator on H with finite real vector (Y, α)-moments to order $N-1$. For $1 \leq k \leq 2n-1$ we define the kth scalar (Y, α)-moment of A by equations (2.13) to (2.15).

Before continuing, we remark that ontologically the scalar (Y, α)-moments of A are functions on $(\mathbb{R}^+)^2$. However, if these functions happen to be given by homogenous polynomials (as e.g. occurs in the case of an HVMS), then there is an obvious

way to extend the moment functions to all of $\mathbb{C}^2$. Concrete formulas for this case would be

$$(2.17) \qquad r_1(z) = \langle R_1(z)\alpha, \alpha \rangle = \langle z_Y^{-1}\alpha, \alpha \rangle \qquad \text{if } k = 1,$$

$$(2.18) \qquad r_k(z) = \langle R_l(z)\alpha, AR_{l-1}(z)^*\alpha \rangle \qquad \text{if } 3 \le k = 2l - 1,$$

$$(2.19) \qquad r_k(z) = \langle R_l(z)\alpha, AR_l(z)^*\alpha \rangle \qquad \text{if } 2 \le k = 2l.$$

Remark 2.20. If $(\{\alpha_n\}, A, Y)$ is a finite HVMS, then by Proposition 2.11 the kth scalar (Y, α)-moments of A are given by

$$r_1(b) = \tfrac{1}{b_1}\langle \alpha_{(1,0)}, \alpha \rangle + \tfrac{1}{b_2}\langle \alpha_{(0,1)}, \alpha \rangle \qquad \text{if } k = 1,$$

$$r_k(b) = \sum_{|m|=l-1, |n|=l} \tfrac{1}{b^{m+n}} \langle \alpha_n, Y\alpha_{m+e_1} + (1-Y)\alpha_{m+e_2} \rangle \quad \text{if } 3 \le k = 2l - 1,$$

$$r_k(b) = \sum_{|m|=l, |n|=l} \tfrac{1}{b^{m+n}} \langle \alpha_n, Y\alpha_{m+e_1} + (1-Y)\alpha_{m+e_2} \rangle \qquad \text{if } 2 \le k = 2l.$$

In particular, they only depend on the Gram matrices $a^1 = \langle Y\alpha_n, \alpha_m \rangle$ and $a^2 = \langle (1-Y)\alpha_n, \alpha_m \rangle$.

Theorem 2.21. *Let $\mathcal{H}$ be a Hilbert space, $\alpha \in \mathcal{H}$, and $N \ge 1$. Assume that Y is a positive contraction on $\mathcal{H}$ and A is a symmetric operator on $\mathcal{H}$. There exists an indexed sequence $\{\alpha_n\}_{n \in I_N}$ of vectors in $\mathcal{H}$ such that $(\{\alpha_n\}, A, Y)$ is an HVMS of size N and*

$$(2.22) \qquad \alpha = \alpha_{(1,0)} + \alpha_{(1,0)}$$

if and only if A has finite real vector (Y, α)-moments to order $N - 1$, and for each $k \le 2N - 1$, the kth scalar (Y, α)-moment of A is a homogeneous polynomial in $\frac{1}{b}$ of order k.

Proof. The necessity of the homogeneity condition follows by the discussion leading up to Definition 2.16. To prove the sufficiency we proceed by induction on N.

When $N = 1$, there is only one scalar moment given by

$$r_1(b) = \langle b_Y^{-1}\alpha, \alpha \rangle.$$

If r_1 is homogenous of degree one, then there exist constants a_1 and a_2 such that

$$(2.23) \qquad \langle b_Y^{-1}\alpha, \alpha \rangle = a_1 \frac{1}{b_1} + a_2 \frac{1}{b_2}.$$

We analyze (2.23) by making the substitutions

$$(2.24) \qquad b_1 = x \quad \text{and} \quad b_2 = \frac{t}{t-1}x.$$

Noting that in the new variables x and t,

$$(2.25) \qquad b_Y^{-1} = \frac{t-1}{x}(t-Y)^{-1},$$

one computes that (2.23) becomes

$$(2.26) \qquad \langle (t-Y)^{-1}\alpha, \alpha \rangle = \frac{a_1}{t-1} + \frac{a_2}{t}.$$

Now, (2.26) implies that the scalar spectral measure of Y w.r.t. α is supported in the set $\{0, 1\}$, which in turn implies that $Y(1-Y)\alpha = 0$. Letting $\alpha_{(1,0)} = Y\alpha$ and $\alpha_{(0,1)} = (1-Y)\alpha$, we see immediately that (2.22) holds. As

$$(1-Y)\alpha_{(1,0)} = (1-Y)Y\alpha = 0$$

and
$$Y\alpha_{(1,0)} = Y(1-Y)\alpha = 0,$$
we see that (2.2) holds. Finally, as (2.3) and (2.4) are both vacuous when $N = 1$, the theorem is proved for the special case when $N = 1$.

Now suppose the sufficiency of the homogeneity conditions whenever A has finite vector (Y, α)-moments to order $N-1$ and $2N-1$ homogenous scalar (Y, α)-moments. Fix $A, Y,$ and α with the properties that A has finite real vector (Y, α)-moments to order N and $2N + 1$ homogenous real scalar (Y, α)-moments, $r_k(b)$. We need to show that there exists an indexed sequence $\{\alpha_n\}_{n \in I_{N+1}}$ in $\mathcal{H}$ such that $(\{\alpha_n\}, A, Y)$ is an HVMS of size $N + 1$ and such that $\alpha = \alpha_{(1,0)} + \alpha_{(1,0)}$. By Proposition 2.11 this will be accomplished if we can construct an indexed set $\{\alpha_n\}_{n \in I_{N+1}}$ in $\mathcal{H}$ such that

$$(2.27) \qquad \{\alpha_n\}_{n \in I_N} \subset \mathrm{Dom}(A)$$

and

$$(2.28) \qquad R_l(b)\alpha = \sum_{|n|=l} \frac{1}{b^n}\alpha_n \text{ for } l = 1, \ldots, N+1.$$

By the induction hypothesis, there exists an indexed set of vectors in $\mathcal{H}$, $\{\alpha_n\}_{n \in I_N}$, such that (2.22) holds and such that $(\{\alpha_n\}_{n \in I_N}, A, Y)$ is an HVMS of size N. By the homogeneity of $r_{2N+1}(b)$, there exist scalars $\rho_n, |n|= 2N + 1$, such that

$$(2.29) \qquad r_{2N+1}(b) = \sum_{|n|=2N+1} \frac{\rho_n}{b^n}.$$

On the other hand, by the definition of the odd scalar moments, (2.14),

$$(2.30) \qquad r_{2N+1}(b) = \langle b_Y^{-1} A R_N(b)\alpha, A R_N(b)\alpha \rangle.$$

Finally, Proposition 2.5 implies that

$$(2.31) \qquad R_N(b)\alpha = \sum_{|n|=N} \frac{1}{b^n}\alpha_n.$$

The remainder of the proof consists of employing the substitutions, (2.24), to make various deductions from (2.29), (2.30), and (2.31) pertinent to establishing (2.27) and (2.28). To facilitate our calculations we shall employ the notation,

$$t^n = t^{n_1}(t-1)^{n_2}.$$

Making the substitutions, (2.24), in (2.31), we obtain that

$$(2.32) \qquad R_N(b)\alpha = (tx)^{-N} \sum_{|n|=N} t^n \alpha_n.$$

As A is assumed to have finite real vector (Y, α)-moments to order N, the left side of (2.32) is in the domain of A for all $b \in (\mathbb{R}^+)^2$. Hence, the right side of (2.32) is in the domain of A for all $t \in \mathbb{R}^+$. Noting that the set $\{t^n \mid |n|= N\}$ is a basis for the polynomials of degree less than or equal to N, it follows that $\alpha_n \in \mathrm{Dom}(A)$ whenever $|n|= N$. On the other hand, as $(\{\alpha_n\}_{n \in I_N}, A, Y)$ is an HVMS of size N, it follows from (2.3) that $\alpha_n \in \mathrm{Dom}(A)$ whenever $|n|< N$. Thus, we have shown that (2.27) holds.

In order to verify (2.28) we must first explain how α_n is defined when $|n| = N+1$. Substitute (2.31) into (2.30) and then equate the right-hand sides of (2.29) and (2.30) to obtain

$$(tx)^{-(2N+1)} \sum_{|n|=2N+1} \rho_n t^n$$

$$= \frac{t-1}{x} \langle (t-Y)^{-1} (tx)^{-N} \sum_{|m|=N} t^m A\alpha_m, (tx)^{-N} \sum_{|m|=N} t^m A\alpha_m \rangle,$$

which simplifies to

$$(2.33) \qquad \frac{p(t)}{t(t-1)} = \langle (t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle,$$

where p is the polynomial of degree less than or equal to $2N+1$ defined by

$$(2.34) \qquad p(t) = \sum_{|n|=2N+1} \rho_n t^n.$$

For m a multi-index, we define $Q_m(t)$, an operator valued polynomial of degree $|m| - 1$, by the formula

$$Q_m(t) = \frac{t^m - Y^m}{t - Y}.$$

Computing with the right side of (2.33) yields that

$$\langle (t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$= \langle (t-Y)^{-1} \sum_{|m|=N} [(t-Y)Q_m(t) + Y^m] A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$= \langle \sum_{|m|=N} Q_m(t) A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$\qquad + \langle (t-Y)^{-1} \sum_{|m|=N} Y^m A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$= \langle \sum_{|m|=N} Q_m(t) A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$\qquad + \langle \sum_{|m|=N} Y^m A\alpha_m, (t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m \rangle$$

$$= \langle \sum_{|m|=N} Q_m(t) A\alpha_m, \sum_{|m|=N} t^m A\alpha_m \rangle + \langle \sum_{|m|=N} Y^m A\alpha_m, \sum_{|m|=N} Q_m(t) A\alpha_m \rangle$$

$$\qquad + \langle \sum_{|m|=N} Y^m A\alpha_m, (t-Y)^{-1} \sum_{|m|=N} Y^m A\alpha_m \rangle.$$

As the first two terms of this last expression are polynomials of degree less than or equal to $2N-1$ and $N-1$ respectively, recalling that p has degree less than or equal to $2N+1$, we see that the third term in the above expression must have the form

$$(2.35) \qquad \langle \sum_{|m|=N} Y^m A\alpha_m, (t-Y)^{-1} \sum_{|m|=N} Y^m A\alpha_m \rangle = \frac{c_1}{t} + \frac{c_1}{t-1} + q(t),$$

where c_1 and c_2 are scalars and q is a polynomial of degree less than or equal to $2N - 1$. (2.35) implies that if we set

$$\beta = \sum_{|m|=N} Y^m A\alpha_m$$

and E is the spectral measure for Y, then $dE_{\beta,\beta}$ is supported in $\{0, 1\}$, which in turn implies that

$$(2.36) \qquad\qquad\qquad Y(Y-1)\beta = 0.$$

Now observe in light of (2.36), that

$$t(t-1)(t-Y)^{-1}\beta = (t+Y-1)\beta.$$

Hence,

$$t(t-1)(t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m$$

$$= t(t-1)(t-Y)^{-1} \sum_{|m|=N} [(t-Y)Q_m(t) + Y^m]A\alpha_m$$

$$= t(t-1) \sum_{|m|=N} Q_m(t)A\alpha_m + t(t-1)(t-Y)^{-1} \sum_{|m|=N} Y^m A\alpha_m$$

$$= t(t-1) \sum_{|m|=N} Q_m(t)A\alpha_m + t(t-1)(t-Y)^{-1}\beta$$

$$= t(t-1) \sum_{|m|=N} Q_m(t)A\alpha_m + (t+Y-1)\beta.$$

As $Q_m(t)$ has degree $N-1$, this implies that

$$t(t-1)(t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m$$

is a vector valued polynomial of degree $N+1$. But the set $\{t^n \mid |n| = N+1\}$ forms a basis for the polynomials of degree less than or equal to $N+1$. Hence, there exist vectors $\alpha_n \in \mathcal{H}$, $|n| = N+1$, such that

$$(2.37) \qquad t(t-1)(t-Y)^{-1} \sum_{|m|=N} t^m A\alpha_m = \sum_{|n|=N+1} t^n \alpha_n.$$

Unraveling the substitutions, (2.24) and (2.37) become

$$b_Y^{-1} A R_N(b)\alpha = \sum_{|n|=N+1} \frac{1}{b^n}\alpha_n$$

or

$$(2.38) \qquad\qquad R_{N+1}(b)\alpha = \sum_{|n|=N+1} \frac{1}{b^n}\alpha_n.$$

In addition, recalling that $(\{\alpha_n\}_{n \in I_N}, A, Y)$ is an HVMS of size N, we see from Proposition 2.5 that

$$(2.39) \qquad\qquad R_l(b)\alpha = \sum_{|n|=l} \frac{1}{b^n}\alpha_n \text{ for } l = 1, \ldots, N.$$

Taken together, (2.38) and (2.39) imply (2.28). $\qquad\qquad\qquad\qquad\qquad \square$

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3. From HVMS's to Loewner functions

We let Π^2 denote the set $\{z \in \mathbb{C}^2 \,|\, \mathrm{Im}(z_1) \geq 0, \mathrm{Im}(z_2) \geq 0\}$. We let $\mathcal{P}$ denote the *Pick class* on Π^2, i.e. the set of holomorphic functions on Π^2 that have a non-negative imaginary part. If $D \subseteq \mathbb{R}^2$, we define the Loewner class, $\mathcal{L}(D)$, by

$$\mathcal{L}(D) = \{h \in \mathcal{P} \,|\, h \text{ is analytic and real valued on } D\}.$$

$\mathcal{L}(D)$, which captures a semi-local version of the notion of inner, arises in a variety of problems involving interpolation, the real edge of the wedge theorem, and the analysis of operator monotone functions; see e.g. [3]. In this section we wish to consider a fully local version of $\mathcal{L}(D)$. To that end we shall require a number of definitions. Let $J_k = I_k \cup \{(0,0)\}$.

Definition 3.1. For $x \in \mathbb{R}^2$ and $S \subseteq \Pi^2$ let us agree to say that S approaches x non-tangentially, $S \xrightarrow{nt} x$, if $x \in S^-$ and there exists a constant c such that

$$\|z - x\| \leq c \min\{\mathrm{Im}(z_1)\mathrm{Im}(z_2)\}$$

for all $z \in S$.

Definition 3.2. Let $h \in \mathcal{P}$ and $x \in \mathbb{R}^2$. We say that x is a C^k-point of h if h is "non-tangentially C^k at x" i.e. there exists an indexed set of scalars, $\delta = \{\delta_n\}_{n \in J_k}$, such that if $S \subset \Pi^2$ and $S \xrightarrow{nt} x$, then

$$(3.3) \qquad \lim_{\substack{z \to x \\ z \in S}} \frac{h(z) - \sum_{n \in J_k} \delta_n z^n}{\|z\|^k} = 0.$$

Evidently, if $h \in \mathcal{L}(D)$, $x \in D$, and x is a C^k-point of h, then δ, as uniquely determined by (3.3), has the property that δ_n is real whenever $n \in J_k$. This suggests the following definition as a reasonable localization of the Loewner class.

Definition 3.4. Let $k \geq 0$ and $x \in \mathbb{R}^2$. If $h \in \mathcal{P}$, we say that h is Loewner to order k at x if x is a C^k-point of h and if δ, as uniquely determined by (3.3), has the property that δ_n is real for all $n \in J_k$.

We introduce in Definition 3.9 below a class of functions, $\mathcal{L}^N$, obtained by adding three extra minor provisos to the notion in Definition 3.4. First we shall assume that $k = 2N - 1$ is odd. Secondly, we wish to consider regularity as z approaches infinity non-tangentially rather than as z approaches a finite point $x \in \mathbb{R}^2$. Finally, we shall normalize h to have the value zero at infinity.

To formalize regularity at ∞, we introduce the following two definitions.

Definition 3.5. If $\{z_n\}$ is a sequence in Π^2, we say $z_n \to \infty$ if $z_n = (\lambda_n, \mu_n)$ and both $\lambda_n \to \infty$ and $\mu_n \to \infty$. For $S \subseteq \Pi^2$ we say that S approaches ∞ non-tangentially, $S \xrightarrow{nt} \infty$, if there is a sequence $\{z_n\}$ in S such that $z_n \to \infty$ and a constant c such that

$$(3.6) \qquad \|z\| \leq c \min\{\mathrm{Im}(z_1)\mathrm{Im}(z_2)\}$$

for all $z \in S$. If $S \xrightarrow{nt} \infty$, we let $\mathrm{adj}(S)$ denote the smallest constant such that (3.6) holds for all $z \in S$.

Definition 3.7. Let Ω be a metric space, $\omega \in \Omega$, and $F : \Pi^2 \to \Omega$ be a map. We say

$$F(z) \to \omega \text{ as } z \xrightarrow{nt} \infty$$

if for each $S \subset \Pi^2$ such that $S \overset{nt}{\to} \infty$,

$$\lim_{\substack{z \to \infty \\ z \in S}} F(z) = \omega.$$

We now can extend the notion of a C^k-point to ∞.

Definition 3.8. If $h \in \mathcal{P}$ we say ∞ is a C^k-point of h if there exists an indexed set of scalars, $\rho = \{\rho_n\}_{n \in J_k}$, referred to as residues, such that

$$\|z\|^k (h(z) - \sum_{n \in J_k} \frac{\rho_n}{z^n}) \to 0 \text{ as } z \overset{nt}{\to} \infty.$$

Finally, notice that the residue, $\rho_{(0,0)}$, when it exists, is the limit of $h(z)$ as $z \to \infty$ non-tangentially, and hence we denote it by $h(\infty)$. Our third proviso is to normalize h by requiring that $h(\infty) = 0$.

Definition 3.9. For N a positive integer, let $\mathcal{L}^N$ denote the set of all $h \in \mathcal{P}$ such that ∞ is a C^{2N-1}-point for h with real residues and $h(\infty) = 0$.

Let us note that Theorem 1.9 implies that any function in $\mathcal{L}^1$ must have a representation as in (1.10). In previous work [3], we required Y to be a projection. This was inspired by representations on the bidisk, as in [6] and [2]. Here, we do not require Y to necessarily be a projection. But, in order for $(\{\alpha_n\}_{n \in I_N}, Y, A)$ to be an HVMS of size N, the operator $Y(1 - Y)$ annihilates $\alpha_{(l,0)}$ and $\alpha_{(0,l)}$ for $1 \leq l \leq N$. So these vectors "think" Y is a projection.

We now can formulate the main result of this section.

Theorem 3.10. *Suppose that $\mathcal{H}$ is a Hilbert space, A is a densely defined self-adjoint operator on $\mathcal{H}$, $\alpha \in \mathcal{H}$, and $h \in \mathcal{P}$ is defined by the type I Nevanlinna representation,*

$$(3.11) \qquad\qquad h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle, \quad z \in \Pi^2.$$

If $(\{\alpha_n\}, A, Y)$ is a of size N and $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, then $h \in \mathcal{L}^N$. Furthermore, if r_l are the scalar (Y, α)-moments of A (as given by the formulas (2.18) and (2.19)) and ρ_n are the residues of h (as given in Definition 3.8), then

$$(3.12) \qquad\qquad \sum_{|n|=l} \frac{\rho_n}{b^n} = -r_l(b)$$

for $l = 1, \ldots, 2N - 1$.

The remainder of the section will be devoted to the proof of Theorem 3.10. Accordingly, fix an HVMS of size N, $(\{\alpha_n\}, A, Y)$, with the property that A is densely defined and self-adjoint, set $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, and assume that h is given by (3.11). The point z will always lie in Π^2, so $(A - z_Y)$ is invertible.

Observe that as

$$(3.13) \qquad A(A - z_Y)^{-1} = (A - z_Y + z_Y)(A - z_Y)^{-1} = 1 + z_Y(A - z_Y)^{-1},$$

the operator $A(A - z_Y)^{-1}$ is bounded. Likewise, the operator $(A - z_Y)^{-1}A$ is bounded. Also, we have the following simple identities involving these operators:

$$(3.14) \qquad\qquad z_Y(A - z_Y)^{-1}A = A(A - z_Y)^{-1}z_Y,$$

$$(3.15) \qquad\qquad (A - z_Y)^{-1} = -z_Y^{-1} + z_Y^{-1}A(A - z_Y)^{-1},$$

$$(3.16) \qquad\qquad (A - z_Y)^{-1} = -z_Y^{-1} + (A - z_Y)^{-1}Az_Y^{-1}.$$

Claim 3.17.

$$\langle (A - z_Y)^{-1}\alpha, \alpha \rangle = -\sum_{k=1}^{2N-1} r_k(z) + \langle A(A - z_Y)^{-1}z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle.$$

Note that as $(\{\alpha_n\}, A, Y)$ is an HVMS, condition (ii) of Proposition 2.11 guarantees that $\alpha \in \mathrm{Dom}(R_N(z))$ and, in addition, that the residues, $r_k(z)$, $k = 1, \ldots, 2N - 1$, are well-defined by equations (2.17) to (2.19). Thus, the expression that appears on the right side of the claim is well-defined.

To prove Claim 3.17 we proceed by induction. Note that when $N = 1$ the claim follows immediately from (3.15). Suppose the claim holds for HVMS's of size N. If $(\{\alpha_n\}, A, Y)$ is an HVMS of size $N + 1$, then as $(\{\alpha_n\}, A, Y)$ is also an HVMS of size N, the inductive hypothesis yields that

$$(3.18) \quad \langle (A - z_Y)^{-1}\alpha, \alpha \rangle = -\sum_{k=1}^{2N-1} r_k(z) + \langle A(A - z_Y)^{-1}z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle.$$

But,

$$\langle A(A - z_Y)^{-1}z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle$$

$$(i) \quad = \langle (A - z_Y)^{-1}z_Y R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$(ii) \quad = \langle (-z_Y^{-1} + (A - z_Y)^{-1}A z_Y^{-1})z_Y R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$= -\langle R_N(z)\alpha, A R_N(z)^*\alpha \rangle + \langle (A - z_Y)^{-1}A R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$(iii) \quad = -\langle R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$\qquad + \langle (-z_Y^{-1} + z_Y^{-1}A(A - z_Y)^{-1})A R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$= -\langle R_N(z)\alpha, A R_N(z)^*\alpha \rangle - \langle z_Y^{-1}A R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$\qquad + \langle z_Y^{-1}A(A - z_Y)^{-1}A R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$= -\langle R_N(z)\alpha, A R_N(z)^*\alpha \rangle - \langle z_Y^{-1}A R_N(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$\qquad + \langle A(A - z_Y)^{-1}z_Y z_Y^{-1}A R_N(z)\alpha, z_Y^{-1*}A R_N(z)^*\alpha \rangle$$

$$(iv) \quad = -\langle R_N(z)\alpha, A R_N(z)^*\alpha \rangle - \langle R_{N+1}(z)\alpha, A R_N(z)^*\alpha \rangle$$

$$\qquad + \langle A(A - z_Y)^{-1}z_Y R_{N+1}(z)\alpha, R_{N+1}(z)^*\alpha \rangle$$

$$(v) \quad = -r_{2N}(z) - r_{2N+1}(z)$$

$$\qquad + \langle A(A - z_Y)^{-1}z_Y R_{N+1}(z)\alpha, R_{N+1}(z)^*\alpha \rangle.$$

Here, the following facts were used.

  (i) as $(\{\alpha_n\}, A, Y)$ is an HVMS of size $N + 1$, $R_N(z)^*\alpha = R_N(\bar{z})\alpha \in \mathrm{Dom}(A)$,

  (ii) (3.16),

  (iii) (3.15),

  (iv) $R_{N+1}(z)\alpha = z_Y^{-1}A R_N(z)$,

  (v) (2.18) and (2.19).

Combining the result of this calculation with (3.18), we deduce that

$$\langle (A - z_Y)^{-1}\alpha, \alpha \rangle = -\sum_{k=1}^{2N+1} r_k(z) + \langle A(A - z_Y)^{-1}z_Y R_{N+1}(z)\alpha, R_{N+1}(z)^*\alpha \rangle,$$

which is (3.18) with N replaced with $N + 1$. This concludes the proof of Claim 3.17.

Now observe that both of the facts we need to prove to establish Theorem 3.10, that $h \in \mathcal{L}^N$ and (3.12), will follow from Claim 3.17 if we can show that

$$(3.19) \qquad \|z\|^{2N-1} \langle A(A - z_Y)^{-1} z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle \to 0 \quad \text{as} \quad z \overset{nt}{\to} \infty.$$

On the other hand, we claim that (3.19) will follow if we can show

Claim 3.20. If $\beta, \gamma \in \mathcal{H}$, then

$$(3.21) \qquad\qquad \langle A(A - z_Y)^{-1}\beta, \gamma \rangle \to 0 \quad \text{as} \quad z \overset{nt}{\to} \infty.$$

To see how Claim 3.20 implies (3.19) we use the following simple property of sets that approach ∞ non-tangentially.

Lemma 3.22. *If n is a multi-index, $S \subset \Pi^2$ and $S \overset{nt}{\to} \infty$, then*

$$|\frac{1}{z^n}| \le (\mathrm{adj}(S))^{|n|}\|z\|^{-|n|}$$

for all $z \in S$.

Proof. If $z \in S$, then Definition 3.5 implies that

$$\|z\| \le \mathrm{adj}(S) \, \min\{\mathrm{Im}(z_1)\mathrm{Im}(z_2)\}$$
$$\le \mathrm{adj}(S) \, \mathrm{Im}(z_1)$$
$$\le \mathrm{adj}(S) \, |z_1|.$$

Hence,

$$|z_1|^{-n_1} \le \mathrm{adj}(S)^{n_1} \, \|z\|^{-n_1}.$$

Likewise,

$$|z_2|^{-n_2} \le \mathrm{adj}(S)^{n_2} \, \|z\|^{-n_2}.$$

The lemma follows by multiplying these last two inequalities together. $\qquad\square$

Now, using Proposition 2.5, if $N \ge 2$,

$$\langle A(A - z_Y)^{-1} z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle$$
$$= \langle A(A - z_Y)^{-1} A R_{N-1}(z)\alpha, R_N(z)^*\alpha \rangle$$
$$= \langle A(A - z_Y)^{-1} A \sum_{|m|=N-1} \frac{1}{z^m}\alpha_m, \sum_{|n|=N} \frac{1}{z^n}\alpha_n \rangle$$
$$= \sum_{\substack{|m|=N-1 \\ |n|=N}} \frac{1}{z^{m+n}} \langle A(A - z_Y)^{-1} A\alpha_m, \alpha_n \rangle.$$

Thus, using Lemma 3.22, we see that if $S \overset{nt}{\to} \infty$ and $z \in S$, then

$$|\langle A(A - z_Y)^{-1} z_Y R_N(z)\alpha, R_N(z)^*\alpha \rangle|$$
$$\le \sum_{\substack{|m|=N-1 \\ |n|=N}} |\frac{1}{z^{m+n}}| \, |\langle A(A - z_Y)^{-1} A\alpha_m, \alpha_n \rangle|$$
$$\le \mathrm{adj}(S)^{2N-1}\|z\|^{-(2N-1)} \sum_{\substack{|m|=N-1 \\ |n|=N}} |\langle A(A - z_Y)^{-1} A\alpha_m, \alpha_n \rangle|.$$

When $N = 1$, we get

$$|\langle A(A - z_Y)^{-1} z_Y R_1(z)\alpha, R_1(z)^*\alpha\rangle|$$

$$= |\sum_{|n|=1} \frac{1}{z^n}\langle A(A - z_Y)^{-1}\alpha, \alpha_n\rangle|$$

$$\leq \text{adj}(S)\|z\|^{-1} \sum_{|n|=1} |\langle A(A - z_Y)^{-1}\alpha, \alpha_n\rangle|.$$

So we see that Claim 3.20 does indeed imply (3.19).

There remains to prove Claim 3.20. For this we shall require three lemmas. These lemmas involve the notion of a *proximity estimate*, an idea which we make precise in the following definition.

Definition 3.23. Let Ω be a metric space and $F : \Pi^2 \to \Omega$ a map. We say that F is proximal (or more precisely, proximal at ∞) if for each $S \subset \Pi^2$ such that $S \overset{nt}{\to} \infty$, there exists a constant c such that

$$(3.24) \qquad d(F(z), F(w)) \leq c\frac{\|z - w\|}{\|z\|}$$

for all $z, w \in S$. We refer to the inequality (3.24) as a proximity estimate.

It turns out that frequently, as a consequence of various forms of the Schwarz Lemma, quantities that are formed from holomorphic functions satisfy proximity estimates. In such cases, the following lemma greatly simplifies the analysis of non-tangential regularity.

Lemma 3.25. *Let Ω be a metric space, let $\omega \in \Omega$ and let $F : \Pi^2 \to \Omega$ be a proximal map. $F(z) \to \omega$ as $z \overset{nt}{\to} \infty$ if and only if for each $\delta \in \Pi^2$,*

$$(3.26) \qquad \lim_{s \to \infty} F(s\delta) = \omega.$$

Proof. Clearly, if $F(z) \to \omega$ as $z \overset{nt}{\to} \infty$, then (3.26) holds. To prove the converse we argue by contradiction. Suppose (3.26) holds. If it is false that $F(z) \to \omega$ as $z \overset{nt}{\to} \infty$, then there exist $\epsilon > 0$, $S \subset \Pi^2$, and a sequence $\{z_l\}$ in S such that $S \overset{nt}{\to} \infty$, $z_l \to \infty$, and

$$(3.27) \qquad d(F(z_l), \omega) \geq \epsilon$$

for all positive l. By compactness, there exist $\delta \in \mathbb{C}^2$ and a subsequence z_{l_j}, such that $\|z_{l_j}\|^{-1} z_{l_j} \to \delta$ as $j \to \infty$. In fact, $\delta \in \Pi^2$. To see this, let $\delta = (\delta_1, \delta_2)$ and $z_{l_j} = (\lambda_{l_j}, \mu_{l_j})$ and observe that

$$(3.28) \qquad \text{Im}(\delta_1) = \lim_{j \to \infty} \frac{\text{Im}(\lambda_{l_j})}{\|z_{l_j}\|} \geq \lim_{j \to \infty} \frac{\min\{\text{Im}(\lambda_{l_j})\text{Im}(\mu_{l_j})\}}{\|z_{l_j}\|} \geq \frac{1}{\text{adj}(S)} > 0.$$

Likewise, $\text{Im}(\delta_2) > 0$ and we conclude that $\delta \in \Pi^2$.

Now, let $w_j = z_{l_j}$ and $s_j = \|z_{l_j}\|$. By construction we have that $w_j - s_j\delta = o(s_j)$ so that the proximity estimate gives that

$$(3.29) \qquad d(F(w_j), F(s_j\delta)) \leq c\frac{\|w_j - s_j\delta\|}{\|s_j\delta\|} \to 0.$$

Also, (3.26) implies that

$$(3.30) \qquad d(F(s_j\delta), \omega) \to 0.$$

Given (3.29) and (3.30), the triangle inequality gives that

$$d(F(w_j), \omega)) \to 0,$$

contradicting (3.27). □

Lemma 3.31. *Let* $\mathcal{L}(\mathcal{H})$ *denote the algebra of bounded operators on* $\mathcal{H}$ *equipped with the operator norm.* $F : \Pi^2 \to \mathcal{L}(\mathcal{H})$, *defined by*

$$F(z) = A(A - z_Y)^{-1}, \quad z \in \Pi^2,$$

is proximal.

Proof. Fix $S \subset \Pi^2$ with $S \xrightarrow{nt} \infty$. For $z \in \Pi^2$ we have that

$$(3.32) \qquad \|z_Y\| \leq \max\{|z_1|, |z_2|\} \leq \sqrt{2}\|z\|.$$

Also, as $\text{Im}(A - z_Y) = -\text{Im}(z_Y) \leq -\min\{\text{Im}(z_1)\text{Im}(z_2)\}$, we have that if, in addition, $z \in S$, then

$$(3.33) \qquad \|(A - z_Y)^{-1}\| \leq \frac{1}{\min\{\text{Im}(z_1)\text{Im}(z_2)\}} \leq \frac{\text{adj}(S)}{\|z\|}.$$

Now, using (3.13), we get

$$\begin{aligned}
F(z) &- F(w) \\
&= A(A - z_Y)^{-1} - A(A - w_Y)^{-1} \\
&= (1 + z_Y(A - z_Y)^{-1}) - (1 + w_Y(A - w_Y)^{-1}) \\
&= z_Y(A - z_Y)^{-1} - w_Y(A - w_Y)^{-1} \\
&= (z_Y - w_Y)(A - z_Y)^{-1} + w_Y((A - z_Y)^{-1} - (A - w_Y)^{-1}) \\
&= (z_Y - w_Y)(A - z_Y)^{-1} + w_Y(A - w_Y)^{-1}(z_Y - w_Y)(A - z_Y)^{-1}.
\end{aligned}$$

Hence using (3.32) and (3.33),

$$\begin{aligned}
\|F(z) &- F(w)\| \\
&\leq \ \|z_Y - w_Y\| \, \|(A - z_Y)^{-1}\| + \|w_Y\| \, \|(A - w_Y)^{-1}\| \, \|z_Y - w_Y\| \, \|(A - z_Y)^{-1}\| \\
&\leq \ \sqrt{2} \, \|z - w\| \, \frac{\text{adj}(S)}{\|z\|} + \sqrt{2} \, \|w\| \, \frac{\text{adj}(S)}{\|w\|} \, \sqrt{2} \, \|z - w\| \, \frac{\text{adj}(S)}{\|z\|} \\
&= \ (\sqrt{2} \, \text{adj}(S) + 2 \, \text{adj}(S)^2) \frac{\|z - w\|}{\|z\|},
\end{aligned}$$

which is (3.24) with $c = \sqrt{2}\text{adj}(S) + 2\text{adj}(S)^2$. □

Lemma 3.34. *If* $\beta, \gamma \in \mathcal{H}$ *and* $\delta \in \Pi^2$, *then*

$$\lim_{s \to \infty} \langle A(A - t\delta_Y)^{-1}\beta, \gamma \rangle = 0.$$

Proof. We claim that for each vector $u \in \mathcal{H}$,

$$(3.35) \qquad \delta_Y(\epsilon A - \delta_Y)^{-1}u \to -u \quad \text{weakly in } \mathcal{H}$$

as $\epsilon \to 0$. To prove this claim, first notice that as $\text{Im}(\delta_Y) \geq \min\{\text{Im}(\delta_1)\text{Im}(\delta_2)\}$, we have both that δ_Y is invertible and that $(\epsilon A - \delta_Y)^{-1}$ is uniformly bounded. In particular, as A is densely defined, $\mathcal{M} = \delta_Y \text{Dom}(A)$ is dense in $\mathcal{H}$. If $u = \delta_Y v \in \mathcal{M}$, then as $v \in \text{Dom}(A)$ and $(\epsilon A - \delta_Y)^{-1}$ is uniformly bounded,

$$(\epsilon A - \delta_Y)^{-1}\epsilon Av \to 0$$

as $\epsilon \to 0$. Hence,

$$
\begin{aligned}
(\epsilon A - \delta_Y)^{-1} u &= (\epsilon A - \delta_Y)^{-1} \delta_Y v \\
&= (\epsilon A - \delta_Y)^{-1} ((\delta_Y - \epsilon A) + \epsilon A) v \\
&= -v + (\epsilon A - \delta_Y)^{-1} \epsilon A v \\
&\to -v \\
&= -\delta_Y^{-1} u.
\end{aligned}
$$

Applying the bounded operator δ_Y yields that (3.35) holds whenever $u \in \mathcal{M}$. As $\mathcal{M}$ is dense and $\delta_Y (\epsilon A - \delta_Y)^{-1}$ is uniformly bounded, it follows that (3.35) holds for all $u \in \mathcal{H}$. This proves the claim.

Now notice that if in the claim we substitute $\epsilon = s^{-1}$, we deduce that for all $u \in \mathcal{H}$,

$$
s \delta_Y (A - s \delta_Y)^{-1} u \to -u \quad \text{weakly in } \mathcal{H}
$$

as $s \to \infty$. Hence, for all $u \in \mathcal{H}$,

$$
1 + s \delta_Y (A - s \delta_Y)^{-1} u \to 0 \quad \text{weakly in } \mathcal{H}
$$

as $s \to \infty$. The lemma now follows by observing that from (3.13)

$$
1 + s \delta_Y (A - s \delta_Y)^{-1} = A (A - s \delta_Y)^{-1}.
$$

$\square$

Armed with the above lemmas it is a simple matter to prove Claim 3.20 and thereby complete the proof of Theorem 3.10. If $\beta, \gamma \in \mathcal{H}$, then by Lemma 3.31 $F(z) = \langle A(A - z_Y)^{-1} \beta, \gamma \rangle$ is proximal. As Lemma 3.34 gives that $\lim_{s \to \infty} F(s\delta) = 0$ whenever $\delta \in \Pi^2$, Lemma 3.25 yields that $F(z) \to 0$ as $z \overset{nt}{\to} \infty$, as was to be proved.

4. From Loewner functions to HVMS's

In this section we shall formulate and then prove a converse to Theorem 3.10, using Theorem 2.21. If $h \in \mathcal{L}^N$, then it is easy to check that h is type I and accordingly has a Nevanlinna representation of the form

$$
(4.1) \qquad h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle, \quad z \in \Pi^2,
$$

where A and Y are operators acting on a Hilbert space $\mathcal{H}$, A is densely defined and self-adjoint, Y is a positive contraction, and $\alpha \in \mathcal{H}$.

Theorem 4.2. *If $h \in \mathcal{L}^N$ and A, Y, and α are such that (4.1) holds, then A has real vector (Y, α)-moments to order $N - 1$ and homogenous scalar (Y, α)-moments to order $2N - 1$. Furthermore,*

$$
(4.3) \qquad \sum_{|n|=l} \frac{\rho_n}{b^n} = -r_l(b)
$$

whenever $1 \le l \le 2N - 1$ and $b \in (\mathbb{R}^+)^2$, where ρ_n are the residues of h.

Proof. We proceed by induction. Let $N = 1$ and assume that $h \in \mathcal{L}^N$ has a Nevanlinna representation as in (4.1). As $N = 1$, the assertion that A have real vector (Y, α)-moments to order $N - 1$ is vacuous. To see that A has homogenous

scalar (Y, α)-moments to order $2N - 1$, first note that since ∞ is a C^1-point for h with real residues, we have that there exist $\rho_{(1,0)}, \rho_{(0,1)} \in \mathbb{R}^2$ such that

$$(4.4) \qquad \langle (A - z_Y)^{-1} \alpha, \alpha \rangle = h(z) = \frac{\rho_{(1,0)}}{z_1} + \frac{\rho_{(0,1)}}{z_2} + o(\|z\|^{-1}),$$

non-tangentially at ∞. Fixing $b \in (\mathbb{R}^+)^2$ and setting $z = isb$ in (4.4) gives that

$$(4.5) \qquad is \langle (A - isb_Y)^{-1} \alpha, \alpha \rangle \rightarrow \frac{\rho_{(1,0)}}{b_1} + \frac{\rho_{(0,1)}}{b_2}$$

as $s \to \infty$ in $\mathbb{R}^+$. Noting that for $b \in (\mathbb{R}^+)^2$, b_Y is strictly positive definite and hence, invertible, we define a self-adjoint operator, X_b, by the formula

$$(4.6) \qquad X_b = b_Y^{-\frac{1}{2}} A b_Y^{-\frac{1}{2}}.$$

Noting that

$$
\begin{aligned}
is \langle (A - isb_Y)^{-1} \alpha, \alpha \rangle &= is \langle (b_Y^{\frac{1}{2}} (X_b - is) b_Y^{\frac{1}{2}})^{-1} \alpha, \alpha \rangle \\
&= is \langle b_Y^{-\frac{1}{2}} (X_b - is)^{-1} b_Y^{-\frac{1}{2}} \alpha, \alpha \rangle \\
&= is \langle (X_b - is)^{-1} b_Y^{-\frac{1}{2}} \alpha, b_Y^{-\frac{1}{2}} \alpha \rangle \\
&= is \langle \frac{X_b + is}{X_b^2 + s^2} b_Y^{-\frac{1}{2}} \alpha, b_Y^{-\frac{1}{2}} \alpha \rangle \\
&= \langle \frac{-s^2 + is X_b}{X_b^2 + s^2} b_Y^{-\frac{1}{2}} \alpha, b_Y^{-\frac{1}{2}} \alpha \rangle,
\end{aligned}
$$

we see upon taking real parts in (4.5) that

$$(4.7) \qquad -\langle \frac{s^2}{X_b^2 + s^2} b_Y^{-\frac{1}{2}} \alpha, b_Y^{-\frac{1}{2}} \alpha \rangle \rightarrow \frac{\rho_{(1,0)}}{b_1} + \frac{\rho_{(0,1)}}{b_2}$$

as $s \to \infty$ in $\mathbb{R}^+$. Now, the Lesbesgue Dominated Convergence Theorem guarantees that

$$\frac{s^2}{X_b^2 + s^2} b_Y^{-\frac{1}{2}} \alpha \to b_Y^{-\frac{1}{2}} \alpha$$

as $s \to \infty$ in $\mathbb{R}^+$. Hence,

$$(4.8) \qquad -\langle b_Y^{-1} \alpha, \alpha \rangle = -\langle b_Y^{-\frac{1}{2}} \alpha, b_Y^{-\frac{1}{2}} \alpha \rangle = \frac{\rho_{(1,0)}}{b_1} + \frac{\rho_{(0,1)}}{b_2}.$$

As (4.8) holds for all $b \in (\mathbb{R}^+)^2$, we conclude that A has homogenous scalar (Y, α)-moments to order 1 as was to be shown. Also note that (4.8) implies that (4.3) holds.

We now turn to the inductive step of the proof. Accordingly, assume that

$(4.9) \qquad A$ has real vector (Y, α)-moments to order $N - 1$,

$(4.10) \qquad A$ has homogenous scalar (Y, α)-moments to order $2N - 1$, and

$$(4.11) \qquad 1 \leq l \leq 2N - 1, b \in (\mathbb{R}^+)^2 \implies \sum_{|n|=l} \frac{\rho_n}{b^n} = -r_l(b)$$

whenever $h \in \mathcal{L}^N$ and has a representation as in (4.1). Fix h with a representation as in (4.1) and assume that $h \in \mathcal{L}^{N+1}$. We need to show that (4.9), (4.10), and (4.11) hold with N replaced with $N+1$. However, as $h \in \mathcal{L}^{N+1} \subset \mathcal{L}^N$, the inductive

hypothesis implies that 4.9, 4.10, and 4.11 hold for N. Therefore, the induction will be complete if we can show the following three conditions:

$$(4.12) \qquad\qquad \alpha \in \mathrm{Dom}((Ab_Y^{-1})^N),$$

$$(4.13) \qquad\qquad r_{2N+1}(b) = - \sum_{|n|=2N+1} \frac{\rho_n}{b^n}, \quad b \in (\mathbb{R}^+)^2,$$

and

$$(4.14) \qquad\qquad r_{2N}(b) = - \sum_{|n|=2N} \frac{\rho_n}{b^n}, \quad b \in (\mathbb{R}^+)^2.$$

First note that as $h \in \mathcal{L}^{N+1}$ and (4.1) holds, there exist scalar residues, ρ_n, $n \in I_{2N+1}$, such that

$$(4.15) \qquad \langle (A - z_Y)^{-1}\alpha, \alpha \rangle = \sum_{n \in I_{2N+1}} \frac{\rho_n}{z^n} + o(\|z\|^{-(2N+1)})$$

as $z \to \infty$ non-tangentially in Π^2. Fixing $b \in (\mathbb{R}^+)^2$ and setting $z = isb$ in (4.15) we deduce that

$$\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle = \sum_{l=1}^{2N+1} (is)^{-l} \sum_{|n|=l} \frac{\rho_n}{b^n} + o(s^{-(2N+1)}),$$

as $s \to \infty$ in $\mathbb{R}^+$, which, upon taking the imaginary parts, yields that

$$(4.16) \qquad \mathrm{Im}(\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle) = \sum_{k=1}^{N+1} \frac{(-1)^k}{s^{2k-1}} \sum_{|n|=2k-1} \frac{\rho_n}{b^n} + o(s^{-(2N+1)})$$

as $s \to \infty$ in $\mathbb{R}^+$. Finally, upon multiplying (4.16) by the factor s^{2N+1}, we deduce the limit,

$$(4.17) \qquad \lim_{s\to\infty} G_b(s) = (-1)^{N+1} \sum_{|n|=2N+1} \frac{\rho_n}{b^n},$$

where for $s \in \mathbb{R}^+$ and $b \in (\mathbb{R}^+)^2$, $G_b(s)$ is defined by

$$(4.18) \quad G_b(s) = s^{2N+1}\mathrm{Im}(\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle) - \sum_{k=1}^{N} (-1)^k s^{2(N-k+1)} \sum_{|n|=2k-1} \frac{\rho_n}{b^n}.$$

We now compute $G_b(s)$ using the substitution of (4.6). We set

$$(4.19) \qquad\qquad \gamma_b = b_Y^{-\frac{1}{2}}\alpha.$$

Note that (4.9) implies that $\gamma_b \in \mathrm{Dom}(X_b^l)$ for $l = 1, \ldots, N-1$. Using (4.11) and (2.14) we see for $k = 1, \ldots, N$, that

$$\sum_{|n|=2k-1} \frac{\rho_n}{b^n} = -r_{2k-1}(b)$$

$$= -\langle b_Y^{-1}(Ab_Y^{-1})^{k-1}\alpha, (Ab_Y^{-1})^{k-1}\alpha \rangle$$

$$= -\langle (X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle.$$

Also, just as in the calculation leading up to (4.7), we compute that

$$\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle = \langle \frac{X_b + is}{X_b^2 + s^2} b_Y^{-\frac{1}{2}}\alpha, b_Y^{-\frac{1}{2}}\alpha \rangle,$$

so that

$$\mathrm{Im}(\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle) = \langle \frac{s}{X_b^2 + s^2} b_Y^{-\frac{1}{2}}\alpha, b_Y^{-\frac{1}{2}}\alpha \rangle$$

$$= \langle \frac{s}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle.$$

Hence we have that

$$G_b(s) = \langle \frac{s^{2N+2}}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle + \sum_{k=1}^{N}(-1)^k s^{2(N-k+1)}\langle (X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle.$$

We claim that the above sum telescopes. Indeed, using the fact that

$$\langle (X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle$$

$$= \langle (\frac{X_b^2}{X_b^2 + s^2} + \frac{s^2}{X_b^2 + s^2})(X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle$$

$$= \langle \frac{X_b^2}{X_b^2 + s^2}(X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle + \langle \frac{s^2}{X_b^2 + s^2}(X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle,$$

we compute that

$$G_b(s) = \langle \frac{s^{2N+2}}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle + \sum_{k=1}^{N}(-1)^k s^{2(N-k+1)}\langle (X_b)^{k-1}\gamma_b, (X_b)^{k-1}\gamma_b \rangle$$

$$= \langle \frac{s^{2N+2}}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle$$

$$- s^{2N}(\langle \frac{X_b^2}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle + \langle \frac{s^2}{X_b^2 + s^2}\gamma_b, \gamma_b \rangle)$$

$$+ s^{2N-2}(\langle \frac{X_b^2}{X_b^2 + s^2}X_b\gamma_b, X_b\gamma_b \rangle + \langle \frac{s^2}{X_b^2 + s^2}X_b\gamma_b, X_b\gamma_b \rangle)$$

$$\vdots$$

$$+ (-1)^N s^2(\langle \frac{X_b^2}{X_b^2 + s^2}(X_b)^{N-1}\gamma_b, (X_b)^{N-1}\gamma_b \rangle$$

$$+ \langle \frac{s^2}{X_b^2 + s^2}(X_b)^{N-1}\gamma_b, (X_b)^{N-1}\gamma_b \rangle)$$

$$= (-1)^N s^2\langle \frac{X_b^2}{X_b^2 + s^2}(X_b)^{N-1}\gamma_b, (X_b)^{N-1}\gamma_b \rangle.$$

This last calculation makes sense since

$$\frac{X_b^2}{X_b^2 + s^2}$$

is a bounded operator and $\gamma_b \in \mathrm{Dom}(X_b{}^l)$ for $l = 1, \ldots, N - 1$.

Now recall (4.17). From the formula for $G_b(s)$ just derived, we see that

$$(4.20) \qquad \lim_{s\to\infty} \langle \frac{s^2 X_b^2}{X_b^2 + s^2}(X_b)^{N-1}\gamma_b, (X_b)^{N-1}\gamma_b \rangle = - \sum_{|n|=2N+1} \frac{\rho_n}{b^n}.$$

As X_b is self-adjoint and $\gamma_b \in \mathrm{Dom}(X_b{}^{N-1})$, we can apply the spectral theorem to X_b and thereby obtain the scalar spectral measure of γ_b, μ. Analyzing the very existence of the limit on the left side of (4.20) in the space $L^2(\mu)$ yields via the Lesbesgue Dominated Convergence Theorem that

$$(4.21) \qquad\qquad\qquad \gamma_b \in \mathrm{Dom}(X_b{}^N).$$

Unraveling (4.21) via (4.6) and (4.19) gives that

$$\alpha \in \mathrm{Dom}((Ab_Y^{-1})^N),$$

which is (4.12). Note also from (4.20) we have that

$$\langle (X_b)^2 (X_b)^{N-1}\gamma_b, (X_b)^{N-1}\gamma_b \rangle = - \sum_{|n|=2N+1} \frac{\rho_n}{b^n},$$

which unravels to

$$(4.22) \qquad\qquad r_{2N+1}(b) = - \sum_{|n|=2N+1} \frac{\rho_n}{b^n},$$

which is (4.13).

There remains to check (4.14). This is done by following the same line of reasoning that led from (4.15) to (4.20). One starts with (4.15) but with $2N+1$ replaced with $2N$:

$$(4.23) \qquad \langle (A - z_Y)^{-1}\alpha, \alpha \rangle = \sum_{n\in I_{2N}} \frac{\rho_n}{z^n} + o(\|z\|^{-(2N)}).$$

Proceeding as before, for a fixed $b \in (\mathbb{R}^+)^2$ and $s \in \mathbb{R}^+$ we set $z = isb$ in (4.23). However, unlike before, where we took imaginary parts to obtain (4.16), we now take real parts. This results in

$$(4.24) \qquad \mathrm{Re}(\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle) = \sum_{k=1}^{N} \frac{(-1)^k}{s^{2k}} \sum_{|n|=2k} \frac{\rho_n}{b^n} + o(s^{-2N})$$

as $s \to \infty$ in $\mathbb{R}^+$. Finally, upon multiplying (4.23) by the factor s^{2N} (rather than s^{2N+1} as before), we deduce the limit,

$$(4.25) \qquad\qquad \lim_{s\to\infty} F_b(s) = (-1)^N \sum_{|n|=2N} \frac{\rho_n}{b^n},$$

where for $s \in \mathbb{R}^+$ and $b \in (\mathbb{R}^+)^2$, $F_b(s)$ is defined by

$$(4.26) \qquad F_b(s) = s^{2N}\mathrm{Re}(\langle (A - isb_Y)^{-1}\alpha, \alpha \rangle) - \sum_{k=1}^{N-1} (-1)^k s^{2(N-k)} \sum_{|n|=2k} \frac{\rho_n}{b^n}.$$

Carrying out the telescoping argument, one computes that

$$F_b(s) = (-1)^{N-1} s^2 \langle \frac{X_b^2}{X_b^2 + s^2}(X_b)^{N-2}\gamma_b, (X_b)^{N-1}\gamma_b \rangle,$$

which implies via (4.25) the existence of the limit

$$(4.27) \qquad \lim_{s\to\infty} \langle \frac{s^2 X_b^2}{X_b^2 + s^2}(X_b)^{N-2}\gamma_b, (X_b)^{N-1}\gamma_b \rangle = -\sum_{|n|=2N} \frac{\rho_n}{b^n}.$$

As (4.21) holds, (4.27) implies that

$$\langle (X_b)^2 (X_b)^{N-2}\gamma_b, (X_b)^{N-1}\gamma_b \rangle = -\sum_{|n|=2N} \frac{\rho_n}{b^n}.$$

As this last equation unravels via (4.6) and (4.19) to

$$r_{2N}(b) = -\sum_{|n|=2N} \frac{\rho_n}{b^n},$$

the proof that (4.14) holds is complete. □

5. Finite Hankel pairs

In this section we give an alternate matrix theoretic treatment of HVMS's based on the fact that it is possible to cleanly characterize the Gram matrix formed from the moment vectors of an HVMS.

For X a set, we let $\ell^2(X)$ denote the Hilbert space of square summable complex valued functions on X. If $f \in \ell^2(X)$, we let $\mathrm{supp}(f)$, *the support of f*, denote the subset of X defined by

$$\mathrm{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

By a *matrix on X* we mean a square array of scalars, doubly indexed by the elements of X. If $a = [a_{x,y}]$ is a matrix on X, then a induces a densely defined linear operator, also denoted by a, on the finitely supported functions in $\ell^2(X)$ by the formula

$$(af)(x) = \sum_{y\in\mathrm{supp}(f)} a_{x,y} f(y).$$

If $a = [a_{x,y}]$ is a matrix on X, then we say that *a is symmetric* if

$$a_{x,y} = \overline{a}_{y,x} \text{ for all } x,y \in X,$$

and we say that *a is positive semi-definite* if for each (finite) choice of elements, $x_1, x_2, \ldots, x_l \in X$, and each choice of scalars, $c_1, c_2, \ldots, c_l \in \mathbb{C}$,

$$\sum_{i,j=1}^{l} a_{x_i,x_j} c_j \overline{c}_i \geq 0.$$

In this section we shall be exclusively interested in the case where $X = I_N$, for N a positive integer. Note that naturally, if $M \leq N$, then $\ell^2(I_M) \subseteq \ell^2(I_N)$, and in addition, that there is a pair of shift operators, $S_1, S_2 : \ell^2(I_{N-1}) \to \ell^2(I_N)$, defined by

$$(S_1 f)(n) = \begin{cases} f(n-e_1) & n-e_1 \in I_{N-1}, \\ 0 & \text{else}, \end{cases}$$

$$(S_2 f)(n) = \begin{cases} f(n-e_2) & n-e_2 \in I_{N-1}, \\ 0 & \text{else}. \end{cases}$$

If $(\{\alpha_n\}, A, Y)$ is an HVMS of size N, then we may define a pair of matrices $a = (a^1, a^2)$ on I_N by

(5.1) $\qquad a^1_{m,n} = \langle Y\alpha_n, \alpha_m \rangle$ and $a^2_{m,n} = \langle (1 - Y)\alpha_n, \alpha_m \rangle$ for $m, n \in I_N$.

Definition 5.2. We say that $a = (a^1, a^2)$ is a finite Hankel pair of size N if a^1 and a^2 are matrices on I_N and there exists an HVMS of size N such that (5.1) holds.

In Theorem 5.7, we give a characterization of when a pair of matrices is a finite Hankel pair. To see how this is a two variable version of Theorem 1.7, let us restate that theorem more abstractly. Let $S : \ell^2(\{0, 1, \ldots, N - 2\}) \to \ell^2(\{0, 1, \ldots, N - 1\})$ be the shift defined by $Sf(j) = f(j - 1), j > 0$, and $Sf(0) = 0$.

Theorem 5.3. *Let H be an N-by-N matrix. There is a self-adjoint operator A and a vector α with $\alpha \in \mathrm{Dom}(A^k)$ for $1 \leq k \leq N - 1$ such that*

$$H_{ij} \;=\; \langle A^j\alpha, A^i\alpha \rangle, \qquad 0 \leq i, j \leq N - 1,$$

if and only if the following three conditions obtain:

(5.4) $\qquad\qquad\qquad H$ *is positive semi-definite,*

(5.5) $\qquad\qquad\qquad H_{i+1,j} \;=\; H_{i,j+1}, \qquad 0 \leq i, j \leq N - 2,$

(5.6) $\qquad\qquad \mathrm{supp}(f) \in \{0, \ldots, N - 2\}$ *and $Hf = 0 \Rightarrow HSf = 0$.*

Here is our two variable version of Hamburger's Theorem 1.7.

Theorem 5.7. *Let a be a pair of matrices on I_N. Then a is a finite Hankel pair of size N if and only if the following four conditions obtain:*

(5.8) $\qquad\qquad\qquad a^1$ *and a^2 are positive semi-definite,*

(5.9) $\qquad a^1_{m+e_1,n} + a^2_{m+e_2,n} = a^1_{m,n+e_1} + a^2_{m,n+e_2}$ *whenever $m, n \in I_{N-1}$,*

(5.10) $\qquad\qquad a^1_{(0,l),(0,l)} = a^2_{(l,0),(l,0)} = 0$ *for $l = 1, \ldots, N$,*

(5.11) $\qquad\qquad \mathrm{supp}(f) \in I_{N-1}$ *and $(a^1 + a^2)f = 0 \Rightarrow (a^1 S_1 + a^2 S_2)f = 0$.*

Proof. (Necessity). Assume that $(\{\alpha_n\}_{n \in I_N}, Y, A)$ is an HVMS and (5.1) holds. Then (5.8) holds because Y and $1 - Y$ are positive operators. (5.9) holds because the left-hand side is

$$\langle Y\alpha_n, \alpha_{m+e_1} \rangle \;+\; \langle (1 - Y)\alpha_n, \alpha_{m+e_2} \rangle \;=\; \langle \alpha_n, A\alpha_m \rangle,$$

by (2.4). But the right-hand side of (5.9) by a similar calculation is $\langle A\alpha_n, \alpha_m \rangle$, which is equal to $\langle \alpha_n, A\alpha_m \rangle$ because A is self-adjoint and α_m, α_n are in its domain for $m, n \in I_{N-1}$. Condition (5.10) follows from (2.2).

Finally, if $\mathrm{supp}(f) \in I_{N-1}$ and $(a^1 + a^2)f = 0$, this says that

$$\Big\langle \sum_{n \in I_{N-1}} f(n)\alpha_n, \alpha_m \Big\rangle \;=\; 0$$

for all $m \in I_N$. But

$$
\begin{aligned}
(a^1 S_1 + a^2 S_2) f(m) &= \sum_{n \in I_{N-1}} f(n) \langle Y \alpha_{n+e_1} + (1-Y) \alpha_{n+e_2}, \alpha_m \rangle \\
&= \langle A(\sum_{n \in I_{N-1}} f(n) \alpha_n), \alpha_m \rangle \\
&= 0,
\end{aligned}
$$

so (5.11) holds.

(Sufficiency). Assume (5.8) — (5.11) hold. Choose vectors α_n in a Hilbert space $\mathcal{H}$ so that their Grammian equals the matrix $a^1 + a^2$:

$$
\langle \alpha_n, \alpha_m \rangle = a^1_{m,n} + a^2_{m,n}.
$$

Since $a^1 \leq \langle \alpha_n, \alpha_m \rangle$, there is a positive operator Y satisfying (5.1). Equation (2.2) follows from (5.10).

If $N = 1$, we can define A arbitrarily, e.g. by $A = 0$.

If $N \geq 2$, we define A on the span of $\{\alpha_n\}_{n \in I_{N-1}}$ by

$$
A \alpha_n = Y \alpha_{n+e_1} + (1-Y) \alpha_{n+e_2}.
$$

To check that this is a well-defined linear operator, we need to know that if

$$
\sum_{n \in I_{N-1}} c_n \alpha_n = 0,
$$

then

$$
\sum_{n \in I_{N-1}} c_n (Y \alpha_{n+e_1} + (1-Y) \alpha_{n+e_2}) = 0.
$$

This follows from (5.11). It follows from (5.9) that A is symmetric. $\qquad \square$

6. Infinite sequences

As in one variable, passage from the finite to the infinite case is straightforward and leads to some simplifications. Let $\mathcal{I}$ denote the set of pairs of non-negative integers, excluding $(0, 0)$.

Definition 6.1. An infinite Hankel vector moment sequence is a 3-tuple, $(\{\alpha_n\}_{n \in \mathcal{I}}, Y, A)$, where: $\{\alpha_n\}_{n \in \mathcal{I}}$ is a sequence of vectors in some Hilbert space $\mathcal{H}$; Y is a positive contraction acting on $\mathcal{H}$, satisfying for each $l \geq 1$

$$
Y \alpha_{(0,l)} = 0 = (1-Y) \alpha_{(l,0)} = 0;
$$

A is a densely defined self-adjoint operator on $\mathcal{H}$ with the property that

$$
\{\alpha_n \,|\, n \in \mathcal{I}\} \subset \mathrm{Dom}(A);
$$

and for each $n \in \mathcal{I}$,

$$
A \alpha_n = Y \alpha_{n+e_1} + (1-Y) \alpha_{n+e_2}.
$$

Theorem 1.15 becomes a description of functions in $\mathcal{L}^\infty$.

Theorem 6.2. *A Pick function h of two variables has an asymptotic expansion*

$$
h(z) = \sum_{n \in \mathcal{I}} \frac{\rho_n}{z^n}
$$

as $z \xrightarrow{nt} \infty$, *for some real numbers ρ_n, if and only if it has a representation as in* (1.10) *and for every such representation there is an infinite HVMS* $(\{\alpha_n\}_{n \in \mathcal{I}}, Y, A)$ *with $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$. Moreover, $\{\rho_n\}$ are given by*

$$\sum_{|n|=l} \frac{\rho_n}{b^n} = -r_l(b)$$

whenever $l \geq 1$ and $b \in (\mathbb{R}^+)^2$.

Sufficiency of the condition follows from Theorem 3.10; necessity follows from the constructive proof of Theorem 4.2.

We define an infinite Hankel pair by

Definition 6.3. We say that $a = (a^1, a^2)$ is an infinite Hankel pair if a^1 and a^2 are matrices on $\mathcal{I}$ and there exists an infinite HVMS such that

(6.4) $\qquad a^1_{m,n} = \langle Y\alpha_n, \alpha_m \rangle$ and $a^2_{m,n} = \langle (1-Y)\alpha_n, \alpha_m \rangle$ for $m, n \in \mathcal{I}$.

If (5.9) holds for all N, then (5.11) holds automatically. So the infinite Hamburger Theorem becomes

Theorem 6.5. *Let a be a pair of matrices on $\mathcal{I}$. Then a is an infinite Hankel pair if and only if the following three conditions obtain:*

$$a^1 \text{ and } a^2 \text{ are positive semi-definite,}$$

$$a^1_{m+e_1,n} + a^2_{m+e_2,n} = a^1_{m,n+e_1} + a^2_{m,n+e_2} \quad \text{whenever} \quad m, n \in \mathcal{I},$$

$$a^1_{(0,l),(0,l)} = a^2_{(l,0),(l,0)} = 0 \quad \text{for} \quad l \geq 1.$$

Here is a two variable version of Kronecker's theorem.

Theorem 6.6. *Let $h \in \mathcal{L}^\infty$. Then there is an infinite HVMS $(\{\alpha_n\}_{n \in \mathcal{I}}, Y, A)$ with $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, satisfying $\mathrm{rank}(a^1 + a^2) < \infty$ and*

(6.7) $$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle$$

if and only if h is a rational function.

Proof. If h is rational of degree (d_1, d_2), then by Theorem 8.12 h has a representation (6.7) on a Hilbert space $\mathcal{H}$ of dimension at most $d = d_1 + d_2$. Since $h \in \mathcal{L}^\infty$, by Theorem 6.2 there is an infinite HVMS $(\{\alpha_n\}_{n \in \mathcal{I}}, Y, A)$ on $\mathcal{H}$. So

$$(a^1 + a^2)_{m,n} = \langle \alpha_n, \alpha_m \rangle_{\mathcal{H}}$$

has rank at most d.

Conversely, suppose there is an infinite HVMS $(\{\alpha_n\}_{n \in \mathcal{I}}, Y, A)$ with $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$, satisfying $\mathrm{rank}(a^1 + a^2) = d < \infty$ and (6.7). Then one can choose vectors β_n in a space $\mathcal{H}$ of dimension d such that

(6.8) $$\langle \beta_n, \beta_m \rangle_{\mathcal{H}} = \langle \alpha_n, \alpha_m \rangle$$

and so that the vectors $\{\beta_n\}$ span $\mathcal{H}$. Define a positive contraction X on $\mathcal{H}$ by

(6.9) $$\langle X\beta_n, \beta_m \rangle = \langle Y\alpha_n, \alpha_m \rangle.$$

Define B by

$$B\beta_n = X\beta_{n+e_1} + (1-X)\beta_{n+e_2}.$$

We claim that B extends by linearity to a well-defined linear operator on $\mathcal{H}$. Indeed, suppose $\sum c_n \beta_n = 0$. Then by (6.8),

$$\left\langle \sum c_n \alpha_n, \sum c_m \alpha_m \right\rangle \;=\; \left\langle \sum c_n \beta_n, \sum c_m \beta_m \right\rangle \;=\; 0.$$

So $\sum c_n \alpha_n = 0$, and therefore by (6.9),

$$\left\langle \sum c_n [X\beta_{n+e_1} + (1-X)\beta_{n+e_2}], \beta_m \right\rangle \;=\; \left\langle \sum c_n [Y\alpha_{n+e_1} + (1-Y)\alpha_{n+e_2}], \alpha_m \right\rangle$$
$$=\; \left\langle A \sum c_n \alpha_n, \alpha_m \right\rangle$$
$$=\; 0.$$

As $\{\beta_m\}$ span $\mathcal{H}$, this means $\sum c_n [X\beta_{n+e_1} + (1-X)\beta_{n+e_2}] = 0$; so B is well defined, and hence $(\{\beta_n\}_{n\in\mathcal{I}}, X, B)$ is an infinite HVMS on $\mathcal{H}$. Let $\beta = \beta_{(1,0)} + \beta_{(0,1)}$.

By Remark 2.20 the scalar (X, β) moments of B agree with the scalar (Y, α) moments of A to all orders. Therefore by Theorem 6.2, the rational function g of degree at most d in each variable given by

$$(6.10) \qquad\qquad g(z) = \langle (B - z_X)^{-1}\beta, \beta \rangle_{\mathcal{H}}$$

has the same asymptotic expansion at ∞ as h. By Lemma 6.11, we are done. $\qquad\square$

Lemma 6.11. *Let g, h be in $\mathcal{L}^\infty$ and have the same asymptotic expansion at ∞. Assume in addition that g is rational. Then g and h are equal.*

Proof. For each fixed w in $\mathbb{R}$, the functions $g(z, z+w)$ and $h(z, z+w)$ are in the one variable Pick class and have the same asymptotic expansions at ∞. By Theorem 1.4, they must be Cauchy transforms of measures with the same moments. Moreover, $g(z, z+w)$ is rational. Therefore by [17, Thm. 1.2], the one variable moment problem is in this case determinate, so the two measures must be equal. Therefore $g(z, z+w) = h(z, z+w)$ for all $z \in \Pi$, $w \in \mathbb{R}$, and so the two functions are identically equal. $\qquad\square$

Corollary 6.12. *Let $h \in \mathcal{L}^N$ have an asymptotic expansion*

$$h(z) \;=\; \sum_{|n| \le 2N-1} \frac{\rho_n}{z^n} \;+\; o(\|z\|^{-(2N-1)})$$

as $z \xrightarrow{nt} \infty$. Then there is a rational function g in $\mathcal{L}^\infty$ that has the same asymptotic expansion to order $2N-1$.

Proof. Let $(\{\alpha_n\}_{n\in I_N}, Y, A)$ be a finite HVMS corresponding to h as in Theorem 4.2. Choose vectors $\{\beta_n\}_{n\in I_N}$ in a finite dimensional space $\mathcal{H}$ so that (6.8) holds, and define X and B as in the proof of Theorem 6.6. Then g given by (6.10) has the same asymptotic expansion. $\qquad\square$

7. AN EXAMPLE

Let $\{w_j\}_{j=1}^\infty$ be a summable sequence of non-negative numbers, and let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of real numbers. Let t_j be numbers in the interval $[0, 1]$. Define

$$A \;=\; \bigoplus \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix},$$

$$Y \;=\; \bigoplus \begin{pmatrix} t_j^2 & t_j\sqrt{1-t_j^2} \\ t_j\sqrt{1-t_j^2} & 1-t_j^2 \end{pmatrix},$$

$$\alpha_{(1,0)} \;\; = \;\; \bigoplus \sqrt{w_j} \begin{pmatrix} t_j \\ \sqrt{1-t_j^2} \end{pmatrix},$$

$$\alpha_{(0,1)} \;\; = \;\; \bigoplus \sqrt{w_j} \begin{pmatrix} \sqrt{1-t_j^2} \\ -t_j \end{pmatrix}.$$

If $\alpha = \alpha_{(1,0)} + \alpha_{(0,1)}$ and $h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle$, then $h(z)$ is given by

$$(7.1) \qquad h(z) \;\; = \;\; \sum_{j=1}^{\infty} w_j \frac{4t_j\sqrt{1-t_j^2}\,\lambda_j + z_1 + z_2}{\lambda_j^2 - \lambda_j(2t_j^2-1)(z_1-z_2) - z_1 z_2}.$$

If $\sum w_j \lambda_j^2 < \infty$, then one can extend the HVMS by

$$\alpha_{(2,0)} \;\; = \;\; \bigoplus \sqrt{w_j}\,\lambda_j\,(2t_j^2-1) \begin{pmatrix} t_j \\ \sqrt{1-t_j^2} \end{pmatrix},$$

$$\alpha_{(1,1)} \;\; = \;\; \bigoplus 2\sqrt{w_j}\,\lambda_j \begin{pmatrix} t_j - t_j^3 + t_j^2\sqrt{1-t_j^2} \\ t_j - t_j^3 - t_j^2\sqrt{1-t_j^2} \end{pmatrix},$$

$$\alpha_{(0,2)} \;\; = \;\; \bigoplus \sqrt{w_j}\,\lambda_j\,(1-2t_j^2) \begin{pmatrix} \sqrt{1-t_j^2} \\ -t_j \end{pmatrix}.$$

Calculating, one gets that

$$r_1(z) \;\; = \;\; \left(\sum w_j\right)\left[\frac{1}{z_1} + \frac{1}{z_2}\right]$$

$$r_2(z) \;\; = \;\; \sum w_j \lambda_j \left[\frac{2t_j^2-1}{z_1^2} + \frac{4t_j\sqrt{1-t_j^2}}{z_1 z_2} + \frac{1-2t_j^2}{z_2^2}\right]$$

$$r_3(z) \;\; = \;\; \sum w_j \lambda_j^2 \left[\frac{(2t_j^2-1)^2}{z_1^3} + \frac{4(t_j^2-t_j^4) + 4t_j(2t_j^2-1)\sqrt{1-t_j^2}}{z_1^2 z_2}\right.$$

$$\left. + \frac{4(t_j^2-t_j^4) + 4t_j(1-2t_j^2)\sqrt{1-t_j^2}}{z_1 z_2^2} + \frac{(2t_j^2-1)^2}{z_2^3}\right].$$

These are (up to a minus sign) the first three terms in the asymptotic expansion of (7.1) at infinity. If one assumes that $\sum w_j \lambda_j^4 < \infty$, then one gets two more terms, and so on.

In the special case that every $t_j = 1/\sqrt{2}$, the formulas simplify. Then

$$h(z) \;\; = \;\; \sum w_j \frac{2\lambda_j + z_1 + z_2}{\lambda_j^2 - z_1 z_2},$$

$$r_1(z) \;\; = \;\; \left(\sum w_j\right)\frac{z_1 + z_2}{z_1 z_2},$$

$$r_2(z) = \left(\sum w_j \lambda_j\right) \frac{2}{z_1 z_2},$$
$$r_3(z) = \left(\sum w_j \lambda_j^2\right) \frac{z_1 + z_2}{z_1^2 z_2^2}.$$

8. MODELS

A *model* for h is a reproducing kernel space $\mathcal{M}$ on Π^2 and a positive contraction Y on $\mathcal{M}$ so that, if the reproducing kernel K for $\mathcal{M}$ is written as

$$(8.1) \qquad\qquad K(z, w) = \langle v_z, v_w \rangle_{\mathcal{M}}$$

with v_z analytic in z, then

$$(8.2) \qquad h(z) - \overline{h(w)} = (z_1 - \bar{w}_1)\langle Y v_z, v_w \rangle + (z_2 - \bar{w}_2)\langle (I - Y)v_z, v_w \rangle.$$

Using our earlier notation $z_Y = z_1 Y + z_2(I - Y)$, (8.2) becomes

$$(8.3) \qquad\qquad h(z) - \overline{h(w)} = \langle (z_Y - w_Y^*)v_z, v_w \rangle.$$

The existence of models for functions in the Pick class was proved in [1]. Indeed, it was shown there that for every h in the Pick class, there are analytic functions $v^1(z)$ and $v^2(z)$ taking values in Hilbert spaces $\mathcal{M}^1$ and $\mathcal{M}^2$ so that

$$h(z) - \overline{h(w)} = (z_1 - \bar{w}_1)\langle v^1(z), v^1(w) \rangle_{\mathcal{M}^1} + (z_2 - \bar{w}_2)\langle v^2(z), v^2(w) \rangle_{\mathcal{M}^2}.$$

Let

$$K(z, w) = \langle v^1(z), v^1(w) \rangle_{\mathcal{M}^1} + \langle v^2(z), v^2(w) \rangle_{\mathcal{M}^2}.$$

This is a kernel, so it can be written as in (8.1) for some other Hilbert space $\mathcal{M}$, and there is a positive contraction Y on $\mathcal{M}$ so that

$$\langle v^1(z), v^1(w) \rangle_{\mathcal{M}^1} = \langle Y v_z, v_w \rangle.$$

This yields (8.3).

Write $\mathbf{i}$ for the point (i, i) in $\mathbb{C}^2$. The equivalence of *(ii)* - *(iv)* in the following theorem was first proved in [4].

Theorem 8.4. *Let $h : \Pi^2 \to \overline{\Pi}$ be in the Pick class, and not identically zero. The following are equivalent.*

(i) For some/every model with reproducing kernel as in (8.1), there is a vector α in $\mathcal{M}$ such that

$$(8.5) \qquad\qquad h(z) = \langle v_z, \alpha \rangle.$$

(ii) There exists a self-adjoint operator A on a Hilbert space $\mathcal{H}$ and a vector α in $\mathcal{H}$ such that

$$(8.6) \qquad\qquad h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha \rangle.$$

(iii) There exists $c > 0$ such that

$$(8.7) \qquad\qquad \lim_{s \to \infty} s h(s\mathbf{i}) = ic.$$

(iv) We have

$$(8.8) \qquad\qquad \liminf_{s \to \infty} |s h(s\mathbf{i})| < \mathbf{i}.$$

Proof. $(i) \Rightarrow (ii)$: Define B by

$$B : v_z \mapsto z_Y v_z + \alpha.$$

Equations (8.3) and (8.5) imply that

$$(8.9) \qquad\qquad \langle B v_z, v_w \rangle \;=\; \langle v_z, B v_w \rangle.$$

Extend B to finite linear combinations of vectors v_{z_j} by linearity, and (8.9) says that B is well defined and symmetric. Indeed, if some linear combination $\sum c_j v_{z_j} = 0$, then for every w we have

$$\left\langle \sum c_j((z_j)_Y v_{z_j} + \alpha), v_w \right\rangle \;=\; \left\langle \sum c_j v_{z_j}, w_Y v_w + \alpha \right\rangle \;=\; 0,$$

so $B(c_j v_{z_j}) = 0$.

If the defect indices of the closure of B match, then B can be extended to a self-adjoint operator on $\mathcal{M}$. If not, B can be extended to a self-adjoint operator on a superspace of $\mathcal{M}$. In either event, we can assume that there is a self-adjoint A on $\mathcal{H} \supseteq \mathcal{M}$ such that

$$A : v_z \mapsto z_Y v_z + \alpha.$$

Therefore $v_z = (A - z_Y)^{-1}\alpha$, and (8.6) follows from (8.5).

$(ii) \Rightarrow (iii)$: By the spectral theorem,

$$s\,h(s\mathbf{i}) \;=\; \int \frac{s}{t - is} d\mu(t),$$

where μ is the finite measure that is the scalar spectral measure of A for α. As the integrand is bounded by 1 in modulus and tends pointwise to i, the dominated convergence theorem implies

$$\lim_{s\to\infty} sh(s\mathbf{i}) \;=\; i\|\alpha\|^2.$$

$(iii) \Rightarrow (iv)$: Obvious.

$(iv) \Rightarrow (i)$ By (8.3),

$$(8.10) \qquad\qquad 2\mathrm{Im}h(s\mathbf{i}) \;=\; 2is\langle v_{s\mathbf{i}}, v_{s\mathbf{i}}\rangle.$$

By (8.8) and (8.10), there is a sequence s_n such that $-is_n v_{s_n\mathbf{i}}$ has a weak limit. Call this limit α. By (8.3) we have

$$(8.11) \qquad h(z) - \overline{h(s_n\mathbf{i})} \;=\; \langle z_y v_z, v_{s_n\mathbf{i}}\rangle + \langle v_z, -is_n v_{s_n\mathbf{i}}\rangle.$$

Take the limit in (8.11) as $s_n \to \infty$ to get (8.5). $\qquad\qquad\square$

Theorem 8.12. *Let h be in the Pick class of two variables, and assume h satisfies (8.8). There exists a representation as in (8.6) with $\mathcal{H}$ finite dimensional if and only if h is rational and real-valued on the complement in $\mathbb{R}^2$ of its polar set.*

Proof. If h has a representation as in (8.6) with $\mathcal{H}$ d dimensional, it is clear that h is rational of degree at most d in each variable, and that h is real on $\mathbb{R}^2$ of its polar set.

For the converse, let

$$\alpha(\lambda) \;=\; i\frac{1+\lambda}{1-\lambda}$$

be a linear fractional map that maps the unit disk $\mathbb{D}$ to Π, and let

$$\beta(z) \;=\; \frac{z-i}{z+i}$$

be its inverse. Let

$$\phi(\lambda_1, \lambda_2) \; = \; \beta \circ h(\alpha(\lambda_1), \alpha(\lambda_2)).$$

This is a function in the unit ball of $H^\infty(\mathbb{D}^2)$, the space of bounded analytic functions on the bidisk. Moreover, ϕ is rational if and only if h is, in which case they have the same bidegree, and ϕ is inner if and only if h is real valued a.e. on $\mathbb{R}^2$.

Assume h is rational and non-constant of bidegree (d_1, d_2). By a result of G. Knese [10], there are Hilbert spaces $\mathcal{M}^1$ and $\mathcal{M}^2$ of dimension d_1 and d_2 respectively, and analytic functions $u^1 : \mathbb{D}^2 \to \mathcal{M}^1$ and $u^2 : \mathbb{D}^2 \to \mathcal{M}^2$ so that

$$(8.13) \quad 1 - \phi(\lambda)\overline{\phi(\zeta)} \; = \; (1 - \lambda_1\overline{\zeta_1})\langle u^1(\lambda), u^1(\zeta)\rangle \; + \; (1 - \lambda_2\overline{\zeta_2})\langle u^2(\lambda), u^2(\zeta)\rangle.$$

Define functions $v^r : \Pi^2 \to \mathcal{M}^r$ for $r = 1, 2$ by

$$v^r(z) \; = \; \frac{h(z) + i}{z_r + i} u^r_{\beta(z)}.$$

Then an algebraic manipulation transforms (8.13) into

$$(8.14) \qquad h(z) - \overline{h(w)} \; = \; (z_1 - \bar{w}_1)\langle v^1(z), v^1(w)\rangle \; + \; (z_2 - \bar{w}_2)\langle v^2(z), v^2(w)\rangle.$$

Let

$$K(z, w) \; = \; \langle v^1(z), v^1(w)\rangle_{\mathcal{M}^1} \; + \; \langle v^2(z), v^2(w)\rangle_{\mathcal{M}^2}.$$

This has rank less than or equal to $d = d_1 + d_2$, so it is the reproducing kernel for some Hilbert function space $\mathcal{M}$ on Π^2 of dimension less than or equal to d. By $(iv) \Rightarrow (i)$ of Theorem 8.4, we have a vector α such that (8.5) holds. Now follow the proof of $(i) \Rightarrow (ii)$, and observe that since B is defined on a finite dimensional space, its defect indices must match, and so it can be extended to a self-adjoint operator A on $\mathcal{M}$. $\qquad \square$

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