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Maxwell's Equations

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Generating Electromagnetic Waves

1. Where do we start?

When SIW first encountered electromagnetic fields at *U. Chicago* during the depression, textbooks used two different systems of units. First, electrical phenomena were described using a *CGS*-based system of units known as the Gaussian or ESU system. Then magnetic phenomena were described using a *CGS*-based system of units known as the electromagnetic (EMU) units. Then both the ESU and EMU units were used to describe phenomena involving both electric and magnetic forces. Remnants of this schizophrenia are present in all modern *E&M* textbooks.

Electric and magnetic fields were mostly theoretical concepts until the 1880s. About 1880 T. A. Edison, *invented* (at his Menlo Park, New Jersey lab) a carbon filament electric lamp. He realized that he could sell one of these lamps to everyone in the USA *if* he could provide them with the requisite electrical excitation. (His early experiments were powered by batteries which were too expensive for homeowners to buy or maintain!) Elecric and magnetic fields had to become household entities. So he hired European educated engineers to build a commercial source of electricity. The Pearl Street Station was the first central power plant in the world. It was located at 255-257 Pearl Street in Manhattan on a site measuring 15 m by 30 m, just south of Fulton Street and fired by coal. It began with one direct current generator, and it started generating electricity on September 4, 1882, serving an initial load of 400 lamps of 82 customers. By 1884, the Pearl Street Station was serving 508 customers with 10,164 lamps. The station was built by the Edison Illuminating Company, headed by Thomas Edison, which evolved into the present *Consolidated Edison* company.

The Pearl Street Station not only holds the distinction of being the world's first central power plant, but it was also the world's first cogeneration plant. While the steam engines provided grid electricity, Edison made use of the thermal byproduct by distributing steam to local manufacturers, and warming nearby buildings on the same Manhattan block. The station burned down in 1890, destroying all but one dynamo that is now kept in the Greenfield Village Museum in Dearborn, Michigan.

Generally, the builders of the Pearl Street Station decribed their work in mathematical terms that were beyond Edison's comprehension. But when it came to *marketing* their product (electricity), Edison forced them to introduce a new unit of current. He was set up to manufactutre essentially 100 W electric light bulbs. His engineers built generators that produced a potential on the power lines of ≈ 100 V. But when the engineers told Edison that it would take $\approx 10^7$ *stat amperes* (the ESU/EMU unit for current) to light the bulbs, he had a fit: If a bulb required that much current, no one would buy one. So he made the engineers define a new unit (a *practical* ampere) with a size that it would only require 1 ampere of current to light up his bulbs. Thus a third set of units ("*practical* units") came into existence.

By the turn of the century, university textbooks came in three flavors: ESU or EMU units used in Arts and Science departments, and Practical units used in engineering departments. The book publishers, however, realized that this meant that each flavor of book had a limited audience. They informed authors that they would stop publishing *E&M* books until a new set of units was developed that could be used by everyone. The *IUPAP* and *IUPAC* organizations formed committees to address this issue. Basically they produced a new system (ultimately the *SI* system) out of the old systems by making two changes to the old. (1°) In 1900, Giovanni Giorgi published a paper in which he noted that the 10^7 factor would go away if one went from $CGS \rightarrow MKS$ units. [Giorgi read a description of this shift at a scientific meeting in St. Louis sponsored by the 1904 St. Louis Worlds Fair!] (2°) Academic scientists had noted that the old systems were *irrational* (4π's were present in descriptions of systems with no rotational symmetry but absent in descriptions of systems with rotational symmetry) but a careful addition of extra 4π factors could rationalize the fundamental equations.

The *SI* system of units was first used in 1960 and dominated all textbooks by 1970. It remains detested by most people working in *quantum field theory*.

Maxwell formalized his description of electrical and magnetic phenomena by giving a set of equations governing electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields. In *SI* units, these equations are

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \tag{1a}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial B}{\partial t} = 0 \tag{1b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \tag{1d}$$

$\boldsymbol{E}(\boldsymbol{r}, t)$ is the electric field at $\boldsymbol{r}, t$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ is the magnetic field at $\boldsymbol{r}, t$. These fields produce a force

$$\boldsymbol{F} = Q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \tag{2}$$

on a particle of charge Q located at $\boldsymbol{r}$ and moving with velocity $\boldsymbol{v}$ at time t. The sources for the fields are the charge density $\rho(\boldsymbol{r}, t)$ and the current density $\boldsymbol{J}(\boldsymbol{r}, t)$ in the system. The eight equations in (1) determine what the fields $\boldsymbol{E}(\boldsymbol{r}, t), \boldsymbol{B}(\boldsymbol{r}, t)$ are in the presence of these sources. The constants ϵ_0 and μ_0 fix the units of charge and current: ρ is the charge density in $Coulombs/meter^3$ and $\boldsymbol{J}$ is the current density in $amperes/meter^2$ with 1 $ampere$ = 1 $Coulomb/second$.

2. Introduce Potentials

There are two general theorems of vector calculus that allow one to simplify the relations in Eq.(1). 1° If a field $\boldsymbol{B}$ satisfies $\nabla \cdot \boldsymbol{B} = 0$, then one can always find another field $\boldsymbol{A}$ from which $\boldsymbol{B}$ can be derived,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

2° If a field $\boldsymbol{F}$ satisfies $\nabla \times \boldsymbol{F} = 0$, then one can always find another field ϕ from which $\boldsymbol{F}$ can be derived,

$$\boldsymbol{F} = \nabla\phi.$$

Thus, if we represent

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{2a}$$

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{2b}$$

then Eqs.(1b,1c) will be satisfied for *any* $\boldsymbol{A}$, ϕ fields. $\boldsymbol{A}$ is called the vector potential of the system and ϕ is called the scalar potential of the system. Using these representations in the remaining equations in Eq.(1) gives

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = \rho/\epsilon_0 \tag{3a}$$

$$\nabla \times (\nabla \times \boldsymbol{A}) + \epsilon_0\mu_0\left(\frac{\partial}{\partial t}\nabla\phi + \frac{\partial A}{\partial t}\right) = \mu_0 \boldsymbol{J} \tag{3b}$$

While specifying $\nabla \times \boldsymbol{A}$ provides an $\boldsymbol{A}$ that makes (1c) correct, one can also prescribe $\nabla \cdot \boldsymbol{A}$. It turns out, however, that Eq.(1c) remains correct no matter what values are prescribed for $\nabla \cdot \boldsymbol{A}$. Thus we can prescribe any value for $\nabla \cdot \boldsymbol{A}$ that simplifies the formulation. Explicitly, we will now prescribe (the "Lorentz gauge")

$$\nabla \cdot \boldsymbol{A} = -\epsilon_0\epsilon_0\frac{\partial\phi}{\partial t}. \tag{4}$$

Eqs.(3) are then reduced to

$$-\nabla^2\phi + \epsilon_0\mu_0\frac{\partial^2\phi}{\partial t^2} = \rho/\epsilon_0, \tag{5a}$$

$$-\nabla^2\boldsymbol{A} + \epsilon_0\mu_0\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \mu_0\boldsymbol{J}. \tag{5b}$$

In the absence of sources ρ and $\boldsymbol{J}$, Eqs.(5) are just wave equations associated with a speed

$$c = \frac{1}{\sqrt{\epsilon_0\,\mu_0}}$$

that we associate with the speed of light. So we can identify $\epsilon_0\,\mu_0$ with $1/c^2$ and rewrite Eqs.(5) as

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \rho/\epsilon_0 \tag{6a}$$

$$-\nabla^2\boldsymbol{A} + \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \mu_0\boldsymbol{J} \tag{6b}$$

3. In summary

Given ρ and $\boldsymbol{J}$, we can solve Eqs.(6) for the potentials ϕ and $\boldsymbol{A}$. [We must also apply the *bounary condition* $\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$.]

Then we can deduce $\boldsymbol{E}$ and $\boldsymbol{B}$ from Eqs.(2).

4. The Poynting vector

In 1884, John Henry Poynting, wrote a paper "On the Transfer of Energy in the Electromagnetic Field" (*Philosophical Transactions of the Royal Society of London*) in which he identified the energy flux associated with electromagnetic fields with the vector (now called the Poynting vector)

$$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B} \qquad (7)$$

Thus an identification of the generation of electromagnetic waves involves the calculation route

$$\rho, \boldsymbol{J} \to \boldsymbol{A}, \phi \to \boldsymbol{E}, \boldsymbol{B} \to \boldsymbol{S}$$

and then examining $\boldsymbol{S}$ at large distances from the sources.

5. The Hertz calculation

In 1879, H. Helmholtz suggesgted to his doctoral student H. Hertz to include a test of Maxwell's 1865 prediction that there were *electromagnetic waves*. The experimental details were complex, but Hertz *explained* the results by calculating the electromagnetic field produced by an oscillating electric dipole moment. What follows here is the original calculation translated into *SI* units.

Suppose that a charge Q_0 oscillates along the z axis. Let

$$\boldsymbol{r}_+(t) = \frac{1}{2}\ell \sin(\omega t)\,\hat{\boldsymbol{k}}$$

locate the charge at time t. And suppose that another charge $-Q_0$ (to keep the total charge zero) is located at

$$\boldsymbol{r}_-(t) = -\frac{1}{2}\ell \sin(\omega t)\,\hat{\boldsymbol{k}}.$$

Then

$$\rho(\boldsymbol{r}, t) = Q_0\left[\delta\left(\boldsymbol{r} - \boldsymbol{r}_+(t)\right) - \delta\left(\boldsymbol{r} - \boldsymbol{r}_-(t)\right)\right] \qquad (8)$$

Knowing the right hand side of (6a), we can solve (6a) for $\phi(r,t)$ *

$$\begin{aligned}\phi(r,t) &= -\frac{1}{4\pi\epsilon_0}\int dr'\,\frac{\rho(r',t-\frac{|r-r'|}{c})}{|r-r'|}\\ &= -\frac{Q_0}{4\pi\epsilon_0}\left[\frac{1}{|r-r_+(t-\frac{|r-r_+|}{c})|}-\frac{1}{|r-r_-(t-\frac{|r-r_-|}{c})|}\right]\end{aligned} \tag{9}$$

Finally, we expand Eq.(9) in ℓ about $\ell=0$ (anticipating that we're only interested in the $\ell\to 0$ limit)

$$\begin{aligned}\phi(r,t) =&\frac{Q_0}{4\pi\epsilon_0}\left[\frac{1}{|r-r_+(t-\frac{r}{c})|}-\frac{1}{|r-r_-(t-\frac{r}{c})|}\right]\\ =&\frac{Q_0}{4\pi\epsilon_0\,r}\left[-\frac{r\cdot(r_+(t-\frac{r}{c})-r_-(t-\frac{r}{c}))}{r^2}+\cdots\right]\\ =&\frac{Q_0}{4\pi\epsilon_0\,r}\left[-\frac{z*\ell*\sin(\omega*(t-\frac{r}{c})}{r^2}+\cdots\right]\end{aligned} \tag{10}$$

If $\ell * Q_0 \to d$ in the limit that $\ell\to 0$, the field associated with the charges $Q_0, -Q_0$ becomes *dipolar* and d is the *dipole moment*. Setting

$$d(t) = Q_0 * \ell * \sin(\omega t)\,\hat{k}$$

the potential in (10) becomes

$$\phi(r,t) = -\frac{d(t-\frac{r}{c})\cdot r}{4\pi\epsilon_0 r^3} \tag{11}$$

The electrical currents in a system are just a description of the moving charges in the system. In fact, they must satisfy the *continuity* equation

$$\frac{\partial\rho(r,t)}{\partial t} + \nabla\cdot J(r,t) = 0. \tag{12}$$

Since

$$\rho(r,t) = Q_0\delta(x)\,\delta(y)\,[\delta(z-z_+(t)) - \delta(z-z_-(t))] \tag{13}$$

* https://www.phy.duke.edu/~rgb/Class/phy319/phy319/node12.html

$$\begin{aligned}\frac{\partial\rho(r,t)}{\partial t} &= -\frac{1}{2}\ell\omega\cos(\omega t)Q_0\delta(x)\,\delta(y)\left[\delta'(z-z_+(t))+\delta'(z-z_-(t))\right]\\ &= -\frac{1}{2}\ell\omega\cos(\omega t)Q_0\delta(x)\,\delta(y)\,\nabla\cdot[\delta(z-z_+(t))+\delta(z-z_-(t))]\,\hat{\boldsymbol{k}}\end{aligned} \tag{14}$$

and we can identify the left hand side of Eq.(14) as $-\nabla\cdot\boldsymbol{J}(r,t)$ with

$$\boldsymbol{J}(r,t) = \frac{1}{2}\ell\omega\cos(\omega t)Q_0\delta(x)\,\delta(y)\,[\delta(z-z_+(t))+\delta(z-z_-(t))]\,\hat{\boldsymbol{k}} \tag{15}$$

Knowing the right hand side of Eq.(6b), we can evaluate $\boldsymbol{A}(\boldsymbol{r},t)$:

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r},t) &= -\frac{\mu_0}{4\pi}\int d\boldsymbol{r}'\;\frac{\boldsymbol{J}(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}_-|}{c})}{|\boldsymbol{r}-\boldsymbol{r}'|}\\ &= -\frac{\mu_0\ell\omega\,Q_0}{8\pi}\left[\frac{\cos(\omega\left((t-\frac{|\boldsymbol{r}-\boldsymbol{r}_+|}{c})\right)}{|\boldsymbol{r}-\boldsymbol{r}_+(t-\frac{|\boldsymbol{r}-\boldsymbol{r}_+|}{c})|}+\frac{\cos(\omega\left(t-\frac{|\boldsymbol{r}-\boldsymbol{r}_+|}{c})\right)}{|\boldsymbol{r}-\boldsymbol{r}_-(t-\frac{|\boldsymbol{r}-\boldsymbol{r}_-|}{c})}\right]\hat{\boldsymbol{k}}\end{aligned} \tag{16}$$

Again, expanding in ℓ leads to

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi r^2}\,\boldsymbol{r}\cdot\dot{\boldsymbol{d}}(t-\frac{r}{c})\,\hat{\boldsymbol{k}} \tag{17}$$

From $\boldsymbol{A},\phi$ we deduce (neglecting terms of order $1/r^2$) that

$$\begin{aligned}\boldsymbol{E} &= -\frac{\omega^2 d}{4\pi\epsilon_0 c^2}\sin\theta\frac{sin[\omega(t-r/c)]}{r}\hat{\theta}\\ \boldsymbol{B} &= -\frac{\omega^2 d}{4\pi\epsilon_0 c^3}\sin\theta\frac{sin[\omega(t-r/c)]}{r}\hat{\phi}\end{aligned}$$

Next, evaluating Eq.(7) gives

$$\begin{aligned}\boldsymbol{S}(r,t)) &= \frac{\omega^4 d^2}{32\pi^2\epsilon_0 c^3}\,\sin^2[\omega(t-r/c)]\frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}\\ &= \frac{\omega^2 \boldsymbol{d}(t)^2}{32\pi^2\epsilon_0 c^3}\,\frac{\sin^2\theta}{r^2}\,\hat{\boldsymbol{r}}\end{aligned}$$

with $\hat{\boldsymbol{r}}$ a unit vector in the $\boldsymbol{r}$-direction. All that remains is the replacement of $\sin^2[\omega(t-r/c)]$ by $1/2$ (i.e., to time average the energy flux).

Appendix

1. The Problem

In this *Appendix* we examine how to solve the *wave equation*. This is a general problem with many practical applications: Basically, we want to construct functions $F(\boldsymbol{r}, t)$ that solve the partial differential equation

$$-\nabla^2 F + \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = S(\boldsymbol{r}, t) \qquad (A1)$$

F is the (sought) amplitude of the wave, a quantity that varies in space and time. $S(\boldsymbol{r}, t)$ is the *source* of the wave.

In the absence of a source, Eq.(A1) reduces to

$$-\nabla^2 F + \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0, \qquad (A2)$$

a ***linear equation*** that has ***free waves*** as solutions. These could be any freely propagating waves. Because of the ***linearity*** of (A1), we can add any solution to (A2) to a solution to (A1) and get another solution to (A1). Generally, we're not interested in waves passing us by associated with initial conditions or boundary conditions. We're interested in waves that are *caused* by the source $S(\boldsymbol{r}, t)$. These will be solutions that are, in fact, proportional to $S(\boldsymbol{r}, t)$ and will go away if $S(\boldsymbol{r}, t) \to 0$.

2. If the source is *localized*

Let us start by supposing that the source is localized at $\boldsymbol{r} = 0$, a point in some 3D space. Solutions to

$$-\nabla^2 F + \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \delta(\boldsymbol{r})\, S(t) \qquad (A3)$$

will have spherical symmetry around $\boldsymbol{r} = 0$, so they will only depend on r and t.

If we suppose that

$$F(r, t) = \frac{G(r, t)}{r} \qquad (A4)$$

then (A3) becomes

$$-\frac{\partial^2 G(r, t)}{\partial r^2} + \frac{1}{c^2}\frac{\partial^2 G(r, t)}{\partial t^2} = r\, \delta(r)\, S(t) = 0. \qquad (A5)$$

But this equation is just the wave equation in *one dimension* and the general solution is known to be

$$G(r, t) = h_1(r - c\,t) + h_2(r + c\,t) \qquad (A6)$$

with $h_1(x), h_2(x)$ *any* functions that have second derivatives.

Thus the general solution to (A3) will look like

$$F(r,t) = \frac{h_1(r-ct)}{r} + \frac{h_2(r+ct)}{r} \tag{A7}$$

The two terms in (A7) correspond to *outward* and *inward* propagating waves. Since we want a solution to (A5) that looks like an outward propagating wave, we drop the h_2 term in (A7) and know that $F(r,t)$ will have the form

$$F(r,t) = \begin{cases} \frac{h(r-ct)}{r}, & \text{if } r > ct, \\ 0, & \text{otherwise.} \end{cases} \tag{A8}$$

3. Fooling the singularity

The $1/r$ in Eq.(A8) leads to a singularity at $r = 0$. This may be assessed quantitatively by using Gauss' Theorem to calculate

$$\int_{|r|<R} d\boldsymbol{r}\; \nabla^2 \frac{1}{r} = \int_{|r|=R} d\boldsymbol{S}\cdot\nabla\frac{1}{r} = -\int_{|r|=R} d\boldsymbol{S}\cdot\frac{\boldsymbol{r}}{r^3} = -4\pi$$

for all $R > 0$. This result is equivalent to

$$\nabla^2 \frac{1}{r} = -4\,\pi\,\delta(\boldsymbol{r}) \tag{A9}$$

We can use (A8) and (A9) to rewrite (A3) as

$$-4\pi h(r-ct) = S(t)$$

whence we can identify $h(x)$ as being $\propto\ S(t)$, just the solution we seek. The result is that the solution to (A3) is

$$F(r,t) = -\frac{S(t-\frac{r}{c})}{4\pi\, r}. \tag{A10}$$

If the source is distributed over space (if $S = S(\boldsymbol{r}', t)$) then superposition gives

$$F(\boldsymbol{r},t) = -\int d\boldsymbol{r}'\, \frac{S(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{A11}$$