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Henon-Heiles Model

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The Henon-Heiles Model

Overview

Henon and Heiles† introduced a simple mechanical model to describe the motion of stars in a galaxy. Their model galaxy has an axial symmetry so that only two coordinates are required to locate a star in the galaxy. Explicitly, they considered the motion of stars generated by the Hamiltonian

$$H(x, y, p_x, p_y) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3, \tag{1}$$

the potential energy modeling the mean gravitational energy in the galaxy.

The equations of motion are non-linear because of the cubic terms in (1). No formal *solution* to the equations of motion is known so Henon and Heiles determined the motion in phase space (the space with coordinates x, y, p_x and p_y) *numerically* (using two different computers and two different numerical algorithms to check their results).

The numerical results exhibited a structure that surprised them, but they recognized that the structure could be understood in terms of abstract theoretical analyses of what was possible in classical mechanics. Most of their paper is devoted to describing the qualitative behavior of the solutions they saw. The model has played an important pedagogical role by illustrating what behaviors can be exhibited by mechanical systems.

The potential energy

Rendering the potential energy in cylindrical coordinates makes it easier to picture the forces in the model:

$$\frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3 \to V(R, \theta) = \frac{1}{2}R^2 - \frac{1}{3}R^3 \sin(3\theta) \tag{2}$$

Here are plots of the radial dependence of $V(R, \theta)$ in two extreme θ directions:

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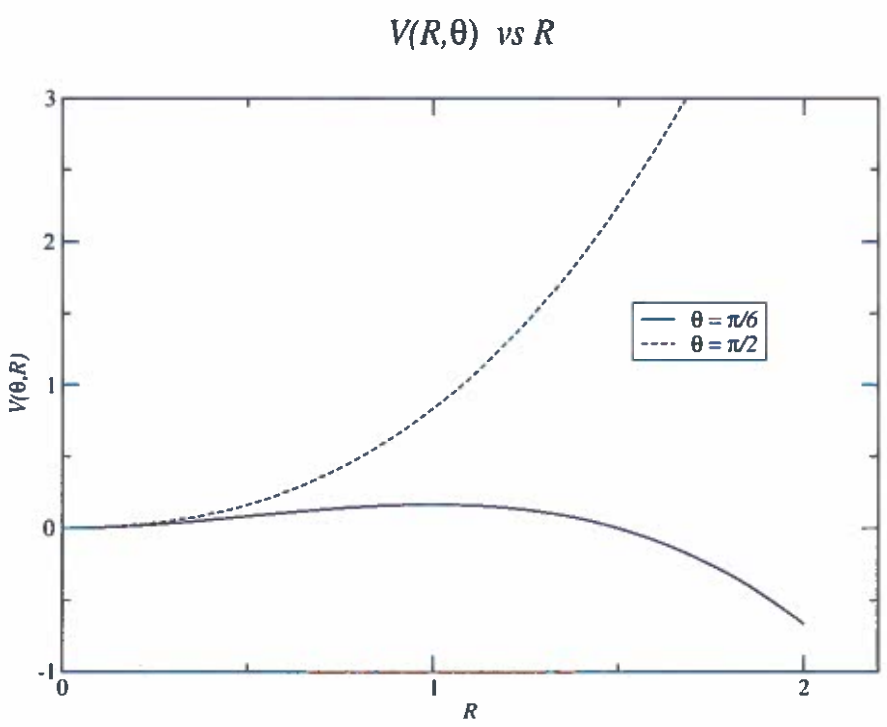
† "The applicability of the third integral of motion: Some numerical experiments," M. Henon and C. Heiles, *The Astrophysical Journal* **69**, 73-79 (1964).



The term $R^2/2$ dominates in all directions when R is small. When the energy is low, the motion is close to that of a two dimensional harmonic oscillator. Because of the $R^3/3$ term, however, the potential goes through a maximum (at $V = 1/6$) as R increases in three directions. At energies $> 1/6$, the motion is unbounded.

By increasing the energy of the system, one can qualitatively change the nature of the system's time evolution, the topology of the system's dynamical trajectories in phase space.

The expectations of those who have had a course in classical mechanics!

The mehanical state of a one dimensional system can be specified by giving a point in the two dimensional phase space x, p_x of the system and dynamical evolution maps out a curve in this space. But the motion is also confined to the ergodic surface

$$H_1(x, p_x) = E, \tag{3}$$

which is generally a simple curve in the phase space. These curves are the same in one dimensional systems, so specifying the system's energy *solves* the mechanical problem.†

Mathematically, the dynamics in phase space is determined by two first order (in time) differential equations. The general solution requires two constants of integration. If these are $x_0 = x(t = 0)$ and $p_0 = p(t = 0)$, the solution to the differential equations are functions

$$\begin{cases} x = x(t, x_0, p_0) \\ p = p(t, x_0, p_0) \end{cases} \tag{4}$$

Using (3) and (4) to eliminate t, x_0 and p_0 identifies the closed curve on which the motion takes place.

Newton's picture of a planet moving around the sun has essentially three degrees of freedom. The mechanical trajectory is thus a curve in a six dimensional phase space. Again, energy is conserved and hence the trajectory is restricted to a simple surface in phase space,

$$H_3(x, y, z, p_x, p_y, p_z) = E \tag{5}$$

But the z component of the angular momentum L_z and the total angular momentum L^2 are also conserved. The motion is also restricted to two other surfaces in phase space. The *intersection* of these surfaces confines the motion to a three dimensional *torus* in the six dimensional phase space. Qualitatively, the motion exhibits oscillation in three directions. Topologically, the motion is equivalent to the motion of a harmonic oscillator in (a strange) three dimensional space.

The motion is thus fixed by knowing three constants of the motion. Explicit formulae can be developed because the constants of the motion are simple functions of the coordinates of phase space.

One can generalize this picture to a system with f degrees of freedom: The trajectory of the system is a curve in a $2f$ dimensional phase space. There are (including the energy)

† Well, one still needs to determine how fast the the state changes in time, how fast this curve in phase space is traversed.

f constants of the motion. Thus the trajectory in phase space is restricted to the intersection of f surfaces in phase space. This is (topologically) an f dimensional torus in phase space.

We could deduce the topology just as in the one dimensional case by looking at the f dimensional extensions of (3) and (4) and trying to solve for the initial conditions (since these must be constants of the motion) as a functions of the coordinates of the phase space. If, however, this elimination requires finding the inverse of some periodic function in phase space, the constraint does not lead to a restriction that is topologically equivalent to the restriction to the ergodic surface. The motion can come back to a point in phase space that is very close to the starting point.

The Statistical Mechanical Significance

In simple mechanical systems (in *solvable* mechanical systems) the state of the system is restricted to points on the ergodic surface that also lie on $f - 1$ other surfaces. If these other surfaces are simple surfaces, Gibbs' rule about how to identify thermodynamic properties can not be true. Knowing the energy is insufficient to determine the mechanical properties of the system. In fact, simulation would not show anything like "relaxatiion to equilibrium." All points on the ergodic surface will not be equally likely to be seen.

In 1887, to celebrate the 60^{th} birthday of Oscar II, King of Sweden, a prise was offered for anyone *solving* (finding functions that solved Newton's differential equations) the three-body problem, the problem of how three bodies would move under the influence of Newton's model of the gravitational forces.

H. Poincaré won the prise by showing that constraints that limited access to the points on the ergodic surface were, in fact, multivalued and that his prevented the construction of reasonable functions that could describe the dynamical trajectory on the ergodic surface. The topologically complex constraints led to the expectation that a uniform distribution on the ergodic surface may be realized.

The Henon-Heiles model exhibits both these possibilities!

The Poincaré Section

How can one learn the nature of the dynamical solution in a phase space with dimensions > 3 (in a higher dimensional space than the one we live in)? Poincaré's solution was to simply look at a two dimensional interasection of the phase space.

So Henon and Heiles considered the $x = 0$ plane in the four dimensional phase space of their system. Actually, they focused on the y, p_y-plane, y and p_y being half the coordinates of the phase space. They *started* their system by *choosing* a $y(0), p_y(0)$ point in this plane (and also a total energy E for the system). They then assigned $x(0) = 0$ and (knowing E) they deduced $p_x(0)$. So the initial conditions necessary to start a trajectory in the four dimensional phase space were known.

Then they solved for the $x(t), y(t), p_x(t)$ and $p_y(t)$ *numerically*. They drew a picture of the y, p_y plane and placed a dot in the plane at the point corresponding to the initial conditions. While $x(0) = 0$, the $x(t)$ moves in time. Because the motion is bounded in phase space, however, the trajectory eventually passes through $x = 0$ again. At that moment, they placed another point in the y, p_y plane with the coordinates that located where the trajectory in phase space recrossed through the $x = 0$ plane.

Continuing this way, they developed a two dimensional plot that represented where a curve in a four dimensional space crossed through the $x = 0$ plane.

They ultimately ran a lot of *batch* jobs generating these plots with different initial conditions and used them to depict the "structure" (topology) of the motion.

My Program

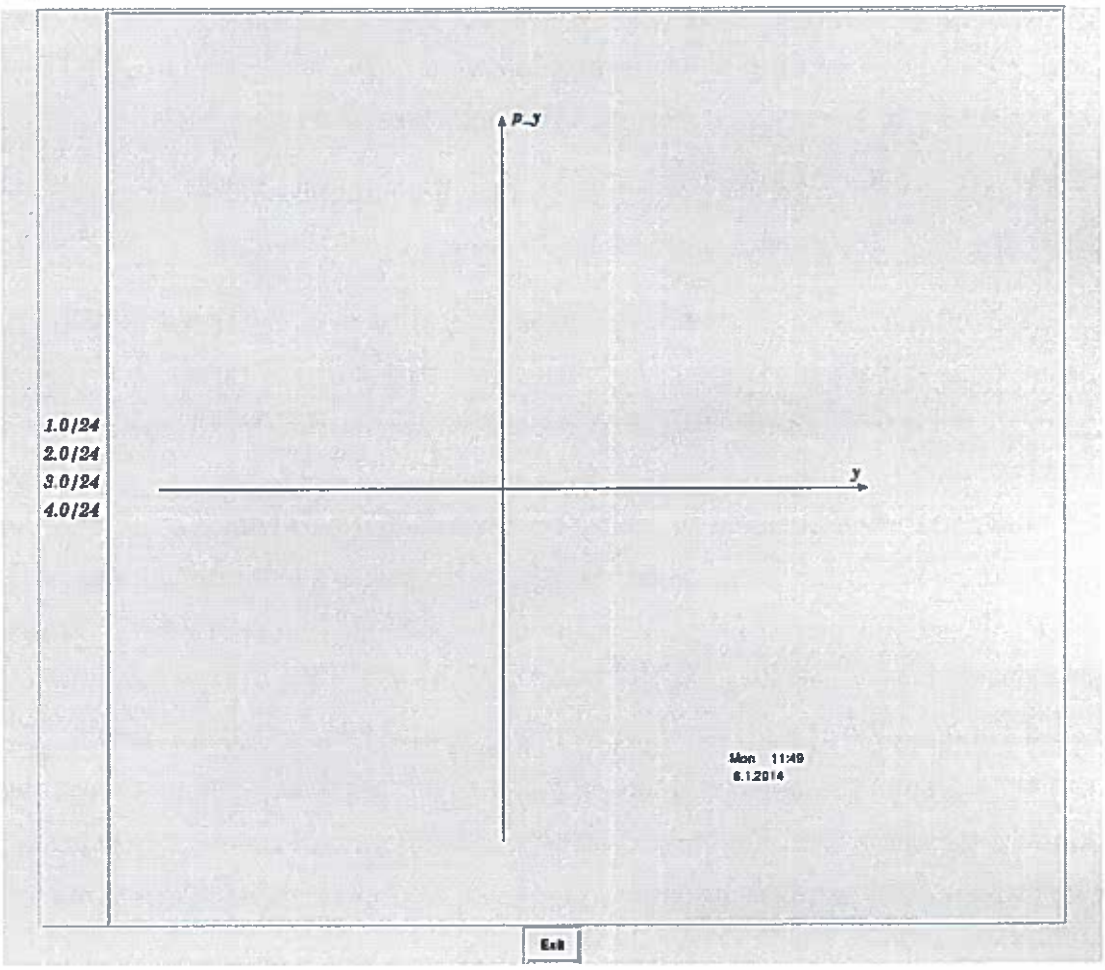
I have written a program to realize the Henon-Heiles calculation more-or-less as a video game. When the program is started, a y, p_y plane is exhibited. If the mouse is "clicked" over this plane, a dot is drawn on the plane at that spot and the trajectory that goes through the $x = 0$ plane with those y, p_y coordinates as starting coordinates is determined numerically. When the trajectory returns to the $x = 0$ plane, another dot is added to the y, p_y plane to show the values of y and p_y at that time. The numerical determination of the trajectory in phase space is then continued until the next crossing is found. The program ultimately stops after some prescribed number of crosses thru the $x = 0$ plane are made.

Interacting with the program

Since there is a dynamic interaction with the screen, I wrote my program in *Perl* supplemented with the *Tk* bindings. I wrote the program in a *Unix* environment, but Walter Reviol was able to transfer the program to the computer in the RadioChemistry's conference room. He followed Greg's instructions to place the program on drive C in a directory *Temp*. There are several programs there but Walter can identify the correct one.

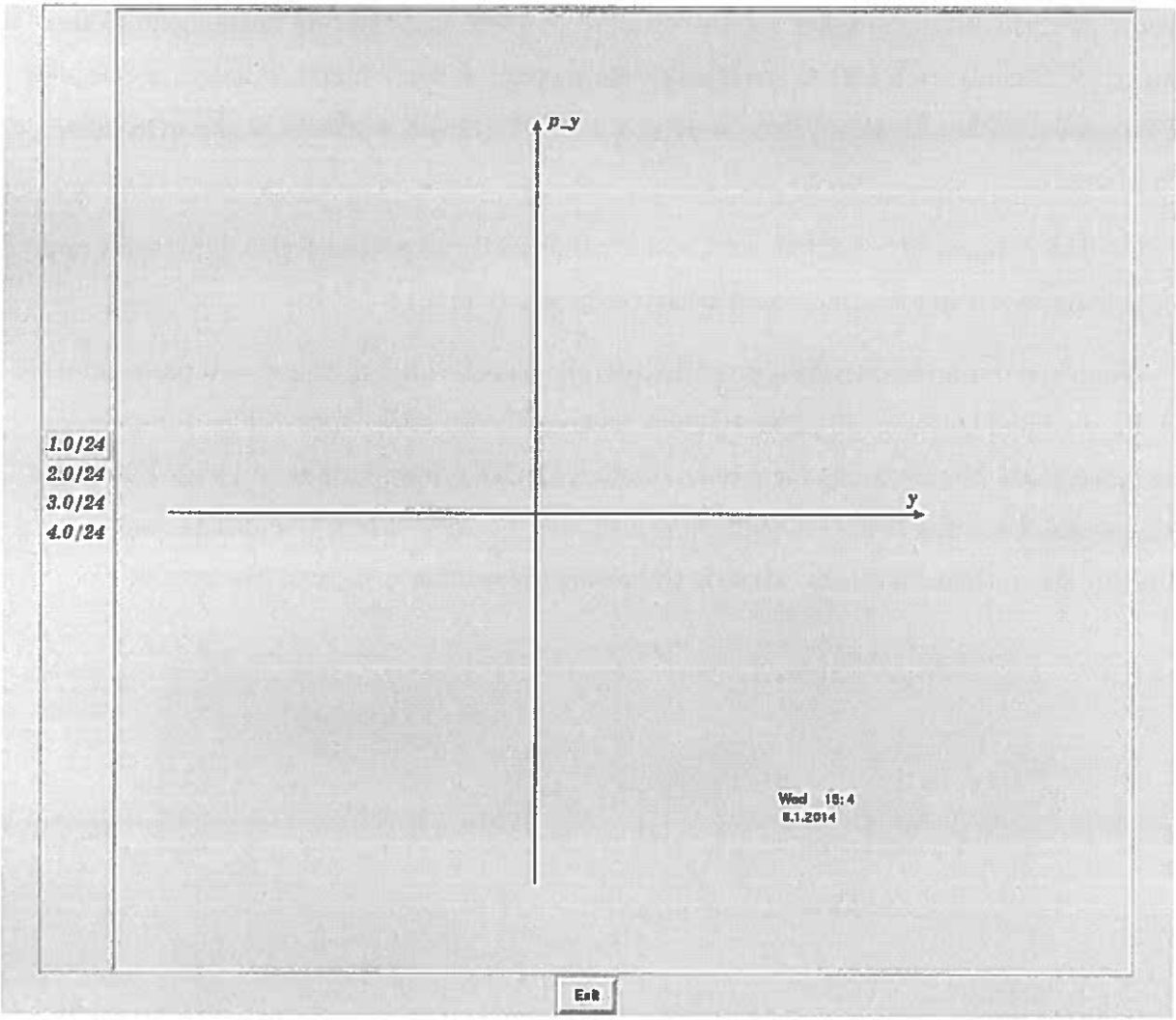
When the program is started, it opens up a window that shows three things:

1. A button "Exit" which is the only way the program can be stopped.
2. A pane on which a y and a p_y axis are drawn. This is the $x = 0$ plane. Dots are placed in the plane when a trajectory passes through the plane.
3. A list of 4 energies in a column on the left hand side. The values span an "interesting" range.



First examples

Start the program. Then, with the mouse, select the lowest energy, 1.0/24. The background to the selected energy will be made *green* and a curve in the y, p_y-plane will be drawn in green.



The points inside the green curve represent system configurations that are bound to the origin. Outside this curve, the system has enough energy to escape from the origin.

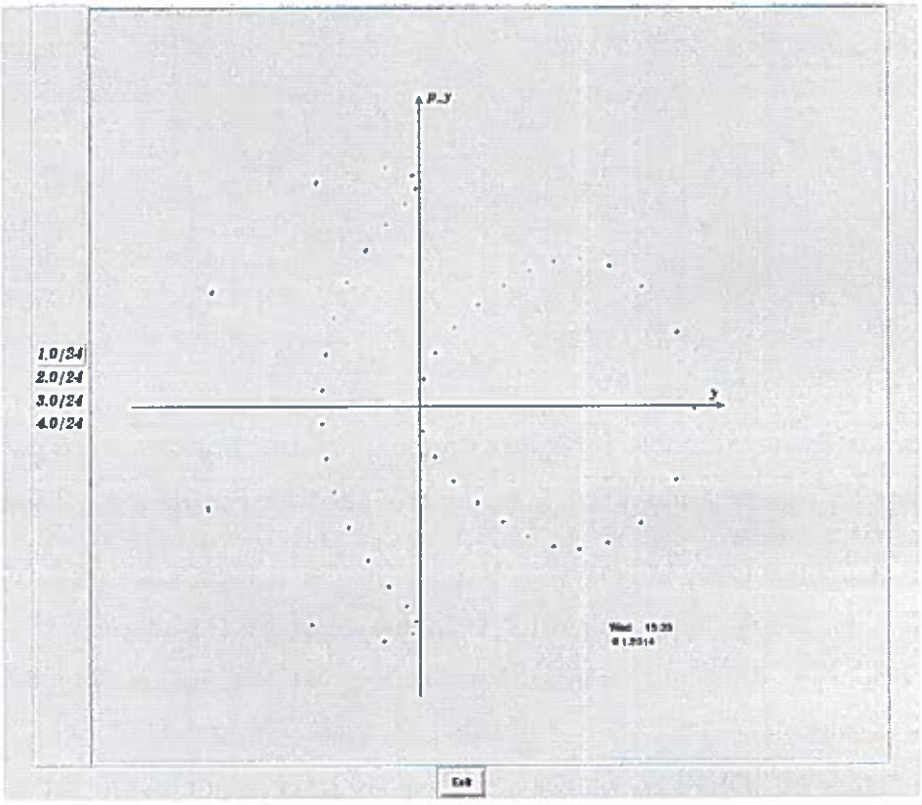
Some initial energy must be selected. If a new initial energy is selected, the old green curve is erased and a new curve is drawn. At higher energies, the shape of the green boundary shifts from elliptical to a figure with a cusp. Nothing in the displayed y, p_y plane indicates the actual scale of the plot so one doesn't know quantitatively what values for y and p_y are displayed. One only knows qualitatively the relation to the green curve. In

fact, at higher energies the range of bound configurations decreases, but the program changes the scale of the display so that the spatial resolution of the display is made as large as is practical.

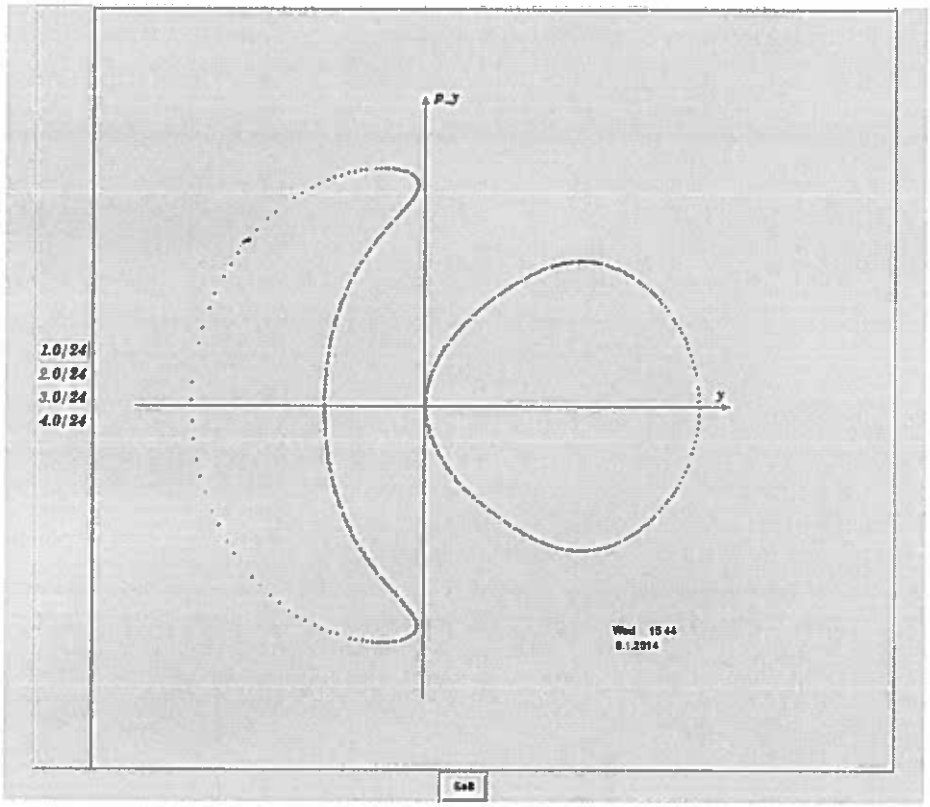
After an energy has been chosen, the mouse can be used to select a point inside the green curve, the program takes the corresponding y and p_y to be the initial conditions of a trajectory. Since $x = 0$ and E are prescribed, p_x can be determined. That is, a complete initial configuration for the system is determined. A red dot is placed in the y, p_y-plane at this point.

The the dynamical equations are solved numerically to produce $x(t), y(t), p_x(t)$ and $p_y(t)$. [The Verlet algorithm is used with a step size 0.001]

When the dynamical trajectory subsequently passes through the $x = 0$ plane another red dot is added to the y, p_y plane to show where this crossing occurs. The dynamics is continued until 200 crossings have been made. It takes a few seconds to locate 200 crossings because the time step has been chosen small – because I didn't want the results to exhibit any discretization errors. Here is the result of starting y, p_y near the origin.



All the crossing points in the y, p_y-plane lie on curves in that plane. These curves are, of course, just intersections of a "torus-like" surface with the y, p_y-plane. In fact, the trajectories actually fill up these surfaces. To illustrate, here is an execution of the program with the number of points in the y, p_y-plane increased to 500:

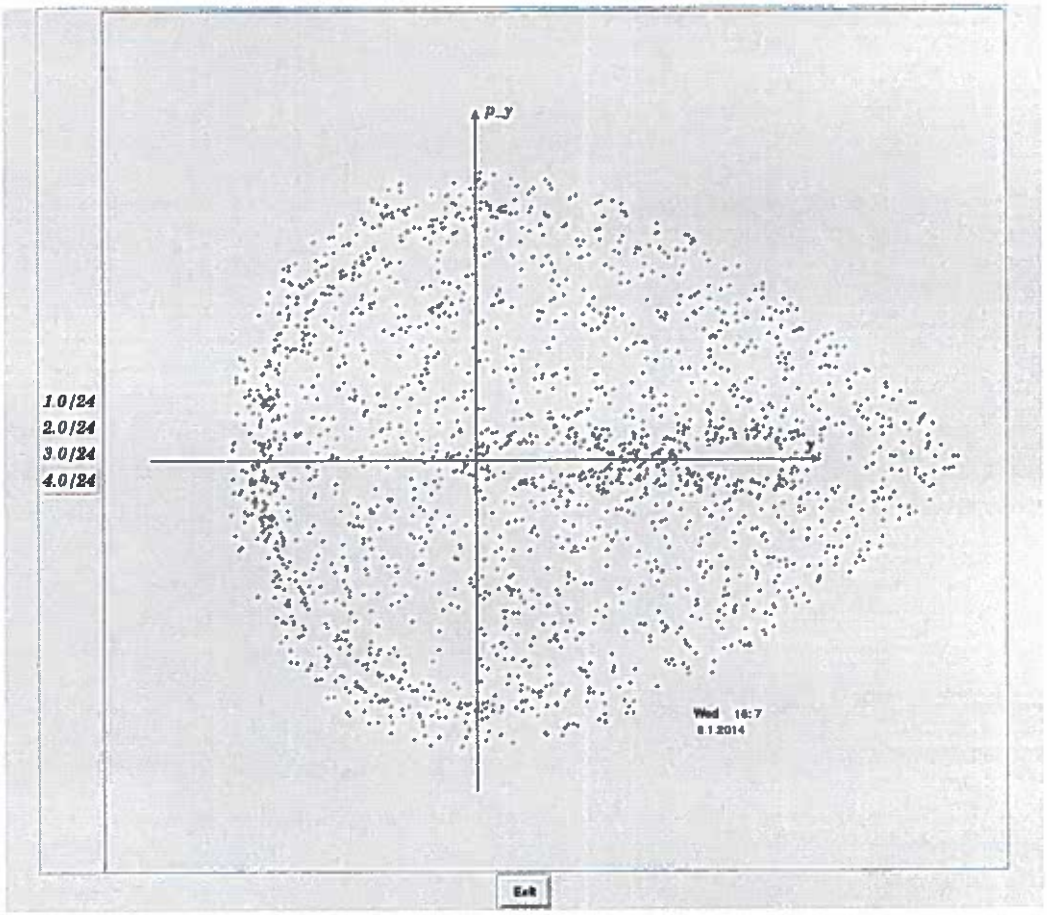


If the y, p_y-plane is clicked again with the mouse, the points associated with the previous trajectory are erased and the display cycle is restarted.

At this low energy, all dynamical solutions conform to the 'torus-like' structure familiar from mechanics classes. Further, you will be able to identify limiting cases where the diameter of the tori goes to zero (look on the y-axis and on the p_y-axis!).

At Energy = 4.0/24

Here is the result of running the program with $E = 4.0/24$ and the starting configuration essentially the origin of the y, p_y plane. In this run, the number of points was increased to 2000.



While there is a weak suggestion of some structure, the structure associated with soluble problems is missing. One sees that, in fact, the trajectory is not confined to some low dimensional surface. The trajectory is actually coming close to filling up all of phase space! This system would exhibit relaxation to equilibrium and Gibbs' measure for thermodynamic properties would be relevant to real experience.