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Shining a Hilbertian lamp on the bidisk

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Abstract. Four lectures on how different aspects of the function theory of the bidisk can be illuminated by using Hilbert spaces and operator theory.

1. Lecture 1: Model Theory

The basic idea behind model theory is to associate a Hilbert space construction with a function, and then use Hilbert space theory to illuminate the function theory.

In one variable, one approach is to study the de Branges-Rovnyak space associated with a function ϕ in the ball of $H^\infty(\mathbb{D})$. This is the Hilbert space of analytic functions on the disk $\mathbb{D}$ with reproducing kernel

$$\frac{1-\overline{\phi(\lambda)}\phi(\zeta)}{1-\bar{\lambda}\zeta}. \tag{1.1}$$

A nice exposition is in the book [**20**] by D. Sarason.

Definition 1.2. We will say that k is a kernel on X, or equivalently that k is positive semi-definite on X, written $k \geq 0$, if k is a function from $X \times X$ to $\mathbb{C}$ such that, for any finite set of distinct points $x_1, \dots, x_N$ in X, the matrix $[k(x_i, x_j)]$ is positive semi-definite, which means that for any complex numbers $c_1, \dots, c_N$ we have

$$\sum_{i,j=1}^{N} c_i \bar{c}_j k(x_i, x_j) \ \geq \ 0.$$

Notice that saying that (1.1) is a kernel on $\mathbb{D}$ is equivalent to saying that ϕ is in the (closed) unit ball of $H^\infty(\mathbb{D})$. Indeed, let H^2 be the Hardy space, and

$$k^S(\zeta, \lambda) \ = \ k^S_\lambda(\zeta) \ = \ \frac{1}{1-\bar{\lambda}\zeta} \tag{1.3}$$

be the Szegő kernel on H^2. Let M_ϕ be the operator of multiplication by ϕ. It is straightforward to check that

$$M_\phi^* k_\lambda^S \ = \ \overline{\phi(\lambda)} k_\lambda^S.$$

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We have

$$
\begin{aligned}
\|\phi\| \leq 1 \quad &\Leftrightarrow \quad \|M_\phi\| \leq 1 \\
&\Leftrightarrow \quad I - M_\phi M_\phi^* \ \geq \ 0 \\
&\Leftrightarrow \quad \langle (I - M_\phi M_\phi^*) \sum_i c_i k^S_{\lambda_i}, \sum_j c_j k^S_{\lambda_j} \rangle \ \geq \ 0 \quad \forall\ c_i, \lambda_i \\
&\Leftrightarrow \quad \sum_{i,j} c_i \bar{c}_j \left(1 - \overline{\phi(\lambda_i)}\phi(\lambda_j)\right) \langle k^S_{\lambda_i}, k^S_{\lambda_j} \rangle \ \geq \ 0 \quad \forall\ c_i, \lambda_i \\
&\Leftrightarrow \quad \left[\frac{1 - \overline{\phi(\lambda_i)}\phi(\lambda_j)}{1 - \bar{\lambda}_i \lambda_j} \right] \ \geq \ 0 \qquad \forall\ \lambda_i. \qquad \lhd
\end{aligned}
$$

Given a kernel k on X, it is an important fact that one can always realize it as a Grammian, *i.e.* one can find a Hilbert space $\mathcal{H}$ and a map $u : X \to \mathcal{H}$ so that

$$k(x, y) \ = \ \langle u(x), u(y) \rangle \ := \ \langle u_x, u_y \rangle.$$

So if (1.1) is positive semidefinite, we can write

$$\frac{1 - \overline{\phi(\lambda)}\phi(\zeta)}{1 - \bar{\lambda}\zeta} \ = \ \langle u_\zeta, u_\lambda \rangle_{\mathcal{H}}. \tag{1.4}$$

Now inside (1.4) lurks an isometry. Indeed, define $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$ by

$$V \ : \ \begin{pmatrix} 1 \\ \zeta u_\zeta \end{pmatrix} \ \mapsto \ \begin{pmatrix} \phi(\zeta) \\ u_\zeta \end{pmatrix}.$$

Then equation (1.4) is equivalent to the assertion that V is an isometry on the linear span of vectors of the form

$$\begin{pmatrix} 1 \\ \zeta_i u_{\zeta_i} \end{pmatrix} \qquad \zeta_i \ \in \ \mathbb{D}.$$

If the codimension of the range is at least as large as the codimension of the domain, then V can be extended to an isometry on all of $\mathbb{C} \oplus \mathcal{H}$. If the codimension is smaller, the same effect can be achieved by adding an infinite dimensional summand to $\mathcal{H}$. Thus we have essentially proved the following realization formula; see *e.g.* [**9**] or [**5**] for full details.

THEOREM 1.5. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D})$ if and only if there is a Hilbert space $\mathcal{H}$ and an isometry $V : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$, such that, writing V as*

$$V \ = \ \begin{matrix} & \begin{matrix} \mathbb{C} & \mathcal{H} \end{matrix} \\ \begin{matrix} \mathbb{C} \\ \mathcal{H} \end{matrix} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{matrix}, \tag{1.6}$$

one has

$$\phi(\lambda) \ = \ A + \lambda B (I - \lambda D)^{-1} C. \tag{1.7}$$

This theory was generalized to the bidisk by Jim Agler [**3**]. We shall use superscripts to denote coordinates; so a point λ in $\mathbb{D}^2$ will be written $\lambda = (\lambda^1, \lambda^2)$. In lieu of studying the positive semi-definite form (1.1), Agler proved:

THEOREM 1.8. *Let $\phi : \mathbb{D}^2 \to \mathbb{D}$ be a function. Then ϕ is analytic iff there are kernels Γ and Δ on $\mathbb{D}^2$ so that*

$$1 - \overline{\phi(\mu)}\phi(\lambda) \;=\; (1 - \bar{\mu}^1\lambda^1)\Gamma(\lambda,\mu) \,+\, (1 - \bar{\mu}^2\lambda^2)\Delta(\lambda,\mu). \tag{1.9}$$

The realization formula becomes:

THEOREM 1.10. *The function ϕ is in the closed unit ball of $H^\infty(\mathbb{D}^2)$ if and only if there are auxiliary Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and an isometry*

$$V \,:\, \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \;\to\; \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$$

such that, if $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$, V is written as

$$V \;=\; \begin{array}{c c} & \begin{array}{cc} \mathbb{C} & \mathcal{H} \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathcal{H} \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}, \tag{1.11}$$

and $\mathcal{E}_\lambda = \lambda^1 I_{\mathcal{H}_1} \oplus \lambda^2 I_{\mathcal{H}_2}$, then

$$\phi(\lambda) \;=\; A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C. \tag{1.12}$$

1.1. Proofs.

DEFINITION 1.13. A kernel k on $\mathbb{D}^2$ is called *admissible* if

$$(1 - \zeta^1\bar{\lambda}^1)\, k(\zeta,\lambda) \;\geq\; 0 \tag{1.14}$$

and

$$(1 - \zeta^2\bar{\lambda}^2)\, k(\zeta,\lambda) \;\geq\; 0. \tag{1.15}$$

If k is an admissible kernel, then the operators T_1 and T_2 defined by

$$T_r \,:\, k_\lambda \;\mapsto\; \bar{\lambda}^r k_\lambda, \qquad r = 1, 2$$

are a pair of commuting contractions on $\mathcal{H}(k)$, the Hilbert function space on the bidisk for which k is the reproducing kernel. The adjoints T_1^* and T_2^* are the operators of multiplication by the coordinate functions, and (1.14) and (1.15) are just the statements that $I - T_1^*T_1$ and $I - T_2^*T_2$ are positive — *i.e.* that T_1 and T_2 are contractions.

Suppose g is a self-adjoint function on $\mathbb{D}^2 \times \mathbb{D}^2$ that has the property that its Schur product with every admissible kernel is positive semi-definite (*i.e.*

$$\sum \bar{c}_i c_j \, g(\lambda_i,\lambda_j)\, k(\lambda_i,\lambda_j) \;\geq\; 0$$

for every admissible kernel k and every finite set of points $\{\lambda_i\}$ and scalars $\{c_i\}$). One way this could happen is if there were a representation

$$g(\zeta,\lambda) \;=\; (1 - \zeta^1\bar{\lambda}^1)\Gamma(\zeta,\lambda) \,+\, (1 - \zeta^2\bar{\lambda}^2)\Delta(\zeta,\lambda), \tag{1.16}$$

for some kernels Γ and Δ. Indeed, by the Schur Product Theorem, the Schur product of any admissible kernel with the right-hand side of (1.16) is automatically positive. The following structure theorem says that g having the form of (1.16) is not only sufficient, but also necessary.

THEOREM 1.17. *Let $g : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ be self-adjoint (i.e. $g(\lambda, \zeta) = \overline{g(\zeta, \lambda)}$). Suppose that*

$$g \cdot k : (\zeta, \lambda) \mapsto g(\zeta, \lambda) k(\zeta, \lambda)$$

is positive semi-definite for every admissible kernel k. Then there are positive semi-definite functions Γ and Δ such that

$$g(\zeta, \lambda) = (1 - \zeta^1 \bar{\lambda}^1)\Gamma(\zeta, \lambda) + (1 - \zeta^2 \bar{\lambda}^2)\Delta(\zeta, \lambda). \tag{1.18}$$

For a proof, see [**5**]. (The idea of the proof is to argue by contradiction. If g does not have the desired form, then by the Hahn-Banach theorem one can separate everything on the right-hand-side of (1.18) from g by a linear functional. One uses this to produce an admissible kernel whose Schur product with g is not positive).

PROOFS OF THEOREMS 1.8 AND 1.10.

(Necessity) Suppose ϕ is in the closed unit ball of $H^\infty(\mathbb{D}^2)$, which we shall write as $H_1^\infty(\mathbb{D}^2)$. For simplicity, we shall assume furthermore that ϕ is continuous on the closed bidisk, so it lies in the bidisk algebra $A(\mathbb{D}^2)$. (This restriction can be dropped by using a limiting argument, which we shall omit). Let k be any admissible kernel. The fact that k is admissible means that the operators T_1 and T_2, defined by

$$T_r : k_\lambda \mapsto \bar{\lambda}^r k_\lambda, \qquad r = 1, 2,$$

are commuting contractions on $\mathcal{H}_k$. We want to use Andô's inequality [**8**] to conclude that $\phi(T_1, T_2)$ is a contraction. Andô's inequality, which will be discussed in detail in Section 3, says that if T_1 and T_2 are commuting contractions, and ϕ is in the bidisk algebra $A(\mathbb{D}^2)$, the uniform closure of the polynomials in the supremum norm on the bidisk, then

$$\|\phi(T_1, T_2)\| \leq \|\phi\|_{\mathbb{D}^2}.$$

We must make a technical adjustment: we must work not with ϕ but with $\breve{\phi}$ (we define $\breve{\phi}$ by $\breve{\phi}(\lambda^1, \lambda^2) := \phi(\bar{\lambda}^1, \bar{\lambda}^2)^*$, and so it is also in the closed unit ball of $A(\mathbb{D}^2)$).

Then, by Andô's inequality, $\breve{\phi}(T_1, T_2)$ is a contraction, so for every finite set of points $\{\lambda_i\}$ in $\mathbb{D}^2$ and scalars c_i, we have

$$\begin{aligned} 0 \quad &\leq \quad \Big\langle (I - \breve{\phi}(T_1, T_2)\breve{\phi}(T_1, T_2)^*) \sum_j c_j k_{\lambda_j}, \sum_i c_i k_{\lambda_i} \Big\rangle \\ &= \quad \sum_{i,j} \bar{c}_i c_j (1 - \phi(\lambda_i)\overline{\phi(\lambda_j)}) \langle k_{\lambda_j}, k_{\lambda_i} \rangle. \end{aligned}$$

Therefore $1 - \phi(\zeta)\overline{\phi(\lambda)}$ satisfies the hypotheses in Theorem 1.17, and so there is a representation

$$1 - \phi(\zeta)\overline{\phi(\lambda)} = (1 - \zeta^1 \bar{\lambda}^1)\Gamma(\zeta, \lambda) + (1 - \zeta^2 \bar{\lambda}^2)\Delta(\zeta, \lambda) \tag{1.19}$$

for some kernels Γ and Δ.

These kernels can be represented as

$$\begin{aligned} \Gamma(\zeta, \lambda) &= \langle g_1(\zeta), g_1(\lambda) \rangle_{\mathcal{H}_1} \\ \Delta(\zeta, \lambda) &= \langle g_2(\zeta), g_2(\lambda) \rangle_{\mathcal{H}_2} \end{aligned}$$

for some functions $g_r : \mathbb{D}^2 \to \mathcal{H}_r$ and some auxiliary Hilbert spaces $\mathcal{H}_r$.

Using these representations, (1.19) becomes

$$1 - \phi(\zeta)\overline{\phi(\lambda)} = (1 - \zeta^1 \bar{\lambda}^1)\langle g_1(\zeta), g_1(\lambda) \rangle_{\mathcal{H}_1} + (1 - \zeta^2 \bar{\lambda}^2)\langle g_2(\zeta), g_2(\lambda) \rangle_{\mathcal{H}_2} \tag{1.20}$$

and so

$$\begin{aligned} & 1 + \zeta^1\bar{\lambda}^1\langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} + \zeta^2\bar{\lambda}^2\langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2} \\ & \quad = \phi(\zeta)\overline{\phi(\lambda)} + \langle g_1(\zeta), g_1(\lambda)\rangle_{\mathcal{H}_1} + \langle g_2(\zeta), g_2(\lambda)\rangle_{\mathcal{H}_2}. \end{aligned} \tag{1.21}$$

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and let $g(\lambda) = g_1(\lambda) \oplus g_2(\lambda)$. Then (1.21) says that if V is defined by

$$V : \begin{pmatrix} 1 \\ \mathcal{E}_\lambda g(\lambda) \end{pmatrix} \mapsto \begin{pmatrix} \phi(\lambda) \\ g(\lambda) \end{pmatrix}, \tag{1.22}$$

then V extends linearly to an isometry on the span of these elements, and, adding an infinite-dimensional summand to $\mathcal{H}$ if necessary, can then be extended to an isometry from $\mathbb{C} \oplus \mathcal{H}$ to $\mathbb{C} \oplus \mathcal{H}$. Writing V as in (1.11) and solving for ϕ in (1.22), we get that

$$\phi(\lambda) = A + B\mathcal{E}_\lambda(I_{\mathcal{H}} - D\mathcal{E}_\lambda)^{-1}C,$$

as desired.

(Sufficiency) Suppose ϕ can be written as in (1.12), which we have shown is equivalent to (1.9). By expanding $(I - D\mathcal{E}_\lambda)^{-1}$ in a Neumann series, it is clear that ϕ can be written as a power series that converges in $\mathbb{D}^2$, so is analytic there.

To prove that $\|\phi\|$ is bounded by 1, we use the fact that V is an isometry to get

$$\begin{aligned} & 1 - \phi(\lambda)^*\phi(\lambda) \\ & \quad = I - A^*A - A^*B\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*B^*A \\ & \qquad - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*B^*B\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C \\ & \quad = C^*C + C^*D\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C + C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*D^*C \\ & \qquad - C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\mathcal{E}_\lambda^*(I - D^*D)\mathcal{E}_\lambda(I - D\mathcal{E}_\lambda)^{-1}C \\ & \quad = C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\left[(I - \mathcal{E}_\lambda^*D^*)(I - D\mathcal{E}_\lambda) + (I - \mathcal{E}_\lambda^*D^*)D\mathcal{E}_\lambda\right. \\ & \qquad \left. + \mathcal{E}_\lambda^*D^*(I - D\mathcal{E}_\lambda) - \mathcal{E}_\lambda^*(I - D^*D)\mathcal{E}_\lambda\right](I - D\mathcal{E}_\lambda)^{-1}C \\ & \quad = C^*(I - \mathcal{E}_\lambda^*D^*)^{-1}\left[I - \mathcal{E}_\lambda^*\mathcal{E}_\lambda\right](I - D\mathcal{E}_\lambda)^{-1}C. \qquad (1.23) \end{aligned}$$

The last expression (1.23) is positive when λ is in $\mathbb{D}^2$, so $\|\phi\|$ is bounded by 1 in the bidisk, as desired. □

2. Lecture 2: Interpolation and Interpolating sequences

The Pick problem on the disk is to determine, given N points $\lambda_1, \ldots, \lambda_N$ in $\mathbb{D}$ and N complex numbers $w_1, \ldots, w_N$, whether there exists $\phi \in H_1^\infty(\mathbb{D})$ such that

$$\phi(\lambda_i) = w_i, \qquad i = 1, \ldots, N.$$

G. Pick proved [**19**] that the answer is yes if and only if the N-by-N matrix

$$\left(\frac{1 - w_i\bar{w}_j}{1 - \lambda_i\bar{\lambda}_j}\right) \tag{2.1}$$

is positive semi-definite.

Pick's theorem on the bidisk was proved by J. Agler [**2**].

THEOREM 2.2. *Given points $\lambda_1, \dots, \lambda_N$ in $\mathbb{D}^2$ and complex numbers $w_1, \dots, w_N$, there is a function $\phi \in H_1^\infty(\mathbb{D}^2)$ that maps each λ_i to the corresponding w_i if and only if there are positive semi-definite matrices Γ and Δ such that*

$$1 - w_i \bar{w}_j \;=\; (1 - \lambda_i^1 \bar{\lambda}_j^1)\Gamma_{ij} + (1 - \lambda_i^2 \bar{\lambda}_j^2)\Delta_{ij}. \tag{2.3}$$

Theorem 2.2 can be proved by representing the matrices Γ and Δ as Grammians, as in the transition from (1.19) to (1.20), rearranging the equation as in (1.21), and then introducing the lurking isometry V as in (1.22). Writing this V as in (1.6), the function ϕ from (1.7) can be shown to solve the interpolation problem (and also to be a rational inner function).

Given a sequence $\{\lambda_i\}_{i=1}^\infty$ in the polydisk $\mathbb{D}^d$, we say it is interpolating for $H^\infty(\mathbb{D}^d)$ if, for any bounded sequence $\{w_i\}_{i=1}^\infty$, there is a function ϕ in $H^\infty(\mathbb{D}^d)$ satisfying $\phi(\lambda_i) = w_i$. L. Carleson characterized interpolating sequences on $\mathbb{D}$ in [**12**].

Before stating his theorem, let us introduce some definitions. Given any kernel k on $\mathbb{D}^d$, a sequence $\{\lambda_i\}_{i=1}^\infty$ has an associated Grammian G^k, where

$$[G^k]_{ij} \;=\; \frac{k(\lambda_i, \lambda_j)}{\sqrt{k(\lambda_i, \lambda_i)\, k(\lambda_j, \lambda_j)}}.$$

We think of G^k as an infinite matrix, representing an operator on ℓ^2 (that is not necessarily bounded). When k is the Szegő kernel on $\mathbb{D}^d$,

$$k^S(\zeta, \lambda) \;=\; \frac{1}{(1 - \zeta^1 \bar{\lambda}^1)(1 - \zeta^2 \bar{\lambda}^2) \cdots (1 - \zeta^d \bar{\lambda}^d)}, \tag{2.4}$$

we call the associated Grammian the *Szegő Grammian*. The Szegő kernel is the reproducing kernel for the Hardy space $H^2(\mathbb{D}^d)$.

An analogue on the polydisk of the pseudo-hyperbolic metric is the *Gleason distance*, defined by

$$\rho(\zeta, \lambda) \;:=\; \sup\{|\phi(\zeta)| : \|\phi\|_{H^\infty(\mathbb{D}^d)} \leq 1, \phi(\lambda) = 0\}.$$

We shall call a sequence $\{\lambda_i\}_{i=1}^\infty$ *weakly separated* if there exists $\varepsilon > 0$ such that, for all $i \neq j$, the Gleason distance $\rho(\lambda_i, \lambda_j) \geq \varepsilon$. We call the sequence *strongly separated* if there exists $\varepsilon > 0$ such that, for all i, there is a function ϕ_i in $H_1^\infty(\mathbb{D})$ such that

$$\phi_i(\lambda_j) \;=\; \begin{cases} \varepsilon, & j = i \\ 0, & j \neq i \end{cases}$$

In $\mathbb{D}$, a straightforward argument using Blaschke products shows that a sequence is strongly separated if and only if

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall\ i.$$

We can now state Carleson's theorem. He proved it using function theoretic methods, but later H. Shapiro and A. Shields [**22**] found a Hilbert space approach, which has proved to be more easily generalized, *e.g.* to characterizing interpolating sequences in the multiplier algebra of the Dirichlet space [**18**]. For a unified treatment, see the lovely monograph [**21**] by K. Seip.

THEOREM 2.5. *On the unit disk, the following are equivalent:*

(1) *There exists $\varepsilon > 0$ such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall\ i.$$

(2) *The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is an interpolating sequence for $H^\infty(\mathbb{D})$.*

(3) *The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is weakly separated and the associated Szegő Grammian is a bounded operator on ℓ^2.*

In 1987 B. Berndtsson, S.-Y. Chang and K.-C. Lin proved the following theorem [**10**]:

THEOREM 2.6. *Let $d \geq 2$. Consider the three statements*

(1) *There exists $\varepsilon > 0$ such that*

$$\prod_{j \neq i} \rho(\lambda_i, \lambda_j) \geq \varepsilon \qquad \forall i.$$

(2) *The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is an interpolating sequence for $H^\infty(\mathbb{D}^d)$.*

(3) *The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is weakly separated and the associated Szegő Grammian is a bounded operator on ℓ^2.*

Then (1) *implies* (2) *and* (2) *implies* (3). *Moreover the converses of these implications are false.*

For the following theorem, which was proved in [**4**], let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for ℓ^2. Recall from Definition 1.13 that a kernel k on $\mathbb{D}^2$ is admissible if the function $(1 - \zeta^r \bar{\lambda}^r)k(\zeta, \lambda)$ is positive semidefinite for r equal to 1 and 2.

THEOREM 2.7. *Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{D}^2$. The following are equivalent:*

(i) *$\{\lambda_i\}_{i=1}^{\infty}$ is an interpolating sequence for $H^\infty(\mathbb{D}^2)$.*

(ii) *The following two conditions hold.*

(a) *For all admissible kernels k, their normalized Grammians are uniformly bounded:*

$$G^k \ \leq \ MI$$

for some positive constant M.

(b) *For all admissible kernels k, their normalized Grammians are uniformly bounded below:*

$$NG^k \ \geq \ I$$

for some positive constant N.

(iii) *The sequence $\{\lambda_i\}_{i=1}^{\infty}$ is strongly separated and condition* (a) *alone holds.*

(iv) *Condition* (b) *alone holds.*

Moreover, Condition (a) *is equivalent to both* (a′) *and* (a″)*:*

(a′)*: There exists a constant M and positive semi-definite infinite matrices Γ^1 and Γ^2 such that*

$$M\delta_{ij} - 1 \ = \ \Gamma^1_{ij}(1 - \bar{\lambda}^1_i \lambda^1_j) + \Gamma^2_{ij}(1 - \bar{\lambda}^2_i \lambda^2_j).$$

(a″)*: There exists a function Φ in $H^\infty(\mathbb{D}^2, B(\ell^2, \mathbb{C}))$ of norm at most $\sqrt{M}$ such that $\Phi(\lambda_i)e_i = 1$.*

Condition (b) *is equivalent to both* (b′) *and* (b″)*:*

(b′)*: There exists a constant N and positive semi-definite infinite matrices Δ^1 and Δ^2 such that*

$$N - \delta_{ij} \ = \ \Delta^1_{ij}(1 - \bar{\lambda}^1_i \lambda^1_j) + \Delta^2_{ij}(1 - \bar{\lambda}^2_i \lambda^2_j).$$

(b$''$): *There exists a function* Ψ *in* $H^\infty(\mathbb{D}^2, B(\mathbb{C}, \ell^2))$ *of norm at most* $\sqrt{N}$ *such that* $\Psi(\lambda_i) = e_i$.

Neither Theorem 2.6 nor 2.7 are fully satisfactory. For example, the following is still an unsolved problem:

QUESTION 2.8. If a sequence on $\mathbb{D}^2$ is strongly separated, is it an interpolating sequence?

3. Lecture 3: Distinguished Varieties and Andô's Inequality

Let $\mathbb{E}$ be the exterior of the closed disk, $\mathbb{C} \setminus \overline{\mathbb{D}}$. We call an algebraic set V a distinguished variety if

$$V \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2.$$

Von Neumann's inequality [**24**] says that if T is a contraction (a Hilbert space operator of norm at most one), then for any polynomial p,

$$\|p(T)\| \leq \|p\|_{\mathbb{D}}.$$

Andô's inequality [**8**] is a two-variable analogue. It says that if $T = (T_1, T_2)$ is a pair of commuting contractions, then

$$\|p(T)\| \leq \|p\|_{\mathbb{D}^2}. \tag{3.1}$$

Both von Neumann's and Andô's inequality extend automatically to functions in the norm-closure of the polynomials, *viz.* the disk and bidisk algebras respectively. Provided one sticks to operators for which the H^∞ functional calculus makes sense, the inequalities also extend to H^∞.

In [**6**] it was shown that if T is a pair of commuting contractive *matrices*, then there is a distinguished variety V so that (3.1) can be sharpened to

$$|p(T)\| \leq \|p\|_{V\cap\mathbb{D}^2}.$$

Distinguished varieties turn out to be intimately connected to function theory on $\mathbb{D}^2$.

3.1. Representing Distinguished Varieties. For positive integers m and n, let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C}^m \oplus \mathbb{C}^n \to \mathbb{C}^m \oplus \mathbb{C}^n \tag{3.2}$$

be an $(m+n)$-by-$(m+n)$ unitary matrix. Let

$$\Psi(z) = A + zB(I - zD)^{-1}C \tag{3.3}$$

be the m-by-m matrix valued function defined on the unit disk $\mathbb{D}$ by the entries of U. This is called the *transfer function* of U. Because $U^*U = I$, a calculation (essentially the same as (1.23), but with $\mathcal{E}_\lambda$ replaced by λI) yields

$$I - \Psi(z)^*\Psi(z) = (1 - |z|^2)\, C^*(I - \bar{z}D^*)^{-1}(I - zD)^{-1}C, \tag{3.4}$$

so $\Psi(z)$ is a rational matrix-valued function that is unitary on the unit circle and contractive on the unit disk. Such functions are called rational matrix inner functions, and it is well-known that all rational matrix inner functions have the form (3.3) for some unitary matrix decomposed as in (3.2) — see *e.g.* [**5**] for a proof.

Let V be the set

$$V = \{(z, w) \in \mathbb{D}^2 : \det(\Psi(z) - wI) = 0\}. \tag{3.5}$$

We shall show that V is a distinguished variety, and that every distinguished variety arises this way — Theorem 3.12 below.

LEMMA 3.6. *Let*

$$U' = \begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} : \mathbb{C}^n \oplus \mathbb{C}^m \to \mathbb{C}^n \oplus \mathbb{C}^m,$$

let

$$\Psi'(z) = D^* + zB^*(I - zA^*)^{-1}C^*,$$

and let

$$V' = \{(z,w) \in \mathbb{D}^2 : \det(\Psi'(w) - zI) = 0\}.$$

Then $V = V'$.

PROOF. The point $(z,w) \in \mathbb{D}^2$ is in V iff there is a non-zero vector v_1 in $\mathbb{C}^m$ such that

$$\left[A + zB(1 - zD)^{-1}C\right] v_1 = w v_1. \tag{3.7}$$

Claim: (3.7) holds if and only if there is a non-zero vector v_2 in $\mathbb{C}^n$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ z\, v_2 \end{pmatrix} = \begin{pmatrix} w\, v_1 \\ v_2 \end{pmatrix}. \tag{3.8}$$

PROOF OF CLAIM: If (3.8) holds, then solving gives (3.7). Conversely, if (3.7) holds, define

$$v_2 = (I - zD)^{-1}Cv_1.$$

Then (3.8) holds. Moreover, if v_2 were 0, then v_1 would be in the kernel of C and be a w-eigenvector of A. As $A^*A + C^*C = I$, this would force $|w| = 1$, contradicting the fact that $(z,w) \in \mathbb{D}^2$. $\triangleleft$

Given the claim, the point (z,w) is in V' iff there are non-zero vectors v_1 and v_2 such that

$$\begin{pmatrix} D^* & B^* \\ C^* & A^* \end{pmatrix} \begin{pmatrix} v_2 \\ w\, v_1 \end{pmatrix} = \begin{pmatrix} z\, v_2 \\ v_1 \end{pmatrix}. \tag{3.9}$$

Interchanging coordinates, (3.9) becomes

$$\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} w\, v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ z\, v_2 \end{pmatrix}. \tag{3.10}$$

Clearly, (3.8) and (3.10) are equivalent. □

Note that if C has a non-trivial kernel $\mathcal{N}$, then (3.4) shows that $\Psi(z)$ is isometric on $\mathcal{N}$ for all z, so by the maximum principle is equal to a constant isometry with initial space $\mathcal{N}$. If C has a trivial kernel, we say Ψ is *pure*. Every rational inner function decomposes into the direct sum of a pure rational inner function and a unitary matrix — see *e.g.* [**23**]. Since $A^*A + C^*C = I$, we see that C has no kernel iff $\|A\| < 1$. Since $AA^* + BB^* = I$, this in turn is equivalent to B^* having no kernel. Therefore Ψ is pure iff Ψ' is.

Let V be a distinguished variety. We say a function f is *holomorphic on* V if, for every point of V, there is an open ball B in $\mathbb{C}^2$ containing the point, and a holomorphic function ϕ of two variables on B, such that $\phi|_{B \cap V} = f|_{B \cap V}$. We shall use $A(V)$ to denote the Banach algebra of functions that are holomorphic on V and continuous on $\overline{V}$. This is a uniform algebra on ∂V, *i.e.* a closed unital subalgebra of $C(\partial V)$ that separates points. The maximal ideal space of $A(V)$ is $\overline{V}$.

If μ is a finite measure on a distinguished variety V, let $H^2(\mu)$ denote the closure in $L^2(\mu)$ of the polynomials. We say a point λ is a bounded point evaluation for $H^2(\mu)$ if evaluation at λ, *a priori* defined only for a dense set of analytic functions, extends continuously to the whole Hilbert space. If λ is a bounded point evaluation, we call the function k_λ that has the property that

$$\langle f, k_\lambda \rangle \;=\; f(\lambda)$$

the evaluation functional at λ.

For the proof of the following lemma, see [**6**].

LEMMA 3.11. *Let V be a distinguished variety. There is a measure μ on ∂V such that every point in V is a bounded point evaluation for $H^2(\mu)$, and such that the span of the evaluation functionals is dense in $H^2(\mu)$.*

THEOREM 3.12. *The set V, defined by (3.5) for some rational matrix inner function Ψ, is a distinguished variety. Moreover, every distinguished variety can be represented in this form.*

PROOF. Suppose V is given by (3.5), and that (z, w) is in $\overline{V}$. Without loss of generality, we can assume that Ψ is pure. Indeed, any unitary summand of Ψ would add sheets to the variety $\det(\Psi(z) - wI) = 0$ of the type $\mathbb{C} \times \{w_0\}$, for some unimodular w_0. These sheets are all disjoint from the open bidisk $\mathbb{D}^2$.

If $|z| < 1$, equation (3.4) then shows that $\Psi(z)$ is a strict contraction, so all its eigenvalues must have modulus less than 1, and so $|w| < 1$ also. To prove that $|w| < 1$ implies $|z| < 1$, just apply the same argument to V'. Therefore V is a distinguished variety.

To prove that all distinguished varieties arise in this way, let V be a distinguished variety. Let μ be the measure from Lemma 3.11, and let $H^2(\mu)$ be the closure of the polynomials in $L^2(\mu)$. The set of bounded point evaluations for $H^2(\mu)$ is precisely V. (It cannot be larger, because $\overline{V}$ is polynomially convex, and Lemma 3.11 ensures that it is not smaller).

Let $T = (T_1, T_2)$ be the pair of operators on $H^2(\mu)$ given by multiplication by the coordinate functions. They are pure commuting isometries[1] because the span of the evaluation functionals is dense. The joint eigenfunctions of their adjoints are the evaluation functionals.

By the Sz.-Nagy-Foiaş model theory [**23**], T_1 can be modelled as M_z, multiplication by the independent variable z on $H^2 \otimes \mathbb{C}^m$, a vector-valued Hardy space on the unit circle. In this model, T_2 can be modelled as M_Ψ, multiplication by $\Psi(z)$ for some pure rational matrix inner function Ψ. A point (z, w) in $\mathbb{D}^2$ is a bounded point evaluation for $H^2(\mu)$ iff $(\bar{z}, \bar{w})$ is a joint eigenvalue for (T_1^*, T_2^*). In terms of the unitarily equivalent Sz.-Nagy-Foiaş model, this is equivalent to $\bar{w}$ being an eigenvalue of $\Psi(z)^*$.

Therefore

$$V = \{(z, w) \in \mathbb{D}^2 \;:\; \det(\Psi(z) - wI) = 0\},$$

as desired. □

[1] A pure isometry S is one that has no unitary summand; this is the same as requiring that $\cap_{i=1}^{\infty} \mathrm{ran}(S^i) = \{0\}$.

G. Kneses gives a more constructive proof of Theorem 3.12 in [**17**].

If Ψ is the transfer function of a unitary U as in (3.2), and Ψ is pure, we shall say that V is of *rank* (m, n). This means that generically there are m sheets above each z, and n sheets above each w.

3.2. A sharpening of Andô's inequality.

THEOREM 3.13. *Let T_1 and T_2 be commuting contractive matrices, neither of which has eigenvalues of modulus 1. Then there is a distinguished variety V such that, for any polynomial p in two variables, the inequality*

$$\|p(T_1, T_2)\| \;\leq\; \|p\|_V \tag{3.14}$$

holds.

PROOF. Let the dimension of the space on which the matrices act be N.

(i) First, let us assume that each T_r has N linearly independent unit eigenvectors, $\{v_j\}_{j=1}^N$. So we have

$$T_r v_j \;=\; \lambda_j^r v_j, \qquad r = 1, 2 \quad 1 \leq j \leq N,$$

for some set of scalars $\{\lambda_j^r\}$. As each T_r is a contraction, we have $I - T_r^* T_r$ is positive semidefinite, so

$$\langle (I - T_r^* T_r) v_j, v_i \rangle \;=\; (1 - \overline{\lambda_i^r} \lambda_j^r) \langle v_j, v_i \rangle \;\geq\; 0. \tag{3.15}$$

As the matrix in (3.15) is positive semidefinite, it can be represented as the Grammian of vectors u_j^r, which can be chosen to lie in a Hilbert space of dimension d_r equal to the defect of T_r (the defect of T_r is the rank of $I - T_r^* T_r$). So we have

$$(1 - \overline{\lambda_i^1} \lambda_j^1) \langle v_j, v_i \rangle \quad = \quad \langle u_j^1, u_i^1 \rangle \tag{3.16}$$

$$(1 - \overline{\lambda_i^2} \lambda_j^2) \langle v_j, v_i \rangle \quad = \quad \langle u_j^2, u_i^2 \rangle. \tag{3.17}$$

Multiplying the first equation by $(1 - \overline{\lambda_i^2} \lambda_j^2)$ and the second equation by $(1 - \overline{\lambda_i^1} \lambda_j^1)$, we see that they are equal. Therefore

$$(1 - \overline{\lambda_i^1} \lambda_j^1) \langle u_j^2, u_i^2 \rangle \;=\; (1 - \overline{\lambda_i^2} \lambda_j^2) \langle u_j^1, u_i^1 \rangle. \tag{3.18}$$

Reordering equation (3.18), we get

$$\langle u_j^1, u_i^1 \rangle \;+\; \overline{\lambda_i^1} \lambda_j^1 \langle u_j^2, u_i^2 \rangle \;=\; \langle u_j^2, u_i^2 \rangle \;+\; \overline{\lambda_i^2} \lambda_j^2 \langle u_j^1, u_i^1 \rangle. \tag{3.19}$$

Equation 3.19 says that there is some unitary matrix

$$U \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;:\; \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \to \mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2} \tag{3.20}$$

such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u_j^1 \\ \lambda_j^1 \, u_j^2 \end{pmatrix} \;=\; \begin{pmatrix} \lambda_j^2 u_j^1 \\ u_j^2 \end{pmatrix}. \tag{3.21}$$

If the linear span of the vectors $u_j^1 \oplus \lambda_j^1 u_j^2$ is not all of $\mathbb{C}^{d_1} \oplus \mathbb{C}^{d_2}$, then U will not be unique. In this event, we just choose one such U. Define the $d_1 \times d_1$ matrix-valued analytic function Ψ by

$$\Psi(z) \;=\; A + zB(1 - zD)^{-1} C. \tag{3.22}$$

For any function Θ of two variables, scalar or matrix-valued, define

$$\Theta^{\cup}(Z, W) \;:=\; [\Theta(Z^*, W^*)]^*.$$

Let $\Phi = \Psi^{\cup}$, so

$$\Phi(z) \;=\; A^* + zC^*(1 - zD^*)^{-1}B^*.$$

Equation 3.21 implies that

$$\Psi(\lambda_j^1)u_j^1 \;=\; \left[\Phi(\overline{\lambda_j^1})\right]^* \; u_j^1 \;=\; \lambda_j^2 u_j^1. \tag{3.23}$$

Let s be the Szegő kernel in the Hardy space H^2 of the unit disk (which we called k^S in (1.3)), so

$$s_\lambda(z) \;=\; \frac{1}{1 - \overline{\lambda} z}. \tag{3.24}$$

Let k_j be the vector in $H^2 \otimes \mathbb{C}^{d_1}$ given by

$$k_j \;:=\; s_{\overline{\lambda_j^1}} \otimes u_j^1.$$

Consider the pair of isometries (M_z, M_Φ) on $H^2 \otimes \mathbb{C}^{d_1}$, where M_z is multiplication by the coordinate function (times the identity matrix on $\mathbb{C}^{d_1}$) and M_Φ is multiplication by the matrix function Φ. Then

$$\begin{aligned} M_z^* &\;:\; k_j \mapsto \lambda_j^1 k_j \\ M_\Phi^* &\;:\; k_j \mapsto \lambda_j^2 k_j. \end{aligned}$$

Therefore the map that sends each v_j to k_j gives a unitary equivalence between (T_1, T_2) and the pair (M_z^*, M_Φ^*) restricted to the span of the vectors $\{k_j\}_{j=1}^N$. Therefore the pair (M_z^*, M_Φ^*), acting on the full space $H^2 \otimes \mathbb{C}^{d_1}$, is a co-isometric extension of (T_1, T_2).

Let p be any polynomial (scalar or matrix valued) in two variables. We have

$$\begin{aligned} \|p(T_1, T_2)\| &= \|p(M_z^*, M_\Phi^*)|_{\vee\{k_j\}}\| \\ &\le \|p(M_z^*, M_\Phi^*)\|_{H^2 \otimes \mathbb{C}^{d_1}} \\ &= \|p^{\cup}(M_z, M_\Phi)\|_{H^2 \otimes \mathbb{C}^{d_1}} \\ &\le \|p^{\cup}(M_z, M_\Phi)\|_{L^2 \otimes \mathbb{C}^{d_1}} \\ &= \|p^{\cup}\|_{\partial V^{\cup}} \end{aligned} \tag{3.25}$$

where $V^{\cup}$ and V are the sets

$$\begin{aligned} V^{\cup} &= \{(z, w) \in \mathbb{D}^2 \;:\; \det(\Phi(z) - wI) = 0\} \\ V &= \{(z, w) \in \mathbb{D}^2 \;:\; \det(\Psi(z) - wI) = 0\}. \end{aligned} \tag{3.26}$$

Equality (3.25) follows from the observation that

$$\|p^{\cup}(M_z, M_\Phi)\|_{L^2 \otimes \mathbb{C}^{d_1}} \;=\; \sup_\theta \|p^{\cup}(e^{i\theta} I, \Phi(e^{i\theta}))\|, \tag{3.27}$$

where the norm on the right is the operator norm on the $d_1 \times d_1$ matrices. Equation (3.4) shows that, except possibly for the finite set $\sigma(D) \cap \mathbb{T}$, the matrix $\Phi(e^{i\theta})$ is unitary, and so the norm of any polynomial applied to $\Phi(e^{i\theta})$ is just the maximum value of the norm of the polynomial on the spectrum of $\Phi(e^{i\theta})$. By continuity, we obtain (3.25). Taking complex conjugates, (3.25) gives

$$\|p(T_1, T_2)\| \;\le\; \|p\|_V,$$

the desired inequality.

By Theorem 3.12, we see that V and $V^{\cup}$ are distinguished varieties, and by construction, V contains the points $\{(\lambda_j^1, \lambda_j^2) : 1 \le j \le N\}$.

(ii) Now, we drop the assumption that $T = (T_1, T_2)$ be diagonizable. J. Holbrook proved that the set of diagonizable commuting matrices is dense in the set of all commuting matrices [**14**]. So we can assume that there is a sequence $T^{(n)} = (T_1^{(n)}, T_2^{(n)})$ of commuting matrices that converges to T in norm and such that each pair satisfies the hypotheses of (i), *i.e.* each $T^{(n)}$ is a pair of commuting contractions that have N linearly independent eigenvectors and no unimodular eigenvalues. Each $T^{(n)}$ has a unitary U_n associated to it as in (3.20). By passing to a subsequence if necessary, we can assume that the defects d_1 and d_2 are constant, and that the matrices U_n converge to a unitary U. The corresponding functions Ψ_n from (3.22) will converge to some function Ψ. Let $q_n(z,w) = \det(\Psi_n(z) - wI)$, and $q(z,w) = \det(\Psi(z) - wI)$. Let V be defined by (3.26) for this Ψ, and V_n be the variety corresponding to Ψ_n. Notice that the degrees of q_n are uniformly bounded.

Claim: V is non-empty.

Indeed, otherwise it would contain no points of the form $(0, w)$ for $w \in \mathbb{D}$. That would mean that $\sigma(A) \subseteq \mathbb{T}$, and so B and C would be zero. That in turn would mean that the submatrices A_n in U_n would have all their eigenvalues tending to $\mathbb{T}$, and hence by (3.21), the eigenvalues of $T_2^{(n)}$ would all tend to $\mathbb{T}$. Therefore T_2 would have a unimodular eigenvalue, contradicting the hypotheses.

Claim: V is a distinguished variety.

This follows from Theorem 3.12.

Claim: Inequality (3.14) holds.

This follows from continuity. Indeed, fix some polynomial p. For every $\varepsilon > 0$, for every $n \geq n(\varepsilon)$, we have

$$\begin{aligned} \|p(T)\| &\leq \varepsilon + \|p(T^{(n)})\| \\ &\leq \varepsilon + \|p\|_{V_n}. \end{aligned}$$

We wish to show that

$$\lim_{n\to\infty} \|p\|_{V_n} \leq \|p\|_V.$$

Suppose not. Then there is some sequence (z_n, w_n) in V_n such that

$$|p(z_n, w_n)| \geq \|p\|_V + \varepsilon \tag{3.28}$$

for some $\varepsilon > 0$. Moreover, we can assume that (z_n, w_n) converges to some point (z_0, w_0) in $\overline{\mathbb{D}^2}$. The point (z_0, w_0) is in the zero set of q, so if it were in $\mathbb{D}^2$, then it would be in V. Otherwise, (z_0, w_0) must be in $\mathbb{T}^2$. To ensure that (z_0, w_0) is in $\overline{V}$, we must rule out the possibility that some sheet of the zero set of q just grazes the boundary of $\mathbb{D}^2$ without ever coming inside.

But this cannot happen. For every z in $\mathbb{D}$, there are d_1 roots of $\det(\Psi(z) - wI) = 0$, and *all* of these occur in $\mathbb{D}$. So as z tends to z_0 from inside $\mathbb{D}$, one of the d_1 branches of w must tend to w_0 from inside the disk too. Therefore (z_0, w_0) is in the closure of V, and (3.28) cannot happen. ☐

Remark 1. Once one knows Andô's inequality for matrices, then it follows for all commuting contractions by approximating them by matrices — see [**13**] for an explicit construction. Of course, the set V must be replaced by the limit points of the sets that occur at each stage of the approximation, and in general this may be the whole bidisk.

Remark 2. In the proof, we actually constructed a co-isometric extension of T that is localized to V, and a unitary dilation of T with spectrum contained in ∂V.

4. Lecture 4: Angular derivatives

The following theorem, called the Julia-Carathéodory theorem, was originally proved by G. Julia [**16**] and C. Carathéodory [**11**].

THEOREM 4.1. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Let τ be a point on the unit circle $\mathbb{T}$. The following conditions are equivalent:*

(A) *there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}$ tending to τ such that*

$$\frac{1 - |\phi(\lambda_n)|}{1 - |\lambda_n|}$$

is bounded;

(B) *for every sequence $\{\lambda_n\}$ tending to τ nontangentially,* (4.1) *is bounded;*

(C) *there exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}$ such that*

$$\lim_{\lambda \overset{\mathrm{nt}}{\to} \tau} \frac{|\phi(\lambda) - \omega - \eta(\lambda - \tau)|}{|\lambda - \tau|} \ = \ 0; \tag{4.2}$$

(D) *there exist $\omega \in \mathbb{T}$ and $\eta \in \mathbb{C}$ such that $\phi(\lambda) \to \omega$ and $\phi'(\lambda) \to \eta$ as $\lambda \to \tau$ nontangentially.*

In two variables, there are natural analogues of conditions (A) - (D). K. Wlodarczyk [**25**], F. Jafari [**15**] and M. Abate [**1**] obtained generalizations of Theorem 4.1, showing that (A) implies (B) (this is Theorem 4.7 below) and (B) does not imply (C). In [**7**], it was shown that on the bidisk (C) and (D) are equivalent (where derivatives are replaced by gradients, and in the numerator of (4.2) η becomes a 2-vector whose scalar product is taken with the 2-vector $\lambda - \tau$).

4.1. Non-tangential Approach. If $\{\lambda_n\}$ is a sequence in $\mathbb{D}$ and $\tau \in \mathbb{T}$, we say that λ_n *approaches* τ *nontangentially* if λ_n tends to τ and there exists a constant c such that, for all n,

$$|\tau - \lambda_n| \leq c(1 - |\lambda_n|).$$

We shall make use of a similar notion for the bidisk: if $\{\lambda_n\}$ is a sequence in $\mathbb{D}^2$ and $\tau \in \mathbb{T}^2$, we say that λ_n *approaches* τ *nontangentially* if λ_n tends to τ and there exists a constant c such that, for all n,

$$||\tau - \lambda_n|| \leq c(1 - ||\lambda_n||). \tag{4.3}$$

We write $\lambda_n \overset{\mathrm{nt}}{\to} \tau$. Here and throughout the section $||\cdot||$ on $\mathbb{C}^2$ denotes the ℓ^∞ norm:

$$||\lambda|| = \max\{|\lambda^1|, |\lambda^2|\}.$$

We say that a set S in $\mathbb{D}^2$ approaches a point τ on the torus non-tangentially if τ is in the closure of S and there exists a constant c such that, for all $\lambda \in S$,

$$||\tau - \lambda|| \leq c(1 - ||\lambda||).$$

4.2. Results for functions on $\mathbb{D}^2$.

DEFINITION 4.4. Let $\phi \in H_1^\infty(\mathbb{D}^2)$ and let $\tau \in \mathbb{T}^2$. We say that τ is a *B-point* for ϕ if there exists a sequence $\{\lambda_n\}$ in $\mathbb{D}^2$ such that

$$\lambda_n \to \tau \text{ and} \tag{4.5}$$

$$\frac{1 - |\phi(\lambda_n)|}{1 - ||\lambda_n||} \text{ is bounded.} \tag{4.6}$$

THEOREM 4.7. *Let ϕ be in $H_1^\infty(\mathbb{D}^2)$. The following are equivalent:*
(A) *the point τ in $\mathbb{T}^2$ is a B-point for ϕ;*
(B) *for every sequence $\{\lambda_n\}$ in $\mathbb{D}^2$ that converges* nt *to τ the statement* (4.6) holds.

When (A) *and* (B) *are satisfied there exists $\omega \in \mathbb{T}$ such that $\phi(\lambda) \to \omega$ as $\lambda_n \overset{\text{nt}}{\to} \tau$.*

There are various ways in which ϕ can have a form of one-sided differentiability at a boundary point. One is for the directional derivative of ϕ at τ in the direction $-\tau\delta$,

$$D_{-\tau\delta}\phi(\tau) = \lim_{t \to 0+} \frac{\phi(\tau - t\tau\delta) - \phi(\tau)}{t} \tag{4.8}$$

to exist whenever δ^1 and δ^2 are in the right half-plane $\mathbb{H}$ (for then $\tau(1 - t\delta) \in \mathbb{D}^2$ for small $t > 0$ and the right-hand side of (4.8) makes sense).

Consider the function

$$\psi(\lambda) = \frac{\frac{1}{2}\lambda^1 + \frac{1}{2}\lambda^2 - \lambda^1\lambda^2}{1 - \frac{1}{2}\lambda^1 - \frac{1}{2}\lambda^2}. \tag{4.9}$$

The point $\tau = (1,1)$ is a B-point for ψ, and the nontangential limit there is 1. For every $\delta \in \mathcal{H}$, the directional derivative $D_{-\delta}\psi(1,1)$ exists and

$$D_{-\delta}\psi(1,1) = -\frac{2\,\delta^1\delta^2}{\delta^1 + \delta^2}. \tag{4.10}$$

Notice that the right-hand side of (4.10) is not linear in δ, but is analytic. For a function holomorphic at τ the directional derivative is of course linear in the direction, and so ψ is not regular at $(1,1)$.

(4.10) is typical of behavior at a B-point. In particular, we have:

THEOREM 4.11. *Let τ be a B-point of $\phi \in H_1^\infty(\mathbb{D}^2)$. For any $\delta \in \mathbb{H}^2$ the directional derivative $D_{-\tau\delta}\phi(\tau)$ exists and is an analytic function of δ.*

We say that ϕ *has a holomorphic differential on S at τ* if $S \subset \mathbb{D}^2$, the closure of S contains τ and there exist $\omega, \eta^1, \eta^2 \in \mathbb{C}$ such that, for $\lambda \in S$,

$$\phi(\lambda) = \omega + \eta^1(\lambda^1 - \tau^1) + \eta^2(\lambda^2 - \tau^2) + e(\lambda) \tag{4.12}$$

where

$$\lim_{\lambda \to \tau,\ \lambda \in S} \frac{e(\lambda)}{||\lambda - \tau||} = 0.$$

We say that $\tau \in \mathbb{T}^2$ is a *C-point for ϕ* if, for every set S that approaches τ nontangentially, ϕ has a holomorphic differential on S and ω in (4.12) is unimodular.

It is clear that, when τ is a C-point for ϕ, the quantities ω, η^1, η^2 in equation (4.12) are the same for every nontangential approach region S, and so we may define the *angular gradient* $\nabla\phi(\tau)$ of ϕ at τ to be the vector $\left(\eta^1\ \eta^2\right)^t$.

If τ is a C-point of ϕ then the directional derivative $D_{-\tau\delta}\phi(\tau)$ exists for $\delta \in \mathcal{H}$ and

$$D_{-\tau\delta}\phi(\tau) = \delta \cdot \nabla\phi(\tau).$$

Every C-point is a B-point, and in one variable Theorem 4.1 states that the two notions are equivalent. However, the function ψ of equation (4.9) shows that, for functions of two variables, *not every B-point is a C-point*: the relation (4.12) fails to hold for $\phi = \psi$ and $\tau = (1,1)$. Nonetheless, we still have equivalence of the two-variable analogues of conditions (C) and (D) from Theorem 4.1:

THEOREM 4.13. *Let $\tau \in \mathbb{T}^2$ be a C-point for $\phi \in H_1^\infty(\mathbb{D}^2)$. Then*

$$\lim_{\lambda \overset{\text{nt}}{\to} \tau} \nabla\phi(\lambda) \ = \ \nabla\phi(\tau).$$

Points at which ϕ is regular are of course C-points, and the assertion of the theorem is trivial for such C-points, but there are examples of functions in $H_1^\infty(\mathbb{D}^2)$ that have singular C-points. One example is the rational inner function

$$\phi(\lambda) = \frac{-4\lambda^1(\lambda^2)^2 + (\lambda^2)^2 + 3\lambda^1\lambda^2 - \lambda^1 + \lambda^2}{(\lambda^2)^2 - \lambda^1\lambda^2 - \lambda^1 - 3\lambda^2 + 4},$$

which has a C-point at $(1,1)$, despite being singular there (ϕ cannot be extended continuously to $\mathbb{D}^2 \cup \{(1,1)\}$).

Proofs of all the results in this section can be found in [**7**]. The proofs rely very heavily on modelling functions as in (1.20).

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