Pseudospectra of elements of reduced Banach algebras

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Let A be a Banach algebra with identity 1. For $\epsilon > 0$, the ϵ-pseudospectrum of an element $a \in A$ is defined as $\Lambda_\epsilon(A, a) = \sigma(A, a) \cup \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\}$, where $\sigma(A, a)$ denotes the spectrum of a in A.

Let $p \in A$ be a non-trivial idempotent. Let $q = 1 - p$. Then pAp and qAq are Banach algebras with identities p and q respectively. These algebras are called reduced Banach algebras. Suppose $a \in A$ such that $ap = pa$. We consider the relationship between the pseudospectrum of $a \in A$ and the pseudospectra of $pap \in pAp$ and $qAq \in qAq$. We show the following:

1. $\sigma(A, a) = \sigma(pAp, pap) \cup \sigma(qAq, qaq)$.
2. For $\epsilon > 0, \Lambda_{\frac{\epsilon}{2}}(A, a) \subseteq \Lambda_\epsilon(pAp, pap) \cup \Lambda_\epsilon(qAq, qaq) \subseteq \Lambda_{\max\{\|p\|, \|q\|\}\epsilon}(A, a)$. In particular, if $\|p\| = \|q\| = 1, \Lambda_{\frac{\epsilon}{2}}(A, a) \subseteq \Lambda_\epsilon(pAp, pap) \cup \Lambda_\epsilon(qAq, qaq) \subseteq \Lambda_\epsilon(A, a)$.
3. If $A = B(H)$ for a Hilbert space H, and $p \in A$ is an orthogonal projection ($p = p^* = p^2$), then $\Lambda_\epsilon(A, a) = \Lambda_\epsilon(pAp, pap) \cup \Lambda_\epsilon(qAq, qaq)$.
4. If $a \in A$ is of G_1 class, i.e., $\|(\lambda - a)^{-1}\| = \frac{1}{d(\lambda, \sigma(A, a))} \, \forall \lambda \notin \sigma(A, a)$, and $\|p\| = \|q\| = 1$, then $\Lambda_\epsilon(A, a) = \Lambda_\epsilon(pAp, pap) \cup \Lambda_\epsilon(qAq, qaq)$.
5. Suppose $p_1, \cdots, p_n$ are idempotents in A such that $ap_i = p_i a \, \forall i$ and $\sum_{i=1}^n p_i = 1$. Then $\sigma(A, a) = \cup_{i=1}^n \sigma(p_i A p_i, p_i a p_i)$ and $\Lambda_{\frac{\epsilon}{n}}(A, a) \subseteq \cup_{i=1}^n \Lambda_\epsilon(p_i A p_i, p_i a p_i) \subseteq \Lambda_{\max\limits_i\{\|p_i\|\}\epsilon}(A, a)$.

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