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Electron in a Gravitational Field

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An Electron in a Gravitational Field

Airy

A. The Problem

Suppose that an electron is in a one-dimensional box in which there is a gravitational field. If z measures the distance from the bottom of the box and $p = m\,\dot{z}$, this is a system with Hamiltonian

$$H = \frac{p^2}{2m} + m\,g\,z \tag{1}$$

The energy eigenstates of this system are solutions to

$$\begin{cases} -\frac{\hbar^2}{2m}\,\psi''(z) + m\,g\,z\,\psi(z) = E\,\psi(z) \\ \psi(0) = 0 \\ \psi(z) \to 0 \text{ as } z \to \infty \end{cases}$$

B. The momentum representation

Let us represent these states as a superposition of momentum eigenstates (we'll, as a superposition of plane waves)

hunting for amplitude of plane waves

Dirac Eq

$$\psi(z) = \int_C dp\, w(p)\, e^{i\,p\,z/\hbar} \tag{2}$$

Hilbert Space

Forms a basis set

This changes the problem from determining $\psi(z)$ to determining the amplitude $w(p)$ of the plane waves in the superposition. These are fixed by

$$0 = (\hat{H} - E)\psi(z) = \int_C dp\, w(p)\, e^{i\,p\,z/\hbar}\left[\frac{p^2}{2m} + m\,g\,z - E\right] \tag{3}$$

C. An operator that reproduces z

The factor z in (3) can be introduced by differentiating with respect to p!

$$0 = \int_C dp\, w(p)\left[\frac{p^2}{2m} + m\,g\left(\frac{\hbar}{i}\frac{\partial}{\partial p}\right) - E\right] e^{i\,p\,z/\hbar} \tag{4}$$

Debroglie Wave Length

Amplitude

Intgegrating by parts on the derivative with respect to p gives

$$0 = \frac{m\,g\,\hbar}{i}\,w(p)\,e^{i\,p\,z/\hbar}\Big|_C + \int_C dp\,\left\{\left[\frac{p^2}{2m} - E\right]\,w(p) - \frac{m\,g\,\hbar}{i}\,w'(p)]\right\}\,e^{i\,p\,z/\hbar} \qquad (5)$$

The term in $\{\cdots\}$ is just

$$(\hat{H} - E)\,w(p)$$

operating on amplitude of z component

in the momentum representation.

D. Solution for $w(p)$

If we set the term in $\{\cdots\}$ to zero, we have a simple differential equation for $w(p)$. The solution is

$$w(p) = c\,e^{\frac{i\,p}{m\,g\,\hbar}\left(\frac{p^2}{6m} - E\right)} \qquad (6)$$

with c a constant of integration.

With this choice for $w(p)$, (5) reduces to

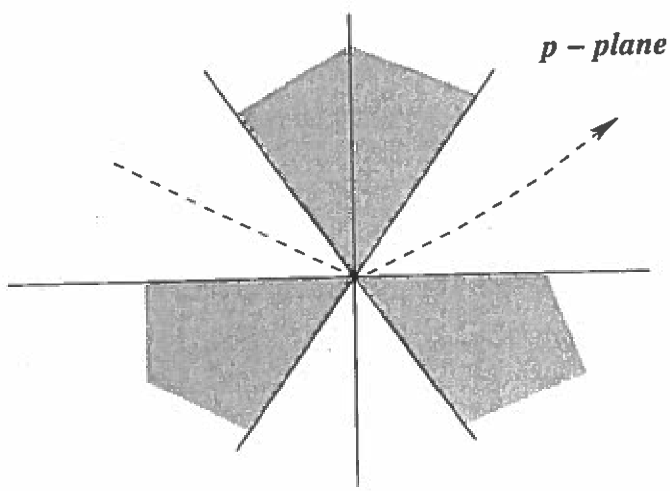
$$0 = c\,\frac{m\,g\,\hbar}{i}\,e^{\frac{i\,p}{m\,g\,\hbar}\left[\frac{p^2}{6m} + m\,g\,z - E\right]}\Bigg|_C \qquad (7)$$

which we'll make zero by choosing a particular contour C.

At large p, this term is dominated by

$$e^{\frac{i\,p^3}{6m^2\,g\,\hbar}}$$

which goes to zero as $|p| \to \infty$ in the unshaded sectors of the figure. Thus Eq(7) will be satisfied on the contour that follows the dotted line in the figure.



Dropping the c, this argument leads to

$$\psi(z) = \int_{-\infty}^{0} dp\, e^{\frac{ipe^{i\pi/6}}{mg\hbar}\left(\frac{p^2 e^{i\pi/3}}{6m} - E + mgz\right)} + \int_{0}^{\infty} dp\, e^{\frac{-ipe^{i\pi/6}}{mg\hbar}\left(\frac{p^2 e^{i\pi/3}}{6m} - E + mgz\right)} \tag{8}$$

As the first integral in (8) is the complex conjugate of the second, (8) gives an imaginary function of z. Adding a phase factor to make $\psi(z)$ real, we can identify

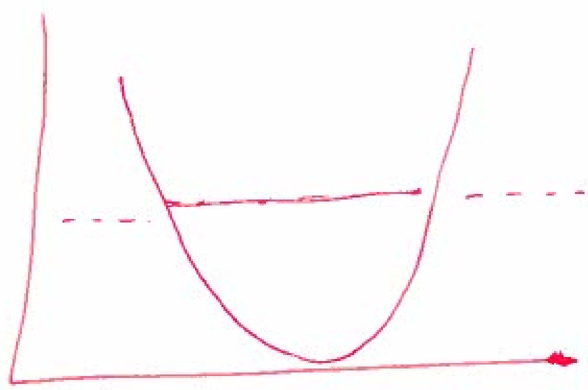
$$\psi(z) = \Im\left\{\int_{0}^{\infty} dp\, e^{\frac{-ipe^{i\pi/6}}{mg\hbar}\left(\frac{p^2 e^{i\pi/3}}{6m} - E + mgz\right)}\right\} \tag{9}$$

E. An approximate evaluation of (9) Asymtotic Expansion

Eigler

Ammonia maser

microwave amplification
by stimulated emission
of radiation.



Problem Set #3

1. Show that, if an electron is started in a single energy eigenstate, then the statistical distribution on finding the electron in space does not change in time. We call such states *stationary states.*

2. Suppose $\psi_0(x)$ is the (real) ground state of a one-dimensional system and $\psi_1(x)$ is the (real) first excited state. Suppose the corresponding energy eigenvalues are E_0 and E_1. If

$$\psi(x,0) = \frac{1}{\sqrt{2}}[\psi_0(x) + \psi_1(x)],$$

what is $\psi(x,t)$? Give an expression for $|\psi(x,t)|^2$ that involves only real quantities. Show that the time evolution is periodic in time with period

$$T = \frac{h}{E_1 - E_0} = \frac{1}{\text{the Einstein frequency}}.$$

3. As a numerical example, consider the $\psi_0(x), \psi_1(x)$ energy eigenstates of the double well problem (Problem Set#1). Make a list of 20 plots showing the evolution of the probability distribution over one period.

> lst=Table[Plot[$f[x,t]$, $\{x, x_{min}, x_{max}\}$, PlotRange -> $\{0, 0.55\}$], $\{t, 0, 19\}$] will generate a list of plots. The option PlotRange -> { , } forces *Mathematica* to use the same vertical scale in all the plots.

An *animated* image of the time evolution can be created by displaying these plots sequentially in time.

> If you select *all* the plots (click with the mouse on the bracket at the right hand side of the window that groups the set of plots) and choose "Animate Selected Graphics" in the Graph menu, *Mathematica* cycles through the plots automatically. "Command <" slows down the animation. Clicking anywhere with the mouse stops the animation.

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An *animated* image of the time evolution can be created by displaying these plots sequentially in time. *Mathematica* has a command *Animate* that will prepare the list of plots and then cycle through, displayinig them sequentially in time:

> In[]:= Animate[Plot[$f[x,t]$, $\{x, x_{min}, x_{max}\}$, PlotRange -> $\{0, 0.55\}$], $\{t, 0, 19\}$]
> The option PlotRange -> { , } forces *Mathematica* to use the same vertical scale in all the plots. The list of plots is prescribed with the standard list generating rule, $\{t, 0, 19\}$.

Make an animated image of (the time evolution of) $|\psi(x,t)|^2$.