2005

The Jacobian conjecture: ideal membership questions and recent advances

David Wright
Washington University in St Louis

Follow this and additional works at: https://openscholarship.wustl.edu/math_facpubs

Recommended Citation
https://openscholarship.wustl.edu/math_facpubs/6

This Article is brought to you for free and open access by the Mathematics and Statistics at Washington University Open Scholarship. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of Washington University Open Scholarship. For more information, please contact digital@wumail.wustl.edu.
The Jacobian Conjecture: ideal membership questions and recent advances

David Wright

CONTENTS

1. The Jacobian Conjecture
2. Ideals Defining the Jacobian Condition
3. Formulas for the Formal Inverse
4. Ideal Membership Results
References

1. The Jacobian Conjecture

1.1. The General Assertion. The Jacobian Conjecture can be stated as follows:

CONJECTURE 1.1 (JC). For any integer \( n \geq 1 \) and polynomials \( F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n] \), the polynomial map \( F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n \) is an automorphism if the determinant \(|JF|\) of the Jacobian matrix \( J = (D_iF_j) \) is a nonzero constant.

Here and throughout this paper we write \( D_i \) for \( \partial / \partial X_i \). We will continue to write \( JF \) for the Jacobian matrix of a polynomial map \( F \), and the determinant of this matrix will be denoted by \(|JF|\).

1.2. Specific Assertions for Fixed Degree and Dimension. A number of reductions and partial solutions of the problem lead us to formulate the following more specific statements. Note that under the hypothesis of each of these conjectures the condition \(|JF|\) is a nonzero constant" is equivalent to \(|JF| = 1\) (This can be seen by evaluating at the origin). The following definitions will be useful in stating the conjectures in the section.

DEFINITION 1.2. By the degree of a polynomial map \( F = (F_1, \ldots, F_n) \) we mean the maximum of total degrees of the coordinate functions \( F_1, \ldots, F_n \) in the variables \( X_1, \ldots, X_n \).

1991 Mathematics Subject Classification. Primary 14R15, 14R10, 13A10; Secondary 05C05.
Key words and phrases. Jacobian Conjecture, radical ideal, power series, polynomial.

© 2005 American Mathematical Society

261
**Definition 1.3.** We say that a polynomial map \( F = (F_1, \ldots, F_n) \) is of *special type* if it has the form \( F_i = X_i - H_i \) with \( H_i \) having only homogeneous summands of degree 2 and higher.

**Definition 1.4.** We say that a polynomial map \( F = (F_1, \ldots, F_n) \) is of *homogeneous type* if it has the form \( F_i = X_i - H_i \) with \( H_i \) homogeneous of the same degree \( d \geq 2 \).

**Definition 1.5.** We say that a polynomial map \( F = (F_1, \ldots, F_n) \) is of *\( d \)-fold linear type*, for \( d \geq 2 \), if there are linear forms \( L_1, \ldots, L_n \in \mathbb{C}[X_1, \ldots, X_n] \) such that \( F \) has the form \( F_i = X_i - H_i \) with \( H_i = L_i^d \).

**Definition 1.6.** We say that a polynomial map \( F = (F_1, \ldots, F_n) \) is of *symmetric type* if it is of special type and if the Jacobian matrix \( JF \) is a symmetric matrix.

**Definition 1.7.** We say that a polynomial map \( F = (F_1, \ldots, F_n) \) is of *symmetric homogeneous type* if it is both of symmetric type and of homogeneous type.

Given a fixed dimension \( n \geq 1 \) and a fixed degree bound \( d \geq 2 \):

**Conjecture 1.8 (JC\(_{n,d}\)).** Suppose \( F = (F_1, \ldots, F_n) \) is a polynomial map of special type having degree \( \leq d \) with \( |JF| = 1 \). Then \( F \) is an automorphism.

**Conjecture 1.9 (JC\(_{n,[d]}\)).** Suppose \( F = (F_1, \ldots, F_n) \) is a polynomial map of homogeneous type having degree \( \geq d \) with \( |JF| = 1 \). Then \( F \) is an automorphism.

**Conjecture 1.10 (LJC\(_{n,[d]}\)).** Suppose \( F = (F_1, \ldots, F_n) \) is a polynomial map of \( d \)-fold linear type with \( |JF| = 1 \). Then \( F \) is an automorphism.

**Conjecture 1.11 (SJC\(_{n,d}\)).** Suppose \( F = (F_1, \ldots, F_n) \) is of symmetric type having degree \( \geq d \). Then \( F \) is an automorphism.

**Conjecture 1.12 (SJC\(_{n,[d]}\)).** Suppose \( F = (F_1, \ldots, F_n) \) is of symmetric homogeneous type having degree \( d \). Then \( F \) is an automorphism.

**Remark 1.13.** In the \( n \)-dimensional case where \( F = X - H \) with \( H \) homogeneous we have these equivalences:

\[
|JF| = 1 \iff JH \text{ is nilpotent} \iff (JH)^n = 0
\]

(See, for example, [W1].)

**1.3. Reductions and Known Cases.** The following reduction, now standard knowledge, is proved in [BCW]:

**Theorem 1.14 (Cubic Reduction).** For any fixed integer \( d \geq 3 \) we have

\[
JC \iff JC_{n,d} \text{ for all } n
\]

\[
\iff JC_{n,[d]} \text{ for all } n
\]

In particular, proving the Jacobian Conjecture is reduced to proving \( JC_{n,[3]} \) for all \( n \).

An even stronger reduction was proved by Drużkowski in [D1]:

**Theorem 1.15 (Cubic Linear Reduction).** For any fixed integer \( d \geq 3 \) we have

\[
JC \iff LJC_{n,[d]} \text{ for all } n
\]

In particular, proving the Jacobian Conjecture is reduced to proving \( LJC_{n,[3]} \) for all \( n \).
In a recent breakthrough, Michiel de Bondt and Arno van den Essen proved this intriguing reduction ([BE1]):

**Theorem 1.16 (Symmetric Reduction).** For any fixed integer \( d \geq 3 \) we have

\[
\text{JC} \iff \text{SJC}_{n,d} \text{ for all } n
\]
\[
\iff \text{SJC}_{n,[d]} \text{ for all } n
\]

In particular, proving the Jacobian Conjecture is reduced to proving \( \text{HJC}_{n,[3]} \) for all \( n \).

Some known cases of the Jacobian Conjecture are:

1. **Theorem 1.17 (Known Cases).** The following assertions hold:

   - \( \text{JC}_{1,d} \) for all \( d \) (trivial)
   - \( \text{JC}_{2,d} \) for \( d \leq 100 \) (Moh [Mo])
   - \( \text{JC}_{2,[d]} \) for all \( d \) (trivial)
   - \( \text{JC}_{n,2} \) for all \( n \) (Wang [Wa])
   - \( \text{JC}_{3,3} \) (Moh, Sathaye [MoS])
   - \( \text{JC}_{3,[3]} + \text{linear triangularization} \) (Wright [W3])
   - \( \text{JC}_{3,[d]} \) for all \( d \), + linear triangularization (de Bondt, van den Essen [BE3])
   - \( \text{JC}_{4,[3]} \) (Hubbers [H])
   - \( \text{LJC}_{n,[3]} \) for \( n \leq 5 \) (Drużkowski [D2])
   - \( \text{SJC}_{n,d} \) for \( n \leq 4 \), all \( d \) (de Bondt, van den Essen [BE2])
   - \( \text{SJC}_{5,[d]} \) for \( d \) (de Bondt, van den Essen [BE2])

(We always assume \( d \geq 2 \).)

### 1.4. The Potential Function in the Symmetric Case

The condition that \( JF \) is symmetric says that \( D IF_j = D_j F_i \) for each \( i, j \). This means there exists \( P \in \mathbb{C}[X_1, \ldots, X_n] \) such that \( F_i = D_i P \) for \( i = 1, \ldots, n \), i.e., \( F = \nabla P \), the gradient of \( P \). The polynomial \( P \) is called the potential function for \( F \). Thus the symmetric case occurs precisely when the Jacobian matrix of \( F \) is the Hessian matrix of \( P \):

\[
JF = 
\n\text{Hess } P = D_i D_j P
\]

For \( F = X - H \), the existence of a potential function for \( F \) is equivalent to the existence of one for \( H \). For if \( F \) has potential function \( P \), then \( H \) has potential function \( \sum \frac{X_i^2}{2} - P \). It will be convenient to take \( P \) to be the potential function for \( H \) instead of \( F \) in the symmetric situation.

### 2. Ideals Defining the Jacobian Condition

---

1 The sixth and seventh statements assert more strongly that in the \( n = 3 \) homogenous situation, maps satisfying the hypothesis of the Jacobian Conjecture can be linearly triangularized (see [W3]). In the second from last assertion one does not need to assume maps are of special type.
2.1. The Formal Map and the Formal Inverse. Let $F = (F_1, \ldots, F_n) = X - H$ be the formal map, meaning that each $F_i$ is a formal power series of the form

$$F_i = X_i - \sum_{|k| \geq 2} a_i^k X^k = X_i - H_i$$

Here $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ (we will consistently write $\mathbb{N}$ for the set of nonnegative integers), $|k| = k_1 + \cdots + k_n$, and

$$a_i^k X^k = a_i^{(k_1, \ldots, k_n)} X_1^{k_1} \cdots X_n^{k_n}$$

The “coefficients” $a_i^k$ of $F$ are viewed as indeterminates lie in the polynomial ring $\mathbb{Z}[\{a_i^k\}]$.

It is known (and easily proved) that such an $F$ has formal power series inverse $G = (G_1, \ldots, G_n)$ where each $G_j$ has the form $X_j +$ higher degree terms.

**Definition 2.1 (Inverse Coefficients).** Letting $G = (G_1, \ldots, G_n) = X + N$ be the inverse of the formal map $F$, we define the elements $b_j^q \in \mathbb{Z}[\{a_i^k\}]$, for $j = 1, \ldots, n$ and $q \in \mathbb{N}^n$ with $|q| \geq 2$, by writing

$$(2.1) \quad G_j = X_j + \sum_{|q| \geq 2} b_j^q X^q = X_j + N_j.$$  

It has been shown that $b_j^q$ can be written as a sum of monomials in the indeterminates $\{a_i^k\}$ which are parameterized by certain combinatorial objects:

$$b_j^q = \sum_{T \in T_j^q} m_T \in \mathbb{Z}[\{a_i^k\}].$$

Here $T_j^q$ is the set of isomorphism classes of labeled, planar trees with root label $j$ and leaf-type $q$. The reader is referred to [W2] for an explanation of these terms, the definition of the monomials $m_T$, and the proof of this formula.

2.2. The Formal Symmetric Map and Its Inverse. Now we let $P = \sum_{|k| \geq 3} c^k X^k$, where $c^k$ are indeterminates over $\mathbb{Z}$, and let $F = (F_1, \ldots, F_n)$ be the formal symmetric map, meaning that

$$F_i = X_i - D_i P.$$  

We call $P$ the formal potential function. Clearly $JF = I - \text{Hess} P$, a symmetric matrix.

There is a unique ring homomorphism $\mathbb{Z}[\{a_i^k\}] \rightarrow \mathbb{Z}[\{c^k\}]$ sending $a_i^k$ to $(k_i + 1) c^{k+e_i}$, where $e_i$ is the element of $\mathbb{N}^n$ whose $i$th coordinate is 1 and all other coordinates are zero. This, when extended to the power series rings in $X_1, \ldots, X_n$, sends the coordinate functions of the formal map defined in Section 2.1 to the coordinate functions of the formal symmetric map. Identifying the inverse coefficients $b_j^q$ with their images in $\mathbb{Z}[\{c^k\}]$, and letting $F = X - H$ be the formal symmetric map, we again have $G = F^{-1}$ given by formula 2.1.

**Proposition 2.2 (Potential Function for the Inverse).** With $N = G - X$ as in 2.1, viewing each $b_j^q$ as an element of $\mathbb{Z}[\{c^k\}]$ as above, there is a power series $Q \in \mathbb{Z}[\{c^k\}][[X_1, \ldots, X_n]]$ such that $\forall Q = N$. 

PROOF. We have $H = \nabla P$, and therefore $F = \nabla \tilde{P}$ where $\tilde{P} = \left( \frac{1}{2} \sum_{j=1}^{n} X_j^2 \right) - P$. Let $\tilde{Q} = \sum_{j=1}^{n} X_j G_j - \tilde{P}(G)$. We have

$$D_i \tilde{Q} = D_i \left( \sum_{j=1}^{n} X_j G_j \right) - D_i \left( \tilde{P}(G) \right)$$

$$= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} (D_j \tilde{P})(G) D_i G_j$$

$$= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} F_j(G) G_j$$

$$= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} X_j D_i G_j = G_i,$$

which shows that $\nabla \tilde{Q} = G$. Therefore $Q = \tilde{Q} - \frac{1}{2} \sum_{j=1}^{n} X_j^2$ has the desired property. \qed

DEFINITION 2.3 (Potential Inverse Coefficients). Letting $Q$ be the potential function whose existence is asserted in Proposition 2.2, we define elements $d^Q \in \mathbb{Z}\{a_k\}$, for $q \in \mathbb{N}^n$ with $|q| \geq 3$, by writing

$$Q = \sum_{|q| \geq 3} d^Q X^q.$$  \hfill (2.2)

2.3. The Jacobian and Nilpotency Relations. Let $F = X - H$ be the formal map. We write $JF$ for the Jacobian matrix of $F$ and $|JF|$ for its determinant. We have $|J(F)| = |I_n - J(H)|$, which lies in $\mathbb{Z}\{a_k\}[[X_1, \ldots, X_n]]$ having the form $1 - (\text{higher degree terms})$.

DEFINITION 2.4 (Jacobian Relations). Letting $F$ be the formal map, we define elements $h^\ell \in \mathbb{Z}\{a_k\}$, for $\ell \in \mathbb{N}^n$ with $|\ell| \geq 1$, by writing

$$|J(F)| = 1 - \sum_{|\ell| \geq 1} h^\ell X^\ell.$$  \hfill (2.3)

It is shown in [L] that the polynomial $h^\ell$ can be written as a sum of monomials parameterized by certain combinatorial objects called cycle sets. Specifically,

$$h^\ell = \sum_{C \in \mathcal{C}^\ell} (-1)^{e(C)} m_C \in \mathbb{Z}\{a_k\}$$

In this formula $\mathcal{C}^\ell$ is the set of cycle sets $C = \{c_1, \ldots, c_r\}$ with leaf type $\ell$. We refer the reader to [L] for the specifics.

Finally, motivated by Remark 1.13, we make the following definition.

DEFINITION 2.5 (Nilpotency Relations). Letting $F = X - H$ be the formal map we write the matrix $(JH)^n$ as $(M_{ij})$ and define the elements $n_{ij}^\ell \in \mathbb{Z}\{a_k\}$, for $1 \leq i, j \leq n$ and $\ell \in \mathbb{N}^n$ with $|\ell| \geq n$, by

$$M_{ij} = \sum_{|\ell| \geq 1} n_{ij}^\ell X^\ell.$$
2.4. Ideal Membership Reformulation of the Jacobian Conjecture.

For integers \( n \geq 1 \) and \( d \geq 2 \), the following definitions define formal degree \( d \) polynomial maps \( F \) of various types in dimension \( n \), the polynomial rings over \( \mathbb{Q} \) generated by their coefficients, and ideals which impose the condition \( |JF| = 1 \).

**Definition 2.6. (Nonhomogeneous)** We set
\[
R_{n,d} = \mathbb{Q}\{a_k^i \mid k \in \mathbb{N}^n, 2 \leq |k| \leq d \},
\]
i.e., the polynomial ring over \( \mathbb{Q} \) generated by the indeterminates \( \{a_k^i\} \) for which \( |k| \leq d \). It can be viewed as the quotient of \( \mathbb{Q}\{a_k^i\} \) which sets \( a_k^i = 0 \) if \( |k| > d \).

We define the **formal polynomial map of degree** \( d \) to be the map \( F = X - H \) where \( H_i = \sum_{2 \leq |k| \leq d} a_k^i X^k \), for \( 1 = 1, \ldots, n \).

**Definition 2.7. (Homogeneous)** We set
\[
R_{n,d} = \mathbb{Q}\{a_k^i \mid k \in \mathbb{N}^n, |k| = d \},
\]
i.e., the polynomial ring over \( \mathbb{Q} \) generated by the indeterminates \( \{a_k^i\} \) for which \( |k| = d \). It can be viewed as the quotient of \( \mathbb{Q}\{a_k^i\} \) which sets \( a_k^i = 0 \) if \( |k| \neq d \).

We define the **formal degree** \( d \) **polynomial map of homogeneous type** to be the map \( F = X - H \) where \( H_i = \sum_{|k| = d} a_k^i X^k \), for \( 1 = 1, \ldots, n \).

To parameterize the \( d \)-fold linear situation we have

**Definition 2.8. (d-Fold Linear)** Let \( l_{ij}, 1 \leq i, j \leq n \) be indeterminates. (They are the entries of the generic \( n \times n \) matrix.) We set
\[
R_{n,d} = \mathbb{Q}\{l_{ij} \mid 1 \leq i, j \leq n \},
\]
i.e., the polynomial ring over \( \mathbb{Q} \) generated by these indeterminates. For \( i = 1, \ldots, n \), let \( L_i = \sum l_{ij} X_j \in R_{n,d}[X_1, \ldots, X_n] \). We define the **formal \( d \)-linear polynomial map** to be the map \( F = X - H \) where \( H_i = L_i^d \), for \( 1 = 1, \ldots, n \).

To parameterize the symmetric situation we recall indeterminates \( \{c_k\} \) where \( k \in \mathbb{N}^n \) with \( 3 \leq |k| \). These are the coefficients of the formal potential function defined in section 1.4.

**Definition 2.9. (Symmetric Nonhomogeneous)** We set
\[
R_{n,d} = \mathbb{Q}\{c_k \mid k \in \mathbb{N}^n, 3 \leq |k| \leq d + 1 \},
\]
i.e., the polynomial ring over \( \mathbb{Q} \) generated by the indeterminates \( \{c_k\} \) for which \( |k| \leq d + 1 \). It can be viewed as the quotient of \( \mathbb{Q}\{c_k\} \) which sets \( c_k = 0 \) for \( |k| > d + 1 \).

We define the **generic potential function of degree** \( d + 1 \) to be \( P = \sum_{3 \leq |k| \leq d+1} c_k X^k \). We define the **formal degree** \( d \) **polynomial map of symmetric type** to be the map \( F = X - H \) where \( H_i = D_i P \), for \( 1 = 1, \ldots, n \) (i.e., \( H = \nabla P \)).

**Definition 2.10. (Symmetric Homogeneous)** We set
\[
R_{n,d} = \mathbb{Q}\{c_k \mid k \in \mathbb{N}^n, |k| = d + 1 \},
\]
i.e., the polynomial ring over \( \mathbb{Q} \) generated by the indeterminates \( \{c_k\} \) for which \( |k| = d + 1 \). We define the **generic homogeneous potential function of degree** \( d + 1 \) to be \( P = \sum_{|k| = d+1} c_k X^k \). We define the **formal degree** \( d \) **polynomial map of homogeneous symmetric type** to be the map \( F = X - H \) where \( H_i = D_i P \), for \( 1 = 1, \ldots, n \) (i.e., \( H = \nabla P \)).
Conjecture. 

For fixed \( n \) and \( d \) we have this commuting diagram of specialization homomorphisms amongst the rings defined above

\[
\begin{array}{ccc}
Q[\{a^k_i\}] & \rightarrow & R_{n,d} \\
\downarrow & & \downarrow \\
Q[\{c^k_i\}] & \rightarrow & R_{n,d}^{\text{sym}} \\
\text{defunct map} & \rightarrow & \text{defunct map} \\
\end{array}
\]  

(2.4)

where the maps are as follows. The map \( Q[\{a^k_i\}] \rightarrow Q[\{c^k_i\}] \) was defined in section 2.2. There is a natural surjective homomorphism \( Q[\{a^k_i\}] \rightarrow R_{n,d} \) which sets \( a^k_i = 0 \) if \( |k| > d \). Similarly, there is a surjection \( Q[\{c^k_i\}] \rightarrow R_{n,d}^{\text{sym}} \). Setting \( a^k_i = 0 \) if \( |k| \neq d \) defines a surjective homomorphism \( R_{n,d} \rightarrow R_{n,d}^{\text{sym}} \). We get a surjective homomorphism \( R_{n,d} \rightarrow R_{n,d}^{\text{sym}} \) by sending \( a^k_i \) to \((k+1)c^k+i\), where \( c_i \) is the element of \( \mathbb{N}^n \) whose \( i \)-th coordinate is 1 and all other coordinates are zero. Note that this sends the coefficient of \( X^k \) in \( H_i \) to the coefficient of \( X^k \) in \( D_i P \), where \( X - H \) is the formal polynomial map of degree \( d \) and \( P \) is the formal potential function of degree \( d+1 \). In similar fashion we get a homomorphism \( R_{n,d}^{\text{sym}} \rightarrow R_{n,d}^{\text{sym}} \). Finally we have \( R_{n,d}^{\text{sym}} \rightarrow R_{n,d}^{\text{lin}} \) (non-surjective) defined by sending the coefficient \( a^k_i \), \( |k| = d \), of \( X^k \) in the \( i \)-th coordinate function of the formal polynomial map of homogeneous type, to the coefficient of \( X^k \) in the \( i \)-th coordinate function of the formal \( d \)-fold linear polynomial map.

The image of the homomorphism

\[ Q[\{a^k_i\}][X_1, \ldots, X_n]] \rightarrow R_{n,d}[[X_1, \ldots, X_n]] \]

induced from (2.4) sends the coordinate functions of the formal map defined in section 2 to those of the formal degree \( d \) polynomial map defined in Definition 2.6. Following this by the other homomorphisms of power series rings induced by the homomorphisms in the diagram (2.4), we get the coordinates of the other formal polynomial maps defined in Definitions 2.7, 2.8, 2.9, and 2.10.

In the following definition, and in for the remainder of this paper, we identify the Jacobian relations \( h^f \in Q[\{a^k_i\}] \) of Definition 2.4, the nilpotency relations \( n^f_{ij} \) of Definition 2.5, and the inverse coefficients \( b^i_j \) of Definition 2.1 with their images in the various rings in diagram 2.4.

**Definition 2.12 (The Jacobian and Nilpotency Ideals).**

1. We let \( J_{n,d} \) be the ideal in \( R_{n,d} \) generated by Jacobian relations \( \{h^f\} \).
2. We let \( J_{n,d}^{\text{lin}} \) be the ideal in \( R_{n,d}^{\text{lin}} \) generated by the Jacobian relations \( \{h^f\} \). We let \( N_{n,d}^{\text{lin}} \) be the ideal in \( R_{n,d}^{\text{lin}} \) generated by the nilpotency relations \( \{n^f_{ij}\} \).
3. We let \( J_{n,d}^{\text{sym}} \) be the ideal in \( R_{n,d}^{\text{sym}} \) generated by the Jacobian relations \( \{h^f\} \). We let \( N_{n,d}^{\text{sym}} \) be the ideal in \( R_{n,d}^{\text{sym}} \) generated by the nilpotency relations \( \{n^f_{ij}\} \).
4. We let \( J_{n,d}^{\text{sym}} \) be the ideal in \( R_{n,d}^{\text{sym}} \) generated by Jacobian relations \( \{h^f\} \).
(5) We let $J_{n,d}^{\text{sym}}$ be the ideal in $\mathbb{R}_{n,d}^{\text{sym}}$ generated by the Jacobian relations \{\alpha_i \}. We let $N_{n,d}^{\text{sym}}$ be the ideal in $\mathbb{R}_{n,d}^{\text{sym}}$ generated by the nilpotency relations \{\nu_i \}.

In the homogeneous situations, the Jacobian ideals and the nilpotency ideals have the following relationships:

**Proposition 2.13.** We have

\[
J_{n,d} \supseteq N_{n,d}, \quad J_{n,d}^{\text{lin}} \supseteq N_{n,d}^{\text{lin}}, \quad J_{n,d}^{\text{sym}} \supseteq N_{n,d}^{\text{sym}},
\]
the containments all being proper for $n \geq 2$. Moreover,

\[
\sqrt{J_{n,d}} = \sqrt{N_{n,d}}, \quad \sqrt{J_{n,d}^{\text{lin}}} = \sqrt{N_{n,d}^{\text{lin}}}, \quad \sqrt{J_{n,d}^{\text{sym}}} = \sqrt{N_{n,d}^{\text{sym}}},
\]
for all $n$.

**Proof.** The first containments follow easily from some basic properties of nilpotent matrices. (See [W1], for example.) The proofs will also appear in [W5]. The latter containments follow from Remark 1.13. \(\square\)

Putting together the Degree Bound\(^2\), the Gap Theorem\(^3\), and Hilbert’s Nullstellensatz\(^4\), we conclude:

**Theorem 2.14.** We have these equivalences:

\[
J_{n,d} \iff b_j^q \in \sqrt{J_{n,d}}, \quad j = 1, \ldots, n, \quad q \in \mathbb{N}^n, \quad d^{n-1} < |q| \leq d^n
\]

\[
J_{n,d} \iff b_j^q \in \sqrt{J_{n,d}}, \quad j = 1, \ldots, n, \quad q \in \mathbb{N}^n, \quad d^{n-1} < |q| \leq d^n
\]

\[
LJ_{n,d} \iff b_j^q \in \sqrt{J_{n,d}^{\text{lin}}}, \quad j = 1, \ldots, n, \quad q \in \mathbb{N}^n, \quad d^{n-1} < |q| \leq d^n
\]

\[
SJC_{n,d} \iff d^q \in \sqrt{J_{n,d}^{\text{sym}}}, \quad q \in \mathbb{N}^n, \quad d^{n-1} + 1 < |q| \leq d^n + 1
\]

\[
SJC_{n,d} \iff d^q \in \sqrt{J_{n,d}^{\text{sym}}}, \quad q \in \mathbb{N}^n, \quad d^{n-1} + 1 < |q| \leq d^n + 1
\]

Theorem 3.11 will allow us to replace the condition $d^{n-1} < |q| \leq d^n$ by $d^{n-1} < |q| \leq 2d^{n-1} - 1$ in each of the three the homogeneous cases above, making it sufficient to establish the ideal membership condition over a much smaller “gap” in those cases.

\(^2\) due to O. Gabber, which asserts that the degree of the inverse of a polynomial automorphism of degree $d$ is bounded by $d^{n-1}$ (see [BCW] or [E]).

\(^3\) which says if those homogeneous summands of the inverse having degree $d^{n-1} + 1, d^{n-1} + 2, \ldots, d^n$ are all zero, then the inverse is a polynomial.

\(^4\) Hilbert’s Nullstellensatz assumes the ground field is algebraically closed. Our result follows using the following elementary fact: Given $A \subset k$ an affine domain over a field $k$, $I \subset A$ an ideal, $a \in A$, and $k'$ a separable algebraic extension of $k$, if $a \in \sqrt{I \otimes_k k'}$, then $a \in \sqrt{I}$.
Questions 2.15. These equivalences immediately impose a number of questions about the ideals \( J_{n,d} \), \( J_{n,d[j]} \), and the other Jacobian ideals, for given \( n \) and \( d \), including:

1. Is \( \sqrt{J_{n,d}} = J_{n,d[j]} \)? Similarly for \( J_{n,d[j]} \)?

2. If \( \sqrt{J_{n,d}} \subseteq \sqrt{J_{n,d[j]}} \), can generators be given for \( \sqrt{J_{n,d[j]}} \) which can be realized combinatorially in some manner along the lines of formula 2.3 for \( h^f \)?

3. Is \( \sqrt{J_{n,d[j]}} \) a prime ideal? If so, what is the geometry (e.g., singular locus) of the irreducible variety it defines? Similarly for \( J_{n,d[j]} \) (which is a homogeneous ideal hence defines a projective locus)?

4. If \( \sqrt{J_{n,d[j]}} \) is not prime, what are its prime components, and what is their geometry? Similarly for \( J_{n,d[j]} \)?

5. In any of the equivalences of Theorem 2.14, can the radical be dropped?

6. Even if the radicals of Theorem 2.14 cannot be dropped, does \( JC_{n,d[j]} \) imply \( b_j^q \in J_{n,d[j]} \) for sufficiently large \( q \)? Similarly for \( JC_{n,d[j]} \) ?

7. Given an integer with \( N > d^n \), does the condition \( b_j^q \in J_{n,d[j]} \) for all \( q \in \mathbb{N}^n \) with \( |q| = N \) imply \( b_j^q \in J_{n,d[j]} \) for all \( q \in \mathbb{N}^n \) with \( |q| > N \)? Similarly for \( J_{n,d[j]} \)? In other words, does it suffice to prove the generic vanishing of one homogeneous summand of the formal inverse?

Question (6) has an affirmative answer, as seen in the following theorem.

Theorem 2.16. For given \( n \geq 1 \), \( d \geq 2 \), \( JC_{n,d[j]} \) is equivalent to the following assertion: There exists an integer \( N_{n,d} > d^n-1 \) such that \( b_j^q \in J_{n,d[j]} \) for \( j = 1, \ldots, n \), \( q \in \mathbb{N}^n \), \( |q| > N_{n,d} \). An analogous statement holds for each of \( JC_{n,d[j]} \), \( LC_{n,d[j]} \), \( SJC_{n,d[j]} \), and \( SJC_{n,d[j]} \).

Proof. This follows from the fact that a polynomial map \( F \) over a ring \( R \) is invertible if (and only if) the map \( F \) over the ring \( R/(\text{Nil} R) \) obtained by base change is invertible, where \( \text{Nil} R \) is the nilradical of \( R \). This is Lemma 1.1.9 of [E]. Applying this to \( \mathfrak{R}_{n,d[j]} \), with \( F \) being the formal degree \( d \) polynomial map, and taking \( N_{n,d} \) to be the degree of \( F^{-1} \) gives the result. Since the degree bound \( d^n-1 \) is attained for certain maps, we must have \( N_{n,d} > d^n-1 \).

Remark 2.17. In answer to Question (5) of 2.15, the author asserted in [W4]: “It is known that the radicals... cannot be removed. For example, computer calculations show that not all \( b_j^q \) lie in \( J_{n,d[j]} \) for \( n = 3 \), \( d = 2 \), and \( |q| = 5 \)” This statement was carelessly based on a vague recollection of a claim by another mathematician, communicated verbally. After a number of inquiries about this statement the author felt behooved to perform some computations to confirm its veracity. Computations using the symbolic algebra program Singular indeed verified this assertion, and produced other results which will be reported in Section 4.

3. Formulas for the Formal Inverse

The formulas for the formal inverse given in this section provide means for the Jacobian Conjecture to be addressed as a problem in combinatorics. They also give specific ways to realize the homogeneous summands of the formal inverse, thereby allowing one to attack the conjecture via Theorem 2.14, and to solve \( JC_{n,d[j]} \) for certain small values of \( n \) and \( d \) using a symbolic algebra computer program.
3.1. The Bass-Connell-Wright Tree Formula. Let $T_{rt}$ be the set of isomorphism classes of finite rooted trees. For $G = F^{-1}$, the Tree Formula of Bass-Connell-Wright states:

**Theorem 3.1 (BCW Tree Formula).** Let $F$ be the formal map defined in Section 2.1, and let $G = (G_1, \ldots, G_n)$ be the formal inverse. Then

$$G_i = X_i + \sum_{T \in T_{rt}} \frac{1}{|Aut T|} P_{T,i},$$

where

$$P_{T,i} = \sum_{\ell: V(T) \to \{1, \ldots, n\}} \prod_{v \in V(T)} D_{\ell(v+)}H_{\ell(v)}.$$

In this expression $v^+$ is the set $\{w_1, \ldots, w_t\}$ of children of $v$ and $D_{\ell(v+)} = D_{\ell(w_1)} \cdots D_{\ell(w_t)}$.

From this and Theorem 2.14 we conclude:

**Corollary 3.3.** The assertion $JC_{n,|d|}$ holds if and only if all coefficients of $N^{(m)}$ lie in $\sqrt{\sigma_{n,|d|}}$ for $\frac{d^{n-1} - 1}{d - 1} < m \leq \frac{d^n - 1}{d - 1}$, where $N^{(m)}$ is given by (3.1).

3.2. The Tree Formula for the Symmetric Case. The formula of Bass-Connell-Wright takes on a simpler form in the symmetric case.

We now let $T$ be the set of isomorphism classes of finite trees (having no designated root).

**Theorem 3.4 (Symmetric Tree Formula).** Let $F = X - H$ be the formal degree $d$ polynomial map of homogeneous type. Then the formal inverse has the form $G = X + N$ where

$$N = N^{(1)} + N^{(2)} + N^{(3)} + \ldots$$

with $N^{(m)}$ homogeneous of degree $m(d - 1) + 1$ and given by the formula

$$N^{(m)} = \sum_{T \in T_{rtm}} \frac{1}{|Aut T|} P_{T,i}.\tag{3.1}$$

From this and Theorem 2.14 we conclude:

**Corollary 3.3.** The assertion $JC_{n,|d|}$ holds if and only if all coefficients of $N^{(m)}$ lie in $\sqrt{\sigma_{n,|d|}}$ for $\frac{d^{n-1} - 1}{d - 1} < m \leq \frac{d^n - 1}{d - 1}$, where $N^{(m)}$ is given by (3.1).
The proof will appear in [W5]. A formula somewhat like the above appears without proof in [Me].

For the symmetric homogeneous case Theorem 3.4 specializes to the following, in the light of Proposition 2.2. Here we let $\mathcal{T}_m$ be the set of isomorphism classes of (non-rooted) trees having $m$ vertices.

**Theorem 3.5 (Symmetric Homogeneous Tree Formula).** Let $F = X - \nabla P$ be the formal degree $d$ polynomial map of symmetric homogeneous type (see Definition 2.10). (Here $P$ is the formal potential function of degree $d + 1$.) Let $G = (G_1, \ldots, G_n)$ be its formal inverse. Then $G = X + \nabla Q$ with

$$Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots$$

and

$$Q^{(m)} = \sum_{T \in \mathcal{T}_m} \frac{1}{|Aut(T)|} Q_T.$$

$Q^{(m)}$ has degree $m(d - 1) + 2$.

Identifying the polynomials $N^{(m)} \in R_{n,d}[X_1, \ldots, X_n]$ with their images in $R_{n,d}[\text{sym}][X_1, \ldots, X_n]$, it is clear from Theorems 3.2 and 3.4 that $N^{(m)} = \nabla Q^{(m)}$.

Note also that the $Q$ in this theorem is the specialization of the $Q$ of Proposition 2.2 and Definition 2.3 to the ring $R_{n,d}[\text{sym}][X_1, \ldots, X_n]$.

**3.3. Zhao's Formulas.** The formulas of Section 3.1 are intriguing from a combinatoric perspective, but they do not give a practical way to calculate summands of the formal inverse, say, using a computer. For such purposes, the following, proved in [Z1], is more useful.

**Theorem 3.6 (Zhao's Formula).** As in Theorem 3.2, let $N^{(m)}$, $m \geq 1$, be the homogeneous summands of the formal inverse of the formal degree $d$ polynomial map $F = X - H$ of homogeneous type. Then $N^{(1)} = H$ and, for $m \geq 2$,

$$N^{(m)} = \frac{1}{m - 1} \sum_{k+\ell=m \atop k, \ell \geq 1} JN^{(k)} \cdot N^{(\ell)}.$$

(In this formula $N^{(\ell)}$ is viewed as a column vector.)

In fact, Zhao's formula gives formally converging summands for $N = G - X$ even when $H$ is not homogeneous, i.e., when $F = X - H$ is the formal (non-homogeneous) map. The summands $N^{(m)}$ are, of course, no longer homogeneous.

So, for example, we have

$$N^{(2)} = JN^{(1)} \cdot N^{(1)}$$

(3.2)

$$N^{(3)} = \frac{1}{2} \left[ JN^{(2)} \cdot N^{(1)} + JN^{(1)} \cdot N^{(2)} \right]$$

A second formula of Zhao ([Z2]) gives the a formula for the inverse potential function in the symmetric situation:

**Theorem 3.7 (Zhao's Formula for the Symmetric Case).** As in Theorem 3.5, let $Q^{(m)}$, $m \geq 1$, be the homogeneous summands of the potential function for $N = X - P$.
$G - X$, where $G$ is formal inverse of the formal degree $d$ polynomial map $F = X - \nabla P$ of symmetric homogeneous type. Then $Q^{(1)} = P$ and, for $m \geq 2$,

$$Q^{(m)} = \frac{1}{2(m - 1)} \sum_{k, \ell \geq 1 \atop k + \ell = m} \left( \nabla Q^{(k)} \cdot \nabla Q^{(\ell)} \right).$$

(Here $(\nabla Q^{(k)} \cdot \nabla Q^{(\ell)})$ denotes the usual dot product of vectors.)

Again it should be noted that this theorem holds in the nonhomogeneous case as well, giving nonhomogeneous converging summands for the potential function for $N$.

The formula of Theorem 3.6 provides a quick proof of the well known fact that the Jacobian Conjecture holds in the case where $F = X - H$ is of homogeneous type and $(JH)^2 = 0$; in fact it gives a stronger ideal membership statement:

**Theorem 3.8.** Let $F = X - H$ be the formal degree $d$ polynomial map of homogeneous type in dimension $n$, and let $\mathfrak{J}$ be the ideal in $\mathfrak{R}_n[d]$ generated by the coefficients of $(JH)^2$. Then all coefficients of $N^{(m)}$, for $m \geq 2$, are in $\mathfrak{J}$.

**Proof.** Since $H$ is homogeneous we have $H = \frac{1}{d} JH \cdot X$ by Euler’s formula. Since $N^{(1)} = H$ we have, by 3.2,

$$N^{(2)} = JN^{(1)} \cdot N^{(1)} = \frac{1}{d} JH \cdot JH \cdot X = \frac{1}{d} (JH)^2 \cdot X$$

which shows the coefficients of $N^{(2)}$ lie in $\mathfrak{J}$. Now we proceed inductively using Zhao’s formula. We have from Theorem 3.6, setting $m = 3$,

$$N^{(3)} = \frac{1}{2} \left[ JN^{(2)} \cdot N^{(1)} + JN^{(1)} \cdot N^{(2)} \right],$$

showing that the coefficients for $N^{(3)}$ are in $\mathfrak{J}$, and similarly for all subsequent $N^{(m)}$. \hfill \square

The following corollary is immediate.

**Corollary 3.9.** If $F = X - H$ is a polynomial map of homogeneous type with $(JH)^2 = 0$, then $F$ is invertible with $F^{-1} = X + H$.

In the case $n = 2$, the ideal $\mathfrak{J}$ coincides with the ideal $\mathfrak{N}_{n,2}$ defined in Definition 2.12. Since $\mathfrak{N}_{n,2} \subset \mathfrak{J}_{n,2}$ (Proposition 2.13), Theorem 3.8 provides a positive answer to Question (5) of 2.15 for $n = 2$:

**Corollary 3.10.** For $j = 1, 2, q \in \mathbb{N}^2$, and $|q| > d$ we have $b_j^q \in \mathfrak{J}_{2,|d|}$. More strongly, $b_j^q \in \mathfrak{N}_{2,|d|}$.

Zhao’s formula leads to an improved gap theorem for polynomial maps of homogeneous type, as follows.

Again letting $F = X - H$ be the formal degree $d$ polynomial map of homogeneous type in dimension $n$, suppose that for some $m \geq 1$, the coefficients of $N^{(m+1)}, N^{(m+2)}, \ldots, N^{(2m)}$ all lie in some ideal $\mathfrak{J} \subset \mathfrak{R}_n[d]$. The Zhao Formula (Theorem 3.6) applied to $N^{(2m+1)}$ shows that its coefficients are in $\mathfrak{J}$ as well. Inductively we conclude that the coefficients of all $N^{(s)}$ with $s > m$ lie in $\mathfrak{J}$.

Now we apply this to the case where $\mathfrak{J} = \mathfrak{J}_{n,|d|}$ and $N^{(m)}$ has degree $d^{m-1}$, the inverse degree bound for polynomial automorphisms. Since $N^{(m)}$ is homogeneous
of degree $m(d-1) + 1$ (see Theorem 3.2) this happens when $m = \frac{d^n-1}{d-1} = 1 + d + d^2 + \cdots + d^{n-2}$, and in this case $N^{(2m)}$ will have degree $2d^{m-1} - 1$. Hence in equivalent condition to $JC_{n,[d]}$ in Theorem 2.14 $d^n$ can be replaced by $2d^{m-1} - 1$, making it sufficient to establish the ideal membership conditions of Theorem 2.14 over a much smaller "gap". We have:

**Theorem 3.11 (Gap Theorem for Maps of Homogeneous Type).** We have these equivalences:

$$
JC_{n,[d]} \iff b_j^q \in \sqrt{J_{n,[d]}}
$$

$$
\text{for } j = 1, \ldots, n, \ q \in \mathbb{N}^n, \ d^{n-1} < |q| \leq 2d^{m-1} - 1
$$

$$
LJC_{n,[d]} \iff b_j^q \in \sqrt{J_{n,[d]}}^{\text{lin}}
$$

$$
\text{for } j = 1, \ldots, n, \ q \in \mathbb{N}^n, \ d^{n-1} < |q| \leq 2d^{m-1} - 1
$$

$$
SJC_{n,[d]} \iff d^q \in \sqrt{J_{n,[d]}}^{\text{sym}}
$$

$$
q \in \mathbb{N}^n, \ d^{n-1} + 1 < |q| \leq 2d^{n-1}
$$

4. Ideal Membership Results

In this section all results obtained by computer were performed using the symbolic algebra program Singular. Inverse coefficients $b_j^q$ were computed using Zhao's Formula (Theorem 3.6).

4.1. Necessity of the Radical. Motivated by Theorem 2.14 and Question (5) of 2.15, we define the ideal membership assertions as follows:

**Assertion 4.1 (IM_{n,d}).** For $j = 1, \ldots, n$, we have $b_j^q \in J_{n,d}$, for all $q \in \mathbb{N}^n$ with $|q| > d^{n-1}$.

**Assertion 4.2 (IM_{n,[d]}).** For $j = 1, \ldots, n$, we have $b_j^q \in J_{n,[d]}$, for all $q \in \mathbb{N}^n$ with $|q| > d^{n-1}$.

**Remark 4.3.** It is not difficult to establish that, for $IM_{n,d}$, it suffices to show the ideal membership condition for $|q| \leq d^n$, and for $IM_{n,[d]}$, it suffices to show the ideal membership condition for $|q| \leq 2d^{m-1} - 1$. The latter uses the improved gap theorem, Theorem 3.11.

The first result shows that the radical is needed in smallest nontrivial nonhomogeneous situation: $n = 2, d = 3$. Since $JC_{2,d}$ for small $d$ (Theorem 1.17) we have $b_j^q \in \sqrt{J_{2,d}}$ for $q \in \mathbb{N}^2$ with $|q| > 1$, by Theorem 2.14. However,

**Proposition 4.4.** The assertion $IM_{2,3}$ fails. Specifically, the coefficient $t = b_1^{(0,4)}$ of $X_1^4$ in $G_1$ has the properties $t^6 \notin J_{2,3}$, $t^7 \in J_{2,3}$.

**Proof.** This was verified by computer.

However, we have:

**Proposition 4.5.** The assertion $IM_{2,[d]}$ holds for all $d \geq 2$.

**Proof.** This follows from Corollary 3.10.
COROLLARY 4.10. If \( F = X - H \) is a polynomial map of homogeneous type having degree \( d = 2 \) with \( (JH)^3 = 0 \), then \( F \) is invertible with

\[
F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)} + N^{(5)}.
\]

In particular, the degree of \( F^{-1} \) is \( \leq 6 \).
The following theorems and their corollaries are new results for the symmetric situation. Their proofs will appear in [W5], and will use the Symmetric Tree Formula (Theorem 3.4).

**Theorem 4.11.** Let \( F = X - H \) be the formal degree \( d \geq 2 \) polynomial map of homogeneous type with symmetric Jacobian matrix in dimension \( n \), and let \( \mathfrak{J} \) be the ideal in \( \mathcal{R}_{n,[d]}^{\text{sym}} \) generated by the coefficients of \((JH)^3\). Then all coefficients \( b_i^3 \) of \( \mathbb{N}^{(m)} \) for \( m \geq 3 \) (hence all \( b_i^3 \) with \( |q| \geq 2d \)) are in \( \mathfrak{J} \).

From this it follows that:

**Corollary 4.12.** If \( F = X - H \) is a polynomial map with symmetric Jacobian matrix of homogeneous type with \((JH)^3 = 0\), then \( F \) is invertible with

\[
F^{-1} = X + N^{(1)} + N^{(2)}.
\]

In particular, the degree of \( F^{-1} \) is \( \leq 2d - 1 \) (independent of \( n \)).

**Theorem 4.13.** Let \( F = X - H \) be the formal degree 2 polynomial map of homogeneous type with symmetric Jacobian matrix in dimension \( n \), and let \( \mathfrak{J} \) be the ideal in \( \mathcal{R}_{n,[2]}^{\text{sym}} \) generated by the coefficients of \((JH)^4\). Then all coefficients \( b_i^4 \) of \( 
\mathbb{N}^{(m)} \) for \( m \geq 5 \) (i.e. \( |q| \geq 6 \)) are in \( \mathfrak{J} \).

Theorem 4.13 improves Wang's result of Theorem 1.17 in the symmetric \( JH^4 = 0 \) case by giving a bound on the degree of the inverse which for large \( n \) is better than the usual degree bound \( d^{n-1} \).

**Corollary 4.14.** If \( F = X - H \) is a degree 2 polynomial map with symmetric Jacobian matrix of homogeneous type with \((JH)^4 = 0\), then \( F \) is invertible with

\[
F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)}.
\]

In particular, the degree of \( F^{-1} \) is \( \leq 5 \) (independent of \( n \)).

**References**


DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST. LOUIS, ST. LOUIS, MO 63130 E-mail: wright@math.wustl.edu