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The Jacobian conjecture: ideal membership questions and recent advances

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CONTENTS

1. The Jacobian Conjecture

1.1. The General Assertion. The Jacobian Conjecture can be stated as follows:

CONJECTURE 1.1 (**JC**). *For any integer $n \geq 1$ and polynomials $F_1, \ldots, F_n \in \mathbb{C}[X_1, \ldots, X_n]$, the polynomial map $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is an automorphism if the determinant $|JF|$ of the Jacobian matrix $JF = (D_iF_j)$ is a nonzero constant.*

Here and throughout this paper we write D_i for $\partial/\partial X_i$. We will continue to write JF for the Jacobian matrix of a polynomial map F, and the determinant of this matrix will be denoted by $|JF|$.

1.2. Specific Assertions for Fixed Degree and Dimension. A number of reductions and partial solutions of the problem lead us to formulate the following more specific statements. Note that under the hypothesis of each of these conjectures the conditon "$|JF|$ is a nonzero constant" is equivalent to $|JF| = 1$ (This can be seen by evaluating at the origin). The following definitions will be useful in stating the conjectures in the section.

DEFINITION 1.2. By the *degree* of a polynomial map $F = (F_1, \ldots, F_n)$ we mean the maximum of total degrees of the coordinate functions $F_1, \ldots, F_n$ in the variables $X_1, \ldots, X_n$.

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DEFINITION 1.3. We say that a polynomial map $F = (F_1, \ldots, F_n)$ is of *special type* if it has the form $F_i = X_i - H_i$ with H_i having only homogeneous summands of degree 2 and higher.

DEFINITION 1.4. We say that a polynomial map $F = (F_1, \ldots, F_n)$ is of *homogeneous type* if it has the form $F_i = X_i - H_i$ with H_i homogeneous of the same degree $d \geq 2$.

DEFINITION 1.5. We say that a polynomial map $F = (F_1, \ldots, F_n)$ is of *d-fold linear type,* for $d \geq 2$, if there are linear forms $L_1, \ldots, L_n \in \mathbb{C}[X_1, \ldots, X_n]$ such that F has the form $F_i = X_i - H_i$ with $H_i = L_i^d$.

DEFINITION 1.6. We say that a polynomial map $F = (F_1, \ldots, F_n)$ is of *symmetric type* if it is of special type and if the Jacobian matrix JF is a symmetric matrix.

DEFINITION 1.7. We say that a polynomial map $F = (F_1, \ldots, F_n)$ is of *symmetric homogeneous type* if it is both of symmetric type and of homogeneous type.

Given a fixed dimension $n \geq 1$ and a fixed degree bound $d \geq 2$:

CONJECTURE 1.8 ($\mathbf{JC}_{n,d}$). *Suppose $F = (F_1, \ldots, F_n)$ is a polynomial map of special type having degree $\leq d$ with $|JF| = 1$. Then F is an automorphism.*

CONJECTURE 1.9 ($\mathbf{JC}_{n,[d]}$). *Suppose $F = (F_1, \ldots, F_n)$ is a polynomial map of homogeneous type having degree d with $|JF| = 1$. Then F is an automorphism.*

CONJECTURE 1.10 ($\mathbf{LJC}_{n,[d]}$). *Suppose $F = (F_1, \ldots, F_n)$ is a polynomial map of d-fold linear type with $|JF| = 1$. Then F is an automorphism.*

CONJECTURE 1.11 ($\mathbf{SJC}_{n,d}$). *Suppose $F = (F_1, \ldots, F_n)$ is of symmetric type having degree $\leq d$. Then F is an automorphism.*

CONJECTURE 1.12 ($\mathbf{SJC}_{n,[d]}$). *Suppose $F = (F_1, \ldots, F_n)$ is of symmetric homogeneous type having degree d. Then F is an automorphism.*

REMARK 1.13. In the n-dimensional case where $F = X - H$ with H homogeneous we have these equivalences:

$$|JF| = 1 \iff JH \text{ is nilpotent} \iff (JH)^n = 0$$

(See, for example, [**W1**].)

1.3. Reductions and Known Cases. The following reduction, now standard knowledge, is proved in [**BCW**]:

THEOREM 1.14 (Cubic Reduction). *For any fixed integer $d \geq 3$ we have*

$$\begin{aligned} \mathbf{JC} &\iff \mathbf{JC}_{n,d} \text{ for all } n \\ &\iff \mathbf{JC}_{n,[d]} \text{ for all } n \end{aligned}$$

In particular, proving the Jacobian Conjecture is reduced to proving $\mathbf{JC}_{n,[3]}$ for all n.

An even stronger reduction was proved by Drużkowski in [**D1**]:

THEOREM 1.15 (Cubic Linear Reduction). *For any fixed integer $d \geq 3$ we have*

$$\mathbf{JC} \iff \mathbf{LJC}_{n,[d]} \text{ for all } n$$

In particular, proving the Jacobian Conjecture is reduced to proving $\mathbf{LJC}_{n,[3]}$ for all n.

In a recent breakthrough, Michiel de Bondt and Arno van den Essen proved this intriguing reduction ([**BE1**]):

THEOREM 1.16 (Symmetric Reduction). *For any fixed integer* $d \geq 3$ *we have*

$$\begin{aligned} \mathbf{JC} &\iff \mathbf{SJC}_{n,d} \text{ for all } n \\ &\iff \mathbf{SJC}_{n,[d]} \text{ for all } n \end{aligned}$$

In particular, proving the Jacobian Conjecture is reduced to proving $\mathbf{HJC}_{n,[3]}$ *for all* n.

Some known cases of the Jacobian Conjecture are:[1]

THEOREM 1.17 (Known Cases). *The following assertions hold:*

$\mathbf{JC}_{1,d}$ *for all* d	*(trivial)*
$\mathbf{JC}_{2,d}$ *for* $d \leq 100$	*(Moh* [**Mo**]*)*
$\mathbf{JC}_{2,[d]}$ *for all* d	*(trivial)*
$\mathbf{JC}_{n,2}$ *for all* n	*(Wang* [**Wa**]*)*
$\mathbf{JC}_{3,3}$	*(Moh, Sathaye* [**MoS**]*)*
$\mathbf{JC}_{3,[3]}$ + *linear triangularization*	*(Wright* [**W3**]*)*
$\mathbf{JC}_{3,[d]}$ *for all* d, + *linear triangularization*	*(de Bondt, van den Essen* [**BE3**]*)*
$\mathbf{JC}_{4,[3]}$	*(Hubbers* [**H**]*)*
$\mathbf{LJC}_{n,[3]}$ *for* $n \leq 5$	*(Drużkowski* [**D2**]*)*
$\mathbf{SJC}_{n,d}$ *for* $n \leq 4$, *all* d	*(de Bondt, van den Essen* [**BE2**]*)*
$\mathbf{SJC}_{5,[d]}$ *for* d	*(de Bondt, van den Essen* [**BE2**]*)*

(We always assume $d \geq 2$.*)*

1.4. The Potential Function in the Symmetric Case. The condition that JF is symmetric says that $D_iF_j = D_jF_i$ for each i, j. This means there exists $P \in \mathbb{C}[X_1, \ldots, X_n]$ such that $F_i = D_iP$ for $i = 1, \ldots, n$, i.e., $F = \nabla P$, the gradient of P. The polynomial P is called the *potential function* for F. Thus the symmetric case occurs precisely when the Jacobian matrix of F is the Hessian matrix of P:

$$JF = \operatorname{Hess} P = D_iD_jP$$

For $F = X - H$, the existence of a potential function for F is equivalent to the existence of one for H. For if F has potential function P, then H has potential function $\sum \frac{X_i^2}{2} - P$. It will be convenient to take P to be the potential function for H instead of F in the symmetric situation.

2. Ideals Defining the Jacobian Condition

[1]The sixth and seventh statements assert more strongly that in the $n = 3$ homogensous situation, maps satisfying the hypothesis of the Jacobian Conjecture can be linearly triangularized (see [**W3**]). In the second from last assertion one does not need to assume maps are of special type.

2.1. The Formal Map and the Formal Inverse. Let $F = (F_1, \ldots, F_n) = X - H$ be the *formal map*, meaning that each F_i is a formal power series of the form

$$F_i = X_i - \sum_{|k|\geq 2} a_i^k X^k = X_i - H_i$$

Here $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ (we will consistently write $\mathbb{N}$ for the set of nonnegative integers), $|k| = k_1 + \cdots + k_n$, and

$$a_i^k X^k = a_i^{(k_1,\ldots,k_n)} X_1^{k_1} \cdots X_n^{k_n}$$

The "coefficients" a_i^k of F are viewed as indeterminates lie in the polynomial ring $\mathbb{Z}[\{a_i^k\}]$.

It is known (and easily proved) that such an F has formal power series inverse $G = (G_1, \ldots, G_n)$ where each G_j has the form $X_j +$ higher degree terms.

Definition 2.1 (Inverse Coefficients). Letting $G = (G_1, \ldots, G_n) = X + N$ be the inverse of the formal map F, we define the elements $b_j^q \in \mathbb{Z}[\{a_i^k\}]$, for $j = 1, \ldots, n$ and $q \in \mathbb{N}^n$ with $|q| \geq 2$, by writing

$$G_j = X_j + \sum_{|q|\geq 2} b_j^q X^q = X_j + N_j\,. \tag{2.1}$$

It has been shown that b_j^q can be written as a sum of monomials in the indeterminates $\{a_i^k\}$ which are parameterized by certain combinatorial objects:

$$b_j^q = \sum_{T\in\mathbb{T}_j^q} m_T \;\in\; \mathbb{Z}[\{a_i^k\}]\,.$$

Here $\mathbb{T}_j^q$ is the set of isomorphism classes of labeled, planar trees with root label j and leaf-type q. The reader is refered to [**W2**] for an explanation of these terms, the definition of the monomials m_T, and the proof of this formula.

2.2. The Formal Symmetric Map and Its Inverse. Now we let $P = \sum_{|k|\geq 3} c^k X^k$, where c^k are indeterminates over $\mathbb{Z}$, and let $F = (F_1, \ldots, F_n)$ be the *formal symmetric map,* meaning that

$$F_i = X_i - D_i P\,.$$

We call P the *formal potential function.* Clearly $JF = I - \operatorname{Hess} P$, a symmetric matrix.

There is a unique ring homomorphism $\mathbb{Z}[\{a_i^k\}] \to \mathbb{Z}[\{c^k\}]$ sending a_i^k to $(k_i + 1)c^{k+e_i}$, where e_i is the element of $\mathbb{N}^n$ whose i^{th} coordinate is 1 and all other coordinates are zero. This, when extended to the power series rings in $X_1, \ldots, X_n$, sends the coordinate functions of the formal map defined in Section 2.1 to the coordinate functions of the formal symmetric map. Identifying the inverse coefficients b_i^q with their images in $\mathbb{Z}[\{c^k\}]$, and letting $F = X - H$ be the formal symmetric map, we again have $G = F^{-1}$ given by formula 2.1.

Proposition 2.2 (Potential Function for the Inverse). *With $N = G - X$ as in 2.1, viewing each b_i^q as an element of $\mathbb{Z}[\{c^k\}]$ as above, there is a power series $Q \in \mathbb{Z}[\{c^k\}][[X_1, \ldots, X_n]]$ such that $\nabla Q = N$.*

PROOF. We have $H = \nabla P$, and therefore $F = \nabla \tilde{P}$ where $\tilde{P} = \left(\frac{1}{2}\sum_{j=1}^{n} X_j^2\right) - P$. Let $\tilde{Q} = \sum_{j=1}^{n} X_j G_j - \tilde{P}(G)$. We have

$$\begin{aligned} D_i\tilde{Q} &= D_i\left(\sum_{j=1}^{n} X_j G_j\right) - D_i\left(\tilde{P}(G)\right) \\ &= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} (D_j\tilde{P})(G) D_i G_j \\ &= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} F_j(G) G_j \\ &= G_i + \sum_{j=1}^{n} X_j D_i G_j - \sum_{j=1}^{n} X_j D_i G_j = G_i\,, \end{aligned}$$

which shows that $\nabla\tilde{Q} = G$. Therefore $Q = \tilde{Q} - \frac{1}{2}\sum_{j=1}^{n} X_j^2$ has the desired property. □

DEFINITION 2.3 (Potential Inverse Coefficients). Letting Q be the potential function whose existence is asserted in Proposition 2.2, we define elements $d^q \in \mathbb{Z}[\{c^k\}]$, for $q \in \mathbb{N}^n$ with $|q| \geq 3$, by writing

$$Q = \sum_{|q|\geq 3} d^q X^q\,. \tag{2.2}$$

2.3. The Jacobian and Nilpotency Relations. Let $F = X - H$ be the formal map. We write JF for the Jacobian matrix of F and $|JF|$ for it's determinant. We have $|J(F)| = |I_n - J(H)|$, which lies in $\mathbb{Z}[\{a_i^k\}]\,[[X_1, \ldots, X_n]]$ having the form 1− (higher degree terms).

DEFINITION 2.4 (Jacobian Relations). Letting F be the formal map, we define elements $h^\ell \in \mathbb{Z}[\{a_i^k\}]$, for $\ell \in \mathbb{N}^n$ with $|\ell| \geq 1$, by writing

$$|J(F)| = 1 - \sum_{|\ell|\geq 1} h^\ell X^\ell\,.$$

It is shown in [**L**] that the polynomial h^ℓ can be written as a sum of monomials parameterized by certain combinatorial objects called *cycle sets.* Specifically,

$$h^\ell = \sum_{C\in\mathbb{C}^\ell} (-1)^{e(C)}\, m_C \;\in\; \mathbb{Z}[\{a_i^k\}] \tag{2.3}$$

In this formula $\mathbb{C}^\ell$ is the set of cycle sets $C = \{c_1 \ldots, c_r\}$ with leaf type ℓ. We refer the reader to [**L**] for the specifics.

Finally, motivated by Remark 1.13, we make the following definition.

DEFINITION 2.5 (Nilpotency Relations). Letting $F = X - H$ be the formal map we write the matrix $(JH)^n$ as (M_{ij}) and define the elements $n_{ij}^\ell \in \mathbb{Z}[\{a_i^k\}]$, for $1 \leq i, j \leq n$ and $\ell \in \mathbb{N}^n$ with $|\ell| \geq n$, by

$$M_{ij} = \sum_{|\ell|\geq 1} n_{ij}^\ell X^\ell\,.$$

2.4. Ideal Membership Reformulation of the Jacobian Conjecture. For integers $n \geq 1$ and $d \geq 2$, the following definitions define formal degree d polynomial maps F of various types in dimension n, the polynomial rings over $\mathbb{Q}$ generated by their coefficients, and ideals which impose the condition $|JF| = 1$.

DEFINITION 2.6. **(Nonhomogeneous)** We set

$$\mathcal{R}_{n,d} = \mathbb{Q}[\{\, a_i^k \mid k \in \mathbb{N}^n, 2 \leq |k| \leq d \,\}],$$

i.e., the polynomial ring over $\mathbb{Q}$ generated by the indeterminates $\{a_i^k\}$ for which $|k| \leq d$. It can be viewed as the quotient of $\mathbb{Q}[\{a_i^k\}]$ which sets $a_i^k = 0$ if $|k| > d$. We define the *formal polynomial map of degree d* to be the map $F = X - H$ where $H_i = \sum_{2 \leq |k| \leq d} a_i^k X^k$, for $1 = 1, \ldots, n$.

DEFINITION 2.7. **(Homogeneous)** We set

$$\mathcal{R}_{n,[d]} = \mathbb{Q}[\{\, a_i^k \mid k \in \mathbb{N}^n, |k| = d \,\}],$$

i.e., the polynomial ring over $\mathbb{Q}$ generated by the indeterminates $\{a_i^k\}$ for which $|k| = d$. It can be viewed as the quotient of $\mathbb{Q}[\{a_i^k\}]$ which sets $a_i^k = 0$ if $|k| \neq d$. We define the *formal degree d polynomial map of homogeneous type* to be the map $F = X - H$ where $H_i = \sum_{|k|=d} a_i^k X^k$, for $1 = 1, \ldots, n$.

To parameterize the d-fold linear situation we have

DEFINITION 2.8. **(d-Fold Linear)** Let l_{ij}, $1 \leq i, j \leq n$ be indeterminates. (They are the entries of the generic $n \times n$ matrix.) We set

$$\mathcal{R}^{\mathrm{lin}}_{n,[d]} = \mathbb{Q}[\{\, l_{ij} \mid 1 \leq i, j \leq n \,\}].$$

i.e., the polynomial ring over $\mathbb{Q}$ generated by these indeterminates. For $i = 1, \ldots, n$, let $L_i = \sum l_{ij} X_j \in \mathcal{R}^{\mathrm{lin}}_{n,[d]}[X_1, \ldots, X_n]$. We define the *formal d-linear polynomial map* to be the map $F = X - H$ where $H_i = L_i^d$, for $1 = 1, \ldots, n$.

To parameterize the symmetric situation we recall indeterminates $\{c^k\}$ where $k \in \mathbb{N}^n$ with $3 \leq |k|$. These are the coefficients of the formal potential function defined in section 1.4.

DEFINITION 2.9. **(Symmetric Nonhomogeneous)** We set

$$\mathcal{R}^{\mathrm{sym}}_{n,d} = \mathbb{Q}[\{\, c^k \mid k \in \mathbb{N}^n, 3 \leq |k| \leq d+1 \,\}],$$

i.e., the polynomial ring over $\mathbb{Q}$ generated by the indeterminates $\{c^k\}$ for which $|k| \leq d+1$. It can be viewed as the quotient of $\mathbb{Q}[\{c^k\}]$ which sets $c^k = 0$ for $|k| > d+1$. We define *the generic potential function of degree $d+1$* to be $P = \sum_{3 \leq |k| \leq d+1} c^k X^k$. We define the *formal degree d polynomial map of symmetric type* to be the map $F = X - H$ where $H_i = D_i P$, for $1 = 1, \ldots, n$ (i.e., $H = \nabla P$).

DEFINITION 2.10. **(Symmetric Homogeneous)** We set

$$\mathcal{R}^{\mathrm{sym}}_{n,[d]} = \mathbb{Q}[\{\, c^k \mid k \in \mathbb{N}^n, |k| = d+1 \,\}],$$

i.e., the polynomial ring over $\mathbb{Q}$ generated by the indeterminates $\{c^k\}$ for which $|k| = d+1$. We define *the generic homogeneous potential function of degree $d+1$* to be $P = \sum_{|k|=d+1} c^k X^k$. We define the *formal degree d polynomial map of homogeneous symmetric type* to be the map $F = X - H$ where $H_i = D_i P$, for $1 = 1, \ldots, n$ (i.e., $H = \nabla P$).

Note that the rings $\mathcal{R}_{n,d}$, $\mathcal{R}_{n,[d]}$, $\mathcal{R}^{\text{lin}}_{n,[d]}$, $\mathcal{R}^{\text{sym}}_{n,d}$, and $\mathcal{R}^{\text{sym}}_{n,[d]}$ are polynomial rings over $\mathbb{Q}$ in finitely many indeterminates.

DEFINITION 2.11 (Specialization Homomorphisms). For fixed n and d we have this commuting diagram of *specialization homomorphisms* amongst the rings defined above

$$\begin{array}{ccccccc} \mathbb{Q}[\{a_i^k\}] & \to & \mathcal{R}_{n,d} & \to & \mathcal{R}_{n,[d]} & \to & \mathcal{R}^{\text{lin}}_{n,[d]} \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Q}[\{c^k\}] & \to & \mathcal{R}^{\text{sym}}_{n,d} & \to & \mathcal{R}^{\text{sym}}_{n,[d]} & & \end{array} \tag{2.4}$$

where the maps are as follows. The map $\mathbb{Q}[\{a_i^k\}] \to \mathbb{Q}[\{c^k\}]$ was defined in section 2.2. There is a natural surjective homomorphism $\mathbb{Q}[\{a_i^k\}] \to \mathcal{R}_{n,d}$ which sets $a_i^k = 0$ if $|k| > d$. Similarly, there is a surjection $\mathbb{Q}[\{c^k\}] \to \mathcal{R}^{\text{sym}}_{n,d}$. Setting $a_i^k = 0$ if $|k| \neq d$ defines a surjective homomorphism $\mathcal{R}_{n,d} \to \mathcal{R}_{n,[d]}$. We get a surjective homomorphism $\mathcal{R}_{n,d} \to \mathcal{R}^{\text{sym}}_{n,d}$ by sending a_i^k to $(k_i+1)c^{k+e_i}$, where e_i is the element of $\mathbb{N}^n$ whose i^{th} coordinate is 1 and all other coordinates are zero. Note that this sends the coefficient of X^k in H_i to the coefficient of X^k in D_iP, where $X - H$ is the formal polynomial map of degree d and P is the formal potential function of degree $d+1$. In similar fashion we get a homomorphism $\mathcal{R}_{n,[d]} \to \mathcal{R}^{\text{sym}}_{n,[d]}$. Finally we have $\mathcal{R}_{n,[d]} \to \mathcal{R}^{\text{lin}}_{n,[d]}$ (non-surjective) defined by sending the coefficient a_i^k, $|k| = d$, of X^k in the i^{th} coordinate function of the formal polynomial map of homogeneous type, to the coefficient of X^k in the i^{th} coordinate function of the formal d-fold linear polynomial map.

The image of the homomorphism

$$\mathbb{Q}[\{a_i^k\}][[X_1, \ldots, X_n]] \to \mathcal{R}_{n,d}[[X_1, \ldots, X_n]]$$

induced from (2.4) sends the coordinate functions of the formal map defined in section 2 to those of the formal degree d polynomial map defined in Definition 2.6. Following this by the other homomorphisms of power series rings induced by the homomorphisms in the diagram (2.4), we get the coordinates of the other formal polynomial maps defined in Definitions 2.7, 2.8, 2.9, and 2.10.

In the following definition, and in for the remainder of this paper, we identify the Jacobian relations $h^\ell \in \mathbb{Q}[\{a_i^k\}]$ of Definition 2.4, the nilpotency relations n_{ij}^ℓ of Definition 2.5, and the inverse coefficients b_i^q of Definition 2.1 with their images in the various rings in diagram 2.4.

DEFINITION 2.12 (The Jacobian and Nilpotency Ideals).

(1) We let $\mathfrak{J}_{n,d}$ be the ideal in $\mathcal{R}_{n,d}$ generated by Jacobian relations $\{h^\ell\}$.
(2) We let $\mathfrak{J}_{n,[d]}$ be the ideal in $\mathcal{R}_{n,[d]}$ generated by the Jacobian relations $\{h^\ell\}$. We let $\mathfrak{N}_{n,[d]}$ be the ideal in $\mathcal{R}_{n,[d]}$ generated by the nilpotency relations $\{n_{ij}^\ell\}$.
(3) We let $\mathfrak{J}^{\text{lin}}_{n,[d]}$ be the ideal in $\mathcal{R}^{\text{lin}}_{n,[d]}$ generated by the Jacobian relations $\{h^\ell\}$. We let $\mathfrak{N}^{\text{lin}}_{n,[d]}$ be the ideal in $\mathcal{R}^{\text{lin}}_{n,[d]}$ generated by the nilpotency relations $\{n_{ij}^\ell\}$.
(4) We let $\mathfrak{J}^{\text{sym}}_{n,d}$ be the ideal in $\mathcal{R}^{\text{sym}}_{n,d}$ generated by Jacobian relations $\{h^\ell\}$.

(5) We let $\mathfrak{J}^{\mathrm{sym}}_{n,[d]}$ be the ideal in $\mathcal{R}^{\mathrm{sym}}_{n,[d]}$ generated by the Jacobian relations $\{h^\ell\}$. We let $\mathfrak{N}^{\mathrm{sym}}_{n,[d]}$ be the ideal in $\mathcal{R}^{\mathrm{sym}}_{n,[d]}$ generated by the nilpotency relations $\{n^\ell_{ij}\}$.

In the homogeneous situations, the Jacobian ideals and the nilpotency ideals have the following relationships:

PROPOSITION 2.13. *We have*

$$\mathfrak{J}_{n,[d]} \supseteq \mathfrak{N}_{n,[d]}\,, \qquad \mathfrak{J}^{\mathrm{lin}}_{n,[d]} \supseteq \mathfrak{N}^{\mathrm{lin}}_{n,[d]}\,, \qquad \mathfrak{J}^{\mathrm{sym}}_{n,[d]} \supseteq \mathfrak{N}^{\mathrm{sym}}_{n,[d]}\,,$$

the containments all being proper for $n \geq 2$. Moreover,

$$\sqrt{\mathfrak{J}_{n,[d]}} = \sqrt{\mathfrak{N}_{n,[d]}}\,, \quad \sqrt{\mathfrak{J}^{\mathrm{lin}}_{n,[d]}} = \sqrt{\mathfrak{N}^{\mathrm{lin}}_{n,[d]}}\,, \quad \sqrt{\mathfrak{J}^{\mathrm{sym}}_{n,[d]}} = \sqrt{\mathfrak{N}^{\mathrm{sym}}_{n,[d]}}$$

for all n.

PROOF. The first containments follow easily from some basic properties of nilpotent matrices. (See [**W1**], for example.) The proofs will also appear in [**W5**]. The latter containments follow from Remark 1.13. □

Putting together the Degree Bound[2], the Gap Theorem[3], and Hilbert's Nullstellensatz[4], we conclude:

THEOREM 2.14. *We have these equivalences:*

$$\begin{aligned}
\mathbf{JC}_{n,d} &\iff b^q_j \in \sqrt{\mathfrak{J}_{n,d}} \\
&\qquad j = 1,\dots,n,\ q \in \mathbb{N}^n,\ d^{n-1} < |q| \leq d^n \\
\mathbf{JC}_{n,[d]} &\iff b^q_j \in \sqrt{\mathfrak{J}_{n,[d]}} \\
&\qquad j = 1,\dots,n,\ q \in \mathbb{N}^n,\ d^{n-1} < |q| \leq d^n \\
\mathbf{LJC}_{n,[d]} &\iff b^q_j \in \sqrt{\mathfrak{J}^{\mathrm{lin}}_{n,[d]}} \\
&\qquad j = 1,\dots,n,\ q \in \mathbb{N}^n,\ d^{n-1} < |q| \leq d^n \\
\mathbf{SJC}_{n,d} &\iff d^q \in \sqrt{\mathfrak{J}^{\mathrm{sym}}_{n,d}} \\
&\qquad q \in \mathbb{N}^n,\ d^{n-1}+1 < |q| \leq d^n+1 \\
\mathbf{SJC}_{n,[d]} &\iff d^q \in \sqrt{\mathfrak{J}^{\mathrm{sym}}_{n,[d]}} \\
&\qquad q \in \mathbb{N}^n,\ d^{n-1}+1 < |q| \leq d^n+1
\end{aligned}$$

Theorem 3.11 will allow us to replace the condition $d^{n-1} < |q| \leq d^n$ by $d^{n-1} < |q| \leq 2d^{n-1} - 1$ in each of the three the homogeneous cases above, making it sufficient to establish the ideal membership condition over a much smaller "gap" in those cases.

[2]due to O. Gabber, which asserts that the degree of the inverse of a polynomial automorphism of degree d is bounded by d^{n-1} (see [**BCW**] or [**E**]).

[3]which says if those homogeneous summands of the inverse having degree $d^{n-1}+1, d^{n-1}+2, \dots, d^n$ are all zero, then the inverse is a polynomial.

[4]Hilbert's Nullensatz assumes the ground field is algebraically closed. Our result follows using the following elementary fact: Given A an affine domain over a field k, $I \subset A$ an ideal, $a \in A$, and k' a seperable algebraic extension of k. If $a \in \sqrt{I \otimes_k k'}$, then $a \in \sqrt{I}$.

QUESTIONS 2.15. These equivalences immediately impose a number of questions about the ideals $\mathfrak{J}_{n,d}$, $\mathfrak{J}_{n,[d]}$, and the other Jacobian ideals, for given n and d, including:

(1): Is $\mathfrak{J}_{n,d} = \sqrt{\mathfrak{J}_{n,d}}$? Similarly for $\mathfrak{J}_{n,[d]}$?

(2): If $\mathfrak{J}_{n,d} \subsetneqq \sqrt{\mathfrak{J}_{n,d}}$, can generators be given for $\sqrt{\mathfrak{J}_{n,d}}$ which can be realized combinatorially in some manner along the lines of formula 2.3 for h^ℓ?

(3): Is $\sqrt{\mathfrak{J}_{n,d}}$ a prime ideal? If so, what is the geometry (e.g., singular locus) of the irreducible variety it defines? Similarly for $\mathfrak{J}_{n,[d]}$ (which is a homogeneous ideal hence defines a projective locus)?

(4): If $\sqrt{\mathfrak{J}_{n,d}}$ is not prime, what are its prime components, and what is their geometry? Similarly for $\mathfrak{J}_{n,[d]}$?

(5): In any of the equivalences of Theorem 2.14, can the radical be dropped?

(6): Even if the radicals of Theorem 2.14 cannot be dropped, does $\mathbf{JC}_{n,d}$ imply $b_j^q \in \mathfrak{J}_{n,d}$ for sufficietly large $|q|$? Similarly for $\mathbf{JC}_{n,[d]}$ $\mathfrak{J}_{n,[d]}$?

(7): Given an integer N with $N > d^n$, does the condition $b_j^q \in \mathfrak{J}_{n,d}$ for all $q \in \mathbb{N}^n$ with $|q| = N$ imply $b_j^q \in \mathfrak{J}_{n,d}$ for all $q \in \mathbb{N}^n$ with $|q| > N$? Similarly for $\mathfrak{J}_{n,[d]}$? In other words, does it suffice to prove the generic vanishing of one homogeneous summand of the formal inverse?

Question **(6)** has an affirmative answer, as seen in the following theorem.

THEOREM 2.16. *For given $n \geq 1$, $d \geq 2$, $\mathbf{JC}_{n,d}$ is equivalent to the following assertion: There exists an integer $N_{n,d} > d^{n-1}$ such that $b_j^q \in \mathfrak{J}_{n,d}$ for $j = 1, \ldots, n$, $q \in \mathbb{N}^n$, $|q| > N_{n,d}$. An analogous statement holds for each of $\mathbf{JC}_{n,[d]}$, $\mathbf{LJC}_{n,[d]}$, $\mathbf{SJC}_{n,d}$, and $\mathbf{SJC}_{n,[d]}$.*

PROOF. This follows from the fact that a polynomial map F over a ring R is invertible if (and only if) the map $\bar{F}$ over the ring $R/(\operatorname{Nil} R)$ obtained by base change is invertible, where $\operatorname{Nil} R$ is the nilradical of R. This is Lemma 1.1.9 of [**E**]. Appling this to $\mathfrak{R}_{n,d}$, with F being the formal degree d polynomial map, and taking $N_{n,d}$ to be the degree of $\bar{F}^{-1}$ gives the result. Since the degree bound d^{n-1} is attained for certain maps, we must have $N_{n,d} > d^{n-1}$. □

REMARK 2.17. In answer to Question **(5)** of 2.15, the author asserted in [**W4**]: "It is known that the radicals... cannot be removed. For example, computer calculations show that not all b_j^q lie in $\mathfrak{J}_{n,[d]}$ for $n = 3$, $d = 2$, and $|q| = 5$." This statement was carelessly based on a vague recollection of a claim by another mathematician, communicated verbally. After a number of inquiries about this statement the author felt behooved to perform some computations to confirm its veracity. Computations using the symbolic algebra program Singular indeed verified this assertion, and produced other results which will be reported in Section 4.

3. Formulas for the Formal Inverse

The formulas for the formal inverse given in this section provide means for the Jacobian Conjecture to be addressed as a problem in combinatorics. They also give specific ways to realize the homogeneous summands of the formal inverse, thereby allowing one to attack the conjecture via Theorem 2.14, and to solve $\mathbf{JC}_{n,[d]}$ for certain small values of n and d using a symbolic algebra computer program.

3.1. The Bass-Connell-Wright Tree Formula. Let $\mathbb{T}_{\mathrm{rt}}$ be the set of isomorphism classes of finite rooted trees. For $G = F^{-1}$, the Tree Formula of Bass-Connell-Wright states:

THEOREM 3.1 (BCW Tree Formula). *Let F be the formal map defined in Section 2.1, and let $G = (G_1, \ldots, G_n)$ be the formal inverse. Then*

$$G_i = X_i + \sum_{T \in \mathbb{T}_{rt}} \frac{1}{|Aut\ T|} \mathcal{P}_{T,i}$$

where

$$\mathcal{P}_{T,i} = \sum_{\substack{\ell: V(T) \to \{1,\ldots,n\} \\ \ell(rt_T) = i}} \prod_{v \in V(T)} D_{\ell(v^+)} H_{\ell(v)} \,.$$

In this expression v^+ is the set $\{w_1, \ldots, w_t\}$ of children of v and $D_{\ell(v^+)} = D_{\ell(w_1)} \cdots D_{\ell(w_t)}$.

In the case where F is specialized to the formal polynomial map of homogeneous type of degree $d \geq 2$, the polynomial $\mathcal{P}_{T,i}$ is homogeneous of degree $m(d-1)+1$ where $m = |V(T)|$, the number of vertices in the tree T. Hence letting $\mathbb{T}_{\mathrm{rt}_m}$ be the set of trees in $\mathbb{T}_{\mathrm{rt}}$ having m vertices, we have:

THEOREM 3.2 (Bass-Connell-Wright Homogeneous Tree Formula). *Let $F = X - H$ be the formal degree d polynomial map of homogeneous type. Then the formal inverse has the form $G = X + N$ where*

$$N = N^{(1)} + N^{(2)} + N^{(3)} + \cdots$$

with $N^{(m)}$ homogeneous of degree $m(d-1)+1$ and given by the formula

$$N^{(m)} = \sum_{T \in \mathbb{T}_{rt_m}} \frac{1}{|Aut\ T|} \mathcal{P}_{T,i} \,. \tag{3.1}$$

From this and Theorem 2.14 we conclude:

COROLLARY 3.3. *The assertion $\mathbf{JC}_{n,[d]}$ holds if an only if all coefficients of $N^{(m)}$ lie in $\sqrt{\mathfrak{J}_{n,[d]}}$ for $\frac{d^{n-1}-1}{d-1} < m \leq \frac{d^n-1}{d-1}$, where $N^{(m)}$ is given by (3.1).*

3.2. The Tree Formula for the Symmetric Case. The formula of Bass-Connell-Wright takes on a simpler form in the symmetric case.

We now let $\mathbb{T}$ be the set of isomorphism classes of finite trees (having no designated root).

THEOREM 3.4 (Symmetric Tree Formula). *Let $F = X - \nabla P$ be the formal symetric map defined in section 2.2, and let $G = (G_1, \ldots, G_n)$ be its inverse. Then $G = X + \nabla Q$ with*

$$Q = \sum_{T \in \mathbb{T}} \frac{1}{|Aut\ T|} \mathcal{Q}_T$$

where

$$\mathcal{Q}_T = \sum_{\ell: E(T) \to \{1,\ldots,n\}} \prod_{v \in V(T)} D_{adj(v)} P \,.$$

Here $adj(v)$ is the set $\{e_1, \ldots, e_s\}$ of edges adjacent to v and $D_{adj(v)} = D_{\ell(e_1)} \cdots D_{\ell(e_s)}$.

The proof will appear in [**W5**]. A formula somewhat like the above appears without proof in [**Me**].

For the symmetric homogeneous case Theorem 3.4 specializes to the following, in the light of Proposition 2.2. Here we let $\mathbb{T}_m$ be the set of isomorphism classes of (non-rooted) trees having m vertices.

THEOREM 3.5 (Symmetric Homogeneous Tree Formula). *Let $F = X - \nabla P$ be the formal degree d polynomial map of symmetric homogeneous type (see Definition 2.10). (Here P is the formal potential function of degree $d+1$.) Let $G = (G_1, \ldots, G_n)$ be its formal inverse. Then $G = X + \nabla Q$ with*

$$Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots$$

and

$$\boldsymbol{Q^{(m)}} = \sum_{T \in \mathbb{T}_m} \frac{\mathbf{1}}{|Aut\ \boldsymbol{T}|} \mathfrak{Q}_{\boldsymbol{T}} .$$

$Q^{(m)}$ has degree $m(d-1)+2$.

Identifying the polynomials $N^{(m)} \in \mathcal{R}_{n,[d]}[X_1, \ldots, X_n]$ with their images in in $\mathcal{R}^{\mathrm{sym}}_{n,[d]}[X_1, \ldots, X_n]$, it is clear from Theorems 3.2 and 3.4 that $N^{(m)} = \nabla Q^{(m)}$. Note also that the Q in this theorm is the specialization of the Q of Proposition 2.2 and Definition 2.3 to the ring $\mathcal{R}^{\mathrm{sym}}_{n,[d]}[[X_1, \ldots, X_n]]$.

3.3. Zhao's Formulas. The formulas of Section 3.1 are intriguing from a combinatoric perspective, but they do not give a practical way to calculate summands of the formal inverse, say, using a computer. For such purposes, the following, proved in [**Z1**], is more useful.

THEOREM 3.6 (Zhao's Formula). *As in Theorem 3.2, let $N^{(m)}$, $m \geq 1$, be the homogeneous summands of the formal inverse of the formal degree d polynomial map $F = X - H$ of homogeneous type. Then $N^{(1)} = H$ and, for $m \geq 2$,*

$$\boldsymbol{N^{(m)}} = \frac{\mathbf{1}}{\boldsymbol{m-1}} \sum_{\substack{k+\ell=m \\ k,\ell \geq 1}} \boldsymbol{JN^{(k)} \cdot N^{(\ell)}} .$$

(In this formula $N^{(\ell)}$ is viewed as a column vector.)

In fact, Zhao's formula gives formally converging summands for $N = G - X$ even when H is not homogeneous, i.e., when $F = X - H$ is the formal (non-homogeneous) map. The summands $N^{(m)}$ are, of course, no longer homogeneous.

So, for example, we have

$$\begin{aligned} N^{(2)} &= JN^{(1)} \cdot N^{(1)} \\ N^{(3)} &= \frac{1}{2}\left[JN^{(2)} \cdot N^{(1)} + JN^{(1)} \cdot N^{(2)}\right] \end{aligned} \tag{3.2}$$

A second formula of Zhao ([**Z2**]) gives the a formula for the inverse potential function in the symmetric situation:

THEOREM 3.7 (Zhao's Formula for the Symmetric Case). *As in Theorem 3.5, let $Q^{(m)}$, $m \geq 1$, be the homogeneous summands of the potential function for $N =$*

$G-X$, where G is formal inverse of the formal degree d polynomial map $F = X-\nabla P$ of symmetric homogeneous type. Then $Q^{(1)} = P$ and, for $m \geq 2$,

$$Q^{(m)} = \frac{1}{2(m-1)} \sum_{\substack{k+\ell=m \\ k,\ell\geq 1}} \left(\nabla Q^{(k)} \cdot \nabla Q^{(\ell)}\right).$$

(Here $(\nabla Q^{(k)} \cdot \nabla Q^{(\ell)})$ denotes the usual dot product of vectors.)

Again it should be noted that this theorem holds in the nonhomogeneous case as well, giving nonhomogeneous converging summands for the potential function for N.

The formula of Theorem 3.6 provides a quick proof of the well known fact that the Jacobian Conjecture holds in the case where $F = X - H$ is of homogeneous type and $(JH)^2 = 0$; in fact it gives a stronger ideal membership statement:

THEOREM 3.8. *Let $F = X - H$ be the formal degree d polynomial map of homogeneous type in dimension n, and let $\mathcal{I}$ be the ideal in $\mathcal{R}_{n,[d]}$ generated by the coefficients of $(JH)^2$. Then all coefficients of $N^{(m)}$, for $m \geq 2$, are in $\mathcal{I}$.*

PROOF. Since H is homegeneous we have $H = \frac{1}{d}JH \cdot X$ by Euler's formula. Since $N^{(1)} = H$ we have, by 3.2,

$$N^{(2)} = JN^{(1)} \cdot N^{(1)} = \frac{1}{d}\,JH \cdot JH \cdot X = \frac{1}{d}\,(JH)^2 \cdot X$$

which shows the coefficients of $N^{(2)}$ lie in $\mathcal{I}$. Now we proceed inductively using Zhao's formula. We have from Theorem 3.6, setting $m = 3$,

$$N^{(3)} = \frac{1}{2}\left[JN^{(2)} \cdot N^{(1)} + JN^{(1)} \cdot N^{(2)}\right],$$

showing that the coefficients for $N^{(3)}$ are in $\mathcal{I}$, and similarly for all subsequent $N^{(m)}$. □

The following corollary is immediate.

COROLLARY 3.9. *If $F = X - H$ is a polynomial map of homogeneous type with $(JH)^2 = 0$, then F is invertible with $F^{-1} = X + H$.*

In the case $n = 2$, the ideal $\mathcal{I}$ coincides with the ideal $\mathcal{N}_{n,[2]}$ defined in Definition 2.12. Since $\mathcal{N}_{n,[2]} \subset \mathcal{J}_{n,[2]}$ (Proposition 2.13), Theorem 3.8 provides a positive answer to Question **(5)** of 2.15 for $n = 2$:

COROLLARY 3.10. *For $j = 1, 2$, $q \in \mathbb{N}^2$, and $|q| > d$ we have $b_j^q \in \mathcal{J}_{2,[d]}$. More strongly, $b_j^q \in \mathcal{N}_{2,[d]}$.*

Zhao's formula leads to an improved gap theorem for polynomial maps of homogeneous type, as follows.

Again letting $F = X - H$ be the formal degree d polynomial map of homogeneous type in dimension n, suppose that for some $m \geq 1$, the coefficients of $N^{(m+1)}, N^{(m+2)}, \ldots, N^{(2m)}$ all lie in some ideal $\mathcal{J} \subset \mathcal{R}_{n,[d]}$. The Zhao Formula (Theorem 3.6) applied to $N^{(2m+1)}$ shows that its coefficients are in $\mathcal{J}$ as well. Inductively we conclude that the coefficients of all $N^{(s)}$ with $s > m$ lie in $\mathcal{J}$.

Now we apply this to the case where $\mathcal{J} = \mathcal{J}_{n,[d]}$ and $N^{(m)}$ has degree d^{n-1}, the inverse degree bound for polynomial automorphisms. Since $N^{(m)}$ is homogeneous of degree $m(d-1)+1$ (see Theorem 3.2) this happens when $m = \frac{d^{n-1}-1}{d-1} = 1+d+d^2+\cdots+d^{n-2}$, and in this case $N^{(2m)}$ will have degree $2d^{n-1}-1$. Hence in equivalent condition to $\mathbf{JC}_{n,[d]}$ in Theorem 2.14 d^n can be replaced by $2d^{n-1}-1$, making it sufficient to establish the ideal membership conditions of Theorem 2.14 over a much smaller "gap". We have:

THEOREM 3.11 (Gap Theorem for Maps of Homogeneous Type). *We have these equivalences:*

$$\begin{aligned}
\mathbf{JC}_{n,[d]} &\iff b_j^q \in \sqrt{\mathfrak{J}_{n,[d]}} \\
&\qquad j = 1,\ldots,n,\ q \in \mathbb{N}^n,\ d^{n-1} < |q| \le 2d^{n-1}-1 \\
\mathbf{LJC}_{n,[d]} &\iff b_j^q \in \sqrt{\mathfrak{J}_{n,[d]}^{\mathrm{lin}}} \\
&\qquad j = 1,\ldots,n,\ q \in \mathbb{N}^n,\ d^{n-1} < |q| \le 2d^{n-1}-1 \\
\mathbf{SJC}_{n,[d]} &\iff d^q \in \sqrt{\mathfrak{J}_{n,[d]}^{\mathrm{sym}}} \\
&\qquad q \in \mathbb{N}^n,\ d^{n-1}+1 < |q| \le 2d^{n-1}
\end{aligned}$$

4. Ideal Membership Results

In this section all resutls obtained by computer were performed using the symbolic algebra program Singular. Inverse coefficients b_i^q were computed using Zhao's Formula (Theorem 3.6).

4.1. Necessity of the Radical. Motivated by Theorem 2.14 and Question **(5)** of 2.15, we define the ideal membership assertions as follows:

ASSERTION 4.1 ($\mathbf{IM}_{n,d}$). For $j = 1,\ldots,n$, we have $b_j^q \in \mathfrak{J}_{n,d}$, for all $q \in \mathbb{N}^n$ with $|q| > d^{n-1}$.

ASSERTION 4.2 ($\mathbf{IM}_{n,[d]}$). For $j = 1,\ldots,n$, we have $b_j^q \in \mathfrak{J}_{n,[d]}$, for all $q \in \mathbb{N}^n$ with $|q| > d^{n-1}$.

REMARK 4.3. It is not difficult to establish that, for $\mathbf{IM}_{n,d}$, it suffices to show the ideal membership condition for $|q| \le d^n$, and for $\mathbf{IM}_{n,[d]}$, it suffices to show the ideal membership condition for $|q| \le 2d^{n-1}-1$. The latter uses the improved gap theorem, Theorem 3.11.

The first result shows that the radical is needed in smallest nontrivial nonhomogeneous situation: $n = 2$, $d = 3$. Since $\mathbf{JC}_{2,d}$ for small d (Theorem 1.17) we have $b_i^q \in \sqrt{\mathfrak{J}_{2,d}}$ for $q \in \mathbb{N}^2$ with $|q| > 2$, by Theorem 2.14. However,

PROPOSITION 4.4. *The assertion $\mathbf{IM}_{2,3}$ fails. Specifically, the coefficient $t = b_1^{(0,4)}$ of X_2^4 in G_1 has the properties $t^6 \notin \mathfrak{J}_{2,3}$, $t^7 \in \mathfrak{J}_{2,3}$.*

PROOF. This was verified by computer. □

However, we have:

PROPOSITION 4.5. *The assertion $\mathbf{IM}_{2,[d]}$ holds for all $d \ge 2$.*

PROOF. This follows from Corollary 3.10. □

The following further result deals with Question **(1)** of 2.15 for the $n = 2$ homogeneous situation. It will be proved in the forthcoming paper [**W5**].

PROPOSITION 4.6. *The ideal $\mathfrak{J}_{2,[d]}$ is a radical ideal only when $d = 2$.*

In [**W5**] it will be shown that the ideals $\mathfrak{J}_{2,[d]}$ define certain classical embeddings of $\mathbb{P}^1_{\mathbb{C}}$ in higher dimensional projective space, and explicit generators for the larger ideal $\sqrt{\mathfrak{J}_{2,[d]}}$ will be given.

Since $\mathbf{JC}_{\boldsymbol{n},[\mathbf{2}]}$ holds for all n (Theorem 1.17) we have $b_i^q \in \sqrt{\mathfrak{J}_{n,[2]}}$ for $q \in \mathbb{N}^n$ with $|q| > 2^{n-1}$, by Theorem 2.14. The following proposition says the radical cannot be removed when $n = 3$, justifying the statement in Remark 2.17.

PROPOSITION 4.7. *The assertion $\mathbf{IM}_{\mathbf{3},[\mathbf{2}]}$ fails. Specifically, every coefficient b_i^q of $N^{(4)}$ and $N^{(5)}$ (here $|q| = 5, 6$, respectively) fails to lie in $\mathfrak{J}_{3,[2]}$. However, all coefficients b_i^q of $N^{(6)}$ (where $|q| = 7$), lie in $\mathfrak{J}_{3,[2]}$.*

PROOF. Verified by computer. □

These results point to the likelihood that the radicals in Theorem 2.14 are indeed needed, exept in the $n = 2$ homogeneous case, for low $|q| > d^{n-1}$. Theorem 2.16 established that if $\mathbf{JC}_{\boldsymbol{n},\boldsymbol{d}}$ (respectively $\mathbf{JC}_{\boldsymbol{n},[\boldsymbol{d}]}$) holds then in fact $b_i^q \in \mathfrak{J}_{n,d}$ (resp. $b_i^q \in \mathfrak{J}_{n,[d]}$) when $|q|$ is sufficiently large.

Finally, we report results in the quadratic linear and cubic linear cases for $n = 3$.

PROPOSITION 4.8. *We have*

(1) *The coefficients of $N^{(4)}$ and $N^{(5)}$ (hence all b_i^q with $5 \le |q| \le 6$) lie in $\mathfrak{J}^{\mathrm{sym}}_{3,[2]}$.*
(2) *The coefficients of $N^{(5)}$ (hence All b_i^q with $|q| = 11$) lie in $\mathfrak{J}^{\mathrm{sym}}_{3,[3]}$.*

PROOF. Verified by computer. In both cases the computation for the coefficients of $N^{(m)}$ for larger m failed due to space limitations. □

Note that in both assertions of Proposition 4.8 the terms $N^{(m)}$ are the lowest ones whose degree is greater than the degree bound d^{n-1}. This proposition raises the question of when the radical is needed in the d-fold linear situation. Could it be that the radical is not needed for b_i^q beyond the degree bound?

4.2. Further Results. The following theorem was essentially proved by Dan Singer in [**Si**].

THEOREM 4.9. *Let $F = X - H$ be the formal polynomial map of homogeneous type of dimension n, degree 2, and let $\mathfrak{I}$ be the ideal in $\mathcal{R}_{n,[2]}$ generated by the coefficients of $(JH)^3$. Then all coefficients b_i^q of $N^{(m)}$ for $m \ge 6$ (i.e. $|q| \ge 7$), are in $\mathfrak{I}$.*

Actually the theorem stated in [**Si**] is the following corollary, but Singer's proof shows the above theorem to be true.

COROLLARY 4.10. *If $F = X - H$ is a polynomial map of homogeneous type having degree $d = 2$ with $(JH)^3 = 0$, then F is invertible with*

$$F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)} + N^{(5)}.$$

In particular, the degree of F^{-1} is ≤ 6.

The following theorems and their corollaries are new results for the symmetric situation. Their proofs will appear in [**W5**], and will use the Symmetric Tree Formula (Theorem 3.4).

THEOREM 4.11. *Let $F = X - H$ be the formal degree $d \geq 2$ polynomial map of homogeneous type with symmetric Jacobian matrix in dimension n, and let $\mathfrak{I}$ be the ideal in $\mathcal{R}^{\mathrm{sym}}_{n,[d]}$ generated by the coefficients of $(JH)^3$. Then all coefficients b_i^q of $N^{(m)}$ for $m \geq 3$ (hence all b_i^q with $|q| \geq 2d$) are in $\mathfrak{I}$.*

From this it follows that:

COROLLARY 4.12. *If $F = X - H$ is a polynomial map with symmetric Jacobian matrix of homogeneous type with $(JH)^3 = 0$, then F is invertible with*

$$F^{-1} = X + N^{(1)} + N^{(2)}\,.$$

In particular, the degree of F^{-1} is $\leq 2d - 1$ (independent of n).

THEOREM 4.13. *Let $F = X - H$ be the formal degree 2 polynomial map of homogeneous type with symmetric Jacobian matrix in dimension n, and let $\mathfrak{I}$ be the ideal in $\mathcal{R}^{\mathrm{sym}}_{n,[2]}$ generated by the coefficients of $(JH)^4$. Then all coefficients b_i^q of $N^{(m)}$ for $m \geq 5$ (i.e. $|q| \geq 6$) are in $\mathfrak{I}$.*

Theorem 4.13 improves Wang's result of Theorem 1.17 in the symmetric $JH^4 = 0$ case by giving a bound on the degree of the inverse which for large n is better than the usual degree bound d^{n-1}.

COROLLARY 4.14. *If $F = X - H$ is a degree 2 polynomial map with symmetric Jacobian matrix of homogeneous type with $(JH)^4 = 0$, then F is invertible with*

$$F^{-1} = X + N^{(1)} + N^{(2)} + N^{(3)} + N^{(4)}\,.$$

In particular, the degree of F^{-1} is ≤ 5 (independent of n).

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