Fall 12-1-2014

Dispersion of EM Waves

Ronald Lovett
Washington University in St. Louis

The Dispersion of EM Waves

Introduction

When light travels through empty space, it travels with a speed

$$c = (2.997925 \pm 0.000003)\ 10^8 m/s.$$

We are taught, however, that light travels in matter with a speed c/n, n the *index of refraction* of the medium. But matter is mostly (how big is an electron?) vacuum! How can light travel with a speed different from c? In fact, it doesn't. To understand this, this analysis presents

(1.) a simple model for the interaction of light with matter;

(2.) a description of an *index of refraction* experiment;

(3.) an explanation of why $n < 1$ for high frequency waves;

(4.) and an explanation of why (3.) doesn't conflict with the rule that nothing can move faster than c.

1. The interaction of light with matter

A simple model

There are many ways in which matter can interact with an electromagnetic (EM) field. What follows is a simple model. All actual interactions are qualitatively similar to that shown in the model. To build the model we will make the following assumptions:

(a) The matter consists of independent "atoms" $\equiv$ boxes of electrons.

(b) These atoms are electrically neutral so, as a first approximation, there would be no interaction with an electric field.

(c) An electric field does, however, induce a dipole moment in the atoms. In a second approximation this dipole moment does interact with an electric field.

(d) The atomic forces opposing this polarization are harmonic.

(e) In the absence of an external electric field, dissipative forces (rotation of atoms or interaction between atoms, typically) relax any atom's dipole $\rightarrow$ zero.

(f) A one-dimensional language will be used even though electric fields are really vector fields. The extension to $3D$ is straight-forward.

Suppose that an electric field moves the charges in an electrically neutral atom so that the atom looks like two charges $\pm q$ separated by a distance ℓ:

In this configuration the atom has a *dipole moment*

$$d = q\,\ell \tag{1}$$

The energy of the dipole in an electric field E is $-d \cdot E = -q\,\ell\,E$. Thus the electric field exerts a *force on* ℓ given by

$$\text{External force} = -\frac{\partial}{\partial \ell}(-q\,\ell\,E) = q\,E$$

If the effective mass of the coordinate ℓ is m, the equation of motion of $\ell(t)$ in a field $E(t)$ will be

$$m\ddot{\ell}(t) + 2\gamma\,\dot{\ell}(t) + k\,\ell(t) = q\,E(t) \tag{2}$$

with γ/m the rate at which the dipole moment relaxes (in the absence of an external field) to zero. k is the harmonic force constant. If there were neither damping nor an external force, $\ell(t)$ would oscillate with a rate

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and it is convenient to replace $k \to m\,\omega_0^2$ in the analysis.

<u>The steady-state motion of $\ell(t)$</u>

If $E(t)$ has been oscillating steadily for a time $\gg \gamma/m$, $\ell(t)$ will forget the initial (starting) conditions and relax to a *steady-state* motion. This motion is easily determined using a complex representation for an oscillating wave. If

$$\begin{aligned} E(t) &= \bar{E}\; e^{-i\omega t} \\ \ell(t) &= \bar{\ell}\; e^{-i\omega t} \end{aligned} \tag{3}$$

then the derivatives in (2) just pick up $-i\omega$ factors and

$$\bar{\ell} e^{-i\omega t} = \frac{q}{-m\omega^2 - 2i\omega\gamma + m\omega_0^2}\,\bar{E} e^{-i\omega t}$$

Thus the induced dipole moment – in this steady-state case – is

$$\bar{d} = \alpha(\omega)\,\bar{E}$$

with

$$\alpha(\omega) = \frac{q^2}{-m\omega^2 - 2i\omega\gamma + m\omega_0^2} \tag{4}$$

the *frequency dependent* polarizability of the atom. As any complex number can be written

$$x + i\,y = \sqrt{x^2 + y^2}\, e^{i \tan^{-1} \frac{y}{x}},$$

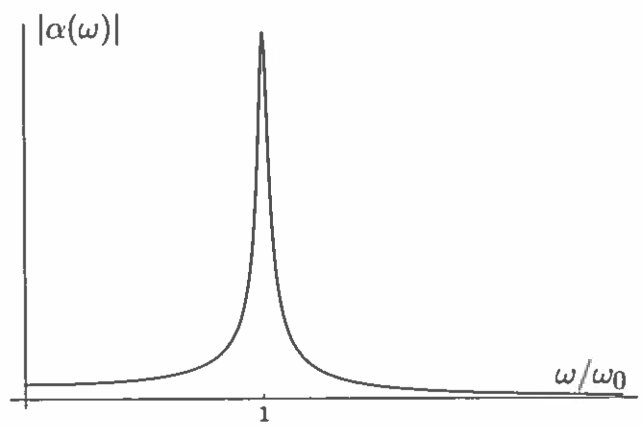
the polarizability can be written

$$\alpha(\omega) = |\alpha(\omega)|\, e^{i\phi(\omega)}$$

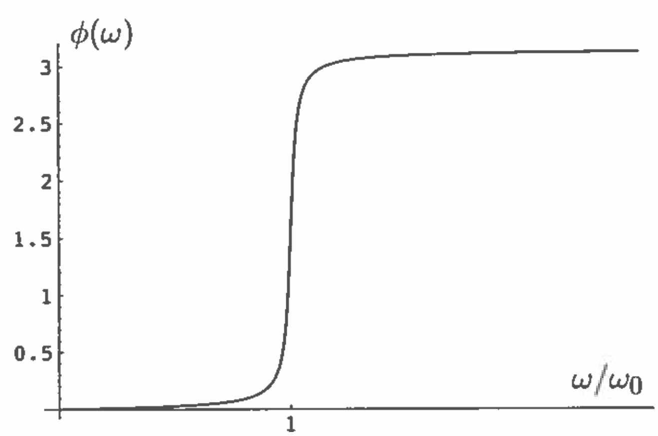
with

$$\begin{aligned} |\alpha(\omega)| &= \frac{q^2}{\sqrt{4\,\gamma^2\,\omega^2 + m^2\,(\omega^2 - \omega_0^2)^2}} \\ \tan\phi(\omega) &= \frac{2\,\gamma\,\omega}{-m\,(\omega^2 - \omega_0^2)} \end{aligned} \tag{5}$$

The amplitude of the response $|\alpha(\omega)|$ shows a large maximum at the resonant frequency $\omega = \omega_0$.



For $\omega \approx 0$ there is no phase shift between the driving field and the induced dipole moment. As ω increases, the phase of the oscillation of the dipole moment falls behind that of $E(t)$. At $\omega = \omega_0$ the phase shift has increased to $\pi/2$. At very high frequencies there is a 180° phase shift.



2. The index of refraction

An incident EM field $\boldsymbol{E}(\boldsymbol{r},t)$ induces a *polarization* (≡ dipole moment per unit volume) in matter. If the driving field oscillates at ω, the polarization will also oscillate at ω. But an oscillating dipole moment (like a radio transmitter's antenna) radiates an EM field. The radiated field also oscillates at ω, but the phase of the radiation varies with ω. All the EM waves in matter travel at a speed c, but the actual field present is a combination of the driving field and the radiated field. The combination of these fields leads to the progressive phase shift that we interpret as a change in the light speed.

Rather than consider the radiation as a separate process, it is easier to simply solve (what E&M texts call) *Maxwell's Equations*. In the absence of free charges or currents, these are (using *SI* units)

$$\begin{aligned} \nabla\cdot\boldsymbol{D} &= 0 \\ \nabla\times\boldsymbol{E}+\frac{\partial\boldsymbol{B}}{\partial t} &= 0 \\ \nabla\cdot\boldsymbol{B} &= 0 \\ \nabla\times\boldsymbol{H} &= \frac{\partial\boldsymbol{D}}{\partial t} \end{aligned} \tag{6}$$

If $\boldsymbol{E}(\boldsymbol{r},t) = \bar{\boldsymbol{E}}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, etc., these differential equations reduce to the algebraic relations

$$\begin{aligned} \boldsymbol{k}\cdot\bar{\boldsymbol{D}} &= 0, \qquad \boldsymbol{k}\times\bar{\boldsymbol{E}}-\omega\bar{\boldsymbol{B}} = 0 \\ \boldsymbol{k}\cdot\bar{\boldsymbol{B}} &= 0, \qquad \boldsymbol{k}\times\bar{\boldsymbol{H}}+\omega\bar{\boldsymbol{D}} = 0 \end{aligned} \tag{7}$$

If there are no magnetic interactions with the matter,

$$\boldsymbol{B} = \mu_0\,\boldsymbol{H}. \tag{8}$$

If there are electrical interactions, then the electric polarization $\boldsymbol{P}$ is non-zero and

$$\boldsymbol{D} = \epsilon_0\,\boldsymbol{E}+\boldsymbol{P}$$

If there are ρ of our atoms per unit volume, the polarization at frequency ω will be

$$\boldsymbol{P} = \rho\,\alpha(\omega)$$

whence

$$\boldsymbol{D} = \epsilon_0\,\chi(\omega)\,\boldsymbol{E} \tag{9}$$

with

$$\chi(\omega) = 1+\rho\,\alpha(\omega)/\epsilon_0, \tag{10}$$

the *complex electric susceptibility*.

It follows from (7) that

$$\begin{aligned} \boldsymbol{k} \times (\boldsymbol{k} \times \bar{\boldsymbol{E}}) =& (\boldsymbol{k} \cdot \bar{\boldsymbol{E}})\boldsymbol{k} - k^2 \bar{\boldsymbol{E}} = -k^2 \bar{\boldsymbol{E}} \\ =& \omega\, \boldsymbol{k} \times \bar{\boldsymbol{B}} = -\mu_0 \epsilon_0 \omega^2 \chi(\omega)\, \bar{\boldsymbol{E}} \end{aligned}$$

which only allows $\bar{\boldsymbol{E}} \neq 0$ if

$$k^2 = \mu_0 \epsilon_0 \omega^2 \chi(\omega) \tag{11}$$

The speed of the wave $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ is

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0\, \chi(\omega)}} \tag{12}$$

In the absence of matter, $\chi(\omega) = 1$, whence

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Thus

$$k\, c = \omega \sqrt{\chi(\omega)} \tag{13}$$

Now $\chi(\omega)$ is complex,

$$\chi(\omega) = \left[1 + \frac{\rho\, q^2}{\epsilon_0} \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega^2 - \omega_0^2)^2 + 4\omega^2\gamma^2}\right] + i \left[\frac{\rho\, q^2}{\epsilon_0} \frac{2\gamma\, \omega}{m^2(\omega^2 - \omega_0^2)^2 + 4\omega^2\gamma^2}\right]$$

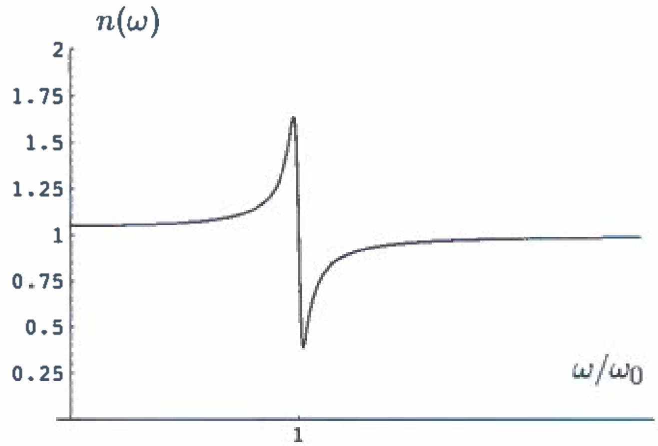
so (13) will assign a complex value to k. To simplify the analysis, suppose that $\rho|\alpha(\omega)|/\epsilon_0 \ll 1$. Then

$$\begin{aligned} \sqrt{\chi(\omega)} \approx& 1 + \frac{\rho\, \alpha(\omega)}{2\epsilon_0} \\ =& \left[1 + \frac{\rho\, q^2}{2\epsilon_0} \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega^2 - \omega_0^2)^2 + 4\omega^2\gamma^2}\right] + i \left[\frac{\rho\, q^2}{2\epsilon_0} \frac{2\gamma\, \omega}{m^2(\omega^2 - \omega_0^2)^2 + 4\omega^2\gamma^2}\right] \end{aligned} \tag{14}$$

The real part of (14) gives the index of refraction

$$n(\omega) = 1 + \frac{\rho\, q^2}{2\epsilon_0} \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega^2 - \omega_0^2)^2 + 4\omega^2\gamma^2}$$

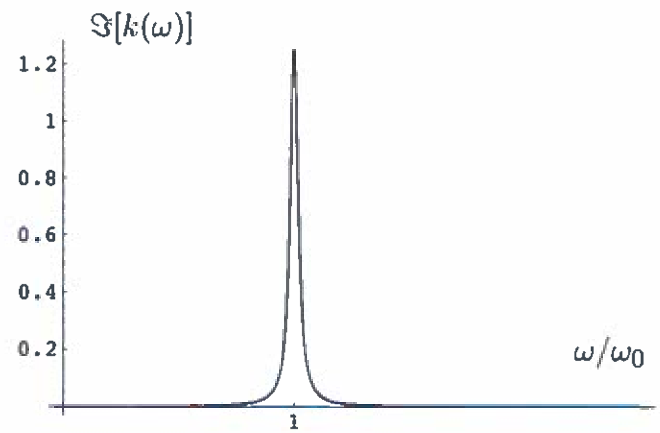
$n(\omega)$ is > 1 for $\omega < \omega_0$, but < 1 for $\omega > \omega_0$. The change reflects the change in the phase shift of the induced dipole moment of the matter. In the first case the induced dipolar radiation is retarded with respect to the driving field. In the second case the radiation is advanced.



If $k = k_R + i\,k_I$,

$$e^{ikx} = e^{ik_Rx - k_Ix}$$

$k_I > 0$ implies that the amplitude of the wave decays as it propagates through the matter (as required by the Beer-Lambert Law). This decay is what is measured in an absorption spectrometer.



3. The index of refraction at high frequencies

The rapid oscillation of $\boldsymbol{E}(\boldsymbol{r}, t)$ at high frequencies means that the actual physical displacement of the electrons in the atoms induced by this field is very small. For this reason, the binding forces on the electrons in the system are irrelevant. The induced dipole moment is just that which would be present in *free* electrons!

For a free electron,

$$m\ddot{\boldsymbol{r}}(t) = -e\,\boldsymbol{E}(\boldsymbol{r}, t)$$

whence

$$\bar{\boldsymbol{r}}e^{-i\omega t} = \frac{e}{m\omega^2}\bar{\boldsymbol{E}}e^{-i\omega t}$$

The induced dipole moment is $-e\,\bar{\boldsymbol{r}}e^{-i\omega t}$ so

$$\alpha(\omega) = -\frac{e^2}{m\omega^2},$$

a quantity < 0. [This is just what (4) gives in the high frequency limit, of course.] The nuclear contribution has the same sign, but the electronic contribution dominates because of the small mass of the electron. The apparent index of refraction of matter for x-rays is < 1, etc.

4. How fast do narrow EM pulses travel?

The examples above are all *steady-state* cases. What if one is dealing with a short pulse of radiation?

The function e^{ikx} describes a simple wave oscillating in the x-direction. A "pulse" can be built up as a superposition of these waves:

$$E(x,0) = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

is a pulse located at $x = 0$. [This is just the limit of the Fourier transform of a unit impulse in the limit that the pulse width $\to 0$.]

At a later time,

$$E(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{i(k\,x - \omega(k)\,t)} \tag{15}$$

The function $\omega(k)$ is implicitly defined by (4), (10) and (13), i.e., by

$$\frac{k^2\,c^2}{\omega^2} = 1 + \frac{\rho}{\epsilon_0} \frac{q^2}{-m(\omega^2 - \omega_0^2) - 2i\,\gamma\,\omega} \tag{16}$$

in our model.

We could evaluate (16) numerically, but consider the following approximate evaluation. First, if the integration variable is changed from $k \to \omega$,

$$E(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, k'(\omega)\, e^{i(k(\omega)\,x - \omega\,t)} \tag{17}$$

Again, if $\rho|\alpha(\omega)|/\epsilon_0 \ll 1$, then the dispersion makes a small correction to $\omega = k/c$. We choose the sign of the square root of (16) so that $k/\omega > 0$, i.e., we take the forward propagating wave. Thus we take

$$k'(\omega) \to 1/c$$

$$e^{i(k(\omega)\,x - \omega\,t)} \to e^{i\,\omega(\frac{x}{c} - t)} \left[1 - \frac{i\,\omega\,x\,\rho\,q^2}{2c\,\epsilon_0} \frac{1}{m(\omega^2 - \omega_0^2) + 2i\,\gamma\,\omega}\right]$$

whence

$$E(x,t) = \frac{1}{2\pi\,c} \int_{-\infty}^{\infty} d\omega\, e^{i\,\omega(\frac{x}{c} - t)} \left[1 - \frac{x\,\rho\,q^2}{2m\,c\,\epsilon_0} \frac{i\,\omega}{\omega^2 + \frac{2i\,\gamma}{m}\,\omega - \omega_0^2}\right] \tag{18}$$

Evaluation of this integral – perhaps not entirely trivial – gives zero if $x > c\,t$: There is no signal at positions in advance of the position $c\,t$.

For $x < c\,t$ the evaluation of this integral gives

$$E(x,t) = \delta(x - c\,t) + \frac{x\,\rho\,q^2}{2m\,c^2\,\epsilon_0\sqrt{\omega_0^2 - \frac{4\gamma^2}{m^2}}}\,e^{\gamma(x-c\,t)/m\,c}\times \left\{\sqrt{\omega_0^2 - \frac{4\gamma^2}{m^2}}\cos\left[\sqrt{\omega_0^2 - \frac{4\gamma^2}{m^2}}(\frac{x}{c} - t)\right] + \frac{\gamma}{m}\sin\left[\sqrt{\omega_0^2 - \frac{4\gamma^2}{m^2}}(\frac{x}{c} - t)\right]\right\} \tag{19}$$

Eq.(19) gives a realistic description of the signal propagated through a sample of finite size. The initial pulse is transmitted with speed c. [In a thicker sample, there would be some absorption and the initial pulse's amplitude would decrease, of course.] This pulse is followed immediately by a ringing signal that is the radiation of the induced dipole moment. The frequency is that of the free dipole moment oscillation and the dampling in time occurs with the rate γ/m.