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# Corrigendum to: Operator monotone functions and Löwner functions of several variables

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## Corrigendum to: Operator monotone functions and Löwner functions of several variables

By JIM AGLER, JOHN E. MCCARTHY, and N. J. YOUNG

### Abstract

We fix a gap in the proof of Theorem 7.24 in *Ann. of Math.* **176** (2012), 1783–1826.

There is a gap in the proof of Theorem 7.24 in [1], though the statement of the theorem is correct.

In the proof of necessity, we argue that  $\Lambda$  is in  $\mathcal{G}$  by contradiction. If it were not, invoking the Hahn-Banach separation theorem would yield a real skew-symmetric matrix  $K$  and a constant  $\delta \geq 0$  such that  $\operatorname{tr}(\Gamma K) \geq -\delta$  for all  $\Gamma$  in  $\mathcal{G}$ , and  $\operatorname{tr}(\Lambda K) < -\delta$ . In the proof we assumed that  $\delta = 0$ , but this assumption is unjustified.

Instead, we argue as follows. Define  $\Delta$  by

$$\Delta_{ij}^r = (x_j^r - x_i^r)K_{ji}, \quad i \neq j,$$

and with the diagonal entries  $\Delta_{ii}^r$  chosen so that each  $\Delta^r \geq 0$  and so that

$$(0.1) \quad \mu^r := \sum_{i=1}^n f_{r,i} \Delta_{ii}^r$$

is minimal over all choices of  $\Delta_{11}^r, \dots, \Delta_{nn}^r$  such that  $\Delta \geq 0$ . (A minimal choice exists, since all the  $f_{r,i}$  are strictly positive by assumption.) Then  $\Delta$  is in  $SAM_n^d$ , and

$$[\Delta^s, S^r]_{ij} = (x_j^s - x_i^s)K_{ji}(x_j^r - x_i^r) = [\Delta^r, S^s]_{ij}.$$

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As  $f$  is locally  $M_n$  monotone, we must have then that  $D_{\Delta}f(S) \geq 0$  by Lemma 7.3. As

$$-\delta > \operatorname{tr}(\Lambda K) = \sum_{1 \leq i, j \leq n} [D_{\Delta}f(S)]_{ij} - \sum_{r=1}^d \sum_{i=1}^n \Delta_{ii}^r f_{r,i},$$

we get that

$$(0.2) \quad \sum_{r=1}^d \mu^r - \delta > \sum_{1 \leq i, j \leq n} [D_{\Delta}f(S)]_{ij} \geq 0.$$

By Duffin's strong duality theorem [2], the minimum  $\mu^r$  in (0.1) satisfies

$$(0.3) \quad -\mu^r = \min_{i \neq j} \sum \Delta_{ij} A^r(i, j),$$

where  $A^r$  range over the set of real positive matrices such that the diagonal entries of  $A^r$  are  $f_{r1}, \dots, f_{rn}$  for each  $r$ .

For each such  $A = (A^1, \dots, A^d)$ , let  $\Gamma$  be the corresponding element of  $\mathcal{G}$ :  $\Gamma_{ii} = 0$  and

$$\Gamma_{ij} = \sum_{r=1}^d (x_j^r - x_i^r) A^r(i, j) \quad \text{for } i \neq j.$$

We have

$$\begin{aligned} -\delta &\leq \operatorname{tr} \Gamma K \\ &= \sum_{i \neq j} \sum_{r=1}^d (x_j^r - x_i^r) A^r(i, j) K_{ji} \\ &= \sum_{r=1}^d \sum_{i \neq j} \Delta_{ij}^r A^r(i, j). \end{aligned}$$

Hence, by equation (0.3),  $-\delta \leq \sum_{r=1}^d (-\mu^r)$ , so  $\sum_{r=1}^d \mu^r \leq \delta$ . This contradicts (0.2), so it follows that  $\Lambda \in \mathcal{G}$ , and necessity is proved.

## References

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