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Three Problems in Operator Theory and Complex Analysis

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Three Problems in Operator Theory and Complex Analysis

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Dedicated to my family.

ABSTRACT OF THE DISSERTATION

Three Problems in Operator Theory and Complex Analysis

by

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Professor John McCarthy, Chair

This thesis concerns three distinct problems in operator theory and complex analysis.

In Chapter 2, we study the following problem: On the Hardy space H^2, when is the product of a Hankel operator and a Toeplitz operator compact? We give necessary and sufficient conditions for when such a product $H_f T_g$ is compact.

In Chapter 3, we discuss hyponormal Toeplitz operators. We show that for those operators, there exists a lower bound for the area of the spectrum. This extends the known estimate for the spectral area of Toeplitz operators with an analytic symbol. This part is joint work with Dmitry Khavinson.

In Chapter 4, we study the Bohr radius R_n for the class of complex polynomials of degree at most n. Bohr's theorem showed that $R_n \to \frac{1}{3}$ as $n \to \infty$. We are interested in the rate of convergence and proved an asymptotic formula that was conjectured by R. Fournier in 2008.

1. Introduction

1.1 Hardy Spaces

The Hardy space plays a key role in real and complex analysis. Its early discoveries were made in 1920s by such mathematicians as Hardy, Littlewood, Privalov, F. and M. Riesz, Smirnov, and Szegő. In this dissertation, $\mathbb{D}$ will denote the open unit disk in the complex plane, and $\partial\mathbb{D}$ will denote the boundary of $\mathbb{D}$. We will focus on the Hardy spaces on $\mathbb{D}$.

For $0 < p < \infty$, the Hardy space H^p is the class of analytic functions f on $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty.$$

The number on the left side of the above inequality is denoted by $||f||_p$. It is a norm when $p \geq 1$.

If $p = \infty$, H^∞ is the space of bounded analytic functions on $\mathbb{D}$ with norm

$$||f||_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Let L^p $(0 < p \leq \infty)$ denote the Lebesgue spaces of $\partial\mathbb{D}$ induced by the measure $\frac{d\theta}{2\pi}$. For $p \geq 1$, by a theorem of F. Riesz (see for example [1, p. 56] for details), we can identify $f(z) \in H^p$ with its boundary function $f(e^{i\theta})$. Then H^p can be viewed as a closed subspace of L^p.

The Hardy space H^2 is a Hilbert space with reproducing kernel

$$K_z(w) = \frac{1}{1 - \bar{z}w},$$

because the formula

$$f(z) = \langle f, K_z \rangle$$

reproduces every function $f \in H^2$. The orthogonal projection from L^2 onto H^2 is called the Szegő projection and is denoted by P. All unspecified norms and inner products used in this dissertation are those on L^2.

1.2 Toeplitz Operators and Hankel Operators

Toeplitz operators and Hankel operators form two of the most significant classes of concrete operators because of their importance both in pure and applied mathematics.

For $\varphi \in L^\infty$, the Toeplitz operator T_φ with symbol φ is defined on H^2 by

$$T_\varphi h = P(\varphi h).$$

In algebra, a Toeplitz matrix is a matrix that is constant on each line parallel to the main diagonal. Thus a Toeplitz matrix is determined by a sequence $\{a_k\}_{k=-\infty}^{\infty}$ of complex numbers, with the entry in row j, column k of the matrix equal to a_{j-k}. The matrix of T_f with respect to the orthonormal basis $\{z^k\}_{k=0}^{\infty}$ in H^2 is the Toeplitz matrix corresponding to the sequence $\{\hat{f}(k)\}_{k=-\infty}^{\infty}$, the Fourier coefficients of f.

The Hankel operator H_φ with symbol $\varphi \in Ł^\infty$ is the operator on H^2 defined by

$$H_\varphi h = U(I - P)(\varphi h), \tag{1.2.1}$$

for $h \in H^2$. Here U is the unitary operator on L^2 defined by

$$Uf(z) = \bar{z}\tilde{f}(z),$$

where $\tilde{f}(z) = f(\bar{z})$. Clearly,

$$H_f^* = H_{f^*},$$

where $f^*(z) = \overline{f(\bar{z})}$. Hankel operators are often defined in an alternative way (see for example [2], [3]). It is easy to see that the two definitions are unitarily equivalent.

1.3 Function Theory on the Unit Circle

The problems discussed in this dissertation belong to the area of function theory on the unit circle, which is a mixture of real and complex analysis, operator theory, harmonic analysis and theory of Banach algebras.

The theory originated with the study of one-dimensional Hardy spaces, and a very rich theory has been developed in the 20th century. We mention some significant highlights of the theory. In 1962, Carleson proved the corona theorem that the unit disk is dense in the maximal ideal space of H^∞ [4]. Carleson's original proof consists of a complicated construction of a system of curves that has found important applications besides the corona theorem. Another deep and well-known result is Fefferman's duality theorem [5], which identifies the real dual space of the real Banach space H^1 as BMO, the space of functions of bounded mean oscillation. The theorem emerged as the birth of real-variable theory of Hardy spaces, which extends the classical Hardy space theory to Euclidean spaces. The breakthrough results of Carleson and Fefferman inspired T. Wolff in 1979 to devise a new and elegant proof of the corona theorem which avoids the Carleson construction.

In this dissertation we prove three results in function theory on the unit circle that are more or less related to Toeplitz operators. In Chapter 2, we investigate when the product of a Hankel operator and a Toeplitz operator is compact. In Chapter 3, a new estimate of the spectral area of certain Toeplitz operators is given. Chapter 4 concerns a nonlinear extremal problem, and its solution relies on the relation between symmetric Toeplitz matrices and analytic functions.

The three chapters to follow are taken from [6], [7], and [8], and are self-contained.

2. Compact Product of Hankel and Toeplitz Operators

2.1 Introduction and Motivation

A linear operator T on a Hilbert space $\mathcal{H}$ is called compact if for any bounded sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, $\{T(a_k)\}_{k=1}^{\infty}$ has a convergent subsequence. We study a mixed compactness problem of Toeplitz and Hankel operators in this chapter.

Let us first look at the compactness of Toeplitz and Hankel operators individually. The only compact Toeplitz operator is the zero operator (see for example [3, p. 194]). For the Hankel operator, we have the following theorem, usually referred to as Hartman's Criterion:

Theorem 2.1.1 *Let $f \in L^{\infty}$. Then the Hankel operator H_f is compact if and only if*

$$f \in H^{\infty} + C.$$

Here C denotes the space of continuous functions on $\partial\mathbb{D}$. $H^{\infty} + C$ is the linear span of H^{∞} and C. It is a closed subalgebra of L^{∞} containing H^{∞} (see [9]).

The proof can be found in [2, p. 26] or [3, p. 198].

The problem of characterizing the compactness for the product of two Hankel operators turns out to be much more difficult. Axler, Chang, Sarason [10], and Volberg [11] gave necessary and sufficient conditions that the product of two Hankel operators is compact. They proved the following result:

Theorem 2.1.2 *Let $f, g \in L^{\infty}$. $H_{\bar{f}} H_g$ is compact if and only if*

$$H^{\infty}[\bar{f}] \cap H^{\infty}[g] \subset H^{\infty} + C. \tag{2.1.1}$$

Here $H^\infty[f]$ denotes the closed subalgebra of L^∞ generated by H^∞ and f.

They also gave a local version of the algebraic condition (2.1.1) using the notion of support sets. We will define the support sets in Section 2.2.

Theorem 2.1.3 *Let $f, g \in L^\infty$.*

$$H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C$$

if and only if for each support set S, either $\bar{f}|_S$ or $g|_S$ is in $H^\infty|_S$.

Later, Zheng in [12] gave the following elementary condition that also characterizes the compactness of $H_{\bar{f}}H_g$.

Theorem 2.1.4 *Let $f, g \in L^\infty$. $H_{\bar{f}}H_g$ is compact if and only if*

$$\lim_{|z|\to 1^-} ||H_{\bar{f}}k_z|| \cdot ||H_g k_z|| = 0.$$

Here k_z denotes the normalized reproducing kernel at z.

The relations between these three conditions in Theorem 2.1.2, 2.1.3 and 2.1.4 can be found in Section 2.3 and 2.4. Inspired by the above theorems, we consider the product of a Hankel operator and a Toeplitz operator in this chapter. The following theorem is our main result:

Theorem 2.1.5 *Let $f, g \in L^\infty$. The product $K = H_f T_g$ of the Hankel operator H_f and the Toeplitz operator T_g is compact if and only if for each support set S, one of the following holds:*

1. $f|_S \in H^\infty|_S$.

2. $g|_S \in H^\infty|_S$ *and* $(fg)|_S \in H^\infty|_S$.

Analogously to Theorem 2.1.2, we also obtain the following algebraic version of Theorem 3.3.5:

Theorem 2.1.6 *Let $f, g \in L^\infty$. $H^\infty[f] \cap H^\infty[g, fg] \subset H^\infty + C$ if and only if for each support set S, one of the following holds:*

1. $f|_S \in H^\infty|_S$.

2. $g|_S \in H^\infty|_S$ *and* $(fg)|_S \in H^\infty|_S$.

Similarly, $H^\infty[u, v]$ denotes the closed subalgebra of L^∞ generated by functions u, v, and H^∞.

2.2 Preliminaries

We begin this section by establishing the relation between Toeplitz operators and Hankel operators. Consider the multiplication operator M_f on L^2 for $f \in L^\infty$, defined by $M_f h = fh$. M_f can be expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus (H^2)^\perp$ as the following:

$$M_f = \begin{pmatrix} T_f & H_{\tilde{f}}U \\ UH_f & UT_{\tilde{f}}U \end{pmatrix}$$

For $f, g \in L^\infty$, $M_{fg} = M_f M_g$, so multiplying the matrices and comparing the entries, we get:

Proposition 2.2.1 *Let f and g be in L^∞. Then*

1. $T_{fg} = T_f T_g + H_{\tilde{f}} H_g$.

2. $H_{fg} = H_f T_g + T_{\tilde{f}} H_g$.

3. *If $g \in H^\infty$, then $H_f T_g = T_{\tilde{f}} H_g$.*

Let $x, y \in L^2$. Define $x \otimes y$ to be the following rank one operator on L^2:

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

Proposition 2.2.2 *[13, p. 29] Let $x, y \in L^2$ and let S, T be operators on L^2. Then*

1. $(x \otimes y)^* = y \otimes x$.

2. $||x \otimes y|| = ||x|| \cdot ||y||$.

3. $S(x \otimes y)T = (Sx) \otimes (T^* y)$.

For each $z \in \mathbb{D}$, let k_z denote the normalized reproducing kernel at z:

$$k_z(w) = \frac{\sqrt{1 - |z|^2}}{1 - \bar{z}w},$$

and ϕ_z be the Möbius transform:

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

We have the following identities:

Lemma 2.2.1 *For $z \in \mathbb{D}$,*

1. $T_{\phi_z} T_{\bar{\phi}_z} = 1 - k_z \otimes k_z$.

2. $T^*_{\tilde{\phi}_z} T_{\tilde{\phi}_z} = 1 - k_{\bar{z}} \otimes k_{\bar{z}}$.

3. $H_{\bar{\phi}_z} = -k_{\bar{z}} \otimes k_z$.

These identities can be found in [12, p. 480] and [14, Lemma 5].

The next lemma in [15] and [14] gives a relation between H_f and H_f^*.

Lemma 2.2.2 *Let* $f \in L^\infty$, $g \in H^2$. *Then* $H_f^* g^* = (H_f g)^*$ *and thus* $||H_f^* g^*|| = ||H_f g||$. *In particular,* $||H_f^* k_{\bar{z}}|| = ||H_f k_z||$.

Proof Notice that for all $g \in L^2$, $(Ug)^* = Ug^*$ and $Pg^* = (Pg)^*$. Thus

$$H_f^* g^* = H_{f^*} g^* = PU(f^* g^*) = P(Ufg)^* = (PUfg)^* = (H_f g)^*$$

Since $||h|| = ||h^*||$ for all $h \in L^2$, we get $||H_f^* g^*|| = ||H_f g||$. ■

To state the local conditions, we need some notation for the maximal ideal space. For a uniform algebra B, let $M(B)$ denote the maximal ideal space of B, the space of nonzero multiplicative linear functionals of B. Given the weak-star topology of B^*, which is called the Gelfand topology, $M(B)$ is a compact Hausdorff space. Identify every element in B with its Gelfand transform, we view B as a uniformly closed algebra of continuous functions on $M(B)$. See [1, Chapter V] for further discussions of uniform algebra.

The space H^∞ is a uniform algebra with pointwise multiplication and the supremum norm $|| \cdot ||_\infty$. For each $\zeta \in \mathbb{D}$, there exists $m_\zeta \in M(H^\infty)$ such that $m_\zeta(z) = \zeta$, where z denotes the coordinate function. It is well known that $\zeta \to m_\zeta$ is a homeomorphic embedding from $\mathbb{D}$ into $M(H^\infty)$ (see for example [1, p. 183] for details), thus we identify $\mathbb{D}$ as a subset of $M(H^\infty)$. By Carleson's Corona Theorem [4], $\mathbb{D}$ is dense in $M(H^\infty)$. Moreover, $M(H^\infty + C) = M(H^\infty) \backslash \mathbb{D}$ (see [9]).

For any m in $M(H^\infty)$, there exists a representing measure μ_m on $M(H^\infty)$ such that $m(f) = \int_{M(H^\infty)} f d\mu_m$, for all $f \in H^\infty$ (see [1, p. 193]). A subset S of $M(H^\infty)$ is called a support set if it is the support of a representing measure for a functional in the "corona" $M(H^\infty + C)$.

2.3 Proof of Theorem 2.1.6

In this section we prove Theorem 2.1.6. The proof we present here is analogous to the proof of [16, Lemma 1.1].

The proof is based on the following two lemmas:

Lemma 2.3.1 *[16, Lemma 1.3] Let A_α be a family of Douglas algebras, the closed subalgebras of L^∞ containing H^∞. Then*

$$M(\cap A_\alpha) = \overline{\cup M(A_\alpha)}.$$

Lemma 2.3.2 *[16, Lemma 1.5] Let $m \in M(H^\infty + C)$ and let S be the support set of m. Then $m \in M(H^\infty[f])$ if and only if $f|_S \in H^\infty|_S$.*

Proof of Theorem 2.1.6. Let

$$A = H^\infty[f] \cap H^\infty[g, fg].$$

By Lemma 2.3.1,

$$M(A) = M(H^\infty[f]) \cup M(H^\infty[g, fg]).$$

Suppose

$$H^\infty[f] \cap H^\infty[g, fg] \subset H^\infty + C.$$

Then $A \subset H^\infty + C$, and $M(H^\infty + C) \subset M(A)$. Lemma 2.3.2 gives that either condition (1) or condition (2) holds.

Conversely, let S be the support set for $m \in M(H^\infty + C)$ and suppose one of the Conditions (1) and (2) holds for m. Then by Lemma 2.3.2, either $m \in M(H^\infty[f])$ or

$m \in M(H^\infty[g, fg])$. Thus, $M(H^\infty + C) \subset M(A)$. By the Chang-Marshall Theorem [17, 18], for two Douglas algebras U and V, $M(U) = M(V)$ if and only if $U = V$. As a consequence $A \subset H^\infty + C$. ■

2.4 Compact Operators and Local Condition

In this section, we present the main tools in the proof of Theorem 3.3.5.

The following lemma in [14, Lemma 9] gives a nice property of compact operators.

Lemma 2.4.1 *If $K : H^2 \to H^2$ is a compact operator, then*

$$\lim_{|z|\to 1^-} ||K - T_{\tilde{\phi}_z} K T_{\bar{\phi}_z}|| = 0. \tag{2.4.1}$$

Remark 2.4.1 *By the Corona Theorem,* (2.4.1) *can be restated as the following: For each $m \in M(H^\infty + C)$, there is a net $z \to m$ such that*

$$\lim_{z\to m} ||K - T_{\tilde{\phi}_z} K T_{\bar{\phi}_z}|| = 0.$$

In [19], Guo and Zheng used the distribution function inequality to prove the following theorem, which can be viewed as a partial converse of Lemma 2.4.1.

Theorem 2.4.1 *Let T be a finite sum of finite products of Toeplitz operators. Then T is a compact perturbation of a Toeplitz operator if and only if*

$$\lim_{|z|\to 1^-} ||T - T^*_{\phi_z} T T_{\phi_z}|| = 0.$$

Remark 2.4.2 *Theorem 2.4.1 cannot be applied directly to $H_f T_g$, since $H_f T_g$ might not be a finite sum of finite products of Toeplitz operators. However, by Proposition 2.2.1,*

$$(H_f T_g)^*(H_f T_g) = T_{\bar{g}} H_f^* H_f T_g = T_{\bar{g}}(T_{\bar{f}f} - T_{\bar{f}} T_f) T_g,$$

thus $(H_f T_g)^*(H_f T_g)$ is a finite sum of finite products of Toeplitz operators.

Remark 2.4.3 *The symbol map σ that sends every Toepltiz operator T_ϕ to its symbol ϕ was introduced in [20] and can be defined on the Toeplitz algebra, the closed algebra generated by Toeplitz operators. Barría and Halmos in [21] showed that σ can be extended to a $*$-homomorphism on the Hankel algebra, the closed algebra generated by Toeplitz and Hankel operators. And they also showed that the symbols of compact operators and Hankel operators are zero. Notice that $(H_fT_g)^*(H_fT_g)$ has symbol zero, so it is a compact perturbation of a Toeplitz operator if and only if it is compact.*

By Theorem 2.4.1 and above remarks, we have

Corollary 2.4.1 *$K = H_fT_g$ is compact if and only if*

$$\lim_{|z|\to 1^-} ||K^*K - T^*_{\phi_z} K^*KT_{\phi_z}|| = 0.$$

The next lemma, stated below, from [16, Lemma 2.5, 2.6] which interprets the local condition in an elementary way, will be used several times later.

Lemma 2.4.2 *Let $f \in L^\infty$, $m \in M(H^\infty + C)$, and let S be the support set of m. Then the following are equivalent:*

1. $f|_S \in H^\infty|_S$.

2. $\underline{\lim}_{z\to m} ||H_f k_z|| = 0.$

3. $\lim_{z\to m} ||H_f k_z|| = 0.$

We also need the following technical lemma:

Lemma 2.4.3 *[14, Lemma 17,18] Let $f, g \in L^\infty$, $m \in M(H^\infty + C)$.*

1. *If*

$$\lim_{z\to m} ||H_f k_z|| = 0,$$

then

$$\lim_{z \to m} ||H_f T_g k_z|| = 0.$$

2. If

$$\lim_{z \to m} ||H_f^* k_{\bar{z}}|| = 0,$$

then

$$\lim_{z \to m} ||H_f^* T_g k_{\bar{z}}|| = 0.$$

2.5 Proof of the Main Theorem

In this section, we prove Theorem 3.3.5. First we set up the following two identities:

Lemma 2.5.1 *[14, Lemma 6] Let $f, g \in L^\infty$ and $z \in \mathbb{D}$. Then*

$$T_{\tilde{\phi}_z} H_f T_g T_{\bar{\phi}_z} = H_f T_g - (H_f T_g k_z) \otimes k_z + (H_f k_z) \otimes (T_{\phi_z} H_f^* k_{\bar{z}}).$$

Lemma 2.5.2 *Let $f, g \in L^\infty$, $z \in \mathbb{D}$ and $K = H_f T_g$. Then*

$$K T_{\phi_z} = T_{\tilde{\phi}_z} K - (H_f k_z) \otimes (H_g^* k_{\bar{z}}).$$

Proof Since $\phi_z \in H^\infty$, by Proposition 2.2.1,

$$T_g T_{\phi_z} = T_{\phi_z} T_g + H_{\tilde{\phi}_z} H_g,$$

and

$$T_{\tilde{\phi}_z} H_g = H_g T_{\phi_z}.$$

Thus,

$$\begin{aligned} K T_{\phi_z} &= H_f T_g T_{\phi_z} = H_f T_{\phi_z} T_g + H_f H_{\tilde{\phi}_z} H_g \\ &= T_{\tilde{\phi}_z} H_f T_g + H_f H_{\tilde{\phi}_z} H_g \\ &= T_{\tilde{\phi}_z} K - (H_f k_z) \otimes (H_g^* k_{\bar{z}}). \end{aligned}$$

The last equality follows from Lemma 2.2.1(3). ■

Now we are ready to prove Theorem 3.3.5.

Proof of Theorem 3.3.5. Necessity: Suppose K is compact. By Lemma 2.4.1, we have:

$$\lim_{|z|\to 1^-} ||H_f T_g - T_{\tilde{\phi}_z} H_f T_g T_{\bar{\phi}_z}|| = 0.$$

By Lemma 2.5.1, we have

$$||H_f T_g - T_{\tilde{\phi}_z} H_f T_g T_{\bar{\phi}_z}|| = ||(H_f T_g k_z) \otimes k_z - (H_f k_z) \otimes (T_{\phi_z} H_g^* k_{\bar{z}})||.$$

Since $k_z \to 0$ weakly as $|z| \to 1$ and $H_f T_g$ is compact,

$$||H_f T_g k_z|| \to 0. \tag{2.5.1}$$

So

$$\lim_{|z|\to 1^-} ||(H_f k_z) \otimes (T_{\phi_z} H_g^* k_{\bar{z}})|| = 0.$$

Since

$$\begin{aligned} ||(H_f k_z) \otimes (H_g^* k_{\bar{z}})|| &= ||((H_f k_z) \otimes (T_{\phi_z} H_g^* k_{\bar{z}}))T_{\phi_z}|| \\ \leq &||(H_f k_z) \otimes (T_{\phi_z} H_g^* k_{\bar{z}})|| \cdot ||T_{\phi_z}|| \leq ||(H_f k_z) \otimes (T_{\phi_z} H_g^* k_{\bar{z}})||, \end{aligned}$$

we get

$$\lim_{|z|\to 1^-} ||(H_f k_z) \otimes (H_g^* k_{\bar{z}})|| = 0.$$

By Lemma 2.2.2,

$$\lim_{|z|\to 1^-} ||H_f k_z|| \cdot ||H_g k_z|| = 0.$$

Let $m \in M(H^\infty + C)$ and let S be the support set of m. By the Corona Theorem, there is a net z converging to m, and

$$\lim_{z\to m} ||H_f k_z|| \cdot ||H_g k_z|| = 0.$$

Thus, either

$$\lim_{\overline{z\to m}} ||H_f k_z|| = 0$$

or

$$\lim_{\overline{z\to m}} ||H_g k_z|| = 0. \tag{2.5.2}$$

By Lemma 2.4.2, we have $f|_S \in H^\infty|_S$ or $g|_S \in H^\infty|_S$. In the second case,we have

$$\begin{aligned}\lim_{\overline{z\to m}} ||H_{fg} k_z|| &= \lim_{\overline{z\to m}} ||H_f T_g k_z + T_{\tilde{f}} H_g k_z|| \\ &\leq \lim_{\overline{z\to m}} ||H_f T_g k_z|| + ||T_{\tilde{f}}|| \cdot \lim_{\overline{z\to m}} ||H_g k_z|| = 0.\end{aligned}$$

The first equality comes from Proposition 2.2.1 and the last equality follows from (2.5.1) and (2.5.2).

Therefore, Lemma 2.4.2 implies $(fg)|_S \in H^\infty|_S$.

Sufficiency: By Corollary 2.4.1, we need to show: for any $m \in M(H^\infty + C)$,

$$\lim_{z\to m} ||K^*K - T^*_{\phi_z} K^* K T_{\phi_z}|| = 0. \tag{2.5.3}$$

Let $F_z = -(H_f k_z) \otimes (H^*_g k_{\bar{z}})$. Lemma 2.5.2 gives

$$KT_{\phi_z} = T_{\tilde{\phi}_z} K + F_z.$$

Then

$$\begin{aligned}T^*_{\phi_z} K^* K T_{\phi_z} &= (KT_{\phi_z})^*(KT_{\phi_z}) = K^* T^*_{\tilde{\phi}_z} T_{\tilde{\phi}_z} K + (K^* T^*_{\tilde{\phi}_z}) F_z + F^*_z (T_{\tilde{\phi}_z} K) + F^*_z F_z \\ &= K^*K + (K^* k_{\bar{z}}) \otimes (K^* k_{\bar{z}}) + (K^* T^*_{\tilde{\phi}_z}) F_z + F^*_z (T_{\tilde{\phi}_z} K) + F^*_z F_z. \end{aligned} \tag{2.5.4}$$

The last equality comes from Lemma 2.2.1 (2).

Let S be the support set of m. If Condition (1) holds, i.e., $f|_S \in H^\infty|_S$, Lemma 2.4.2 and Lemma 2.2.2 give

$$\lim_{z\to m} ||H_f k_z|| = 0,$$

and

$$\lim_{z\to m} ||H_f^* k_{\bar{z}}|| = 0.$$

So

$$\lim_{z\to m} ||F_z|| = 0, \tag{2.5.5}$$

and

$$\lim_{z\to m} ||K^* k_{\bar{z}}|| = \lim_{z\to m} ||T_g^* H_f^* k_{\bar{z}}|| = 0.$$

Since $||K|| < \infty$ and $\sup\limits_{z\in\mathbb{D}} ||F_z|| < \infty$, (2.5.4) implies (2.5.3).

If Condition (2) holds, i.e., $g|_S \in H^\infty|_S$ and $(fg)|_S \in H^\infty|_S$, by Lemma 2.4.2,

$$\lim_{z\to m} ||H_g k_z|| = 0,$$

and

$$\lim_{z\to m} ||H_{fg} k_z|| = 0. \tag{2.5.6}$$

So (2.5.5) also holds. By Proposition 2.2.1,

$$(H_f T_g)^* k_{\bar{z}} = H_{fg}^* k_{\bar{z}} - (T_{\tilde{f}} H_g)^* k_{\bar{z}} = H_{fg}^* k_{\bar{z}} - H_g^* T_{f^*} k_{\bar{z}}.$$

Using (2.5.5) and Lemma 2.4.3, we get

$$\lim_{z\to m} ||K^* k_{\bar{z}}|| = 0.$$

Thus, (2.5.3) holds and $H_f T_g$ is compact. ∎

Notice that $(T_f H_g)^* = H_{g^*} T_{\bar{f}}$. Combining Theorem 3.3.5 and Theorem 2.1.6, we get the following characterization of the compactness of the product $T_f H_g$:

Corollary 2.5.1 *Let $f, g \in L^\infty$. The following are equivalent:*

1. *$T_f H_g$ is compact.*

2. $H^\infty[g^*] \cap H^\infty[\bar{f}, \bar{f}g^*] \subset H^\infty + C$.

3. For each support set S*, one of the following holds:*

(a) $g^*|_S \in H^\infty|_S$.

(b) $\bar{f}|_S \in H^\infty|_S$ *and* $(\bar{f}g^*)|_S \in H^\infty|_S$.

3. Spectral Area Estimate of Hyponormal Toeplitz Operators

3.1 Preliminaries and Definitions

In this chapter, we will focus on the spectrum of hyponormal Toeplitz operators.

The spectrum of a linear operator T, denoted as $sp(T)$, is the set of complex numbers λ such that $T - \lambda I$ is not invertible; here I denotes the identity operator. The following results about the spectrum of Toeplitz operators are well-known (see for instance [20, Chapter 7]).

Theorem 3.1.1 *If $\varphi \in L^\infty$ and T_ϕ is a Toeplitz operator on H^2. Then*

1. *If φ is real-valued, $sp(T_\varphi) = [essinf\varphi, esssup\,\varphi]$.*

   Here $essinf\varphi$ and $esssup\,\varphi$ are the essential infimum and essential supremum of φ respectively.

2. *If φ is analytic, $sp(T_\varphi) = \overline{\varphi(\mathbb{D})}$.*

3. *If φ is continuous, $sp(T_\varphi) = Ran(\varphi) \cup \{\lambda \in \mathbb{C} | i_t(\varphi, \lambda) \neq 0\}$.*

   Here $Ran(\varphi)$ is the range of φ and $i_t(\varphi, \lambda)$ is the winding number of the curve determined by φ with respect to λ.

In the general case, Harold Widom [22] proved the following theorem for arbitrary symbols.

Theorem 3.1.2 *Every Toeplitz operator has a connected spectrum.*

Let $[T^*, T]$ denote the operator $T^*T - TT^*$, called the self-commutator of T. An operator T is called hyponormal if $[T^*, T]$ is positive. Hyponormal operators satisfy the celebrated Putnam inequality [23]

Theorem 3.1.3 *If T is a hyponormal operator, then*

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi}.$$

This result turned out to be very useful for many problems in operator theory and in function theory.

3.2 Motivations

First we notice that Toeplitz operators with an analytic symbol are hyponormal.

Proposition 3.2.1 *Suppose $f \in H^\infty$. Then T_f is hyponormal.*

Proof For every $p \in H^2$,

$$\begin{aligned}\langle [T_f^*, T_f]p, p\rangle &= \langle T_f p, T_f p\rangle - \langle T_f^* p, T_f^* p\rangle \\ &= ||fp||^2 - ||T_{\bar{f}}p||^2 \\ &= ||\bar{f}p||^2 - ||T_{\bar{f}}p||^2 \geq 0\end{aligned}$$

Thus $[T_f^*, T_f]$ is a positive operator. ■

This result enables us to apply Putnam's inequality to T_f where f is analytic. The lower bounds of the area of $sp(T_f)$ were obtained in [24] (see [25], [26] [27] and [28] for generalizations to uniform algebras and further discussions). Together with Putnam's inequality such lower bounds were used to prove the isoperimetric inequality (see [29], [30] and the references there). Recently, there has been revived interest in the topic in the context of analytic Topelitz operators on the Bergman space (cf. [31], [32] and [33]). Together with Putnam's inequality, the latter lower bounds have provided an alternative proof of the celebrated St. Venant's inequality for torsional rigidity.

The main purpose of this chapter is to show that for a rather large class of Toeplitz operators on H^2, hyponormal operators with a harmonic symbol, there is still a lower bound for the area of the spectrum, similar to the lower bound obtained in [24] in the context of uniform algebras.

We shall use the following characterization of the hyponormal Toeplitz operators given by Cowen in [34]

Theorem 3.2.1 *Let $\varphi \in L^\infty$, where $\varphi = f + \bar{g}$ for f and g in H^2. Then T_φ is hyponormal if and only if*

$$g = c + T_{\bar{h}} f,$$

for some constant c and $h \in H^\infty$ with $\|h\|_\infty \leq 1$.

3.3 Main Results

In this section, we obtain a lower bound for the area of the spectrum for hyponormal Toeplitz operators by estimating the self-commutators.

Theorem 3.3.1 *Suppose $\varphi \in L^\infty$ and*

$$\varphi = f + \overline{T_{\bar{h}} f},$$

for $f, h \in H^\infty$, $\|h\|_\infty \leq 1$ and $h(0) = 0$. Then

$$||[T_\varphi^*, T_\varphi]|| \geq \int |f - f(0)|^2 \frac{d\theta}{2\pi} = ||P(\varphi) - \varphi(0)||^2.$$

Proof Let

$$g = T_{\bar{h}} f. \tag{3.3.1}$$

For every p in H^2,

$$\begin{aligned}
\langle [T_\varphi^*, T_\varphi]p, p\rangle &= \langle T_\varphi p, T_\varphi p\rangle - \langle T_\varphi^* p, T_\varphi^* p\rangle \\
&= \langle fp + P(\bar{g}p), fp + P(\bar{g}p)\rangle - \langle gp + P(\bar{f}p), gp + P(\bar{f}p)\rangle \\
&= ||fp||^2 - ||P(\bar{f}p)||^2 - ||gp||^2 + ||P(\bar{g}p)||^2 \\
&= ||\bar{f}p||^2 - ||P(\bar{f}p)||^2 - ||\bar{g}p||^2 + ||P(\bar{g}p)||^2 \\
&= ||H_{\bar{f}}p||^2 - ||H_{\bar{g}}p||^2.
\end{aligned}$$

The third equality holds because

$$\langle fp, P(\bar{g}p)\rangle = \langle fp, \bar{g}p\rangle = \langle gp, \bar{f}p\rangle = \langle gp, P(\bar{f}p)\rangle.$$

By the computation in [34, p. 4], (3.3.1) implies

$$H_{\bar{g}} = T_{\bar{k}} H_{\bar{f}},$$

where $k(z) = \overline{h(\bar{z})}$. Thus

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle = ||H_{\bar{f}}p||^2 - ||T_{\bar{k}} H_{\bar{f}}p||^2, \tag{3.3.2}$$

for $k \in H^\infty$, $\|k\|_\infty \leq 1$ and $k(0) = 0$.

First, we assume k is a Blaschke product vanishing at 0. Then

$$|k| = 1 \text{ on } \partial\mathbb{D}.$$

Let $u = H_{\bar{f}}p \in H^2$. By (3.3.2) we have

$$\langle [T_\varphi^*, T_\varphi]p, p\rangle = ||u||^2 - ||T_{\bar{k}}u||^2 = ||u||^2 - ||\bar{k}u||^2 + ||H_{\bar{k}}u||^2 = ||H_{\bar{k}}u||^2. \tag{3.3.3}$$

Then

$$\begin{aligned}
||H_{\bar{k}}u|| &= ||(I-P)(\bar{k}u)|| = ||k\bar{u} - \overline{P(\bar{k}u)}|| \\
&\geq \sup_{\substack{m\in H^2 \\ m(0)=0}} \frac{|\langle k\bar{u} - \overline{P(\bar{k}u)}, m\rangle|}{||m||} \\
&= \sup_{\substack{m\in H^2 \\ m(0)=0}} \frac{1}{||m||}\left|\int k\bar{u}\bar{m}\,\frac{d\theta}{2\pi}\right|.
\end{aligned}$$

The last equality holds because $m(0) = 0$ implies that $\bar{m}$ is orthogonal to H^2. Since $k(0) = 0$, taking $m = k$, we find

$$||H_{\bar{k}}u|| \geq \left|\int \bar{u}\,\frac{d\theta}{2\pi}\right| = |u(0)|. \tag{3.3.4}$$

Next, suppose k is a convex linear combination of Blaschke products vanishing at 0, i.e.

$$k = \alpha_1 B_1 + \alpha_2 B_2 + ... + \alpha_l B_l,$$

where B_j's are Blaschke products with $B_j(0) = 0$, $\alpha_j \in [0, 1]$ and $\sum_{j=1}^{l} \alpha_j = 1$.

By (4.1.2) and (3.3.4), for each j

$$\begin{aligned}
&||u||^2 - ||T_{\bar{B}_j}u||^2 = ||H_{\bar{B}_j}u||^2 \geq |u(0)|^2 \\
\Longrightarrow\ &||T_{\bar{B}_j}u|| \leq \sqrt{||u||^2 - |u(0)|^2} = ||u - u(0)||.
\end{aligned}$$

Then

$$\begin{aligned}
||u||^2 - ||T_{\bar{k}}u||^2 &= ||u||^2 - ||\alpha_1 T_{\bar{B}_1}u + \alpha_2 T_{\bar{B}_2}u + ... + \alpha_l T_{\bar{B}_l}u||^2 \\
&\geq ||u||^2 - \Big(\alpha_1||T_{\bar{B}_1}u|| + \alpha_2||T_{\bar{B}_2}u|| + ... + \alpha_l||T_{\bar{B}_l}u||\Big)^2 \\
&\geq ||u||^2 - ||u - u(0)||^2 = |u(0)|^2.
\end{aligned} \tag{3.3.5}$$

In general, for k in the closed unit ball of H^∞, vanishing at 0, by Carathéodory's Theorem(cf. [1, p. 6]), there exists a sequence $\{B_n\}$ of finite Blaschke products such that

$$B_n \longrightarrow k \quad \text{pointwise on } \mathbb{D}.$$

Since B_n's are bounded by 1 in H^2, passing to a subsequence we may assume

$$B_n \longrightarrow k \quad \text{weakly in } H^2.$$

Then by [35, Theorem 3.13], there is a sequence $\{k_n\}$ of convex linear combinations of Blaschke products such that

$$k_n \longrightarrow k \quad \text{in } H^2.$$

Since $k(0) = 0$, we can let those k_n's be convex linear combinations of Blaschke products vanishing at 0.

Then

$$||T_{\bar{k}_n} u - T_{\bar{k}} u|| = ||P(\bar{k}_n u - \bar{k} u)|| \leq ||k_n - k|| \cdot ||u|| \to 0.$$

Since (3.3.5) holds for every k_n, we have

$$\begin{aligned}\langle [T_\varphi^*, T_\varphi] p, p\rangle = ||u||^2 - ||T_{\bar{k}} u||^2 &= \lim_{n\to\infty} (||u||^2 - ||T_{\bar{k}_n} u||^2) \\ &\geq |u(0)|^2 = |(H_{\bar{f}} p)(0)|^2.\end{aligned}$$

By the definition of Hankel operator (1.2.1),

$$|(H_{\bar{f}} p)(0)| = |\langle p\bar{f}, \bar{z}\rangle| = \left| \int \bar{f} z p \, \frac{d\theta}{2\pi} \right|.$$

From the standard duality argument (cf. [1, Chapter IV]), we have

$$\begin{aligned}\sup_{\substack{||p||=1 \\ p\in H^2}} \left| \int \bar{f} z p \, \frac{d\theta}{2\pi} \right| &= \sup \left\{ \left| \int \bar{f} p \, \frac{d\theta}{2\pi} \right| : p \in H^2, ||p|| = 1, p(0) = 0 \right\} \\ &= \operatorname{dist}(\bar{f}, H^2) = ||f - f(0)||.\end{aligned}$$

Hence

$$||[T_\varphi^*, T_\varphi]|| = \sup_{\substack{||p||=1 \\ p\in H^2}} |\langle [T_\varphi^*, T_\varphi] p, p\rangle| \geq ||f - f(0)||^2.$$

■

Remark 3.3.1 *For arbitrary h in the closed unit ball of H^∞, it follows directly from* (3.3.2) *that T_φ is normal if and only if h is a unimodular constant. So we made the assumption that $h(0) = 0$ to avoid these trivial cases. Of course, Theorem 3.3.1 implies right away that T_φ is normal if and only if $f = f(0)$, i.e., when φ is a constant, but under more restrictive hypothesis that $h(0) = 0$.*

Applying Theorem 3.1.3 and 3.2.1, we have

Corollary 3.3.1 *Suppose $\varphi \in L^\infty$ and*

$$\varphi = f + \overline{T_{\bar{h}} f},$$

for $f, h \in H^\infty$, $\|h\|_\infty \leq 1$ and $h(0) = 0$. Then

$$Area(sp(T_\varphi)) \geq \pi ||P(\varphi) - \varphi(0)||^2.$$

Remark 3.3.2 *Thus, the lower bound for the spectral area of a general hyponormal Toeplitz operator T_φ still reduces to the H^2 norm of the analytic part of φ. For analytic symbols this is encoded in [24, Theorem 2] in the context of Banach algebras. In other words, allowing more general symbols does not reduce the area of the spectrum.*

4. Asymptotic Bohr Radius for the Polynomials

4.1 Introduction

The Bohr radius R for H^∞ is defined as

$$R = \sup\{r \in (0,1) : \sum_{k=0}^{\infty} |a_k| r^k \leq ||f||_\infty, \text{ for all } f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^\infty\}.$$

Bohr's famous power series theorem [36] shows that $R = \frac{1}{3}$. Actually, Bohr first considered this problem and proved $R \geq \frac{1}{6}$. Later M. Riesz, I. Schur, and F. Wiener solved the problem independently.

Let $\mathcal{P}_n$ denote the subspace of H^∞ consisting of all the complex polynomials of degree at most n. In 2005, Guadarrama [37] considered the Bohr type radius for the class $\mathcal{P}_n$ defined by

$$R_n = \sup\{r \in (0,1) : \sum_{k=0}^{n} |a_k| r^k \leq ||p||_\infty, \text{ for all } p(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n\}, \tag{4.1.1}$$

and gave the estimate

$$\frac{C_1}{3^{n/2}} < R_n - \frac{1}{3} < C_2 \frac{\log n}{n},$$

for some positive constants C_1 and C_2. Later in 2008, Fournier obtained an explicit formula for R_n by using the notion of *bounded preserving* functions. He proved the following theorem [38]

Theorem 4.1.1 *For each $n \geq 1$, let $T_n(r)$ be the following $(n+1) \times (n+1)$ symmetric Toeplitz matrix*

$$\begin{bmatrix} 1 & r & -r^2 & r^3 & \cdots & (-1)^{n-1}r^n \\ r & 1 & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ -r^2 & r & 1 & r & & \\ r^3 & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-1}r^n & & \cdots & & r & 1 \end{bmatrix}. \tag{4.1.2}$$

Then R_n is equal to the smallest root in $(0,1)$ of the equation

$$\det T_n(r) = 0.$$

Based on numerical evidence, he conjectured that

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + \frac{3\pi^4}{4n^4} + \ldots$$

The purpose of this chapter is to provide a positive answer. We shall prove

Theorem 4.1.2 *Let R_n be as in (4.1.1), then*

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + o(\frac{1}{n^2}), \text{ as } n \to \infty.$$

4.2 Main Theorem

In this section, we prove Theorem 4.1.2. The methods we use are similar to those in [39, Chapter 5].

Proof of Theorem 4.1.2. Fix $r \in (0,1)$, we consider the eigenvalues of $T_n(r)$, the symmetric Toeplitz matrix (4.1.2). Let $\Delta_n(\lambda) = \det(T_n(r) - \lambda I)$, the characteristic polynomial of $T_n(r)$.

For $n \geq 2$, multiplying the second row of $\Delta_n(\lambda)$ by r, adding it to the first row and performing a similar operation with the columns, we have

$$\Delta_n(\lambda) = \det \begin{bmatrix} 1-\lambda+(3-\lambda)r^2 & (2-\lambda)r & 0 & & \cdots & 0 \\ (2-\lambda)r & 1-\lambda & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ 0 & r & 1-\lambda & r & & \\ & -r^2 & r & 1-\lambda & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & \cdots & & r & 1-\lambda \end{bmatrix}$$

$$= [(3-\lambda)r^2 + 1 - \lambda]\Delta_{n-1}(\lambda) - (2-\lambda)^2\Delta_{n-2}(\lambda). \tag{4.2.1}$$

If we set $\Delta_{-1}(\lambda) = 1$, then the recurrence relation (4.2.1) holds for all $n \geq 1$.

Consider the function associated with these Toeplitz matrices $T_n(r)$

$$f(x) = 1 + \sum_{|n|>0} (-1)^{n-1} r^n e^{inx} = \frac{3r^2 + 4r\cos x + 1}{r^2 + 2r\cos x + 1}. \tag{4.2.2}$$

Suppose

$$\lambda = f(x),\ x \in [0, \pi].$$

Then the second order recurrence relation (4.2.1) becomes

$$\Delta_n(\lambda) = [-2(2-\lambda)r\cos x]\Delta_{n-1}(\lambda) - (2-\lambda)^2 r^2 \Delta_{n-2}(\lambda).$$

Its characteristic equation has the roots $(\lambda - 2)re^{\pm ix}$. Adding the initial conditions

$$\Delta_{-1}(\lambda) = 1,\ \Delta_0(\lambda) = 1 - \lambda,$$

we have

$$\Delta_n(\lambda) = \frac{[(\lambda-2)r]^{n+1}}{1-r^2}\left(\frac{\sin(n+2)x}{\sin x} + 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}\right).$$

Denote

$$p_n(x) = \frac{\sin(n+2)x}{\sin x} + 2r\frac{\sin(n+1)x}{\sin x} + r^2\frac{\sin nx}{\sin x}.$$

Then it is easy to verify that p_n is a polynomial of degree $n+1$ in $\cos x$. Let

$$t_\nu^{(n)} = \frac{\nu\pi}{n+2}, \quad \nu = 1, 2, \cdots, n+1.$$

Direct computation shows that

$$p_n(t_\nu^{(n)}) = (-1)^{\nu+1} 2r(1 + r\cos\nu),$$

thus

$$\operatorname{sgn} p_n(t_\nu^{(n)}) = (-1)^{\nu+1}.$$

Also

$$\lim_{x\to\pi^-} p_n(x) = 2(-1)^{n+1}(1-r)^2.$$

So p_n has exactly $n+1$ distinct zeros $\{x_\nu^{(n)} | \nu = 1, 2, \cdots, n+1\}$ on $[0, \pi]$, such that

$$0 < t_1^{(n)} < x_1^{(n)} < t_2^{(n)} < x_2^{(n)} < \cdots < t_{n+1}^{(n)} < x_{n+1}^{(n)} < \pi. \tag{4.2.3}$$

That means for each n,

$$\lambda_\nu^{(n)} = f(x_\nu^{(n)}),\ \nu = 1, 2, \cdots, n+1.$$

are all the eigenvalues of $T_n(r)$. Since f is decreasing on $[0, \pi]$, $\lambda_{n+1}^{(n)} = f(x_{n+1}^{(n)})$ is the smallest eigenvalue of $T_n(r)$.

Next, we will find an asymptotic expression for $x_{n+1}^{(n)}$. Notice that

$$\lim_{n\to\infty} (-1)^{n+1} \frac{p_n(\pi - \frac{z}{n+2})}{n+2} = (1-r)^2 \frac{\sin z}{z}, \tag{4.2.4}$$

where (4.2.4) holds uniformly for $|z| < 2\pi$. Let

$$x_{n+1}^{(n)} = \frac{(n+1)\pi + \epsilon_n}{n+2},$$

then $\epsilon_n \in (0, \pi)$ by relation (4.2.3). Thus

$$\begin{aligned} 0 &= \lim_{n\to\infty} (-1)^{n+1} \frac{p_n(x_{n+1}^{(n)})}{n+2} \\ &= \lim_{n\to\infty} (-1)^{n+1} \frac{p_n(\pi - \frac{\pi-\epsilon_n}{n+2})}{n+2} \\ &= \lim_{n\to\infty} (1-r)^2 \frac{\sin(\pi - \epsilon_n)}{\pi - \epsilon_n}. \end{aligned}$$

Hence the accumulation points of $\{\epsilon_n\}$ are either 0 or π. Let

$$y^{(n)} = \frac{t_{n+1}^{(n)} + \pi}{2} = \pi - \frac{\frac{\pi}{2}}{n+2}.$$

Using (4.2.4) again, we have

$$\begin{aligned} &\lim_{n\to\infty} (-1)^{n+1} \frac{p_n(y^{(n)})}{n+2} \\ &= \lim_{n\to\infty} (-1)^{n+1} \frac{p_n(\pi - \frac{\frac{\pi}{2}}{n+2})}{n+2} \\ &= \lim_{n\to\infty} (1-r)^2 \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} > 0. \end{aligned}$$

When n is sufficiently large,

$$\operatorname{sgn} p_n(y^{(n)}) = (-1)^{n+1} = \operatorname{sgn} p_n(\pi^-),$$

which implies $x_{n+1}^{(n)} \in (t_{n+1}^{(n)}, y^{(n)})$. Consequently, $\epsilon_n \to 0$ as $n \to \infty$, and then

$$x_{n+1}^{(n)} = \pi - \frac{\pi}{n} + o(\frac{1}{n}), \quad \text{as}\, n \to \infty. \tag{4.2.5}$$

Now we are ready to find the asymptotic expression for R_n. Notice that $r = R_n$ is the root in $(0, 1)$ of the equation

$$\lambda_{n+1}^{(n)} = f(x_{n+1}^{(n)}) = 0.$$

By (4.2.2), that means

$$3R_n^2 + 4R_n \cos x_{n+1}^{(n)} + 1 = 0.$$

Thus

$$R_n = \frac{1}{3}(-2\cos x_{n+1}^{(n)} - \sqrt{4\cos^2 x_{n+1}^{(n)} - 3}).$$

Using (4.2.5), we have

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + o(\frac{1}{n^2}).$$

■

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