The Discrete Orthonormal Wavelet Transform: An Introduction

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The Discrete Orthonormal Wavelet Transform: An Introduction

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Abstract

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1. Introduction

In this paper we discuss the wavelet transform in a simple but rigorous manner. Two underlying themes are simultaneously developed. The first is that of signal recovery from a sequence of inner products. The second theme is operational, and concerns the design of filters needed to implement the wavelet transform in hardware or software. Necessary and sufficient conditions for orthonormal decomposition and perfect reconstruction are established. Algorithms for the design of wavelet filters are presented together with illustrative examples. A rudimentary knowledge of $z$-transforms, vector spaces, and inner-products, is assumed. The exposition is, otherwise, complete and self-contained.

The cases of one-dimensional, two-dimensional, and multidimensional signals, are treated separately. A direct treatment of the multidimensional case would have obscured the simplicity of the basic ideas in a web of notation. The one-dimensional case, presented first, lays out the themes that are later echoed in the treatment of higher-dimensional signals. The two-dimensional case provides a gentle introduction to the multiindex notation, used extensively in the treatment of multidimensional signals. The two-dimensional case is also of immediate interest, ia so far as applications to picture-compression and picture-processing are concerned.

The theory of wavelets [1]–[6] is related to older ideas in the field of multirate and subband signal processing [7]. In recent years subband filter theory has seen considerable development [8]–[10]. Our own interest in wavelets came about through a study of the Frazier-Jawerth transform (FJT) [11]–[15], which is also related to multirate signal processing.

1.1. Notation

By $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}^+$, are meant the set of complex numbers, the set of real numbers, the set of integers (positive, negative, and zero), and the set of positive integers. $\mathbb{Z}_N$ denotes the ring of integers $\{0,\ldots,N-1\}$. For any discrete set $U$, $l^1(U)$ denotes the vector space of all absolutely summable sequences defined from $U$ to $\mathbb{C}$, and $l^2(U)$ the vector space of all absolutely square summable sequences defined from $U$ to $\mathbb{C}$. If $f$ is a function from $U$ to $\mathbb{C}$, then the support of $f$ is defined as that subset of $U$ where $f$ does not vanish: $\text{supp}(f) = \{u \in U : f(u) \neq 0\}$. 
2. Two–Band Orthonormal Decomposition And Perfect Reconstruction Of Discrete One–Dimensional Signals

In this section we discuss a certain decomposition of a one–dimensional signal (or sequence) \( x \in V \), where \( V \) is the infinite–dimensional vector space \( l^2(\mathbb{Z}) \) or the \( N \)-dimensional vector space \( \mathbb{C}^N \), \( N \) some even positive integer. We will consider the design of orthogonal subspaces \( V_L \) and \( V_H \) of \( V \). The projections \( y_L \) and \( y_H \) of \( x \) into \( V_L \) and \( V_H \) will be computed, and the signal \( x \) will be reconstructed as the vector sum of these projections.

Consider first the arrangement of filters, downsamplers, and upsamplers in Figure 1. This picture depicts the first basic step in an orthogonal wavelet decomposition. By “\( \downarrow 2 \)” we mean downsampling, or the deletion of every other number from the input sequence. By “\( \uparrow 2 \)” we mean upsampling, or the insertion of a zero between every pair of numbers in the input sequence. The symbols \( \mathcal{D} \) and \( \mathcal{U} \) will also be used to denote the down and upsampling operators. The symbols \( f_L \) and \( f_H \) denote analyzing filters, while \( g_L \) and \( g_H \) are synthesizing filters. The subscripts “\( H \)” and “\( L \)” could be thought of as abbreviations for “highpass” and “lowpass”, respectively; because it is usual, though not necessary, that \( f_H \) and \( g_H \) are highpass filters, and \( f_L \) and \( g_L \) lowpass filters.

If the signal space is infinite–dimensional, \( V = l^2(\mathbb{Z}) \), then we will assume that \( f_L, f_H, g_L, g_H \in l^2(\mathbb{Z}) \). It follows that for \( x \in l^2(\mathbb{Z}) \), the sequences \( u_L, u_H, w_L, y_L, u_H, v_H, w_H, \) and \( y_H \), all belong to \( l^2(\mathbb{Z}) \). For the finite–dimensional signal space, \( V = \mathbb{C}^N \), we will assume \( f_L, f_H, g_L, g_H \in \mathbb{C}^N \). Then for \( x \in \mathbb{C}^N \) the sequences \( u_L, w_L, y_L, u_H, w_H, \) and \( y_H \), belong to \( \mathbb{C}^N \); while the sequences \( u_L \) and \( y_H \) belong to \( \mathbb{C}^{N/2} \).

The wavelet transform splits the signal space \( V \) into many orthogonal components through the recursive use of the method used to split \( V \) into \( V_L \) and \( V_H \). This recursion is the subject of Section 4.

We note, by the way, that if the up and downsamplers are dropped from the highpass branch in Figure 1 then we have the Frazier–Jawerth Transform (FJT) [14], [15]. The Frazier–Jawerth decomposition is not orthogonal.

The decomposition and reconstruction of \( x \) in Figure 1 will be analyzed with the help of \( x \)-transforms.
2.1. $z$-Transform Notation

For an infinite sequence $x = (\ldots, x(-1), x(0), x(1), \ldots) \in l^2(\mathbb{Z})$, $x(i) \in \mathbb{C}$ for all $i$, the $z$-transform $\hat{x}(z)$ of $x$ is written:

$$\hat{x}(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}, \quad (1)$$

where the indeterminate $z$ ranges over the unit circle $T$ in the complex plane. In the standard definition of the $z$-transform, the indeterminate $z$ ranges over the entire complex plane. Here we will use restrictions of the standard definition. Note that $x(n)$ is the $n$-th Fourier coefficient of the function that maps $\omega \in [-\pi, \pi]$ to $\hat{x}(e^{-j\omega})$; in particular then,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(e^{-j\omega}) e^{-j\omega n} \, dw. \quad (2)$$

From (2), if $\hat{x}(z) = \hat{y}(z)$ for all $z \in T$, then $x = y$; hence the $z$-transform is one-to-one.

If $x$ is a finite sequence $x = (x(0), \ldots, x(N-1)) \in \mathbb{C}^N$ for some $N \in \mathbb{Z}^+$, then the $z$-transform $\hat{x}(z)$ of $x$ is written:

$$\hat{x}(z) = \sum_{n=0}^{N-1} x(n)z^{-n}. \quad (3)$$

The function $x$ is defined for $z \in W_N = \{e^{-j2\pi m/N} : m \in \mathbb{Z}_N\}$, the set of all the $N$-th roots of unity. The complex number $\hat{x}(e^{-j2\pi m/N})$, $m \in \mathbb{Z}_N$, is the $m$-th Discrete Fourier Transform (DFT) coefficient of $x$. By DFT inversion,

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{x}(e^{-j2\pi m/N})e^{-j2\pi mn/N}. \quad (4)$$

From (4), if $\hat{x}(z) = \hat{y}(z)$ for all $z \in W_N$, then $x = y$.

By $(Fz)(\omega)$ we will mean the Fourier transform of $z$ evaluated at $\omega \in [-\pi, \pi]$, $(Fz)(\omega) = \hat{z}(e^{-j\omega})$, if $z$ lies in $l^2(\mathbb{Z})$. If $z$ lies in $\mathbb{C}^N$, then by $(Fz)(m)$ we will mean the DFT of $z$ evaluated at $m \in \mathbb{Z}_N$; $(Fz)(m) = \hat{z}(e^{-j2\pi m/N})$.

When the range of the index of summation is not specified in some particular equation, it will mean that the equation holds equally for sums over $\mathbb{Z}$ and $\mathbb{Z}_N$. Similarly, when we make a statement about some property of $\hat{z}(z)$ that holds "for all $z$" or "$\forall z$" we will mean "$\forall z \in T$" or "$\forall z \in W_N$" depending on whether $z$ belongs to the signal space $V = l^2(\mathbb{Z})$ or to $V = \mathbb{C}^N$.

In case of finite sequences, since $z \in W_N$, all arithmetic on the powers of the indeterminate $z^{-1}$ in $\hat{z}(z)$ will be done in the ring $\mathbb{Z}_N$. Thus, if $N = 4$, then $az^{-2} + bz^{-3} + cz^{-4} = az^{-2} + bz^{-1} + cz^{-3}$. When, for a finite sequence, its $z$-transform is written only in terms of $z^{-1}$ raised to some number in $\mathbb{Z}_N$, then we will say that the $z$-transform is written in the canonical form. For infinite sequences the $z$-transform is always in the canonical form. If we assume that all $z$-transforms are always written in the canonical form then the mathematical development here is valid, simultaneously, for the finite and infinite-dimensional signal spaces $\mathbb{C}^N$ and $l^2(\mathbb{Z})$.

For $x, y \in l^2(\mathbb{Z})$ or $x, y \in \mathbb{C}^N$, their convolution $(x \ast y)$ is defined by

$$(x \ast y)(n) = \sum_m x(m)y(n-m). \quad (5)$$
If \( x, y \in L^2(\mathbb{Z}) \), then \( n \) and \( m \) range over \( \mathbb{Z} \). If \( x, y \in C^N \), then \( n \) and \( m \) range over \( \mathbb{Z}_N \). In this latter case, in (5), the index \( (n - m) \) into the sequence \( y \) may appear to go out of the domain \( \mathbb{Z}_N \) of \( y \). However, when dealing with finite convolutions, we will do all arithmetic on sequence indices in the ring \( \mathbb{Z}_N \). This is as in the case of the powers of \( z^{-1} \) in \( z \)-transforms. Then there is no problem with indices going out of range, and all finite convolutions are "circular" convolutions in the standard terminology of digital signal processing.

The \( z \)-transform of the convolution \( (x * y) \) of \( x \) and \( y \) is the product of the \( z \)-transforms of \( x \) and \( y \):

\[
(x * y)(z) = \sum_n (x * y)(n)z^{-n}
\]

\[
= \sum_n \left( \sum_m x(m)y(n - m) \right)z^{-n}
\]

\[
= \sum_m x(m) \left( \sum_n y(n - m)z^{-n} \right)
\]

\[
= \sum_m x(m)z^{-m} \left( \sum_n y(n - m)z^{-(n-m)} \right)
\]

\[
= \hat{x}(z)\hat{y}(z).
\]

By \( \text{coeff}_k(\hat{x}(z)) \) we will mean the coefficient of \( x^{-k} \) in the canonical form \( \hat{x}(z) \). By (2) or (4), this coefficient is well defined. Define the inner product of \( f, g \in L^2(\mathbb{Z}) \) or \( f, g \in C^N \) by,

\[
\langle f, g \rangle = \sum_n f(n)\overline{g(n)}
\]

where the index \( n \) runs over \( \mathbb{Z} \) or \( \mathbb{Z}_N \) depending upon the signal space under consideration. This inner product can be written in the \( z \)-transform notation:

**Lemma 1.** If \( f \) and \( g \) are complex sequences and \( z \in \mathbb{T} \) or \( z \in \mathbb{W}_N \), then \( \langle f, g \rangle = \text{coeff}_0 \left( \hat{f}(z)\overline{\hat{g}(z)} \right) \).

**Proof of Lemma 1.**

\[
\hat{f}(z)\overline{\hat{g}(z)} = \left( \sum_n f(n)z^{-n} \right) \overline{\left( \sum_m g(m)z^{-m} \right)}
\]

\[
= \sum_n \sum_m f(n)\overline{g(m)}z^{m-n}.
\]

In going from (12) to (13) we have used the fact that for \( z \in \mathbb{T} \) or \( z \in \mathbb{W}_N \), \( \overline{z^{-1}} = z \). Equation 13 implies the lemma. \( \Box \)
2.2. Orthonormal Decomposition

Let \( R \) denote a rightshift operator that acts upon a sequence \( x \in l^2(\mathbb{Z}) \) or \( x \in \mathbb{C}^N \) such that
\[
(Rx)(n) = x(n-1),
\]
or equivalently,
\[
(Rx)^\ast(z) = z^{-1} \hat{x}(z).
\] (14)

In the finite-dimensional signal space \( \mathbb{C}^N \), the right shift operator defined in (14) wraps sequences around; i.e., when \( x \) is shifted right once, then the number \( x_{N-1} \) moves to occupy the place where \( x_0 \) was. This is a consequence of using \( \mathbb{Z}_N \) arithmetic upon indices for \( x \). The \( z \)-transform notation reflects this wrap-around accurately when the canonical form is employed.

Let \( f \) be the impulse response of a filter with input \( x \) and output \( y \). Define \( \tilde{f}(n) = \overline{f(-n)} \). The sequence \( \tilde{f} \) is the complex conjugate of the time-reversal of the sequence \( f \). Then,
\[
y(k) = \sum_n x(n) f(k-n) = \sum_n x(n) \tilde{f}(n-k) = \sum_n x(n) \left( R^k \tilde{f} \right)(n)
\]
(15)
\[
= \sum_n x(n) \left( \overline{R^k \tilde{f}} \right)(n)
\]
(16)
\[
= \langle x, R^k \tilde{f} \rangle.
\]
(17)

Equation (16) follows from (15) because the rightshifted conjugate of the sequence \( \tilde{f} \) is the same as the conjugate of the rightshifted sequence \( f \). From (15)–(17) it is evident that the process of filtering a signal \( x \) with a filter \( f \) is equivalent to the computation of a sequence of inner products. One inner product is computed for each number \( y(k) \) produced by the filter.

A multiset (or a bag) is a "set" which may contain multiple copies of one or more of its elements. In Figure 1, the sequence \( u_L \) is made up of the inner products of \( x \) against elements in the multiset \( \{ R^k \tilde{f}_L \} \) or \( k \in \mathbb{Z} \). The reason why \( \{ R^k \tilde{f}_L \} \) is declared to be a multiset, and not a set, is that it is possible that for some \( k_1 \neq k_2 \), \( R^{k_1} \tilde{f}_L = R^{k_2} \tilde{f}_L \). The sequence \( v_L \) is obtained from \( u_L \) by discarding every other number, and therefore consists of the inner products of \( x \) against elements in the multiset
\[
\tilde{B}_L = \{ R^{2k} \tilde{f}_L \},
\]
(18)
where \( k \in \mathbb{Z} \) or \( k \in \mathbb{Z}/2 \) depending upon the signal space.

Define also the multisets
\[
\tilde{B}_H = \{ R^{2k} \tilde{f}_H \},
\]
(19)
\[
\tilde{B} = \tilde{B}_L \cup \tilde{B}_H.
\]
(20)
The union in (20) is a multiset union that preserves the multiplicity of multiset elements.

In orthonormal wavelet analysis, \( \tilde{B} \) is required to define an orthonormal basis of the signal space \( V = l^2(\mathbb{Z}) \) or \( V = \mathbb{C}^N \). The multisets \( \tilde{B}_L \) and \( \tilde{B}_H \) are required to define orthonormal bases of the mutually orthogonal subspaces \( V_L \) and \( V_H \) of \( V \). Then \( \tilde{B}_L, \tilde{B}_H, \) and \( \tilde{B} \), must each be a set. We will see that the orthonormality of \( \tilde{B} \) will lead to particularly simple conditions for the decomposition and perfect reconstruction of signals.

We next deduce the consequences of the orthonormality of \( \tilde{B} \). Before that, however, we state a lemma that will result in somewhat simpler notation.
Lemma 2. Define the multisets

\[ B_L = \{ R^{2k}f_L \}_k \]
\[ B_H = \{ R^{2k}f_H \}_k \]
\[ B = B_L \cup B_H. \]

Then \( B_L \) is orthonormal if and only if \( B_L \) is; \( B_H \) is orthonormal if and only if \( B_H \) is; and \( B \) is orthonormal if and only if \( B \) is.

Proof of Lemma 2. For \( f \in \mathcal{F}(Z) \) or \( f \in \mathcal{C}^N \),

\[ (\tilde{f})^\ast(z) = \sum_n \tilde{f}(n)z^{-n} = \sum_{-n}^{n} \tilde{f}(n)z^{-n} = \sum_{m=-n}^{n} \tilde{f}(m)z^{n} \]
\[ = \left( \sum_{m} f(m)z^{-m} \right) = \tilde{f}(z). \]

By Lemma 1 and (25),

\[ \langle R^{2k}\tilde{f}, R^{2l}\tilde{g} \rangle = \text{coeff}_0 \left( z^{-2k}(\tilde{f})^\ast(z)z^{2l}(\tilde{g})^\ast(z) \right) \]
\[ = \text{coeff}_0 \left( z^{-2k}\tilde{f}(z)z^{2l}\tilde{g}(z) \right) \]
\[ = \langle R^{-2l}\tilde{g}, R^{-2k}f \rangle. \]

If \( B_L \) is orthonormal, then

\[ \langle R^{2k}\tilde{f}_L, R^{2l}\tilde{f}_L \rangle = \delta(k - l); \]

where \( \delta \) is the Kronecker delta:

\[ \delta(i) \triangleq \begin{cases} 
1 & , i = 0 \\
0 & , i \neq 0.
\end{cases} \]

Then from (26)–(29), \( \langle R^{-2l}\tilde{f}_L, R^{-2k}f_L \rangle = \delta(k - l) \), and \( B_L \) is orthonormal. The lemma follows by similar arguments.

It is easy to see that the multiset \( B = \{ R^{2k}f_L \}_k \cup \{ R^{2k}f_H \}_k \) (and therefore \( B \)) is orthonormal if and only if the following equations hold \( \forall k \in Z \) or \( \forall k \in Z_{N/2} \):

\[ \langle f_L, R^{2k}f_L \rangle = \delta(k) \]
\[ \langle f_H, R^{2k}f_H \rangle = \delta(k) \]
\[ \langle f_L, R^{2k}f_H \rangle \equiv 0. \]

Equations (31)–(33) will be called the orthonormality conditions.

The two lemmas that follow will be used in the proof of the main theorem governing orthonormal decompositions. A definition is necessary before the statement of the lemmas. A function \( \hat{h}(z) \) is said to be odd in \( z \) if and only if \( \hat{h}(z) = -\hat{h}(-z) \). We will call a function \( \hat{g}(z) \) an almost-odd function of \( z \) if and only if \( \hat{g}(z) \) is the sum of the constant function \( \text{"}1\text{"} \) with an odd function.
Lemma 3. Let \( f \in l^2(\mathbb{Z}) \) or \( \mathbb{C}^N \). The following are equivalent:

P1: For all \( k \in \mathbb{Z} \) or \( k \in \mathbb{Z}_{N/2} \), \( \langle f, R^{2k}f \rangle = \delta(k) \).

P2: \( \forall z, |f^*(z)|^2 \) is an almost-odd function of \( z \).

P3: \( \forall z, |f^*(z)|^2 + |f(-z)|^2 = 2. \)

Proof of Lemma 3. By Lemma 1,
\[
\langle f, R^{2k}f \rangle = \text{coeff}_0 \left( f^*(z) \overline{(R^{2k}f)^*(z)} \right)
\]
\[
= \text{coeff}_0 \left( f^*(z) \overline{z^{-2k}f(z)} \right)
\]
\[
= \text{coeff}_0 \left( z^{2k} |f(z)|^2 \right)
\]
\[
= \text{coeff}_{2k} \left( |f(z)|^2 \right).
\]

Define
\[
\hat{\alpha}(z) = \sum_n \alpha(n)z^{-n} = |f(z)|^2.
\]

By (37), P1 is true if and only if,
\[
\hat{\alpha}(z) = 1 + \sum_n \alpha(2n+1)z^{-(2n+1)}.
\]

Equation (39) implies that \( \hat{\alpha} \) is an almost-odd function of \( z \). Hence P1 \( \Rightarrow \) P2. Conversely, if P2 holds, then (2) and (4) and a symmetry argument show that \( \alpha(2n) = 0 \) for \( n \neq 0 \). Hence (37) shows that P2 \( \Rightarrow \) P1.

By P2, \( |f(z)|^2 = 1 + \hat{\beta}(z) \), where \( \hat{\beta}(z) \) is some function that is odd in \( z \). By direct substitution, P3 is equivalent to \( \hat{\beta}(z) + \hat{\beta}(-z) = 0 \). Hence P2 \( \Leftrightarrow \) P3. \( \square \)

Lemma 4. Let \( f, g \in l^2(\mathbb{Z}) \) or \( \mathbb{C}^N \). The following are equivalent:

P1: For all \( k \in \mathbb{Z} \) or \( k \in \mathbb{Z}_{N/2} \), \( \langle f, R^{2k}g \rangle = 0 \).

P2: \( \forall z, (\hat{f} \overline{g}) \) is an odd function of \( z \).

P3: \( \forall z, \hat{f}(z)\overline{g}(z) + \hat{f}(-z)\overline{g}(-z) = 0. \)

Proof of Lemma 4. From Lemma 1,
\[
\langle f, R^{2k}g \rangle = \text{coeff}_{2k} \left( \hat{f}(z)\overline{g}(z) \right).
\]

Arguing as in the proof of Lemma 3, P1 is true if, and only if, \( (\hat{f} \overline{g}) \) is an odd function of \( z \). Therefore P1 \( \Leftrightarrow \) P2. Also P2 \( \Leftrightarrow \) P3 by definition. \( \square \)

In other words, Lemma 3 says that the sets \( B_L = \{ R^{2k}f_L \}_k \) and \( B_H = \{ R^{2k}f_H \}_k \) (and therefore \( \bar{B}_L \) and \( \bar{B}_H \)) are each orthonormal if, and only if, the polynomials \( \hat{f}_L(z) \) and \( \hat{f}_H(z) \) are almost-odd as functions of \( z \in \mathbb{T} \) or \( z \in \mathbb{W}_N \). Lemma 4 says that the sets \( B_L \) and \( B_H \) (and therefore \( \bar{B}_L \) and \( \bar{B}_H \)) are mutually orthogonal if, and only if, the polynomial \( \hat{f}(z)\overline{g}(z) \) is an odd function of \( z \in \mathbb{T} \) or \( z \in \mathbb{W}_N \). These lemmas give us the following interesting result:
Theorem 1. The set $\tilde{B} = \{R^{2k}\tilde{f}_L\}_k \cup \{R^{2k}\tilde{f}_H\}_k$ is orthonormal if and only if the matrix

$$A(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{f}_L(z) & \tilde{f}_H(z) \\ \tilde{f}_L(-z) & \tilde{f}_H(-z) \end{pmatrix}. \quad (41)$$

is unitary for all $z \in T$ or $W_N$.

Proof of Theorem 1. From the orthonormality conditions (31)–(33), and Lemmas 2–4, we see that the set $\tilde{B}$ is orthonormal if and only if the following equations hold for all $z \in T$ or $W_N$.

$$|\tilde{f}_L(z)|^2 + |\tilde{f}_L(-z)|^2 = 2 \quad (42)$$

$$|\tilde{f}_H(z)|^2 + |\tilde{f}_H(-z)|^2 = 2 \quad (43)$$

$$\tilde{f}_L(z)\overline{\tilde{f}_H(z)} + \tilde{f}_L(-z)\overline{\tilde{f}_H(-z)} = 0. \quad (44)$$

Equations (42)–(44) are equivalent to the unitarity of the matrix (41). \qed

Equations (42)–(44) may in fact be said to define the unitarity of $A(z)$ in (41). Equations (42) and (43) assert that the two columns of $A(z)$ must each have norm (or “size”, or “length”) of unity in some appropriately defined inner product space $W$. Equation (44) says that the columns of $A(z)$ regarded as vectors in $W$ must be orthogonal, in that their inner product in $W$ is required to vanish. The matrix $A(z)$ in (41) will be encountered over and over again, and will be called the system matrix.

In the finite case (41) is required to be unitary only for $z \in W_N$. Since $\tilde{f}(e^{-j2\pi m/N}) = (\mathcal{F}f)(m)$, Theorem 1 shows that the multiset $\tilde{B}$ is orthonormal if and only if

$$A(m) = \frac{1}{\sqrt{2}} \begin{pmatrix} (\mathcal{F}f_L)(m) & (\mathcal{F}f_H)(m) \\ (\mathcal{F}f_L)(m \pm \frac{N}{2}) & (\mathcal{F}f_H)(m \pm \frac{N}{2}) \end{pmatrix} \quad (45)$$

is unitary for $m = 0, \ldots, (N-1)$; where the sign in (45) is chosen such that $m \pm \frac{N}{2} \in \{0, \ldots, (N-1)\}$. In order to determine the validity of a filter pair $(f_L, f_H)$ for the orthonormal decomposition of an $N$–dimensional signal space we only need check the unitarity of $N/2 \times 2$–matrices. The remaining $N/2 \times 2$–matrices are row–reversed copies of the first $N/2$, and their unitarity is automatic.

2.3. Perfect Reconstruction

We have so far established the necessary and sufficient conditions for orthonormal decomposition. We now establish conditions such that the arrangement in Figure 1 will yield a perfect reconstruction of the input signal. The downsampling operation can result in the violation of the Nyquist sampling criterion in each branch of Figure 1. It is remarkable that these violations do not prohibit the noiseless recovery of a signal, as we shall see. From the lowpass (i.e. lower) branch of Figure 1 we have:

$$\hat{u}_L(z) = \hat{f}_L(z)\hat{e}(z) \quad (46)$$

$$\hat{u}_L(z) = (Du_L)(z) = \frac{1}{2} \left( \hat{u}_L(z^{1/2}) + \hat{u}_L(-z^{1/2}) \right) \quad (47)$$

$$\hat{u}_L(z) = (Du_L^*)(z) = \hat{v}_L(z^2) \quad (48)$$

$$\hat{y}_L(z) = \hat{g}_L(z)\hat{w}_L(z). \quad (49)$$
Equations (47) and (48) describe the down and upsampling operations, respectively, and may in fact be considered to define those operations. Alternatively, we may define the downsampling operator \( D : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z}) \) or \( D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/2} \) by
\[
(Du)(n) = u(2n), \quad n \in l^2(\mathbb{Z}) \lor n \in \mathbb{C}^{N/2}.
\] (50)

The definitions of \( D \) in (47) and (50) are equivalent because
\[
\hat{u}(z^{1/2}) + \hat{u}(-z^{1/2}) = \sum_{n} u(n)z^{-n/2} + \sum_{n} u(n)(-1)^n z^{-n/2}
\] (51)
\[
= 2 \sum_{n \text{ even}} u(n)z^{-n/2} = 2 \sum_{m} u(2m)z^{-m}.
\] (52)

Similarly, the upsampling operator \( U : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z}) \) or \( U : \mathbb{C}^{N/2} \rightarrow \mathbb{C}^N \) may be defined as:
\[
(Uv)(n) = \begin{cases} 
  v(n/2), & n \text{ even} \\
  0, & \text{otherwise}.
\end{cases}
\] (53)
The definitions of \( U \) in (48) and (53) are equivalent.

From (46)–(49),
\[
\hat{y}_L(z) = \frac{1}{2} \hat{y}_L(z)\hat{f}_L(z)\hat{\chi}(z) + \frac{1}{2} \hat{y}_L(z)\hat{f}_L(-z)\hat{\chi}(-z).
\] (54)

Similarly, from the highpass branch in Figure 1,
\[
\hat{y}_H(z) = \frac{1}{2} \hat{y}_H(z)\hat{f}_H(z)\hat{\chi}(z) + \frac{1}{2} \hat{y}_H(z)\hat{f}_H(-z)\hat{\chi}(-z).
\] (55)

Perfect reconstruction holds in Figure 1 if and only if \( \hat{\chi}(z) = \hat{y}_L(z) + \hat{y}_H(z) \); i.e. if and only if
\[
\hat{\chi}(z) = \frac{1}{2} \left( \hat{y}_L(z)\hat{f}_L(z) + \hat{y}_H(z)\hat{f}_H(z) \right) \hat{\chi}(z) + \frac{1}{2} \left( \hat{y}_L(z)\hat{f}_L(-z) + \hat{y}_H(z)\hat{f}_H(-z) \right) \hat{\chi}(-z).
\] (56)

We also have the following:

**Lemma 5.** If \( \hat{a}(z), \hat{b}(z) \) are fixed polynomials in \( z \), and if for all \( \hat{\chi}(z) \)
\[
\hat{\chi}(z) = \hat{a}(z)\hat{\chi}(z) + \hat{b}(z)\hat{\chi}(-z),
\] (57)
then \( \hat{a}(z) = 1 \) and \( \hat{b}(z) = 0 \).

**Proof of Lemma 5.** Substituting \( \hat{\chi}(z) = 1 \) and \( \hat{\chi}(z) = z \) into (57) we have
\[
z = z\hat{a}(z) + z\hat{b}(z)
\] (58)
\[
z = z\hat{a}(z) - z\hat{b}(z).
\] (59)

By adding and subtracting (58) and (59) we have the lemma. \( \square \)
From (56) and Lemma 5 we deduce that perfect reconstruction holds if and only if the following equations are true:

\begin{align}
\hat{g}_L(z)\hat{f}_L(z) + \hat{g}_H(z)\hat{f}_H(z) &= 2 \quad (60) \\
\hat{g}_L(z)\hat{f}_L(-z) + \hat{g}_H(z)\hat{f}_H(-z) &= 0 \quad (61)
\end{align}

The system of equations (60) and (61) can be written as the single matrix equation below.

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
\hat{f}_L(z) & \hat{f}_H(z) \\
\hat{f}_L(-z) & \hat{f}_H(-z)
\end{pmatrix} \begin{pmatrix}
\hat{g}_L(z) \\
\hat{g}_H(z)
\end{pmatrix} = A(z) \begin{pmatrix}
\hat{g}_L(z) \\
\hat{g}_H(z)
\end{pmatrix} = \begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix}. \quad (62)
\]

The condition (62) is necessary and sufficient for perfect reconstruction, whether or not \( \hat{B} \) is orthonormal. If \( \hat{B} \) is orthonormal then by Theorem 1 the system matrix \( A(z) \) is unitary and is, therefore, particularly easy to invert: \( A^{-1}(z) = A^*(z) = A^T(z) \). Hence (62) is easily solved:

\[
\begin{pmatrix}
\hat{g}_L(z) \\
\hat{g}_H(z)
\end{pmatrix} = A^*(z) \begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix} = \begin{pmatrix}
\overline{f}_L(z) \\
\overline{f}_H(z)
\end{pmatrix}. \quad (63)
\]

From (63) and (24)–(25), when \( \hat{B} \) is orthonormal, perfect reconstruction requires that \( g_L = \overline{f}_L \) and \( g_H = \overline{f}_H \).

This is hardly a surprising result in view of the fact that given an orthonormal basis \( \hat{B} \) of an inner product space \( V, \langle x, z \rangle \in V \) can be written as a weighted sum of vectors in \( \hat{B} \), where the weights are given by the inner products of \( x \) computed against the basis vectors. Recall from (16)–(17) that \( (x \ast f)(2k) = \langle x, R^{2k} f \rangle \). By perfect reconstruction in Figure 1,

\[
x(n) = (g_L \ast (UD(x \ast f_L)))(n) + (g_H \ast (UD(x \ast f_H)))(n) = \sum_k g_L(n - 2k)\langle x, R^{2k} f_L \rangle + \sum_k g_H(n - 2k)\langle x, R^{2k} f_H \rangle \quad (64)
\]

\[
= \sum_k \langle x, R^{2k} f_L \rangle (R^{2k} g_L)(n) + \sum_k \langle x, R^{2k} f_H \rangle (R^{2k} g_H)(n). \quad (65)
\]

In view of (64)–(66), and in view of the orthonormality of \( \hat{B} \), it is not surprising that \( g_L = \overline{f}_L \) and \( g_H = \overline{f}_H \). Equation (66), along with the equations \( g_L = \overline{f}_L \) and \( g_H = \overline{f}_H \), demonstrates the completeness of \( \hat{B} \). These three equations show that every vector \( x \in L^2(\mathbb{Z}) \) or \( C^N \) can be written as a sum of the elements of \( \hat{B} \). The sequences \( \overline{f}_L \) and \( \overline{f}_H \) are first-generation wavelets.

We have proved:

**Theorem 2.** The following are equivalent:

P1: The set \( \hat{B} = \{R^{2k} \hat{f}_L\}_k \cup \{R^{2k} \hat{f}_H\}_k \) is orthonormal, and we have perfect reconstruction in Figure 1.

P2: The system matrix \( A(z) \) in (41) is unitary for all \( z \in T \) or \( z \in \mathbb{W}_N \), \( g_L = \overline{f}_L \), and \( g_H = \overline{f}_H \).
3. The Design Of Wavelet Filters

In this section we discuss the design of the analysis filters \( f_L \) and \( f_H \), and of the synthesis filters \( g_L \) and \( g_H \), in the signal spaces \( \mathcal{L}(\mathbb{Z}) \) and \( \mathbb{C}^N \). The requirements of orthonormality determine the unitarity of the system matrix \( A(z) \) by Theorem 1. By the unitarity condition the norm of the first column of the system matrix \( A(z) \) must be unity. Therefore,

\[
|\hat{f}_L(z)|^2 + |\hat{f}_L(-z)|^2 = 2. \tag{67}
\]

From (67) and Lemma 3, \( |\hat{f}_L(z)|^2 \) is almost-odd. Then \( |\hat{f}_L(z)|^2 = 1 + \hat{h}(z) \), where \( \hat{h} \) is some odd function of \( z \). Moreover \( \hat{h}(z) \) is real-valued because \( |\hat{f}_L(z)|^2 - 1 \) is. As the squared modulus of a complex number, \( |\hat{f}_L(z)|^2 \) is bounded below by zero for all \( z \in \mathcal{T} \). From (67), \( |\hat{f}_L(z)|^2 \) is also bounded above: \( |\hat{f}_L(z)|^2 \leq 2, \forall z \in \mathcal{T} \) or \( \mathcal{W}_N \). Then \(-1 \leq \hat{h}(z) \leq 1, \forall z \in \mathcal{T} \) or \( \mathcal{W}_N \). These observations yield a recipe for the construction of \( \hat{f}_L(z) \).

Let \( \hat{h}(z) \) be any real-valued function defined upon the complex unit circle \( \mathcal{T} \) or the roots \( \mathcal{W}_N \) of unity, such that \( \hat{h}(z) = -\hat{h}(-z) \); and \(-1 \leq \hat{h}(z) \leq 1 \). Let \( \rho(z) \) be another arbitrary real-valued function defined on \( \mathcal{T} \) or \( \mathcal{W}_N \). Define

\[
\hat{f}_L(z) = \sqrt{1 + \hat{h}(z)} \ e^{j\rho(z)}. \tag{68}
\]

The form of \( \hat{f}_L(z) \) in (67) is the most general possible. Since \( \rho(z) \) need not be a polynomial in \( z \), we do not write \( \rho(z) \).

Because \( A(z) \) is unitary, so is \( A^T(z) \). Because the norm of the first column of \( A^T(z) \) must be unity, \( |\hat{f}_L(z)|^2 + |\hat{f}_H(z)|^2 = 2 \). Then \( \hat{f}_H(z) \) must have the form:

\[
\hat{f}_H(z) = \sqrt{1 - \hat{h}(z)} \ e^{j\sigma(z)}, \tag{69}
\]

where \( \sigma(z) \) is a real-valued function of \( z \).

Substituting (68) and (69) into the system matrix (41), and using the oddness of \( \hat{h}(z) \), we have:

\[
A(z) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sqrt{1 + \hat{h}(z)} \ e^{j\rho(z)} & \sqrt{1 - \hat{h}(z)} \ e^{j\sigma(z)} \\
\sqrt{1 - \hat{h}(z)} \ e^{j\rho(-z)} & \sqrt{1 + \hat{h}(z)} \ e^{j\sigma(-z)}
\end{pmatrix}. \tag{70}
\]

Theorem 1 also requires the orthogonality of the two column vectors in \( A^T(z) \). The inner product of the column vectors in \( A^T(z) \) must vanish, then

\[
\sqrt{1 - \hat{h}^2(z)} \left( e^{j(\sigma(z) - \rho(-z))} + e^{j(\sigma(-z) - \rho(z))} \right) = 0 \tag{71}
\]

If \( \hat{h}(z) \neq \pm 1 \), then (71) requires that:

\[
\sigma(z) - \sigma(-z) = (2k + 1)\pi + \rho(z) - \rho(-z), \tag{72}
\]

for some \( k \in \mathbb{Z} \), \( k \) depending possibly on \( z \). If \( \hat{h}(z) = \pm 1 \), then \( \sigma(z) \) and \( \rho(z) \) are unconstrained.

We have shown that if \( A(z) \) is unitary then \( A(z) \) has the form (70) for \( \hat{h}(z) \), \( \sigma(z) \), and \( \rho(z) \), as described. Conversely, any such matrix is easily seen to be unitary. Finally, the synthesis filter sequences \( g_L \) and \( g_H \) must be chosen so as to satisfy Theorem 2. The three steps S1, S2, and S3, in the construction of the filter sequences \( f_L, f_H, g_L \), and \( g_H \), are summarized below.
S1. Construct an arbitrary real-valued function \( \hat{h}(z), z \in T \) or \( W_N \), such that \(-1 \leq \hat{h}(z) \leq 1\), and \( \hat{h}(z) = -\hat{h}(-z) \). Construct an arbitrary real-valued function \( \rho(z), z \in T \) or \( W_N \). Define \( \tilde{f}_L(z) = \sqrt{1 + \hat{h}(z)e^{j\rho(z)}}, z \in T \) or \( W_N \).

S2. Construct a real-valued function \( \sigma(z), z \in T \) or \( W_N \), such that \( \sigma(z) - \sigma(-z) = (2k + 1)\pi + \rho(z) - \rho(-z) \), if \( \hat{h}(z) \neq \pm 1 \). Here \( k \in Z \) is an arbitrary integer that possibly depends on \( z \).

Define \( \tilde{f}_H(z) = \sqrt{1 - \hat{h}(z)e^{j\sigma(z)}}, z \in T \) or \( W_N \).

S3. Define \( \tilde{g}_L(z) = \overline{\tilde{f}_L(z)}; \tilde{g}_H(z) = \overline{\tilde{f}_H(z)} \).

Theorem 2 shows that any and all wavelet filters at the first stage can be constructed with this algorithm.

3.1. Examples Of Wavelet Filter Construction

Let \( z \in C^4 \). Let \( \omega = e^{-j2\pi/4} = -j \); \( W_4 = \{\omega^0, \omega^1, \omega^2, \omega^3\} = \{1, -j, -1, j\} \). Choose

\[ \hat{h}(1) = 1, \quad \hat{h}(-j) = -1, \quad \hat{h}(-1) = -1, \quad \hat{h}(j) = 1. \] (73)

Choose \( \rho(z) \equiv 0 \). Then from \( \tilde{f}_L(z) = \sqrt{1 + \hat{h}(z)} \) we have

\[ \tilde{f}_L(1) = \sqrt{2}, \quad \tilde{f}_L(-j) = 0, \quad \tilde{f}_L(-1) = 0, \quad \tilde{f}_L(j) = \sqrt{2}. \] (74)

The filter sequence \( f_L \) itself can be computed as the inverse DFT of the vector \( \mathcal{F}f_L = (\sqrt{2}, 0, 0, \sqrt{2}): \)

\[ f_L = 2^{-3/2}(2, (1 + j), 0, (1 - j)). \] Knowing \( f_L \), we can write the polynomial \( \tilde{f}_L(z) \):

\[ \tilde{f}_L(z) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}(1 + j)z^{-1} + \frac{1}{2\sqrt{2}}(1 - j)z^{-3}. \] (75)

In order to compute \( \tilde{f}_H(z) \) we need to choose the function \( \sigma(z) \). Since \( \hat{h}(z) = \pm 1 \) for \( z = 1, j, -1, -j \), the choice of \( \sigma(z) \) is unconstrained by \( \rho(z) \). We choose \( \sigma(z) \equiv 0 \). Then the DFT of \( f_H \), computed from \( \tilde{f}_H(z) = \sqrt{1 - \hat{h}(z)} \) is:

\[ \mathcal{F}f_H = (0, \sqrt{2}, \sqrt{2}, 0), \] (76)

and

\[ \tilde{f}_H(z) = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}(1 + j)z^{-1} - \frac{1}{2\sqrt{2}}(1 - j)z^{-3}. \] (77)

By Step 3 of the filter construction algorithm we have \( \tilde{g}_L(z) = \overline{\tilde{f}_L(z)} = \tilde{f}_L(z) \) and \( \tilde{g}_H(z) = \overline{\tilde{f}_H(z)} = \tilde{f}_H(z) \).
3.1.1. A Second Example. The filters $f_L$ and $f_H$ in the last example were complex sequences. In this example we produce real sequences. Let $x \in \mathbb{C}^4$. Let

\[ \hat{h}(1) = 1, \hat{h}(-j) = 0, \hat{h}(-1) = -1, \hat{h}(j) = 0. \quad (78) \]

Choose

\[ \rho(1) = 0, \rho(-j) = \pi/2, \rho(-1) = 0, \rho(j) = -\pi/2. \quad (79) \]

Then

\[ \hat{f}_L(1) = \sqrt{2}, \hat{f}_L(-j) = j, \hat{f}_L(-1) = 0, \hat{f}_L(j) = -j; \quad (80) \]

or $\mathcal{F}f_L = (\sqrt{2}, j, 0, -j) \Rightarrow f_L = 0.25(\sqrt{2}, \sqrt{2} + 2, \sqrt{2}, \sqrt{2} - 2)$.

In order to compute $f_H$ we need to choose $\sigma(x)$. This time because $\hat{h}(x) \not\equiv \pm 1$ for all $x \in \mathbb{W}_4$, step S2 requires us to exercise a little care in the choice of $\sigma(x)$. Choose $\sigma(x) \equiv 0$. This choice respects the constraints in S2. We obtain $\mathcal{F}f_H = (0, 1, \sqrt{2}, 1) \Rightarrow f_H = 0.25(\sqrt{2} + 2, -\sqrt{2}, \sqrt{2} - 2, -\sqrt{2})$.

Because the filters $f_L$, $f_H$, $g_L$ and $g_H$ are all real filters in this example, all the sequences in Figure 1 will be real if the signal sequence is real.

3.2. Time–Frequency Localization

Wavelets and other methods of time–frequency analysis, like the FJT, have many practical applications which require the filter sequences or “analyzing functions” to possess certain specific properties. These properties concern the frequency–localization or the simultaneous time–frequency–localization [14]–[16] of filter sequences. In the case of the FJT time–frequency localization is easy [16]. In the case of wavelets it is not, because we are hemmed in by the orthogonality conditions. In what follows we formulate the time–frequency localization problem for wavelets.

In Figure 2 are drawn the subspaces $V_T$ and $V_B$ of $V = L^2(\mathbb{Z})$ or $V = C^N$. The subspace $V_T$ consists of all sequences $x \in V$, with a fixed $\text{supp}(x)$, which is some proper subset of $V$. The subspace $V_B$ consists of all sequences $x \in V$, with a fixed support for $\mathcal{F}x$. The only vector common to $V_T$ and $V_B$ is the zero vector. $P_T : V \rightarrow V_T$ and $P_B : V \rightarrow V_B$ are projection operators.

Let $V_S \subset V$ be the set of all valid wavelets $f_L$ and $f_H$. It is easy to check that $V_S$ is neither a subspace nor an affine space of $V$, hence any mapping $P_S : V \rightarrow V_S$ cannot be a projection. $V_S$ is in fact a manifold that lies embedded in the surface of the unit sphere in $L^2(\mathbb{Z})$ or $C^N$. Define $P_S : V \rightarrow V_S$ to be an operator that maps any given $f \in V$ to $\psi = P_S f \in V_S$, such that $\|f - \psi\|_V$ is a minimum. We now construct this operator for $V = C^N$.

Given any $f \in C^N$ we would like to find a wavelet $\psi$ that is closest to $f$. If $\psi$ is a wavelet that is closest to $f$, then $\|f - \psi\|_{C^N}$ is a minimum.

\[ \|f - \psi\|_{C^N}^2 = \frac{1}{N} \|((\mathcal{F}f) - (\mathcal{F}\psi))\|_{C^N}^2 \]

\[ = \frac{1}{N} \sum_{m=0}^{N-1} |(\mathcal{F}f)(m) - (\mathcal{F}\psi)(m)|^2 \]

\[ = \frac{1}{N} \sum_{m=0}^{N/2-1} |(\mathcal{F}f)(m) - (\mathcal{F}\psi)(m)|^2 + |(\mathcal{F}f)(m + N/2) - (\mathcal{F}\psi)(m + N/2)|^2. \]
Figure 2: The signal space $V$, the subspaces $V_T$ and $V_B$, the manifold $V_S$, and mappings from $V$ into $V_T$, $V_B$, and $V_S$.

For $m \in \mathbb{Z}_{N/2}$, define

$$A(m) = |(\mathcal{F}f)(m) - (\mathcal{F}\psi)(m)|^2 + |(\mathcal{F}f)(m + N/2) - (\mathcal{F}\psi)(m + N/2)|^2. \quad (84)$$

From (81)–(84),

$$\|f - \psi\|_{C^N}^2 = \frac{1}{N} \sum_{m=0}^{N/2-1} A(m). \quad (85)$$

In order to minimize $\|f - \psi\|_{C^N}^2$, we minimize $A(m)$ for each $m \in \mathbb{Z}_{N/2}$.

In order for $\psi$ to be a valid wavelet filter $f_L$ or $f_H$, the following equation must hold for $m \in \mathbb{Z}_N$:

$$(\mathcal{F}\psi)(m) = \sqrt{1 + (\mathcal{F}h)(m)} e^{j\rho(e^{-j2\pi m/N})}, \quad (86)$$

by the conditions set forth in step S1 of the wavelet construction algorithm. Also it must be that $(\mathcal{F}h)(m + N/2) = -(\mathcal{F}h)(m) \in [-1, 1]$, and $\rho(e^{-j2\pi m/N})$ real, for $m \in \mathbb{Z}_N$.

Define $a(m)$, $b(m)$, $\theta(m)$, and $\gamma(m)$ by

$$\begin{align*}
(\mathcal{F}f)(m) &= a(m)e^{j\theta(m)}; \quad m = 0, \ldots, (N/2 - 1) \quad (87) \\
(\mathcal{F}f)(m + N/2) &= b(m)e^{j\gamma(m)}; \quad m = 0, \ldots, (N/2 - 1), \quad (88)
\end{align*}$$

where $a(m), b(m) \in \mathbb{R}$; $a(m), b(m) \geq 0$; and $\theta(m), \gamma(m) \in [-\pi, \pi]$; for $m \in \mathbb{Z}_{N/2}$.
From (84) and (86)-(88), for \( m = 0, \ldots, (N/2 - 1) \),

\[
A(m) = \left| a(m)e^{j\theta(m)} - \sqrt{1 + (\mathcal{F}h)(m)} e^{j\rho(e^{-j2\pi m/N})} \right|^2 + \left| b(m)e^{j\gamma(m)} - \sqrt{1 - (\mathcal{F}h)(m)} e^{j\rho(e^{-j2\pi m/N})} \right|^2
\]

\[
= a^2(m) + 1 + (\mathcal{F}h)(m) - 2a(m)\sqrt{1 + (\mathcal{F}h)(m)} \cos(\theta(m) - \rho(e^{-j2\pi m/N})) + b^2(m) + 1 - (\mathcal{F}h)(m) - 2b(m)\sqrt{1 - (\mathcal{F}h)(m)} \cos(\gamma(m) - \rho(e^{-j2\pi m/N}))
\]

\[
= 2 + a^2(m) + b^2(m) - 2a(m)\sqrt{1 + (\mathcal{F}h)(m)} \cos(\theta(m) - \rho(e^{-j2\pi m/N})) - 2b(m)\sqrt{1 - (\mathcal{F}h)(m)} \cos(\gamma(m) - \rho(e^{-j2\pi m/N})).
\]

(89)

(90)

(91)

If \( a(m) = b(m) = 0 \) for some \( m = m_0 \) then (91) tells us that \( A(m_0) = 2 \), and we have all the freedom we want in the choice of \( (\mathcal{F}h)(m_0) \in [-1, 1] \), \( \rho(e^{-j2\pi m_0/N}) \in [-\pi, \pi] \), and \( \rho(e^{-j2\pi m_0/N}) \in [-\pi, \pi] \). It follows that there may be infinitely many wavelets \( \psi \) that are closest to a given \( f \in V \).

Now assume that \( a(m) \) and \( b(m) \) are not both zero. From (91) we note that in order to minimize \( A(m) \) we want \( \rho(e^{-j2\pi m/N}) = \theta(m) \) and \( \rho(e^{-j2\pi m/N}) = \gamma(m) \). Then (91) reduces to:

\[
A(m) = 2 + a^2(m) + b^2(m) - 2\left( a(m)\sqrt{1 + (\mathcal{F}h)(m)} + b(m)\sqrt{1 - (\mathcal{F}h)(m)} \right).
\]

(92)

In (92) \( a(m) \) are fixed, and we wish to choose \( (\mathcal{F}h)(m) \in [-1, 1] \) so as to minimize \( A(m) \). This will be done if we choose \( (\mathcal{F}h)(m) \) so as to maximize \( a(m)\sqrt{1 + (\mathcal{F}h)(m)} + b(m)\sqrt{1 - (\mathcal{F}h)(m)} \).

Consider the following:

**Lemma 6.** Let \( a, b \geq 0 \), fixed; not both \( a \) and \( b \) zero. Let \( f(x) = a\sqrt{1 + x} + b\sqrt{1 - x} \). The function \( f(x) \) attains a maximum on \([-1, 1]\) at

\[
x_0 = \frac{a^2 - b^2}{a^2 + b^2}.
\]

(93)

**Proof of Lemma 6.** For \( x_0 \in (-1, 1) \),

\[
f'(x_0) = \frac{a}{2(1 + x_0)^{1/2}} - \frac{b}{2(1 - x_0)^{1/2}} = 0 \quad \Rightarrow \quad x_0 = \frac{a^2 - b^2}{a^2 + b^2}.
\]

(94)

Because \( f(x_0) = \sqrt{2}\sqrt{a^2 + b^2} \) for \( x_0 \in (-1, 1) \); \( f(-1) = \sqrt{2}b \); and \( f(1) = \sqrt{2}a \); we see that \( f(x) \) is indeed maximum at \( x_0 \in [-1, 1] \).

It follows that, if not both \( a(m) \) and \( b(m) \) are zero, then \( a(m)\sqrt{1 + (\mathcal{F}h)(m)} + b(m)\sqrt{1 - (\mathcal{F}h)(m)} \) is maximised (and \( A(m) \) minimised) by the following choice of \( (\mathcal{F}h)(m) \in [-1, 1] \), \( m \in \mathbb{Z}_{N/2} \):

\[
(\mathcal{F}h)(m) = \frac{a^2(m) - b^2(m)}{a^2(m) + b^2(m)} = \frac{|(\mathcal{F}f)(m)|^2 - |(\mathcal{F}f)(m + N/2)|^2}{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m + N/2)|^2}; \quad m \in \mathbb{Z}_{N/2}.
\]

(95)
Hence the DFT coefficient \((\mathcal{F}\psi)(m)\), \(m \in \mathbb{Z}_{N/2}\), of the wavelet \(\psi\) that is closest to \(f \in V\) is

\[
(\mathcal{F}\psi)(m) = \sqrt{1 + (\mathcal{F}h)(m)} e^{j\theta(m)} = \frac{\sqrt{2}a(m)e^{j\theta(m)}}{\sqrt{a^2(m) + b^2(m)}} \quad (96)
\]

\[
= \frac{\sqrt{2}(\mathcal{F}f)(m)}{\sqrt{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m + N/2)|^2}}; \quad m \in \mathbb{Z}_{N/2}. \quad (97)
\]

For \(N/2 \leq m < N\), step S1 in the wavelet construction algorithm tell us that \((\mathcal{F}h)(m) = -(\mathcal{F}h)(m - N/2)\). Then, from (95), for \(N/2 \leq m < N\),

\[
(\mathcal{F}h)(m) = -(\mathcal{F}h)(m - N/2) = \frac{|(\mathcal{F}f)(m - N/2)|^2 - |(\mathcal{F}f)(m)|^2}{|(\mathcal{F}f)(m - N/2)|^2 + |(\mathcal{F}f)(m)|^2}. \quad (98)
\]

By \(\mathbb{Z}_N\) arithmetic upon indices into \((\mathcal{F}f)\), \((\mathcal{F}f)(m - N/2) = \mathcal{F}(m + N/2)\), and we note from (95) and (98) that (95) and (97) hold not only for \(m \in \mathbb{Z}_{N/2}\) but for the full range \(m \in \mathbb{Z}_N\).

In summary, we present the following simple three-step algorithm for the construction of a wavelet \(\psi\) that is closest to a given vector \(f \in \mathbb{C}^N\), \(f \neq 0\):

S1. If \((\mathcal{F}f)(m) = (\mathcal{F}f)(m + N/2) = 0\), then assign any value \(C \in [0, \sqrt{2}]\) to \(|(\mathcal{F}\psi)(m)|\). To \(|(\mathcal{F}\psi)(m + N/2)|\) assign \(\sqrt{2} - C^2\). To each of the two phase terms \(\rho(e^{-j2\pi m/N})\) and \(\rho(-e^{-j2\pi m/N})\) of \((\mathcal{F}\psi)(m)\) and \((\mathcal{F}\psi)(m + N/2)\) assign any value in \([-\pi, \pi]\).

S2. If not both \((\mathcal{F}f)(m)\) and \((\mathcal{F}f)(m + N/2)\) vanish, then assign

\[
(\mathcal{F}\psi)(m) = \frac{\sqrt{2}(\mathcal{F}f)(m)}{\sqrt{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m + N/2)|^2}}. \quad (99)
\]

S3. From \(\mathcal{F}\psi\) compute \(\psi\) by DFT inversion.

This algorithm is an operational description of the operator \(P_S : V \to V_S\), \(V = \mathbb{C}^N\). A similar description of \(P_S\) is possible for the signal space \(l^2(\mathbb{Z})\).

We note from the above algorithm that if \(\mathcal{F}\) is real-valued, then we can always find a wavelet \(\psi\) closest to \(f\) such that \(\mathcal{F}\psi\) is also real-valued.

In case of FJT analyzing functions, time-frequency localization involves the computation of an eigenvector of the double projection operator \((P_B P_T)\) [16]. This eigenvector may not lie in \(V_S\) and may not be a valid wavelet. One approach to the construction of a localized wavelet may be to map the eigenvector \(\phi\) of \((P_B P_T)\) into \(V_S\) using \(P_S\). Let \(\psi\) be the image of \(\phi\) under \(P_S\). If the support of \(\phi\) is not too severely restricted in the frequency domain, i.e. if not both \((\mathcal{F}\phi)(m)\) and \((\mathcal{F}\psi)(m + N/2)\) vanish for any \(m\), then (99) tells us that \(\psi\) will preserve, precisely, the frequency localization of \(f\). If the support of \(\mathcal{F}\phi\) is severely restricted then we can still, to some extent, control the support of \(\mathcal{F}\psi\) through a judicious choice of the constants \(C\) in S1. It is not, however, possible to say anything about the time-localization of \(\psi\) on the basis of the analysis in Section 3.2.

It is possible to choose \(\phi\) such that both \(\phi\) and \(\mathcal{F}\phi\) are real-valued. Then we can produce a wavelet \(\psi\) closest to \(\phi\) that is also real-valued in both the time and the frequency domains.

Define \((\mathcal{F}g)(m) = \sqrt{2}([(\mathcal{F}\phi)(m)]^2 + [(\mathcal{F}\phi)(m + N/2)]^2)^{-1/2}\). Then, from (99), \(\psi = \phi * g\). It follows that if \(\phi\) and \(g\) are well-localized in time, then \(\psi\) will be also. The function \(\phi\) is well-localized by design, and that leaves us with questions concerning the time-localization of \(g\). These questions remain open.
4. Wavelet Recursion For One–Dimensional Signals

If in Figure 1 we subject the sequence $v_L$ to the same treatment as $x$ was subjected to, then we obtain the situation depicted in Figure 3. If we decompose and reconstruct both $v_L$ and $v_H$, we have the situation in Figure 4. Like the level–1 filters $f_L$, $f_H$, $g_L$, and $g_H$; the level–2 filters $f_{LL}$, $f_{LH}$, $f_{HL}$, $f_{HH}$, $g_{LL}$, $g_{LH}$, $g_{HL}$, and $g_{HH}$ must also obey the requirements set forth in Theorem 2, if we require a perfect and orthogonal decomposition. Then the matrices

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{f}_{LL}(z) & \tilde{f}_{LH}(z) \\ \tilde{f}_{LL}(-z) & \tilde{f}_{LH}(-z) \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{f}_{HL}(z) & \tilde{f}_{HH}(z) \\ \tilde{f}_{HL}(-z) & \tilde{f}_{HH}(-z) \end{pmatrix},$$

must be unitary; and the reconstruction filters $g_{LL}$, $g_{LH}$, $g_{HL}$, and $g_{HH}$, must equal $\tilde{f}_{LL}$, $\tilde{f}_{LH}$, $\tilde{f}_{HL}$, and $\tilde{f}_{HH}$, respectively.

Figure 3: Recursion in the lower branch.

Figure 4: Recursion in both branches.
4.1. The Equivalence Of Recursive And Non-recursive Structures

If $V$ is the vector space $l^2(\mathbb{Z})$ or $C^N$, $N \in 2\mathbb{Z}$, then for $f \in V$ in Figure 1, $y_L$ and $y_H$ are the projections of $f$ onto the orthogonal subspaces $V_L$ and $V_H$. The signal $f$ is recovered as a direct sum of its projections. The sequences $u_L$ and $u_H$ consist of numbers related to the sizes (or norms) of the projections of $f$ onto individual basis vectors that span $V_L$ and $V_H$, respectively. The subspaces $V_L$ and $V_H$ have equal dimensions. In Figures 3 and 4 we obtain projections onto more than two subspaces. In order to see this we draw in Figures 5 and 6 the non-recursive versions of the recursive representations in Figures 3 and 4. These non-recursive representations are obtained by pushing level-2 filters across and through the up or down-sampling operators. The operator $\mathcal{U}$ is the up-sampling operator, so that $(\mathcal{U}f_{LL}(z)) = f_{LL}(z^2)$. The sequences $y_{LL}$ and $y_{LH}$ in Figure 5 belong to the mutually orthogonal subspaces $V_{LL}$ and $V_{LH}$ of $V_L \subseteq V$.

In order to prove, formally, the equivalence between Figures 3 and 5, for example, we show that the two sequences of operations in Figure 7 are equivalent.
Figure 7: Two equivalent sequences of operations.

Figure 8: The sequence of graphs generated in classical wavelet analysis.

For the first arrangement in Figure 7,

\[ \hat{y}(z) = 0.5\hat{u}(z^{1/2}) + 0.5\hat{u}(-z^{1/2}) + 0.5\hat{f}_{LL}(z^{1/2})\hat{g}(z^{1/2}) + 0.5\hat{f}_{LL}(-z^{1/2})\hat{g}(-z^{1/2}) \]  

\[ = 0.25\hat{f}_{LL}(z^{1/2})[\hat{u}(z^{1/4}) + \hat{g}(-z^{1/4})] + 0.25\hat{f}_{LL}(-z^{1/2})[\hat{u}(jz^{1/4}) + \hat{g}(-jz^{1/4})] \]  

\[ = 0.25\hat{f}_{LL}(z^{1/2})[\hat{f}_{L}(z^{1/4})\hat{v}(z^{1/4}) + \hat{f}_{L}(-z^{1/4})\hat{v}(-z^{1/4})] + 0.25\hat{f}_{LL}(-z^{1/2})[\hat{f}_{L}(jz^{1/4})\hat{v}(jz^{1/4}) + \hat{f}_{L}(-jz^{1/4})\hat{v}(-jz^{1/4})]. \]  

For the second arrangement in Figure 7,

\[ \hat{r}(z) = (f_L \ast Uf_{LL})^\ast(z)\hat{v}(z) \]  

\[ = \hat{f}_L(z)(Uf_{LL})^\ast(z)\hat{v}(z) \]  

\[ = \hat{f}_L(z)\hat{f}_{LL}(z^2)\hat{v}(z). \]
Also,
\[
\hat{f}(z) = 0.5\hat{h}(z^{1/4}) + 0.5\hat{h}(-z^{1/4})
= 0.25[\hat{r}(z^{1/4}) + \hat{r}(-z^{1/4})] + 0.25[\hat{r}(jz^{1/4}) + \hat{r}(-jz^{1/4})]
= 0.25\hat{f}_L(z^{1/4})\hat{f}_{LL}(z^{1/2})\hat{z}(z^{1/4}) + 0.25\hat{f}_L(-z^{1/4})\hat{f}_{LL}(z^{1/2})\hat{z}(-z^{1/4})
+ 0.25\hat{f}_L(jz^{1/4})\hat{f}_{LL}(-z^{1/2})\hat{z}(jz^{1/4}) + 0.25\hat{f}_L(-jz^{1/4})\hat{f}_{LL}(-z^{1/2})\hat{z}(-jz^{1/4}).
\] (109)

The equality of (103) and (109) proves the equivalence of the transformation steps in the two arrangements in Figure 7; and therefore in the transformation steps of Figures 3 and 5. The equivalence of the reconstruction steps in Figures 3 and 5 can be similarly established.
4.2. The Orthonormality Of The Recursive Representation

Corollary 1. The set \( \{ R^{4k} (f_L \ast U f_{LL}) : k \in \mathbb{Z} \text{ or } \mathbb{Z}_{N/4} \} \cup \{ R^{4k} (f_L \ast U f_{LH}) : k \in \mathbb{Z} \text{ or } \mathbb{Z}_{N/4} \} \cup \{ R^{2k} f_H : k \in \mathbb{Z} \text{ or } \mathbb{Z}_{N/2} \} \) is orthonormal.

Proof of Corollary 1. We first show that the set \( \{ R^{4k} (f_L \ast U f_{LL}) \} \) is orthonormal. By Lemma 1,

\[
\langle (f_L \ast U f_{LL}), R^{4k} (f_L \ast U f_{LL}) \rangle = \text{coeff}_{4k} \left( |\hat{f}_L(z)|^2 |\hat{f}_{LL}(z^2)|^2 \right).
\]  

(110)

By Lemma 3, the functions \( |\hat{f}_L(z)|^2 \) and \( |\hat{f}_{LL}(z)|^2 \) are almost odd, and we may write:

\[
|\hat{f}_L(z)|^2 = 1 + \sum_i \alpha_{2i+1} z^{-(2i+1)};
\]  

(111)

\[
|\hat{f}_{LL}(z)|^2 = 1 + \sum_i \beta_{2i+1} z^{-(2i+1)}.
\]  

(112)

From (112) it follows that

\[
|\hat{f}_{LL}(z^2)|^2 = 1 + \sum_i \beta_{2i+1} z^{-(2i+1)}, \text{ and}
\]  

(113)

\[
|\hat{f}_L(z)|^2 \hat{f}_{LL}(z^2)^2 = 1 + \sum_i \alpha_{2i+1} z^{-(2i+1)} + \sum_i \beta_{2i+1} z^{-(2i+1)} + \sum_i \sum_j \alpha_{2i+1} \beta_{2i+1} z^{-(2i+4j+3)}.
\]  

(114)

From (110) and (114) we have that:

\[
\langle (f_L \ast U f_{LL}), R^{4k} (f_L \ast U f_{LL}) \rangle = \delta(k).
\]  

(115)

Then \( \{ R^{4k} (f_L \ast U f_{LL}) \} \) is orthonormal. Similarly \( \{ R^{4k} (f_L \ast U f_{LH}) \} \) is orthonormal. \( \{ R^{2k} f_H \} \) is orthonormal by Lemma 3. The cross-orthogonality of these sets can also be established by similar arguments. For example,

\[
\langle (f_L \ast U f_{LL}), R^{4k} (f_L \ast U f_{LH}) \rangle = \text{coeff}_{4k} \left( |\hat{f}_L(z)|^2 \hat{f}_{LL}(z^2) \overline{\hat{f}_{LH}(z^2)} \right) \equiv 0,
\]  

(116)
by the almost-oddness of $|\hat{f}_L(z)|^2$, and the oddness of $\hat{f}_{LL}(z)\overline{\hat{f}_{LH}(z)}$ (see equation (44)).

As in Lemma 2, the set in the statement of Corollary 1 is orthonormal if and only if the set $\hat{D} = \{ R^{4k}(f_L \ast U f_{LL})^*: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/4 \} \cup \{ R^{4k}(f_L \ast U f_{LH})^*: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/4 \} \cup \{ R^{2k}\hat{f}_H: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/2 \}$ is orthonormal. Arguing as in equations (64)–(66), one can see that perfect reconstruction in Figures 3 and 5 is equivalent to the reconstruction of $x \in V$ (where $V$ is $l^2(\mathbb{Z})$ or $C^N$) from its decomposition via the orthonormal basis $\hat{D}$. In Figures 3 and 5 we reconstruct the signal $x$ as the direct sum of its projections upon the three orthogonal subspaces $V_{LL}$, $V_{LH}$, and $V_H$, of $V$. These subspaces are defined, respectively, by the action of the filters $(f_L \ast U f_{LL})$, $(f_L \ast U f_{LH})$, and $f_H$. More precisely,

$$V_{LL} = \text{span} \left( \{ R^{4k}(f_L \ast U f_{LL})^*: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/4 \} \right)$$

$$V_{LH} = \text{span} \left( \{ R^{4k}(f_L \ast U f_{LH})^*: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/4 \} \right)$$

$$V_H = \text{span} \left( \{ R^{2k}\hat{f}_H: k \in \mathbb{Z} \text{ or } \mathbb{Z}N/2 \} \right).$$

4.3. Wavelets, Filter Banks, And Wavelet Packets

In the classical wavelet analysis of Lemarie, Meyer, Mallat, and Daubechies, [1]–[6] recursion is performed only in the lower-most branch. Then wavelet decomposition generates the sequence of graphs in Figure 8, where each node represents a single filter-pair with their associated up-sample-by-2 or down-sample-by-2 operators. The number of filters at each level of analysis and synthesis is constant.

In case of full recursion at every level, we have the sequence of graphs in Figure 9. The number of filters doubles at each level of analysis and synthesis. This is the approach adopted by the filter bank school [7][8]. Yet a third approach to recursion is that of “best–adapted wavelet–packets” pioneered by Wickerhauser and Coifman et al. [17] [18], and characterized by an arrangement of the sort in Figure 10. In the best–adapted wavelet–packet method recursion is or is not performed at a certain level in the transformation tree depending upon a criterion of the optimality of representation. We will not discuss these approaches in detail here.

4.4. Recursion With Repeated Filters

As a final matter concerning recursion, consider the choice of the level–2 filters $f_{LL}$, $f_{LH}$, $f_{HL}$, $f_{HH}$; of the level–3 filters $f_{LLL}$, $f_{LLH}$, $f_{LHL}$, etc.; and so on. It is possible to construct the filters independently and differently at each level according to the prescription in Theorem 1. It is possible also to derive the filters at levels higher than the first from the level–1 filters $f_L$ and $f_H$. In case of filters $f \in l^2(\mathbb{Z})$, we can clearly choose $f_{yL} = f_L$ and $f_{yH} = f_H$, where $y \in \{L, H\}^*$ is some string over the set $\{L, H\}$. This does not quite make sense in the case of finite signals since if $f_{LL} \in C^M$ then $f_L \in C^{2M}$. If $f_L$ and $f_H$ are such that $A(z)$ in (41) is unitary $\forall z \in W_M$, then

$$\frac{1}{\sqrt{2}} \left( \begin{array}{c} f_{LL}(z) \\ f_{LH}(z) \end{array} \right) = \left( \begin{array}{c} f_{LL}(-z) \\ f_{LH}(-z) \end{array} \right)$$

(120)

will be unitary $\forall z \in W_M$ if we can determine $f_{LL}$ and $f_{LH}$ so that $f_{LL}(z) = f_L(z)$ and $f_{LH}(z) = f_H(z)$ for all $z \in W_M$. The next lemma tells us how to do this.
Lemma 7. Let \( f \in C^{2M} \), \( g \in C^M \). Then the following are equivalent:

P1: \( \hat{g}(z) = \hat{f}(z) \), \( \forall z \in W_M \).

P2: \( (\mathcal{F}g)(n) = (\mathcal{F}f)(2n) \), \( \forall n \in Z_M \).

P3: \( g(n) = f(n) + f(n + M) \), \( \forall n \in Z_M \).

Proof of Lemma 7. We have:

\[
(\mathcal{F}g)(n) = \hat{g}(e^{-i2\pi n M}) \quad (121)
\]

\[
(\mathcal{F}f)(2n) = \hat{f}(e^{-i2\pi 2n}) = \hat{f}(e^{-i2\pi n M}). \quad (123)
\]

Hence P1 \( \Leftrightarrow \) P2.

Now,

\[
\hat{f}(z) = \sum_{n=0}^{2M-1} f(n)z^{-n} \quad (124)
\]

\[
= \sum_{n=0}^{M-1} f(n)z^{-n} + \sum_{n=0}^{M-1} f(n + M)z^{-n-M}. \quad (125)
\]

Therefore, for \( z \in W_M \),

\[
\hat{f}(z) = \sum_{n=0}^{M-1} (f(n) + f(n + M)) z^{-n}. \quad (126)
\]

Thus P3 \( \Rightarrow \) P1; and by the fact that the \( z \)-transform is one-to-one on \( Z_M \), P1 \( \Rightarrow \) P3. \( \Box \)

This lemma states that we can obtain \( f_{LL} \in C^{N/2} \) from \( f_L \in C^N \) (and, similarly, \( f_{HH} \) from \( f_H \)) by "folding" \( f_L \); i.e. by breaking \( f_L \) into two halves, placing one half on top of the other, and summing pairwise: \( f_{LL}(n) = f_L(n) + f_L(n + N/2) \).

If \( N \) is divisible by 4, then we can continue by defining \( f_{LLL} \in C^{N/4} \) by \( f_{LLL}(n) = f_{LL}(n) + f_L(n + N/4) \), for \( n \in Z_{N/4} \), and so on for \( f_{LLL} \), etc.
5. Orthonormal Wavelet Analysis Of Two-Dimensional Signals

Let \( x(n) = x(n_1, n_2) \) be a two-dimensional sequence, either in \( l^2(\mathbb{Z}^2) \), or in \( C^{N \times N} \), \( N \) even. The space \( l^2(\mathbb{Z}^2) \) is an infinite-dimensional space, while the space \( C^{N \times N} \) has dimension \( N^2 \). The dimensionality of a sequence and the dimensionality of the vector space in which a sequence lies are two different concepts that must be kept apart. We will develop the conditions for the orthonormal decomposition and perfect reconstruction of \( x \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). We will use multiindex notation when convenient, writing

\[
\hat{x}(z) = \sum_n x(n) z^{-n} \tag{127}
\]

for

\[
\hat{x}(x_1, x_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) x_1^{-n_1} x_2^{-n_2}. \tag{128}
\]

In (127) \( z \) ranges over \( T^2 \) or \( W_N^2 \), and \( n \) ranges over \( Z^2 \) or \( Z_N^2 \) depending upon whether \( x \) belongs to \( l^2(\mathbb{Z}^2) \) or to \( C^{N \times N} \).

The down-sampling operator \( D : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2) \) or \( D : C^{N \times N} \to C_N^{N \times N} \) behaves such that \( (Dx)(n_1, n_2) = x(2n_1, 2n_2) \). In multiindex notation, \( (Dx)(n) = x(2n) \). By way of example, we draw a two-dimensional sequence in \( C^4 \times C^4 \) in Figure 11. The points retained by the downsampling operator are dark–circled.

![Figure 11: The downsampling operator retains the elements with dark circles.](image)

Lemma 8.

\[
(Dx)^* (z) = \frac{1}{4} \left( \hat{x}(1/z_1, z_2) + \hat{x}(-1/z_1, z_2) + \hat{x}(z_1, -1/z_2) + \hat{x}(-z_1, -1/z_2) \right). \tag{129}
\]

Proof of Lemma 8.

\[
\hat{x}(z_1, z_2) = \sum_{n_1} \sum_{n_2} x(n_1, n_2) z_1^{-n_1/2} z_2^{-n_2/2} \tag{130}
\]

\[
\hat{x}(-z_1, z_2) = \sum_{n_1} \sum_{n_2} (-1)^n_1 x(n_1, n_2) z_1^{-n_1/2} z_2^{-n_2/2} \tag{131}
\]
\[ \hat{z}(z_1^{1/2}, -z_2^{1/2}) = \sum_{n_1} \sum_{n_2} (-1)^{n_2} z(n_1, n_2) z_1^{-n_1/2} z_2^{-n_2/2} \]  \hspace{1cm} (132) \\
\[ \hat{z}(-z_1^{1/2}, -z_2^{1/2}) = \sum_{n_1} \sum_{n_2} (-1)^{n_1+n_2} z(n_1, n_2) z_1^{-n_1/2} z_2^{-n_2/2} \]  \hspace{1cm} (133)

Now,

\[ 1 + (-1)^{n_1} + (-1)^{n_2} + (-1)^{n_1+n_2} = \begin{cases} 4, & \text{if } n_1 \text{ and } n_2 \text{ even} \\ 0, & \text{otherwise}. \end{cases} \]  \hspace{1cm} (134)

By (130)–(134), the right side of (129) is:

\[ \sum_{n_1 \text{ even}} \sum_{n_2 \text{ even}} z(n_1, n_2) z_1^{-n_1/2} z_2^{-n_2/2} = \sum_{m_1} \sum_{m_2} z(2m_1, 2m_2) z_1^{-m_1} z_2^{-m_2} = (Dz)'(z), \]  \hspace{1cm} (135)

where the summation ranges \( m_1, m_2 \) in (135) are appropriately adjusted. \( \square \)

Upsampling is no different from the one-dimensional case:

\[ (Uz)(n_1, n_2) = \begin{cases} z(n_1/2, n_2/2), & \text{if } n_1 \text{ and } n_2 \text{ even} \\ 0, & \text{otherwise}. \end{cases} \]  \hspace{1cm} (136)

Then \( (Uz)'(z_1, z_2) = \hat{z}(z_1^2, z_2^2) \), or \( (Uz)'(z) = \hat{z}(z^2) \) in multiindex notation.

The "rightshift" operator will now carry two parameters:

\[ (R^{(k_1, k_2)} z)(n_1, n_2) = z(n_1-k_1, n_2-k_2). \]  \hspace{1cm} (137)

Then \( (R^{(k_1, k_2)} z)'(z_1, z_2) = z_1^{-k_1} z_2^{-k_2} \hat{z}(z_1, z_2) \). In multiindex notation we will write \( (R^k z)'(z) = z^{-k} \hat{z}(z) \). By way of example, Figure 12 shows the effect of the operator \( R^{(2,3)} \) upon a sequence in \( \mathbb{C}^{4 \times 4} \). \( R^{(2,3)} \) shifts the input sequence by two in the "direction" of \( z_1 \) (down) and by three in the "direction" of \( z_2 \) (right). The choice of the "directions" was dictated by the fact that we would like \( x(i, k) \) to be the coefficient of \( z_1^{-i} z_2^{-k} \).

![Diagram](image)

**Figure 12: An Example Of The Shift Operation**

By \( \text{coeff}_k(\hat{z}(z)) \) we will mean the coefficient of \( z^{-k} = z_1^{-k_1} z_2^{-k_2} \) in \( \hat{z}(z) \). Also, for \( z \in \mathbb{T}^2 \) or \( W^2_N \), it is easy to see that:

\[ \langle x, R^k y \rangle = \text{coeff}_k(\hat{z}(z) \hat{y}(z)). \]  \hspace{1cm} (138)
The following lemma is of general utility, and is easily proved by the method used in the proof of Lemma 8.

**Lemma 9.** Let \( x \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). Then

\[
\hat{x}(z_1, z_2) + \hat{x}(-z_1, z_2) + \hat{x}(z_1, -z_2) + \hat{x}(-z_1, -z_2) = 4 \sum_{n_1 \text{ even}} \sum_{n_2 \text{ even}} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}.
\] (139)

Consider the arrangement of filters, downsamplers, and upsamplers in Figure 13. We will establish the conditions for orthonormal decomposition and perfect reconstruction.

![Figure 13: The basic filter bank for 2D signals.](image)

**Lemma 10.** Let \( f \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). The following are equivalent:

P1: \( (f, R^{2k} f) = (f, R^{2k_1, 2k_2} f) = \delta(k) = \delta(k_1)\delta(k_2), k \in \mathbb{Z}^2 \) or \( Z_N^2 \).

P2: \( \forall z \in \mathbb{T}^2 \) or \( W_N^2 \), \( |\hat{f}(z_1, z_2)|^2 + |\hat{f}(-z_1, z_2)|^2 + |\hat{f}(z_1, -z_2)|^2 + |\hat{f}(-z_1, -z_2)|^2 \equiv 4 \).

Proof of Lemma 10. From (138), P1 is equivalent to:

\[
\text{coeff}_{2k} \left( |\hat{f}(z)|^2 \right) = \delta(k).
\] (140)

Define \( \hat{h}(z) = |\hat{f}(z)|^2 \). Equation (140) is equivalent to:

\[
\sum_{n_1 \text{ even}} \sum_{n_2 \text{ even}} h(n_1, n_2) z_1^{-n_1} z_2^{-n_2} = 1.
\] (141)
By Lemma 9, (141) is equivalent to:
\[ h(z_1, z_2) + \bar{h}(-z_1, z_2) + h(x_1, -z_2) + \bar{h}(-x_1, -z_2) = 4, \quad (142) \]
which is the statement P2.

Similarly, we have:

Lemma 11. Let \( f, g \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). The following are equivalent:

P1: \( \langle f, R^{2k}g \rangle = 0, \quad k \in \mathbb{Z}^2 \) or \( \mathbb{Z}_N^2 \).

P2: \( \forall z \in \mathbb{T}^2 \) or \( \mathbb{W}_N^2 \), \( \hat{f}(z_1, z_2)\bar{g}(z_1, z_2) + \hat{f}(-z_1, z_2)\bar{g}(-z_1, z_2) + \hat{f}(z_1, -z_2)\bar{g}(z_1, -z_2) + \hat{f}(-z_1, -z_2)\bar{g}(-z_1, -z_2) \equiv 0 \).

We need one final lemma before the statement of the main theorem governing orthonormal decomposition. The proof of Lemma 12 is similar to the proof of Lemma 2.

Lemma 12. Let \( f_0, f_1, f_2, f_3 \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). Define
\[
\hat{B} = \{ R^{2k}f_0 \}_{k} \cup \{ R^{2k}f_1 \}_{k} \cup \{ R^{2k}f_2 \}_{k} \cup \{ R^{2k}f_3 \}_{k},
\]
where \( k \) runs through \( \mathbb{Z}^2 \) or \( \mathbb{Z}_N^2 \). Then the multiset \( \hat{B} \) is orthonormal if, and only if, the multiset \( B \) is.

Note that, as in the one-dimensional case, \( (\hat{f})^\ast(z_1, z_2) = \bar{f}(z_1, z_2) \).

The following theorem is the discrete analog of a result in Meyer's paper [19] for the continuous signal space \( L^2(\mathbb{R}^N) \).

Theorem 3. Let \( f_0, \ldots, f_3 \in l^2(\mathbb{Z}^2) \) or \( C^{N \times N} \). \( \hat{B} \) is orthonormal if, and only if, the system matrix
\[
A(z_1, z_2) = \frac{1}{2} \begin{pmatrix}
\hat{f}_0(z_1, z_2) & \hat{f}_1(z_1, z_2) & \hat{f}_2(z_1, z_2) & \hat{f}_3(z_1, z_2) \\
\hat{f}_0(-z_1, z_2) & \hat{f}_1(-z_1, z_2) & \hat{f}_2(-z_1, z_2) & \hat{f}_3(-z_1, z_2) \\
\hat{f}_0(z_1, -z_2) & \hat{f}_1(z_1, -z_2) & \hat{f}_2(z_1, -z_2) & \hat{f}_3(z_1, -z_2) \\
\hat{f}_0(-z_1, -z_2) & \hat{f}_1(-z_1, -z_2) & \hat{f}_2(-z_1, -z_2) & \hat{f}_3(-z_1, -z_2)
\end{pmatrix}
\]
is unitary for all \( z \in \mathbb{T}^2 \) or \( \mathbb{W}_N^2 \).

Proof of Theorem 3. By Lemma 12 the orthonormality of \( \hat{B} \) is equivalent to the orthonormality of \( B \). \( B \) is orthonormal if and only if the multiset \( \{ R^{2k}f_i \} \) is orthonormal for any \( i \), and if \( \{ R^{2k}f_i \} \) and \( \{ R^{2k}f_l \} \) are mutually orthogonal for \( i, l \in \{0,1,2,3\} \). By Lemma 10, the orthonormality of the multiset \( \{ R^{2k}f_i \} \) is equivalent to the statement that the norm of the \( i \)-th column of \( A(z_1, z_2) \) is unity. By Lemma 11, the cross-orthogonality of the multisets \( \{ R^{2k}f_i \} \) and \( \{ R^{2k}f_l \} \) is equivalent to the assertion that the inner-product of the columns \( i \) and \( l \) in \( A(z_1, z_2) \) vanish.
5.1. Perfect Reconstruction

**Theorem 4.** We have perfect reconstruction in Figure 13 if, and only if, the following equations hold for all \( z \in \mathbb{T}^2 \) or \( \mathbb{W}_N^2 \):

\[
\sum_{i=0}^{3} \hat{g}_i(z_1, z_2) \hat{f}_i(z_1, z_2) \equiv 4
\]

\[
\sum_{i=0}^{3} \hat{g}_i(z_1, z_2) \hat{f}_i(-z_1, z_2) \equiv 0
\]

\[
\sum_{i=0}^{3} \hat{g}_i(z_1, z_2) \hat{f}_i(z_1, -z_2) \equiv 0
\]

\[
\sum_{i=0}^{3} \hat{g}_i(z_1, z_2) \hat{f}_i(-z_1, -z_2) \equiv 0.
\]

**Proof of Theorem 4.**

\[
(D(f \ast x))^{-1}(z_1, z_2) = \frac{1}{4} \hat{f}_i(z_1^{1/2}, z_2^{1/2}) \hat{x}(z_1^{1/2}, z_2^{1/2}) + \frac{1}{4} \hat{f}_i(-z_1^{1/2}, z_2^{1/2}) \hat{x}(-z_1^{1/2}, z_2^{1/2}) + \frac{1}{4} \hat{f}_i(z_1^{1/2}, -z_2^{1/2}) \hat{x}(z_1^{1/2}, -z_2^{1/2}) + \frac{1}{4} \hat{f}_i(-z_1^{1/2}, -z_2^{1/2}) \hat{x}(-z_1^{1/2}, -z_2^{1/2}).
\]

Then,

\[
(g_i \ast UD(f_i \ast x))^{-1}(z) = \frac{1}{4} \hat{g}_i(z) \hat{f}_i(z_1, z_2) \hat{x}(z_1, z_2) + \frac{1}{4} \hat{g}_i(-z_1, z_2) \hat{x}(-z_1, z_2) + \frac{1}{4} \hat{g}_i(z) \hat{f}_i(-z_1, -z_2) \hat{x}(-z_1, -z_2) + \frac{1}{4} \hat{g}_i(-z_1, -z_2) \hat{x}(-z_1, -z_2).
\]

From Figure 13, perfect reconstruction requires:

\[
\hat{x}(z) = \sum_{i=0}^{3} (g_i \ast UD(f_i \ast x))^{-1}(z)
\]

\[
= \frac{1}{4} \left( \sum_{i=0}^{3} \hat{g}_i(z) \hat{f}_i(z_1, z_2) \right) \hat{x}(z_1, z_2) + \frac{1}{4} \left( \sum_{i=0}^{3} \hat{g}_i(z) \hat{f}_i(-z_1, z_2) \right) \hat{x}(-z_1, z_2) + \frac{1}{4} \left( \sum_{i=0}^{3} \hat{g}_i(z) \hat{f}_i(z_1, -z_2) \right) \hat{x}(z_1, -z_2) + \frac{1}{4} \left( \sum_{i=0}^{3} \hat{g}_i(z) \hat{f}_i(-z_1, -z_2) \right) \hat{x}(-z_1, -z_2).
\]

The equation (154) is of the form

\[
\hat{x}(z) = \hat{C}(z) \hat{x}(z_1, z_2) + \hat{D}(z) \hat{x}(-z_1, z_2) + \hat{E}(z) \hat{x}(z_1, -z_2) + \hat{F}(z) \hat{x}(-z_1, -z_2),
\]
where \( \hat{C}, \ldots, \hat{F} \) are some fixed polynomials in \( z \). Since equation (155) is required to hold for all \( z \), it must, in particular, hold for \( \Hat{z} = 1, \Hat{z} = z_1, \Hat{z} = z_2 \), and \( \Hat{z} = z_1 z_2 \). Substituting these values for \( \Hat{z} \) in (155), we have the following system of equations:

\[
\begin{align*}
\hat{C} + \hat{D} + \hat{E} + \hat{F} &= 1 \quad (156) \\
\hat{C} - \hat{D} + \hat{E} - \hat{F} &= 1 \quad (157) \\
\hat{C} + \hat{D} - \hat{E} - \hat{F} &= 1 \quad (158) \\
\hat{C} - \hat{D} - \hat{E} + \hat{F} &= 1. \quad (159)
\end{align*}
\]

Solving, we have \( \hat{C} = 1, \hat{D} = \hat{E} = \hat{F} = 0 \). This proves the theorem.

In order to summarize the orthonormal decomposition and perfect reconstruction problem for two-dimensional signals, we state:

**Corollary 2.** The following are equivalent:

P1: The set \( \tilde{B} = \{R^{2k} \tilde{f}_0\} \cup \{R^{2k} \tilde{f}_1\} \cup \{R^{2k} \tilde{f}_2\} \cup \{R^{2k} \tilde{f}_3\} \) is orthonormal, and there is perfect reconstruction in Figure 13.

P2: The system matrix \( A \) in (146) is unitary; and \( g_i = \tilde{f}_i \), \( \forall i \in \{0,1,2,3\} \).

**Proof of Corollary 2.**

(P1 \( \Rightarrow \) P2). By Theorem 4, perfect reconstruction is equivalent to

\[
A(z_1, z_2) \begin{pmatrix} \tilde{g}_0(z) \\ \tilde{g}_1(z) \\ \tilde{g}_2(z) \\ \tilde{g}_3(z) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (160)
\]

By Theorem 3, the orthonormality of \( \tilde{B} \) is equivalent to the unitarity of \( A \). \( A \) unitary \( \Rightarrow A^{-1} = A^* = A^T \). Now it is easy to solve (160) for the \( \tilde{g}_i \), and to show that \( \tilde{g}_i = \tilde{f}_i \) which is equivalent to \( g_i = \tilde{f}_i \).

(P1 \( \Leftarrow \) P2). If \( \tilde{g}_i = \tilde{f}_i \), then substituting for \( \tilde{g}_i \) in (147)–(150), we have

\[
\begin{align*}
\sum_{i=0}^{3} |\tilde{f}_i(z_1, z_2)|^2 &\equiv 4 \quad (161) \\
\sum_{i=0}^{3} \tilde{f}_i(z_1, z_2) \tilde{f}_i(-z_1, z_2) &\equiv 0 \quad (162) \\
\sum_{i=0}^{3} \tilde{f}_i(z_1, z_2) \tilde{f}_i(z_1, -z_2) &\equiv 0 \quad (163) \\
\sum_{i=0}^{3} \tilde{f}_i(z_1, z_2) \tilde{f}_i(-z_1, -z_2) &\equiv 0. \quad (164)
\end{align*}
\]

Equations (161)–(164) hold by virtue of the unitarity of \( A \). Therefore, by Theorem 4 we have perfect reconstruction. \( \square \)
5.2. Product Wavelets

Suppose \( \hat{f}_L \) and \( \hat{f}_H \) are one-dimensional filters such that

\[
\frac{1}{\sqrt{2}} \begin{pmatrix}
\hat{f}_L(z) & \hat{f}_H(z) \\
\hat{f}_L(-z) & \hat{f}_H(-z)
\end{pmatrix}
\]

is unitary for all \( z \in \mathbb{T} \) or \( \mathbb{W}_N \). Define

\[
\begin{align*}
f_0(n_1, n_2) &= f_L(n_1)f_L(n_2) \\
f_1(n_1, n_2) &= f_H(n_1)f_L(n_2) \\
f_2(n_1, n_2) &= f_L(n_1)f_H(n_2) \\
f_3(n_1, n_2) &= f_H(n_1)f_H(n_2).
\end{align*}
\]

With this definition, the matrix (146) is a Kronecker product of two matrices of form (165):

\[
A(z_1, z_2) = \frac{1}{2} \begin{pmatrix}
\hat{f}_0(z_1, z_2) & \hat{f}_1(z_1, z_2) & \hat{f}_2(z_1, z_2) & \hat{f}_3(z_1, z_2) \\
\hat{f}_0(-z_1, z_2) & \hat{f}_1(-z_1, z_2) & \hat{f}_2(-z_1, z_2) & \hat{f}_3(-z_1, z_2) \\
\hat{f}_0(z_1, -z_2) & \hat{f}_1(z_1, -z_2) & \hat{f}_2(z_1, -z_2) & \hat{f}_3(z_1, -z_2) \\
\hat{f}_0(-z_1, -z_2) & \hat{f}_1(-z_1, -z_2) & \hat{f}_2(-z_1, -z_2) & \hat{f}_3(-z_1, -z_2)
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
\hat{f}_L(z_2) & \hat{f}_H(z_2) \\
\hat{f}_L(-z_2) & \hat{f}_H(-z_2)
\end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix}
\hat{f}_L(z_1) & \hat{f}_H(z_1) \\
\hat{f}_L(-z_1) & \hat{f}_H(-z_1)
\end{pmatrix}.
\]

It is easy to check directly that the orthonormality of the set \( \{ R^{2k} f_L \}_k \cup \{ R^{2k} f_H \}_k \) implies the orthonormality of the set \( \mathcal{B} \) in Corollary 2, for \( f_0, \ldots, f_3 \) given by (166)–(169). This is the usual way of constructing two-dimensional wavelets from one-dimensional wavelets. We call \( f_0, \ldots, f_3 \) product wavelets. This is consistent with Theorem 3, since the unitarity of the matrix (170) follows from the unitarity of the matrices in the Kronecker product formula in (171).

Thus, two-dimensional wavelet filters can be constructed as products of one-dimensional filters. However, Theorem 3 shows that there exist two-dimensional wavelet filters that are not product filters. This follows from the existence of the following unitary matrix, which cannot be expressed as a Kronecker product:

\[
\frac{1}{2} \begin{pmatrix}
\sqrt{2} & 0 & 0 & \sqrt{2} \\
\sqrt{2} & 0 & 0 & -\sqrt{2} \\
0 & \sqrt{2} & \sqrt{2} & 0 \\
0 & \sqrt{2} & -\sqrt{2} & 0
\end{pmatrix}.
\]
6. Orthonormal Wavelets For Multidimensional Signals

Let \( P \) be a fixed positive integer. Let \( f(n) \) be a \( P \)-dimensional signal in \( l^2(\mathbb{Z}^P) \) or in \( \mathbb{C}^{N^P} \), \( N \) even. The \( z \)-transform \( \tilde{f}(z) \) of \( f \) is defined for \( z \in \mathbb{T}^P \) or \( \mathbb{W}_N^P \). Multiindex notation will be used throughout. The first lemma states a result of general utility.

Lemma 13. For any \( n \in \mathbb{Z}^P \) or \( n \in \mathbb{Z}_N^P \),

\[
\sum_{k \in \{0,1\}^P} (-1)^{(k_1 n_1 + \ldots + k_P n_P)} = \sum_{k \in \mathbb{Z}_2^P} (-1)^{k \cdot n} = \begin{cases} 0, & \text{if for some } i, n_i \text{ is odd} \\ 2^P, & \text{otherwise}; \end{cases} \tag{173}
\]

where \( k \cdot n \) denotes the dot-product \( k \cdot n = k_1 n_1 + \ldots + k_P n_P \).

Proof of Lemma 13. If \( n = 2m = (2m_1, \ldots, 2m_P) \) for \( m \in \mathbb{Z}^P \) or \( m \in \mathbb{Z}_N^P \), then each term in the sum is 1, and the sum is \( 2^P \). Else, if \( n_i \) is odd for some \( i \), write

\[
\sum_{k \in \mathbb{Z}_2^P} (-1)^{k \cdot n} = \sum_{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_P \in \mathbb{Z}_2} (-1)^{(k_1 n_1 + \ldots + k_{i-1} n_{i-1} + k_{i+1} n_{i+1} + \ldots + k_P n_P)} \left( \sum_{k_i = 0}^1 (-1)^{k_i n_i} \right). \tag{174}
\]

Because \( n_i \) is odd, the last sum and, hence, the entire sum vanishes.

Define the downsampling operator \( \mathcal{D} \) by \( (\mathcal{D} f)(n) = f(2n), n \in \mathbb{Z}^P \) or \( n \in \mathbb{Z}_N^P \). \( \mathcal{D} : l^2(\mathbb{Z}^P) \rightarrow l^2(\mathbb{Z}^P) \) or \( \mathcal{D} : \mathbb{C}^{N^P} \rightarrow \mathbb{C}^{(N/2)^P} \).

Lemma 14. For \( z \in \mathbb{T}^P \) or \( \mathbb{W}_N^P \),

\[(\mathcal{D} f)(z) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \tilde{f}\left((-1)^{k_1 z_1^{1/2}} , \ldots , (-1)^{k_P z_P^{1/2}}\right) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \tilde{f}((-1)^{k z^{1/2}}). \tag{175}\]

Proof of Lemma 14. For \( n \in \mathbb{Z}^P \) or \( n \in \mathbb{Z}_N^P \),

\[
\tilde{f}((-1)^{k z^{1/2}}) = \sum_n f(n)((-1)^{k z^{1/2}})^{-n} = \sum_n f(n)(-1)^{k \cdot n} z^{-n/2} = \sum_n f(n)(-1)^{(k_1 n_1 + \ldots + k_P n_P) z_1^{-1/2} \ldots z_P^{-1/2}}. \tag{176}
\]

Therefore,

\[
2^{-P} \sum_{k \in \mathbb{Z}_2^P} \tilde{f}((-1)^{k z^{1/2}}) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \sum_n f(n)(-1)^{k \cdot n} z_1^{-n_1/2} \ldots z_P^{-n_P/2}. \tag{179}\]
Interchanging the order of summation in (179), and using Lemma 13, we have

\[ 2^{-P} \sum_{k \in \mathbb{Z}_2^p} \hat{f}((-1)^k z^{1/2}) = \sum_{n=2m} f(n_1, \ldots, n_P) z_1^{-n_1/2} \cdots z_P^{-n_P/2} \]

(180)

\[ = \sum_m f(2m_1, \ldots, 2m_P) z_1^{-m_1} \cdots z_P^{-m_P} \]

(181)

\[ = \sum_m (Df)(m_1, \ldots, m_P) z_1^{-m_1} \cdots z_P^{-m_P} \]

(182)

\[ = (Df)^*(z). \]

(183)

Define the upsampling operator \( U : l^2(\mathbb{Z}^P) \rightarrow l^2(\mathbb{Z}^P) \) or \( U : C^{N^p} \rightarrow C^{(2N)^p} \) by

\[(Uf)(n) = \begin{cases} f(n/2), & \text{if } n = 2m, \ m \in \mathbb{Z}^P \text{ or } m \in \mathbb{Z}_N^p \\ 0, & \text{otherwise.} \end{cases} \]

(184)

It is easy to see that \((Uf)^*(z) = \hat{f}(z^2)\), i.e. \((Uf)^*(z_1, \ldots, z_P) = \hat{f}(z_1^2, \ldots, z_P^2)\).

Define the shift operator \( R \) so that \((R^k f)(n) = f(n - k)\), or

\[(R^{k_1, \ldots, k_P}) f(n_1, \ldots, n_P) = f(n_1 - k_1, \ldots, n_P - k_P). \]

(185)

It is easy to see that \((R^k f)^*(z) = z^{-k} \hat{f}(z) = z_1^{-k_1} \cdots z_P^{-k_P} \hat{f}(z_1, \ldots, z_P)\).

By \(\text{coeff}_k \hat{f}(z)\); \(k \in \mathbb{Z}^p \) or \(k \in \mathbb{Z}_N^p\); will be meant the coefficient of \(z^{-k} = z_1^{-k_1} \cdots z_P^{-k_P}\) in the canonical form of the polynomial \(\hat{f}(z)\). For \(f = \{f(n)\}_n\) and \(g = \{g(n)\}_n\), \(\langle f, g \rangle = \sum_n f(n)\overline{g(n)}\). It is easy to see that \(\langle f, R^k g \rangle = \text{coeff}_k (\hat{f}(z) \overline{g}(z))z \in \mathbb{T}^P \) or \(\mathbb{W}_N^P\).

**Lemma 15.** Let \(f = f(n)\), \(n \in \mathbb{Z}^P\) or \(n \in \mathbb{Z}_N^P\). Then the following are equivalent:

P1: \(\langle f, R^{2k} f \rangle = \delta(k) = \delta(k_1) \cdots \delta(k_P)\).

P2: \(\sum_{l \in \mathbb{Z}_2^p} |\hat{f}((-1)^l z)^2| = 2^P, \forall z \in \mathbb{T}^P \) or \(\mathbb{W}_N^P\).

Here \(\hat{f}((-1)^l z) = \hat{f}((-1)^l z_1, \ldots, (-1)^l z_P)\) for \(l = (l_1, \ldots, l_P) \in \mathbb{Z}_2^P\).

**Proof of Lemma 15.** P1 is equivalent to:

\[\text{coeff}_{2k} \left(|\hat{f}(z)|^2\right) = \delta(k). \]

(186)

Define

\[h(z) = |\hat{f}(z)|^2. \]

(187)

For \(m \in \mathbb{Z}^P\) or \(m \in \mathbb{Z}_{N/2}^p\), (186) is equivalent to:

\[\sum_{n=2m} h(n)z^{-n} = 1. \]

(188)
But, \( \sum_{n=2m} h(n)x^{-n} = (UDh)^\dagger(z) = (Dh)^\dagger(z^2) \). Therefore, by Lemma 14,
\[
\sum_{n=2m} h(n)x^{-n} = (Dh)^\dagger(z^2) = 2^{-p} \sum_{l \in \mathbb{Z}_2^p} \hat{h}((-1)^l z).
\] (189)

From (189), (188) is equivalent to:
\[
\sum_{l \in \mathbb{Z}_2^p} \hat{h}((-1)^l z) = 2^p.
\] (190)

From (187), \( \hat{h}((-1)^l z) = |\hat{f}((-1)^l z)|^2 \). Substituting for \( \hat{h}((-1)^l z) \) in (190) we have the lemma. \( \Box \)

![Diagram](image_url)  
Figure 14: The basic filter–bank for a \( P \)-dimensional signal.

**Lemma 16.** Let \( f = f(n) \), \( g = g(n) \), \( n \in \mathbb{Z}^P \) or \( n \in \mathbb{Z}_N^P \). Then the following are equivalent:

P1: \( \langle f, R^{2k}g \rangle = 0 \), \( \forall k \).

P2: \( \sum_{l \in \mathbb{Z}_2^P} \hat{f}((-1)^l z)\overline{\hat{g}}((-1)^l z) = 0 \), \( \forall z \in T^P \) or \( W_N^P \).

**Proof of Lemma 16.** The proof is similar to that of Lemma 15. P1 is equivalent to:
\[
\text{coeff}_{-2k} \left( \hat{f}(z)\overline{\hat{g}}(z) \right) = 0.
\] (191)

Define
\[
\hat{h}(z) = \hat{f}(z)\overline{\hat{g}}(z).
\] (192)

From (192), for \( m \in \mathbb{Z}^P \) or \( m \in \mathbb{Z}_N^{P/2} \), (191) is equivalent to:
\[
\sum_{n=2m} h(n)x^{-n} = 0.
\] (193)

By Lemma 14,
\[
\sum_{n=2m} h(n)x^{-n} = (UDh)^\dagger(z) = (Dh)^\dagger(z^2) = 2^{-p} \sum_{l \in \mathbb{Z}_2^p} \hat{h}((-1)^l z).
\] (194)
Substituting (192) in (194) we have the lemma.

The proof of the following lemma is similar to the proof of Lemma 2.

**Lemma 17.** Define

\[
\tilde{f}_i(n) = \overline{f_i(-n)}
\]

(195)

\[
\tilde{B} = \bigcup_{i=0}^{2^p-1} \{R^{2k} \tilde{f}_i : k \in \mathbb{Z}^p \text{ or } k \in \mathbb{Z}_{N/2}^p \}
\]

(196)

\[
B = \bigcup_{i=0}^{2^p-1} \{R^{2k} f_i : k \in \mathbb{Z}^p \text{ or } k \in \mathbb{Z}_{N/2}^p \}
\]

(197)

The multiset \( \tilde{B} \) is orthonormal if and only if the multiset \( B \) is.

Consider now the orthonormal decomposition of the signal \( z \) in Figure 14.

We need some new notation for the next theorem. Let \( j = (j_1, \ldots, j_P) \in \{0,1\}^P = \mathbb{Z}_2^P \). Since \( \text{card}(\{0,1\}^P) = 2^P \), we can enumerate all such \( j \)'s as \( j^{(0)}, \ldots, j^{(2^P-1)} \). Select this enumeration such that \( j^{(0)} = (0,0,\ldots,0) \). Then, we write \( j^{(0)} = (j_1^{(0)}, \ldots, j_P^{(0)}) \), with \( j_k^{(i)} \in \mathbb{Z}_2 \) for all \( k \). We also write \( \hat{f}((-1)^{j^{(0)} z}) = \hat{f}((-1)^{j^{(0)} z_1}, \ldots, (-1)^{j^{(0)} z_P}). \)

**Theorem 5.** Let \( f_0(n) = f_0(n_1, \ldots, n_P), \ldots, f_{2^P-1}(n) = f_{2^P-1}(n_1, \ldots, n_P) \) be \( 2^P \) sequences, with \( n \in \mathbb{Z}^P \) or \( n \in \mathbb{Z}_N^P \). Define a \( 2^P \times 2^P \) matrix called the system matrix \( A(z) = (A_{k,i}(z))_{k,i \in \mathbb{Z}_2^P} \), \( z \in \mathbb{T}^P \) or \( W_N^P \), such that \( A_{k,i}(z) = 2^{-P/2} \hat{f}_i((-1)^{j^{(0)} z}) \), i.e.,

\[
A(z) = 2^{-P/2}
\begin{pmatrix}
\hat{f}_0((-1)^{j^{(0)} z}) & \hat{f}_1((-1)^{j^{(0)} z}) & \cdots & \hat{f}_{2^P-1}((-1)^{j^{(0)} z}) \\
\hat{f}_0((-1)^{j^{(1)} z}) & \hat{f}_1((-1)^{j^{(1)} z}) & \cdots & \hat{f}_{2^P-1}((-1)^{j^{(1)} z}) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{f}_0((-1)^{j^{(2^P-1)} z}) & \hat{f}_1((-1)^{j^{(2^P-1)} z}) & \cdots & \hat{f}_{2^P-1}((-1)^{j^{(2^P-1)} z})
\end{pmatrix}
\]

(198)

Then \( \tilde{B} \) is orthonormal if, and only if, \( A(z) \) is unitary for all \( z \in \mathbb{T}^P \) or \( W_N^P \).

**Proof of Theorem 5.** By Lemma 17, the orthonormality of \( \tilde{B} \) is equivalent to the orthonormality of \( B \). By Lemma 15 the orthonormality of \( \{R^{2k} \tilde{f}_i : k \in \mathbb{Z}^P \text{ or } k \in \mathbb{Z}_{N/2}^P \} \) for every fixed \( i \) is equivalent to the \( i \)-th column of \( A \) having length "1". By Lemma 16 the cross-orthogonality of \( \{R^{2k} \tilde{f}_i : k \in \mathbb{Z}^P \text{ or } k \in \mathbb{Z}_{N/2}^P \} \) and \( \{R^{2k} f_i : k \in \mathbb{Z}^P \text{ or } k \in \mathbb{Z}_{N/2}^P \} \) is equivalent to the orthogonality of the \( i_1 \)-th and the \( i_2 \)-th columns of \( A \).
6.1. Perfect Reconstruction of $P$-Dimensional Signals

From Figure 14 we can see that perfect-reconstruction occurs if and only if the following equation holds for all input signals $x$:

$$
\sum_{i=0}^{2^P-1} (g_i * UD(f_i * x))^{-}(z) = \hat{x}(z).
$$

Equation (199) will be called the perfect reconstruction condition.

**Lemma 18.** The perfect reconstruction condition is satisfied if, and only if,

$$
\sum_{i=0}^{2^P-1} \hat{f}_i((-1)^k z^i)\hat{g}_i(z) = 2^P \delta(k).
$$

**Proof of Lemma 18.** From Lemma 14,

$$
(D(f_i * x))^{-}(z) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \hat{f}_i((-1)^{kz^i})\hat{x}((-1)^{kz^i}).
$$

From (201) it follows that

$$
(UD(f_i * x))^{-}(z) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \hat{f}_i((-1)^{kz^i})\hat{x}((-1)^{kz^i}).
$$

Using (202), the perfect reconstruction condition (199) can be written as:

$$
2^{-P} \sum_{i=0}^{2^P-1} \sum_{k \in \mathbb{Z}_2^P} \hat{f}_i((-1)^{kz^i})\hat{x}((-1)^{kz^i})\hat{g}_i(z) = \hat{x}(z).
$$

Define

$$
\hat{C}_k(z) = \sum_{i=0}^{2^P-1} \hat{f}_i((-1)^{kz^i})\hat{g}_i(z).
$$

The perfect reconstruction condition in (203) now reads:

$$
2^{-P} \sum_{k \in \mathbb{Z}_2^P} \hat{C}_k(z)\hat{x}((-1)^{kz^i}) = \hat{x}(z).
$$

Define $\hat{x}_l(z) = z^l$, for $l \in \mathbb{Z}_2^P$. Since the perfect reconstruction condition must hold for all $\hat{x}(z)$, it must in particular hold for all the $\hat{x}_l(z)$, $l \in \mathbb{Z}_2^P$. Then for every $l$, from (205),

$$
\hat{x}_l(z) = z^l = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} ((-1)^{kz^i})^l \hat{C}_k(z) = 2^{-P} \sum_{k \in \mathbb{Z}_2^P} (-1)^{kz^i} x^l \hat{C}_k(z).
$$
From (206) it follows that, for all \( l \in \mathbb{Z}_2^P \),
\[
2^{-P} \sum_{k \in \mathbb{Z}_2^P} (-1)^{k+l} \hat{C}_k(z) = 1. \tag{207}
\]

Let \( K \in \mathbb{Z}_2^P \) be any fixed binary \( P \)-tuple. Multiplying both sides in (207) by \((-1)^{K-l}\), and summing over all \( l \) we have:
\[
\sum_{l \in \mathbb{Z}_2^P} (-1)^{K-l} = 2^{-P} \sum_{l \in \mathbb{Z}_2^P} \sum_{k \in \mathbb{Z}_2^P} (-1)^{k+l} (-1)^{K-l} \hat{C}_k(z) \tag{208}
\]
\[
= 2^{-P} \sum_{l \in \mathbb{Z}_2^P} \sum_{k \in \mathbb{Z}_2^P} (-1)^{k+l} (-1)^{-K+l} \hat{C}_k(z) \tag{209}
\]
\[
= 2^{-P} \sum_{k \in \mathbb{Z}_2^P} \hat{C}_k(z) \sum_{l \in \mathbb{Z}_2^P} (-1)^{(k-K)+l}. \tag{210}
\]

By Lemma 13, because \( K \in \mathbb{Z}_2^P \),
\[
\sum_{l \in \mathbb{Z}_2^P} (-1)^{K-l} = 2^P \delta(K) = 2^P \delta(K_1) \ldots \delta(K_P). \tag{211}
\]

Also by Lemma 13,
\[
\sum_{l \in \mathbb{Z}_2^P} (-1)^{(k-K)+l} = 2^P \delta(k-K). \tag{212}
\]

From (208)–(212),
\[
\sum_{k \in \mathbb{Z}_2^P} \hat{C}_k(z) \delta(k-K) = 2^P \delta(K), \tag{213}
\]
or
\[
\hat{C}_K(z) = 2^P \delta(K). \tag{214}
\]

From (204) and (214) we have the lemma. \(\square\)

The following theorem is immediate from Lemma 18.

**Theorem 6.** Let \( A(z) \) be the system matrix (198). The system in Figure 14 gives perfect reconstruction if and only if, \( \forall z \in T^P \) or \( C_N^P \),
\[
A(z) \begin{pmatrix}
\hat{g}_0(z) \\
\hat{g}_1(z) \\
\vdots \\
\hat{g}_{2^P-1}(z)
\end{pmatrix} = \begin{pmatrix}
2^{P/2} \\
0 \\
\vdots \\
0
\end{pmatrix}. \tag{215}
\]
In the following corollary we summarize the results of the orthonormal decomposition and perfect reconstruction of signals in $l^2(\mathbb{Z}^P)$ and $\mathbb{C}^{2^P}$.

**Corollary 3** The following are equivalent:

P1: The set $B = \bigcup_{k=0}^{2^P-1} \{ R^{2k} \tilde{f}_i : k \in \mathbb{Z}^P \text{ or } k \in \mathbb{Z}_{N/2}^P \}$ is orthonormal, and we have perfect reconstruction in Figure 14.

P2: The system matrix $A(z)$ in (198) is unitary, and $\forall i \in \mathbb{Z}_{2^P}$, $g_i = \tilde{f}_i$.

**Proof of Corollary 3 (P1 $\Rightarrow$ P2).** The unitarity of $A(z)$ follows from Theorem 5. Hence $A^{-1}(z) = \overline{A(z)}$. From this observation and Theorem 6 follows the fact that, $\forall i \in \mathbb{Z}_{2^P}$, $\tilde{g}_i(z) = \overline{f}_i(z)$; or equivalently $g_i = \tilde{f}_i$.

(P1 $\Leftrightarrow$ P2). By Theorem 5, $B$ is orthonormal. Since $\tilde{g}_i(z) = \overline{f}_i(z)$, the equation

$$
A(z) \begin{pmatrix}
2^{-P/2}\tilde{g}_0(z) \\
2^{-P/2}\tilde{g}_1(z) \\
\vdots \\
2^{-P/2}\tilde{g}_{2^P-1}(z)
\end{pmatrix}
= A(z) \begin{pmatrix}
2^{-P/2}\overline{f}_0(z) \\
2^{-P/2}\overline{f}_1(z) \\
\vdots \\
2^{-P/2}\overline{f}_{2^P-1}(z)
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
$$

(218)

is true, because the rows of $A(z)$ are orthonormal by the unitarity of $A(z)$. Then perfect reconstruction follows by Theorem 6. $\square$
7. Conclusions and Discussion

We have described the orthonormal wavelet transform of discrete signals in the vector spaces $l^2(\mathbb{Z}^P)$ and $C^N$. Specifically, we have established the necessary and sufficient conditions for orthonormal decomposition and perfect reconstruction. We have seen that these conditions translate naturally into conditions upon the the design of the analyzing and synthesizing filters employed in a hardware or software realization of the wavelet transform.

An algorithm is given for the construction of any and all wavelet filters for the decomposition of one-dimensional signals. Filters for higher-dimensional signal spaces can be realized as tensor products of one-dimensional filters, but the direct design of multidimensional filters with desirable properties remains a problem for further research.

The problems of frequency localization and simultaneous time–frequency localization of one-dimensional wavelet filters are discussed because of their importance to data compression and coding. While the mapping of a filter in the signal space to a closest wavelet is seen to preserve its frequency–localization, the mapping destroys time–localization. The problem of simultaneous time–frequency localization is an important open problem so far as wavelets are concerned. This may be a point of advantage for the FJT.

We have not discussed issues relating to the computational complexity of wavelet decomposition and reconstruction. It is, however, easy to see that if the filtering is done in the frequency–domain $O(N \log N)$ algorithms result for the signal space $V = C^N$. It is possible to do better if the filters have a restricted support in the time–domain, and if filtering operations are accomplished through convolutions in the time–domain. Such implementations are easily seen to yield $O(N)$ complexity.

We have not addressed the construction of wavelet filters that have a restricted support in the time–domain. Not only are such filters of interest in fast implementations, they may also have applications to the compression and coding of moving pictures.

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