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We show that any class learnable in the exact model using the MQ and EQ oracles is also learnable in the UAV model using the MQ and UAV-EQ oracles as long as the counterexamples provided by the UAV-EQ oracle have a logarithmic number of unspecified attributes. We also show that any class learnable in the exact model using the MQ and EQ oracles is also learnable in the UAV model using the UAV-MQ and UAV-EQ oracles as well as an oracle to evaluate a given boolean formula on an example with unspecified attributes. (For some hypothesis classes such as decision trees and unate formulas the evaluation can be done in polynomial time without an oracle.) We also study the learnability of a universal class of decision trees under the UAV model and of DNF formulas under a representation-dependent variation of the UAV model.

Keywords: Unspecified or missing attributes, exact learning, membership queries, equivalence queries, DNF formulas, decision trees

1 INTRODUCTION

Most theoretical work on learning boolean functions assumes that the examples are drawn from $\{0, 1\}^n$ and each example is classified as positive ("1") or negative ("0") by the unknown target concept $f$. However, consider the situation of trying to predict whether or not an individual will default on a loan. There are a large number of attributes that one might consider (some about the individual and some about the individual’s employer). Using standard approaches to learning boolean functions, a learning algorithm could consider each loan application as an example from $\{0, 1\}^n$ where $n$ is the number of attributes under consideration for loan applications. Let’s assume that if we had correct data for all of the attributes, then such an algorithm could make a very good prediction as to whether or not the individual will default on the loan. However, in practice some attributes' values will not be provided (or specified). For example, some states may prohibit the release of certain credit information. The learner is to take an example with some unspecified attributes and reply that from the given information either (1) the individual is extremely likely to default on the loan, (2) the individual is extremely likely not to default on the loan, or (3) there is not enough information available to make a decision.

There are three key aspects in this scenario. First, if all attribute values were known then the learner could make perfect predictions after learning the target concept. Second, in any given example provided to the learner, some of the attribute values are likely to be unspecified. Finally, the learner’s goal is to make a correct prediction if the values for the provided attributes are sufficient to do so, or otherwise indicate that insufficient information has been provided. Along with the scenario described above, one could imagine situations
in medical diagnosis in which these aspects occur. Another such setting comes from feature extraction for image processing. Suppose the learner is given the task of predicting whether a given image is a face. The attributes could be provided by the output of feature extractors which sometimes are unable to determine whether or not a given feature is present.

In this paper we introduce a variant of Angluin’s exact learning model [3] in which some of the attribute values are left unspecified. When first defining the PAC model, Valiant [43] also suggested a variation in which some attribute values are unspecified but he did not study this variation in detail (see Section 3). In our new model, the examples are drawn from \( \{0,1,\_\}^n \) where “\_“ denotes unspecified. Given a standard boolean function \( f \) and an example \( x \in \{0,1,\_\}^n \), the classification of \( x \) given by \( f \) can naturally be viewed as either being positive (meaning that \( f \) is true regardless of the values for the unspecified attributes), negative (meaning that \( f \) is false regardless of the values for the unspecified attributes), or unknown (meaning that \( f \) could be true or false depending on the values of the unspecified attributes). Thus in our model the classification of an example is drawn from \( \{+,-,?\} \) where “?“ is used to denote the value of unknown. The goal of our learner is to exactly learn an unknown ternary function \( f \) over the domain \( \{0,1,\_\}^n \). Since there are unspecified attribute values, we refer to this model as the UAV Exact Learning model.

In this model an equivalence query, denoted by UAV-EQ, takes as input a hypothesis \( h \) and returns an example \( x \in \{0,1,\_\}^n \) (that could have any number of unspecified attributes) for which \( h(x) \) is different from the correct classification for \( x \). A corresponding model of UAV PAC Learning can be naturally defined. Namely, the learner is provided with labeled (from \( \{+,-,?\} \) ) examples of \( f \), drawn randomly according to some unknown target distribution \( \mathcal{D} \) over \( \{0,1,\_\}^n \). The learner is also given as input \( \epsilon \) and \( \delta \) such that \( 0 < \epsilon, \delta < 1 \), and an upper bound \( k \) on \( |f| \). The learner’s goal is to output, with probability at least \( 1 - \delta \), a hypothesis \( h \in \mathcal{C}_n \) that has probability at most \( \epsilon \) of disagreeing with \( f \) on a randomly drawn example from \( \mathcal{D} \). In both the exact UAV and PAC UAV models, membership queries can be added. In a UAV membership query, denoted by UAV-MQ, the learner gives the membership oracle an example \( x \) from \( \{0,1,\_\}^n \) and is given the value of \( f(x) \) from \( \{+,-,?\} \). We use MQ to denote the standard membership oracle that requires that the example \( x \in \{0,1\}^n \) (and hence the output is from \( \{+,-\} \)), and EQ to denote the standard equivalence oracle that always returns a counterexample \( x \) from \( \{0,1\}^n \). Hence, the UAV-MQ oracle is a generalization of the standard MQ oracle in which some of the attributes can be unspecified. Since, for some concept classes, an NP-hard problem must be solved by the UAV-MQ oracle, we also explore when a concept class is learnable in the UAV model using the UAV-EQ oracle and the standard membership (MQ) oracle.

We prove that any concept class that is learnable in the exact model using the MQ and EQ oracles is UAV exactly learnable using the MQ and UAV-EQ oracles as long as the counterexamples provided by UAV-EQ have \( O(\log n) \) missing attributes where \( n \) is the number of attributes. As a corollary to this result, we prove that any concept class that is learnable in the exact model
using the MQ and EQ oracles is UAV exactly learnable using the UAV-MQ and UAV-EQ oracles as well as an oracle to evaluate a given boolean formula on an example with unspecified attributes. For some hypothesis classes such as decision trees and unate formulas (a superset of read-once formulas), this "evaluation oracle" is not needed. Also, any monotone class that is exactly learnable using just an EQ oracle is exactly learnable in the UAV model using only the UAV-EQ oracle. It is also straightforward to show that any class learnable in the UAV model with only the UAV-EQ oracle is learnable in the standard model using only an EQ oracle. We also show that there is a class of universal decision trees that can be efficiently learned in the UAV model using only a UAV-MQ oracle (no evaluation oracle is needed regardless of how many attributes are unspecified), yet cannot be efficiently learned in the standard exact learning model using only an MQ oracle. Likewise, known results can be applied to show that read-once formulas are UAV exactly learnable using only a UAV-MQ oracle yet are not exactly learnable in standard model using only an MQ oracle.

For a known DNF formula \( f \), the problem of determining the classification of an example \( x \in \{0, 1, *\}^n \) (with \( \Omega(n^c) \) unknown attributes for some constant \( c \)) is NP-complete since it involves solving a satisfiability and a non-satisfiability problem. For the concept class of DNF formulas, we study a representation-dependent variation of the UAV model (which we call the RUAV model) in which the teacher (and learner) can efficiently classify any example from \( \{0, 1, *\}^n \) with respect to a given DNF formula. In this variation, the classification of an example \( x \in \{0, 1, *\}^n \) for the teacher's representation of the target function \( f \) is defined as follows. If any term in \( f \) is satisfied by the known attributes in \( x \) then \( f(x) = + \). If every term in \( f \) is known not to be satisfied by the known attributes from \( x \) then \( f(x) = - \). Otherwise, \( f(x) = ? \). We study the relationship between the UAV and RUAV models, and then prove that the class of DNF formulas is learnable under the RUAV model using RUAV-MQ and RUAV-EQ oracles (no evaluation oracle is needed).

2 OUR LEARNING MODEL

We first briefly review the exact learning model introduced by Angluin [3]. The learner's goal is to exactly learn (using various types of queries) how an unknown (boolean) target function \( f \), taken from some known concept class \( C \), classifies (from \( \{+,-\} \) all instances from the domain. Many different types of queries have been studied. We now define the most common queries.

Membership Query The learner supplies an example \( x \in \{0, 1\}^n \) and is told \( f(x) \).

Equivalence Query the learner presents a candidate function \( h \) and either is told that \( h \equiv f \) (in which case learning is complete), or else is given a counterexample \( x \in \{+,-\} \) for which \( h(x) \neq f(x) \).
Subset Query The learner presents a candidate function $h$ and is either told
that for all $x \in \{0,1\}^n$, if $h(x) = "+"$ then $f(x) = "+"$ or otherwise is
given an example $x \in \{0,1\}^n$ for which $h(x) = "+"$ and $f(x) = "-"$.

Superset Query The learner presents a candidate function $h$ and is either told
that for all $x \in \{0,1\}^n$, if $h(x) = "-"$ then $f(x) = "-"$ or otherwise is
given an example $x \in \{0,1\}^n$ for which $h(x) = "-"$ and $f(x) = "+"$.

As suggested by Valiant [43], we consider when the target concept is defined
over $n$ boolean variables $x_1, \ldots, x_n$ with an instance space $X_n = \{0,1,\ast\}^n$,
where "$\ast$" indicates that the corresponding variable is unspecified. An example
$x \in \{0,1\}^n$ is called total. We say that a total example $y$ is a completion of
each specified attribute in $x$ has the same value in $x$. Let $f$ be a boolean function defined over $x_1, \ldots, x_n$, and let $x \in \{0,1,\ast\}^n$.
We define $f(x) = +$ if and only if $f(y)$ is positive for all vectors $y$ that are
completions of $x$. Similarly, we define $f(x) = -$ if and only if $f(y)$ is negative for
all vectors $y$ that are completions of $x$. Otherwise, we say that $f(x) = ?$ (for
unknown). Note that we use $x_i$ to denote both the $i$th boolean variable and
the $i$th bit of the example $x$. Thus for an example $x \in \{0,1,\ast\}^n$, $x_i$ gives the
value for variable $x_i$. We will always use $x_1, \ldots, x_n$ as the variables, so for an
example $y \in \{0,1,\ast\}^n$, $y_i$ gives the value for variable $x_i$. We denote the example
obtained from $x$ by setting all the unspecified bits to $b \in \{0,1\}$ by $x_{\ast \rightarrow b}$ and
the example obtained by flipping the $i$th bit of $x$, $x_i$, to $b \in \{0,1,\ast\}$ by $x_{i \rightarrow b}$.

For target concepts $f$ and $x \in \{0,1,\ast\}^n$, UAV-MQ($x$) returns the value of
$f(x)$ from $\{+, -, ?\}$. In response to UAV-EQ($h$), the learner either is told that
$h \equiv f$ or else is given a counterexample $x$ (along with its classification) for which
$h(x) \neq f(x)$. Note that the target function is a ternary (versus the standard binary)
function and all three values are distinct. So, for example, if $h(x) = +$ and
$f(x) = ?$, then $x$ could serve as a counterexample for $h$. When given a
boolean formula $h$ and a total example $x \in \{0,1\}^n$, in polynomial time one
can evaluate $h(x)$. However, when given a boolean formula $h$ from hypothesis
class $H$ and an example $x \in \{0,1,\ast\}^n$, the problem of evaluating $h(x)$ is NP-
complete for many choices for $H$. Thus we introduce an evaluation oracle, where
on input $h$ and $x \in \{0,1,\ast\}^n$, EV($h, x$) outputs the value of $h(x)$. There are
some important cases when the EV oracle is not needed since a polynomial
time algorithm can replace it. First, for a projection closed class if we can
efficiently determine whether two different representations of a concept class are
functionally equivalent then an EV oracle is not needed. Such equivalence tests
exist for monotone DNF, read-once branching programs [36], Horn-sentences,
free-branching programs [14, 21], read-once boolean formulas [28] and read-
twice DNF [34]. Also, if the number of unspecified attributes is $O(\log n)$, then
evaluation can be done in polynomial time by simply considering all completions
of $x$. Likewise, when $h$ is a unate formula, then it is easily seen that the
evaluation (for any example from $\{0,1,\ast\}^n$) can be done in polynomial time.

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1 A concept class $C$ is projection closed if for any $f \in C$, the function obtained by fixing a
variable in $f$ to 0 or 1 is $f' \in C$ for $|f'| \leq |f|$.
Finally, for the class of decision trees itself we can efficiently implement the EV oracle by computing the classification of all nodes in the order given by a post-order traversal of the decision tree.

Notice that for a target function \( f \), the MQ oracle on input \( z \) returns the value of EV\((f, z)\). For applications such as the medical diagnosis example, one view is that an expert serves as the UAV-MQ oracle and thus the computational issues are not a problem. However, when studying the learnability of boolean formulas under the UAV model, it is desirable that the UAV-MQ oracle runs in polynomial time. Thus we introduce variants of the UAV membership and equivalence oracles. The UAV-EQ\(_{\log} \) oracle takes as input a hypothesis \( h \) and returns a counterexample in which there are at most \( O(\log n) \) unspecified attributes. Similarly, the UAV-MQ\(_{\log} \) oracle takes as input any example \( x \) with \( O(\log n) \) unspecified attributes and outputs the classification (\(+\), \(-\), or \(?\)) of \( x \).

As in the standard exact learning model, the learner's goal is to exactly identify the target concept using membership and equivalence queries (i.e. for all \( z \in \{0, 1, *\}^n \), \( h(x) = f(x) \)). Since there are unspecified attribute values, we call this model the UAV Exact Learning model. We also study variations in which only a UAV-EQ oracle or only a UAV-MQ oracle is allowed. In some of our results, the unspecified attributes of the examples given to the UAV-MQ oracle are a subset of the attributes that were unspecified in the recent counterexample that came from the UAV-EQ oracle. In these cases, as long as we use the UAV-EQ\(_{\log} \) oracle (and thus have \( O(\log n) \) unspecified attributes), then we can replace the UAV-MQ oracle by a UAV-MQ\(_{\log} \) oracle, and hence using Observation 2, by a standard MQ oracle. Thus for these cases, we obtain positive results in the UAV model by just using a UAV-EQ\(_{\log} \) oracle and an MQ oracle. Finally, a corresponding model of UAV PAC Learning can be naturally defined.

3 RELATED WORK

Within the exact learning model a number of interesting polynomial time algorithms have been presented to learn target classes such as decision trees \([19]\), deterministic finite automata \([2]\), Horn sentences \([4]\), read-once formulas \([5, 18, 16]\), read-twice DNF formulas \([1]\), \( k \)-term DNF formulas \([13, 20]\), etc. For all of these classes it is known (using information-theoretic arguments), that neither membership queries nor equivalence queries alone suffice.

Although different from the goals of our work, there has been work on learning when the examples may be mislabeled \([3, 32, 41, 29]\) and when there is attribute noise \([40, 23, 33]\). There has also been some work in which the answers to membership queries are noisy or missing \([37, 7, 42, 8, 12]\). Although we can get a "?" response from our membership oracle, this response is not adversarially generated to model an inconclusive outcome, but rather is structurally defined by the target concept and indicates that there is insufficient information to determine the classification of the provided example.

A learning model that has very similar motivations to the UAV model is
the model of RFA (restricted focus of attention) learnability [9, 10]. The $k$-RFA model is a variant of the PAC model in which for each example only $k$ attributes (of the $n$ attributes), as selected by the learner, are specified. Thus, unlike the UAV model in which the unspecified attributes are adversarially selected, in the $k$-RFA model the learner can select which attributes are specified. Another key difference is in the criteria required of the learner. In the $k$-RFA model the requirements for successful learning are exactly as in the PAC model. Namely, the learner is given a random example $x \in \{0, 1\}^n$ and asked to properly classify $x$ with high probability. To explore the differences between these two models, we consider the task of medical diagnosis which has been suggested as an example application for both models. In the $k$-RFA model the view is that for the training data, the learner can select which test to run and hence which data is available. However, since the training data is likely to be medical records from past patients, any data not available (even if a test could have been run) cannot be obtained at a later date. Also, there are some attributes such as those related to family history that may just not be available. The UAV model captures this by having an adversary choose which attributes are not specified. Another key difference between the two models is that in the RFA model the examples to be classified have all attributes specified. However, in medical diagnosis (like many other scenarios), some data for the patient being diagnosed may just not be available. For example, some information about family history may be unavailable, and maybe due to the patient's condition some test(s) could not be run. Thus we feel it is important for algorithms in our model to classify examples that have unspecified attributes. This requirement led us to have a three-valued (versus binary) output.

In the different "learning to reason" framework, Khardon and Roth [31] investigate the construction of a knowledge base for representing "the world", i.e. some boolean function $f$. This knowledge base is then used to deduce (i.e. reason) if $f$ logically implies $\alpha$ where $\alpha$ is a propositional query capturing the situation at hand. They consider various interpretations of applying a boolean function $f$ to a partial instance $x$:

(1) **Universal**: $f(x) = +$ if all completions of $x$ are classified as "+" by $f$, otherwise $f(x) = -$.

(2) **Existential**: $f(x) = +$ if there exists a completion $x'$ of $x$ such that $f(x') = +$, otherwise $f(x) = -$.

(3) **Abbreviated**: $f$ classifies $x$ by treating the "*"s as 0s and then classifying the total instance obtained.

Notice that the universal and existential interpretations are the same as treating the classification "?" in the UAV model as "-" and "+", respectively. The membership and equivalence query oracles under these two interpretations are clearly less informative than their UAV counterparts. The universal interpretation was introduced earlier when Valiant [43] first defined the PAC model.
Clearly, if the target concept is monotone, then both the universal and abbreviated interpretations are equivalent. Using this observation, Valiant constructs an efficient PAC learning algorithm, with the aid of a membership query oracle, for learning monotone DNF under the universal interpretation.

Some work that has similar motivations is the p-concepts model of Kearns and Schapire [30]. In the p-concepts model (when applied to the boolean domain), the learner is given a total example from \(\{0,1\}^n\), yet there is some probabilistic process (or possibly something that appears probabilistic due to the learner being unaware of some important attributes) that determines whether the output is positive or negative. In our setting we assume that the learner knows which attributes it would like to gather, but cannot obtain values for all such attributes. Other key distinctions are that our target concept is ternary versus binary and deterministic versus probabilistic, i.e., given an example \(x\), \(f(x)\) always has the same value drawn from \(\{+,-,?\}\) in the UAV model, whereas in the p-concepts model \(x\) can sometimes have a classification of “+” and sometimes have a classification of “-”.

In other related work, Frazier, Goldman, Mishra, and Pitt [22] introduce a learning model that captures the idea that teachers may have gaps in their knowledge. They consider learning from a teacher who labels each example as “+”, “-”, or “?” in such a way that knowledge of the concept class and all the positive and negative examples are not sufficient to determine the labeling of any example labeled with “?”. Their goal is to PAC-learn the ternary labeling presented by such a consistently ignorant teacher. Similar to their model, we are learning a ternary function. However, in their model all attribute values are known (i.e., the examples come from \(\{0,1\}^n\)). It is easily shown that under the domain \(\{0,1,\ast\}^n\), the “?”'s allowed by our model are placed in such a way to meet the definition of the teacher being consistently ignorant. Namely, no “?”’s value could be determined from the positive examples, negative examples, and knowledge of the concept class. Another key difference between the two models is that instead of having the “?”’s placed adversarially (as in the consistently ignorant teacher model), in our UAV model the “?”’s are defined based on the target concept. Thus in the UAV model, the complexity of the ternary target function has the same complexity as that of the defining boolean function.

There has been some empirical work studying the task of learning from incomplete data [8, 35, 15]. With the goal of giving a theoretical explanation for the observed empirical phenomena, Schuurmans and Greiner [38, 39] studied the problem of learning accurate default concepts to be used when working with incomplete data. Given an example with unspecified attribute values, they call an example \(x \in \{0,1,\ast\}^n\) ambiguous if the classification cannot be determined (i.e., using our terminology the classification of \(x\) is unknown). Their goal is to find a good way to compute a default classifier (which can have nonmonotonic behavior) for making a prediction of + or – for the ambiguous examples. In other words, unlike our UAV model, there are no “?” responses for either the target concept or the hypotheses. Also, instead of an adversary choosing which

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2Although the “*”s in the examples are adversarially placed.
attributes are unspecified, in their model a total example is drawn from some fixed distribution and then some attribute values become unspecified according to some probability distribution. They investigate, under various conditions, whether the strategies [8, 35, 15] that are used in practice converge to some optimum hypothesis in the limit and the sample complexities required to achieve a certain PAC-like learning criterion.

Finally, there have been two oracles studied that are similar to the UAV-MQ oracle: the constrained instance oracle [26] and the projective equivalence oracle [27]. We use CIQ to denote a constrained instance query, and PEQ to denote a projective equivalence query. Let a (which can be viewed as partial assignment or an example in the UAV model) be drawn from \( \{0, 1, *\}^n \), and for the target boolean formula \( f \), let \( f_a \) be the projection defined by \( a \). That is, \( f_a \) is the function defined by \( f \) when the variables assigned in \( a \) are replaced by a constant. Note that

\[
\text{UAV-MQ}(a) = \begin{cases} 
+ & \text{if } f_a \text{ is a tautology} \\
- & \text{if } f_a \text{ is a contradiction} \\
? & \text{otherwise}
\end{cases}
\]

The constrained instance query is defined as follows where \( \ell \in \{-, +\} \): CIQ(\(a, \ell\)) returns “yes” if and only if there is an assignment \( x \) that is a completion of \( a \) for which \( f(x) = \ell \). Observe that UAV-MQ(\(a\)) = + if and only if CIQ(\(a, -\)) = “yes” and CIQ(\(a, +\)) = “yes.” Similarly, UAV-MQ(\(a\)) = − if and only if CIQ(\(a, -\)) = “no” and CIQ(\(a, +\)) = “no.” Finally, UAV-MQ(\(a\)) = ? if and only if CIQ(\(a, -\)) = “yes” and CIQ(\(a, +\)) = “yes.” Thus these two queries are essentially equivalent. The projective equivalence query is defined as follows. For a partial assignment \( a \) and a boolean formula \( f' \), PEQ(\(a, f'\)) replies “yes” if and only if \( f_a \equiv f' \). Similar to the above, it is easily shown that the special case of a projective equivalence oracle in which \( f' \) is either the constant true or false is essentially equivalent to the UAV membership oracle.

It is known that read-once formulas are exactly learnable using only constrained instance queries [26] or using these restricted projective equivalence queries [27]. Thus read-once formulas are exactly learnable in the UAV model using only the UAV-MQ oracle. Furthermore, it is known [5] that a polynomial number of calls to the MQ oracle is not sufficient. This demonstrates (as one would expect) that the UAV-MQ oracle is more powerful than the MQ oracle.

Bshouty, Cleve, Kannan and Tamon [17] show that DNF formulas can be learned by a randomized algorithm in expected polynomial time with equivalence queries and the aid of an NP oracle. Using this result, they also show that DNF formulas can be learned using subset and superset queries. The hypothesis class of their algorithms is the class of depth-3 \( \land \lor \land \lor \land \) formulas. An obvious question to ask is whether the UAV-MQ oracle is powerful enough to simulate the NP oracle and EQ oracle (or the superset and subset oracles) needed by the algorithm of Bshouty et. al. If so, then it would immediately follow that the class of DNF formulas is learnable by a randomized algorithm in expected polynomial time using a UAV-MQ oracle alone. However, achieving either of these two simulation results does not seem possible. It is easily shown that a a
UAV-MQ oracle can simulate subset and superset query oracles where the hypotheses are conjunctions and disjunctions of literals respectively. For example, if a subset query is made with the hypothesis $\ell_1 \land \ell_2 \land \cdots \land \ell_k$ for literals $\ell_i$ then this can be simulated with the UAV-MQ oracle by setting each of the $k$ literals used in the hypothesis to be true and setting the remaining variables to "$s$". If the UAV-MQ oracles returns "$+$" then the answer to the subset query is "$yes$". If the UAV-MQ oracle returns "$-$" then return $x_{s \rightarrow a}$ as a negative counterexample. Finally, if the UAV-MQ oracle responds with "$?$" then a FormNegEx (a variation of FormPosEx from Figure 1 with "$+$" and "$-$" exchanged) can be used to construct a counterexample. However, we believe that the UAV-MQ oracle cannot simulate subset and superset query oracles with the hypothesis class of depth-3 $\land\lor\land$ formulas as would be needed to use the algorithm of Bshouty et. al to learn DNF formulas with only a UAV-MQ oracle.

Recently, Birkendorf, Klasner, Kuhlman and Simon [11] investigated the UAV model further and answered a number of open problems posted in an earlier (conference) version of this paper [24]. They presented lower bound results on the number of UAV-EQs and UAV-MQs required to learn a concept class in terms of its Vapnik Chervonenkis dimension. Further, they extended Angluin’s [3] sunflower lemma (which is useful in proving lower bound results in the exact model) to the UAV setting. In doing so, they establish exponentially large lower bounds on the number of UAV-MQs needed to learn monotone DNF, read-once DNF, $O(n)$-DNF, read-twice DNF and 2-decision lists. In contrast to these negative results, they also present efficient algorithms using UAV-MQs for learning read-once DNF, constant-term DNF and 1-decision lists. They also determine how restrictions on the number of unspecified attributes affects the strength of the UAV oracles. They show that a UAV-EQ oracle that returns counterexamples with at most $r - 1$ unspecified attributes is strictly stronger than one that is allowed to return counterexamples with up to $r$ unspecified attributes. They also establish a similar result for the UAV-MQ oracle. Besides these hierarchical results, they also compare the power among the standard and UAV oracles.

4 RELATIONSHIPS WITH THE STANDARD EXACT LEARNING MODEL

We begin with an observation that directly follows from the fact that within the UAV model the examples given as input to the UAV-MQ oracle can be selected from $\{0, 1, *\}^n \supset \{0, 1\}^n$.

**Observation 1** Let $C$ be a concept class that is exactly learnable using the MQ oracle alone. Then $C$ is UAV exactly learnable using only the UAV-MQ oracle.

Next we consider when a UAV-MQ$_{log}$ oracle can be simulated with a standard membership query oracle. Let $x$ be an example given to the UAV-MQ$_{log}$ oracle (so $x$ has at most $c \log_2 n$ unspecified attributes for some constant $c$).
Since there are at most \( n^c \) completions of \( x \), the value of UAV-MQ_{log}(x) can be computed from making \( n^c \) calls to the MQ oracle and using polynomial time. Thus we get the following observation.

**Observation 2** The UAV-MQ_{log} oracle can be simulated in polynomial time using the standard MQ oracle.

While standard MQ oracle can be trivially simulated in our model, equivalence queries are more complicated since the UAV-EQ oracle returns an example \( x \in \{0, 1, *\}^n \) versus from \( \{0, 1\}^n \). We show the somewhat surprising result that the standard EQ oracle can be simulated by a UAV-EQ_{log} oracle and a standard MQ oracle.

**Theorem 3** Let \( C \) be a concept class that is exactly learnable in polynomial time using the MQ and EQ oracles. Then \( C \) is UAV exactly learnable in polynomial time using the MQ and UAV-EQ_{log} oracles.

**Proof:** Let \( A \) be the exact learning algorithm for \( C \), and let \( f \in C \) be the target function. We construct an algorithm \( A' \) to learn \( C \) in the UAV model. Suppose that \( A \) makes an equivalence query with hypothesis \( h \). Then \( A' \) uses its UAV-EQ_{log} oracle in the following manner. If UAV-EQ_{log}(h) responds "yes" then \( A' \) can output \( h \) and halt. Suppose instead that counterexample \( x \in \{0, 1, *\}^n \) is returned. In each of the following three cases we show how to generate an example \( y \in \{0, 1\}^n \) such that \( h(y) \neq f(y) \). Then \( y \) is given to \( A \) as a counterexample. Note that in all of the cases, the unspecified attributes in any example given to the UAV-MQ oracle are a subset of those unspecified in the counterexample \( x \). Since \( x \) had \( O(\log n) \) unspecified attributes, all examples given to the membership oracle have \( O(\log n) \) unspecified attributes. Thus by Observation 2 we can, in polynomial time, simulate the calls to the UAV-MQ oracle with an MQ oracle.

**Case 1:** \( h(x) = - \). One possibility is that \( f(x) = + \). Let \( y = x_{n \rightarrow 0} \). By the definition of a positive example in the UAV model, we know that \( f(y) = + \). By the definition of a negative example in the UAV model, we know that \( h(y) = - \). Thus \( y \in \{0, 1\}^n \) is a counterexample for \( h \).

The other possibility is that \( f(x) = ? \). By definition, we know there is some completion \( y \) of \( x \) for which \( f(y) = + \). We use membership queries to find such a \( y \). The basic idea is to go through the unspecified attributes setting the next attribute to 1 (or to 0) and checking with the UAV-MQ oracle if this example is positive. If not, since \( f(x) = ? \) at least one of these examples has a "?" classification, and thus we can continue with this new example (which has one less unspecified attribute). Figure 1 describes our procedure FormPosEx that receives an \( x \in \{0, 1, *\}^n \) for which \( h(x) = - \) and \( f(x) = ? \), and returns a total example \( y \in \{0, 1\}^n \) for which \( h(y) = - \) and \( f(y) = + \).
FormPosEx($x$)

1. Let $x_i$ be the first unspecified attribute in $x$
2. $v := x_{x_i \rightarrow 0}$
3. $w := x_{x_i \rightarrow 1}$
4. if UAV-MQ($v$) = ? then
   5. FormPosEx($v$)
5. else if UAV-MQ($v$) = + then
   6. return $v_{x_i \rightarrow 0}$ to $A$
8. else (so UAV-MQ($v$) = -)
   9. if UAV-MQ($w$) = + then
   10. return $w_{x_i \rightarrow 0}$ to $A$
11. else (so UAV-MQ($w$) = ?)
12. FormPosEx($w$)

Figure 1: The algorithm FormPosEx which takes as input an example $x \in \{0, 1, *\}^n$ for which $f(x) = ?$ and $h(x) = -$, and returns a total example $y \in \{0, 1\}^n$ for which $h(y) = -$ and $f(y) = +$.

Case 2: $h(x) = +$. This is handled using a symmetric argument to Case 1. Notice that FormPosEx (and its proof of correctness given below) can be easily modified to create the procedure FormNegEx.

Case 3: $h(x) = ?$. One possibility is that $f(x) = +$. Here we can apply a variation of FormNegEx in which we use the evaluation oracle, EV($h, x$), instead of UAV-MQ($x$). (Recall that since there are $O(\log n)$ unspecified attributes in $x$, the computation of EV($h, x$) can be done in polynomial time.) The other possibility is that $f(x) = -$ in which we apply a variation of FormPosEx where we again use EV($h, x$) instead of UAV-MQ($x$).

We now prove that FormPosEx is correct. Let $x$ be the example given as input to FormPosEx (for which we maintain the invariant that $f(x) = ?$). Let $x_i$ be the first unspecified attribute in $x$, let $v = x_{x_i \rightarrow 0}$, and let $w = x_{x_i \rightarrow 1}$. Notice that if FormPosEx ever reaches the last unspecified attribute, then clearly $f(v)$ and $f(w)$ must be "+" or "-" since $v, w \in \{0, 1\}^n$. Furthermore, since UAV-MQ($x$) = ? then either $f(v) = +$ or $f(w) = +$. Thus eventually FormPosEx returns an example $y \in \{0, 1\}^n$ to $A$. We now argue that $y$ is a counterexample. First, since $h(x) = -$ and $y$ is a completion of $x$, $h(y) = -$. Finally, it follows directly from FormPosEx that UAV-MQ($y$) = $f(y) = +$. The last thing we must argue is that when we reach the final else clause (so $f(v) = -$), then $f(w) \neq -$. That is, if $f(w) \neq +$, then $f(w) = ?$. This is crucial, since it is important that the example for the recursive call to FormPosEx has a classification of "?".
We now prove the following slightly more general result. For \( \epsilon \in \{0, 1, \ast\}^n \) for which \( f(\epsilon) = ? \) and for which \( \epsilon_i \) is unspecified, as we did in Figure 1 define \( v = x_{\epsilon_i \rightarrow 0} \) and \( w = x_{\epsilon_i \rightarrow 1} \). If \( f(v) \neq ? \), we show that \( f(v) \neq f(w) \). For the sake of contradiction, assume that \( f(w) = f(v) = - \). Consider the total vectors \( w' = w_{\ast \rightarrow 0} \) and \( v' = v_{\ast \rightarrow 0} \). Note that \( f(w') = f(v') = - \). Also, the only difference between \( w' \) and \( v' \) is that for the \( i \)-th attribute \( w'_i = 1 \) and \( v'_i = 0 \), yet we are given that \( f(\epsilon) = ? \). Thus there must be some completion \( z \) of \( \epsilon \) for which \( f(z) = + \). Suppose that the \( i \)-th attribute \( z_i = 0 \). Then since \( f(v) = - \), it follows that \( f(z) = - \). Likewise, suppose that the \( i \)-th attribute \( z_i = 1 \). Then since \( f(w) = - \), it follows that \( f(z) = - \). This contradicts the existence of such a \( z \), completing the proof that if \( f(v) \neq ? \), then \( f(v) \neq f(w) \).

Thus the algorithm \texttt{FormPosEx} will, after at most \( 2n \) calls to the UAV-MQ oracle (simulated by a polynomial number of calls to the MQ oracle) and polynomial time, return a counterexample \( y \) to \( A \) as desired. Thus both the membership and equivalence queries requested by \( A \) can be simulated by \( A' \) in polynomial time. \( \square \)

As a corollary to this theorem we consider when the UAV-EQ oracle (versus the UAV-EQ_{\text{exg}} oracle) is used. The only change required is that now the UAV-MQ oracle cannot necessarily be efficiently simulated by the MQ oracle). Also, now the EV oracle is needed since we may not be able to efficiently evaluate the learner’s hypothesis on the examples from \( \{0, 1, \ast\}^n \).

**Corollary 4** Let \( C \) be a concept class that is exactly learnable in polynomial time using the MQ and EQ oracles. Then \( C \) is UAV exactly learnable in polynomial time using the UAV-MQ, UAV-EQ, and EV oracles.

While in general our result requires the use of the EV oracle, it is easily seen that a polynomial time algorithm can simulate the EV oracle for a unate class. More specifically, consider the query \( \text{EV}(f, x) \) where \( x \in \{0, 1, \ast\}^n \). Let \( x_1 \) be the completion of \( x \) obtained by setting the unnegated attributes to \( 1 \) and the negated attributes to \( 0 \). Likewise, let \( x_2 \) be the completion of \( x \) obtained by setting the negated attributes to \( 0 \) and the unnegated attributes to \( 1 \). If \( f(x_1) = f(x_2) \) then output \( f(x_1) \) as the classification for \( f(x) \). Otherwise, \( f(x) = ? \). Thus only two examples from \( \{0, 1\}^n \) need to be classified. So for any learning algorithm that uses a unate class for the hypothesis class, the EV oracle is not needed.

**Corollary 5** Let \( C \) be exactly learnable in polynomial time with a unate hypothesis class using the EQ and MQ oracles. Then \( C \) is UAV exactly learnable in polynomial time using the UAV-EQ and UAV-MQ oracles.

As discussed earlier, for the class of decision trees we can efficiently implement the EV oracle by computing the classification of all nodes in the order given by a post-order traversal of the decision tree. Thus, if the hypothesis class is the class of decision trees, the EV oracle is not needed.
Corollary 8 Let $C$ be exactly learnable in polynomial time with a hypothesis class of decision trees using the EQ and MQ oracles. Then $C$ is UAV exactly learnable in polynomial time using the UAV-EQ and UAV-MQ oracles.

Finally, for a projection closed class for which we can efficiently determine whether two different representations of a concept class are functionally equivalent then an EV oracle is not needed. Such equivalence tests exist for monotone DNF, read-once branching programs [38], Horn-sentences, free-branching programs [14, 21], read-once boolean formulas [28] and read-twice DNF [34].

Corollary 7 Let $C$ be exactly learnable in polynomial time using the EQ and MQ oracles with a hypothesis class $H$ where $H$ is projection closed and for which there is a polynomial time algorithm to determine if $h_1 \in H$ and $h_2 \in H$ are logically equivalent. Then $C$ is UAV exactly learnable in polynomial time using the UAV-EQ and UAV-MQ oracles.

Notice that the simulation used in Theorem 3 requires the use of membership queries by $A'$ even when $A$ uses only equivalence queries. One natural question is: When can a standard exact learning algorithm that uses only equivalence queries be simulated by a UAV algorithm that uses only equivalence queries? We now partly answer that question by arguing that FormPosEx (respectively, FormNegEx) need not use membership queries to simulate the equivalence queries when the concept class is monotone. In particular, given an example $x$ for which $f(x) = ?$ (or $h(x) = ?$), FormPosEx (respectively, FormNegEx) and the variations used in Case 3 need just return $x_{\rightarrow 1}$ (respectively, $x_{\rightarrow 0}$).

Corollary 8 Let $C$ be exactly learnable in polynomial time with a monotone hypothesis class using only the EQ oracle. Then $C$ is UAV exactly learnable in polynomial time using only the UAV-EQ oracle.

All of the above results address the issue of converting an algorithm that works in the standard model to one that works in the UAV model. Since the example $x$ returned from a standard equivalence query is a total example for which $f(x) \neq h(x)$, $x$ can also serve as a response for the UAV equivalence query. Thus we get the following result.

Observation 9 If concept class $C$ is exactly learnable in polynomial time in the UAV model using only the UAV-EQ oracle, then $C$ is exactly learnable in the standard model in polynomial time using only the EQ oracle.

5 LEARNING ORDERED DECISION TREES WITH A UAV-MQ ORACLE

Combining our results from the last section with Bshouty’s decision tree algorithm [19], we can learn the class of decision trees in the UAV model using the UAV-MQ, UAV-EQ, and EV oracles. As discussed in Section 2, for the class of
Tree $\text{BuildTree}(\text{proj})$

1. if $\text{UAV-MQ}(\text{proj} \ast \cdots \ast) \neq \$ then return $\text{tree}(\text{UAV-MQ}(\text{proj} \ast \cdots \ast), \text{nil}, \text{nil})$
2. $T_L := \text{tree}(\text{UAV-MQ}(\text{proj0} \cdots 0), \text{nil}, \text{nil})$
3. $i := i' - 1$
4. while $i > |\text{proj}|$
5. if $\text{IsParent}(\text{proj}, i, T_L) = \text{true}$
6. $T_L := \text{tree}(x_i, T_L, \text{BuildTree}(\text{proj} \overline{0 \cdots 0 1})_{i=|\text{proj}|-1})$
7. $i := i - 1$
8. return $T_L$

Figure 2: The algorithm $\text{BuildTree}$ for learning an arbitrary boolean function $f$ in its canonical ordered decision tree representation $T_f$. The input $\text{proj}$ gives a partial assignment for the first $|\text{proj}|$ attributes. In the initial call $\text{proj}$ is empty. Note that the projection sent recursively in Line 6 uses $\text{proj}$ to assign $x_1, \ldots, x_{|\text{proj}|}$, assigns 0 to $x_{|\text{proj}|+1}, \ldots, x_{i-1}$, and assigns $x_i = 1$.

decision trees we can implement the EV oracle efficiently. However, since the hypothesis class in Bahoutry's algorithm is not a decision tree, the EV oracle must still be provided.

In this section we consider a class $\mathcal{T}$ of "ordered" decision trees defined with respect to a linear ordering of the variables, say $x_1 < \cdots < x_n$, such that the variable in each internal node is greater than the variables in its ancestors in this ordering. We give an algorithm to learn $\mathcal{T}$ in the UAV model using only the UAV-MQ oracle. Since our hypothesis class is also an ordered decision tree, we do not need an EV oracle because we can efficiently evaluate $f(x)$ for a decision tree $T_f$ and example $x \in \{0,1,*\}^n$. Since the class of singletons can be represented by trees in $\mathcal{T}$ with $O(n)$ nodes and the class of singletons cannot be learned with standard membership queries [3], $\mathcal{T}$ cannot be learned using standard membership queries. This provides a second example (along with read-once formulas, see Section 3) demonstrating that the UAV-MQ oracle is more powerful than standard MQ oracle.

We denote the size of a tree $T$ by $|T|$. We also adopt the convention that we branch to the left subtree if the variable associated with the current node is "0". Otherwise we branch to the right. Notice that any boolean concept has a representation in $\mathcal{T}$ although the size of the representation may be exponential in the size of its DNF representation. We now prove that the representation of any boolean function $f$ using $\mathcal{T}$ with the smallest number of nodes is unique.

Lemma 10 The representation of any boolean function $f$ using $\mathcal{T}$ with the smallest number of nodes is unique.

Proof: The proof is by induction on the number of variables $n$. Clearly, it is
true when \( n = 1 \). For the sake of contradiction, suppose \( f \) has two minimum representations \( T_1 \) and \( T_2 \). If the variables at the roots of both trees are the same, then by our inductive hypothesis, both left and right subtrees of both trees are identical. Without loss of generality, suppose variable \( x_i \) at the root of \( T_1 \) is smaller than that of \( T_2 \). Then \( x_i \) must not be relevant to \( f \) meaning that \( T_1 \) could be replaced by either of its subtrees, contradicting the minimality assumption for \( T_1 \). 

We now prove the main result of this section.

**Theorem 11.** The class of ordered decision trees is efficiently exactly learnable in the UAV model using only the UAV-MQ oracle. Specifically, for an arbitrary boolean function \( f \), let \( T_f \) be its canonical (minimal) representation as an ordered decision tree. Then \( T_f \) can be exactly identified using \( O(n^2 |T_f|) \) time and calls to the UAV-MQ oracle.

**Proof:** We present an algorithm BuildTree (see Figure 2) for learning \( T_f \). The basic idea is to first determine the subtree rooted at a node along the path from the root to the leftmost leaf, then determine the label of its parent, and recursively determine its right sibling. We can then iteratively continue this process moving up the tree. More specifically, BuildTree takes as input a partial assignment \( proj \) and builds the canonical ordered decision tree for \( f_{proj} \) (the projection of \( f \) defined by \( proj \)). The assignment \( proj \) assigns a 0 or 1 for attributes \( x_1, \ldots, x_j \) for some \( j \) with the remaining attributes left unspecified. (Initially, all attributes are unspecified.) We denote \( proj \) by a length \( j \) bit string (i.e., \( proj \in \{0,1\}^j \)) where only the specified attributes are included. We let \( proj b \cdots b \) denote the example obtained by padding \( proj \) by repeating \( b \in \{0,1,*\} \) \( n-j \) times.

The procedure BuildTree is called recursively to construct the canonical ordered decision tree for \( f_{proj} \). The terminating condition (see Line 1) is reached when UAV-MQ(\( proj \ast \cdots \ast \)) \( \neq \ast \), in which case we return a leaf with label \( f(\text{proj} \ast \cdots \ast) = \text{UAV-MQ(\text{proj} \ast \cdots \ast)} \). The notation tree(\( x_i, T_L, T_R \)) denotes the tree where the root is labeled \( x_i \), the left subtree is \( T_L \), and the right subtree is \( T_R \). A leaf with label \( \ell \in \{+,-\} \) is denoted as tree(\( \ell, \text{nil}, \text{nil} \)). If the terminating condition does not hold, then the root of the tree is an internal node. Define \( P_L \) as the path from the tree's root to its leftmost leaf. Clearly, the subtree rooted at the last node of \( P_L \) is simply a leaf with label \( f(\text{proj}0 \cdots 0) \). In Line 2 we construct such a leaf (as a single-node subtree) and call it \( T_L \). Next we determine the attribute \( x_i \) associated with the parent of \( T_L \) (see Figure 3). Let \( x_i \) be the attribute associated with the root of \( T_L \). Observe that the index \( i \) for the attribute \( x_i \) must be between 1 and \( j' - 1 \). We begin by setting \( i = i' - 1 \) and testing (in Line 5) whether \( x_i \) is the attribute for the parent of \( T_L \) by calling the function IsParent(\( proj, i, T_L \)). If we find \( x_i \) is not the parent, then we decrement \( i \) and repeat the test. If the test succeeds for \( i \), we then construct (in Line 6) the subtree rooted at the parent which consists of a root labeled \( x_i, T_L \) as its left subtree, and the right subtree \( T_R \) that we recursively learn by calling BuildTree with the projection \( proj' = proj 0 \cdots 0 1 \). (So \( proj'_{i=j'-1} \)
assigns $x_1, \ldots, x_i$ according to $proj$, assigns 0 to the attributes $x_{i+1}, \ldots, x_{i-1}$ and finally assigns 1 for $x_i$.) We then let this newly constructed tree be our next $T_L$ and iteratively repeat the process of finding its parent and a new $T_L$ until we have constructed the entire tree.

We now describe how $\text{IsParent}(proj, i, T_L)$ (see Figure 4) tests whether $x_i$ is the parent of $T_L$ in the subtree $T_{L_{proj}}$. Let $i'$ be the index for the attribute at the root of $T_L$ (see Figure 3). Notice that the way $\text{BuildTree}$ calls $\text{IsParent}$ ensures that the attributes between $x_i$ and $x_{i'}$ have failed the test (and thus cannot appear as the label in the parent of $T_L$). For leaf $\ell$ of $T_L$, let $a_{\ell}$ be the partial assignment for $x_{i-1}, \ldots, x_n$ which defines the path to $\ell$ from the root of $T_L$. First consider the case where $x_i$ is not the parent of $T_L$. Then changing $x_i$ from "*" to "1" does not change its classification. Thus the partial assignment $proj \, 0 \cdots 0 \, 1 \, \cdots \, * \, a\ell$ must be classified consistently with the label of $\ell$. Conversely, if $x_i$ is the parent of $T_L$ then $T_L$ is different from $T_R$. That is there is a leaf $\ell$ in $T_L$ such that the classification of the partial assignment $proj \, 0 \cdots 0 \, 1 \, \cdots \, * \, a\ell$ is inconsistent with the labeling of $\ell$.

We now use induction on $n$ to prove the stated upperbound for the number of calls to the UAV-MQ oracle (and hence the time complexity). Clearly, the bound holds for $n = 1$. For $n > 1$, the subroutine $\text{IsParent}$ makes at most
queries and is called by BuildTree at most \( n \) times. The total number of queries needed in Line 6 to recursively construct the right subtrees of the nodes along the path \( P_L \) is at most \((n - 1)^2\) times the total size of these right subtrees, which is bounded by \( |T_f| \). Together with the two initial queries at Lines 1 and 2, the number of queries made is \( 2 + O(n|T_f|) + O((n - 1)^2|T_f|) = O(n^2|T_f|) \) as desired.

\[ \square \]

6 LEARNING DNF FORMULAS UNDER THE RUAV MODEL

For many concept classes, the problem of determining the classification of an example \( x \in \{0, 1, *\}^n \) for a known target function \( f \) is NP-complete since it involves solving a satisfiability and a non-satisfiability problem. As we have noted, for unate formulas and decision trees this evaluation problem can be efficiently solved. On the other hand, for the concept class of DNF formulas, SAT can trivially be reduced to our evaluation problem and thus the evaluation problem is NP-complete. For the concept class of DNF formulas, we introduce a representation-dependent variation of the UAV model in which the learner and teacher can efficiently classify any example from \( \{0, 1, *\}^n \). Let \( f \) be the target DNF formula. For a moment suppose a computationally unbounded teacher builds a DNF formula \( f' = f \) that has one term for each (of the possibly exponentially many) prime implicants\(^3\) of \( f \). The teacher could now compute \( f(x) \) for \( x \in \{0, 1, *\}^n \) as follows. If any term in \( f' \) is satisfied by the known attributes in \( x \), then \( f(x) = + \). If every term in \( f' \) is known not to be satisfied by the known attributes in \( x \), then \( f(x) = - \). Otherwise \( f(x) = * \).

Using the above observation as a motivation, we now define the following representation-dependent version of the UAV model (which we call the RUAV model) in which a computationally bounded agent can evaluate the target formula \( f \) on example \( x \in \{0, 1, *\}^n \). The idea is to approximate the above procedure directly using \( f \). Namely, in the RUAV model, if any term in \( f \) is satisfied by the known attributes in \( x \), then \( f(x) = + \). If every term in \( f \) is known not to be satisfied by the known attributes in \( x \), then \( f(x) = - \). Otherwise \( f(x) = * \). Note that it is possible for \( f(x) = * \) in the RUAV model when \( f(x) = + \) in the UAV model. For example, suppose \( f = x_1x_2 + \overline{x_3} \). So \( f' = x_1x_2 + \overline{x_1}x_2 + x_2x_3 \). Now consider the example \( x = 11 \). Notice that both \( x_1x_2 \) and \( \overline{x_1}x_2 \) individually could be satisfied or not satisfied. Thus in the RUAV model, \( f(11) = * \). However, the prime implicant \( x_2x_3 \) is satisfied by \( 11 \) and thus in the UAV model, \( f(11) = + \). Since the classification of some examples depend on the teacher’s representation of the target concept, we refer to this as a representation-dependent model.

It is common for a human expert to (maybe subconsciously) encode a scheme

\(^3\) A prime implicant of a boolean formula \( f \) is a conjunction \( t \) (not containing contradictory literals) such that \( t \) implies \( f \), but no proper subset of \( t \) implies \( f \).
Boolean IsParent(proj, i, TL)

1. \(i' := \text{index for the attribute at the root of } TL \ (n + 1 \text{ if } TL \text{ is a leaf})\)
2. for each leaf \(\ell \text{ in } TL\)
3. \(\text{Let } a_\ell \text{ be the partial assignment for } x_{i'}, \ldots, x_n \text{ that defines the path to } \ell \text{ in } TL\)
4. \(\text{if UAV-MQ(proj, 0 \cdots 0, 1 \ast \cdots \ast a_\ell) \neq \text{label of } \ell \text{ then return true}}\)
5. \(i' := i + 1\)

\(\text{return false}\)

Figure 4: The Algorithm IsParent which tests whether \(x_i\) is the parent of \(TL\) in the subtree \(T_{proj}\). The assignment \(a_\ell\) assigns all variables from \(x_{i'}, \ldots, x_n\) on the path from the root of \(TL\) to the leaf \(\ell\) according to the path, and the rest are left unspecified.

For classifying examples in \(\{0, 1, \ast\}^n\) as a collection of rules (i.e. a DNF formula). However, as we have noted, evaluating DNF formulas on \(\{0, 1, \ast\}^n\) examples is NP-complete. To circumvent this problem, a typical human expert response is likely to evaluate the formula as described above in the definition of the RUAV model. Thus, the RUAV model captures the situation where we are trying to acquire knowledge from such an expert.

Although the class of universal decision trees \(\mathcal{T}\) considered in the previous section can be learned in the UAV model using membership queries only, the size of the representation can be exponential in the size of the DNF representation. We now show that DNF formulas are learnable in the RUAV model.

**Theorem 12** The class of DNF boolean formulas can be exactly learned in the RUAV model using \(O(n^2m)\) time, \(O(n^2m)\) calls to the RUAV-MQ oracle, and at most \(m\) calls to the RUAV-EQ oracle where \(m\) is the number of terms in the target concept.

**Proof:** Our algorithm LearnDNF (see Figure 5) maintains a hypothesis \(h\) for the target concept \(f\) (where initially \(h = \text{false}\)) with the invariant that the terms of \(h\) are a subset of those of \(f\). Thus we do not receive any negative counterexamples and we never incorrectly predict positive (i.e. if \(h(x) = +\) then \(f(x) = +\)). LearnDNF processes a counterexample \(x\) as follows:

**Case 1:** Suppose \(f(x) = +\). Then we know that there is at least one term which \(x\) satisfies but is not present in \(h\). We use a subroutine ExtractTerm\((x)\) to construct \(t\) and replace \(h\) by \(h \lor t\).

**Case 2:** Suppose \(f(x) = -\). Since we never incorrectly predict positive, \(h(x) = -\). Thus some term \(t\) in \(f\) is missing from \(h\) where \(t(x) = +\). If \(f(x_{\ast \rightarrow 0}) = +\), the instance \(x_{\ast \rightarrow 0}\) satisfies \(f\) but not \(h\). Thus, as in Case 1, we construct \(t\) by calling the subroutine ExtractTerm\((x_{\ast \rightarrow 0})\). Now suppose \(f(x_{\ast \rightarrow 0}) = +\)
DNF LearnDNF

1 h := false
2 while RUAV-EQ(h) \neq "yes" do
3     Let \( x \) be the countereexample provided
4     if \( f(x) = + \) then \( h := h \lor \text{ExtractTerm}(x) \)
5     else if \( f(x_{\rightarrow 0}) = + \) then \( h := h \lor \text{ExtractTerm}(x_{\rightarrow 0}) \)
6     else
7         \( x^+ := \text{PositiveCompletion}(x) \)
8         \( h := h \lor \text{ExtractTerm}(x^+) \)
9 return \( h \)

Figure 5: The algorithm LearnDNF that learns the class of DNF formulas in the RUAV model using membership and equivalence queries. The model is defined in such a way that no evaluation oracle is needed.

Term ExtractTerm(\( x \))

1 \( t := true \)
2 for \( i = n \) downto 1
3     if RUAV-MQ(\( x_{\rightarrow i} \)) = + then \( x := x_{\rightarrow i} \)
4     else if \( x_i = 1 \) then \( t := t \land x_i \)
5     else \( t := t \land \overline{x_i} \)
6 return \( t \)

Figure 6: The algorithm ExtractTerm that takes a positive example \( x \) and constructs a term \( t \) in \( f - h \).

- Then we use PositiveCompletion(\( x \)) to construct a positive example \( y \) that is a completion of \( x \). We then call ExtractTerm(\( y \)) to construct \( t \).

Given an example \( x \) for which \( f(x) = + \), the procedure ExtractTerm (see Figure 6) determines a term \( t \) in \( f - h \) for which \( t(x) = + \). It achieves this goal by flipping as many attributes in \( x \) to \( * \) as possible without making the example non-positive. The remaining bits then specify \( t \). We now argue that ExtractTerm is correct. If \( f(x_{\rightarrow i}) = + \) then there still exists some term in \( f \) that is satisfied by \( x_{\rightarrow i} \) and hence we can replace \( x \) by \( x_{\rightarrow i} \). Now if \( f(x_{\rightarrow i}) = - \) then depending on the value of \( x_i \), either \( x_i \) or \( \overline{x_i} \) must appear in all the terms in \( f \) that are satisfied by \( x \). Thus after all attributes have been considered, there is only one term in \( f \) that satisfies \( x \). Otherwise some non-*
Figure 7: The algorithm PositiveCompletion that takes an example $x$ for which $f(x) = ?$ and $f(x \rightarrow 0) = -$ and constructs a positive example $y$ that is a completion of $x$.

It remains to show that given a "?" instance $x$ where $f(x \rightarrow 0) = -$ the algorithm PositiveCompletion (see Figure 7) constructs a positive instance $y$ that is a completion of $x$. Since $x$ is a "?" instance but $h(x) = -$, we know there is at least one term in $f - h$ not falsified by $x$. Denote the set of such terms by $T'$. Let $P$ be the set of positive literals from $T'$ that are unspecified in $x$. Since $f(x \rightarrow 0) = -$ we know that each term in $T'$ has at least one positive literal in $P$. The idea here is to find all the positive literals of one term in $T'$ and then derive a positive counterexample by setting these literals to 1 and the "*" bits to 0. PositiveCompletion begins with $y = x \rightarrow 0$ (and since PositiveCompletion was called we know that $f(y) = -$). Next, the unspecified bits of $x$, now 0 in $y$, are flipped back to "*" one at a time starting from $x_n$ until we find a "?" example $y$. This will eventually occur when we reach the variable $x_i$ where

$$i = \max_{t \in \mathcal{T}'} (\min \text{ index of the positive literals in } t).$$

After this, PositiveCompletion flips $y_i$ back to 1 and all the "*" bits to 0. Let $j_i$ denote the value of $i$ when we exit the for loop (via the goto in Line 9) for the $i$th time and let $\mathcal{L}_i$ be the set of literals in $P - \{x_{j_1}, \ldots, x_{j_i}\}$ that are also in $t$. Note that if all the positive literals $t$ are in $\{x_{j_1}, \ldots, x_{j_i}\}$ then the label of $y$ created after the $i$th execution of the for loop will be positive, and thus we will return $y$ in Line 2. Otherwise, for every term in $T'$ at least one positive literal is missing from $\{x_{j_1}, \ldots, x_{j_i}\}$. In this case,

$$j_{i+1} = \max_{t \in \mathcal{T}'} \min_{x_i \in \mathcal{L}_i} i$$

and hence we will reach a "?" example and go to Line 2.
Finally, since PositiveCompletion and ExtractTerm make $O(n^2)$ and $O(n)$ membership queries respectively, we have the desired result.

7 CONCLUSION

In this paper, we introduced a variant of Angluin’s [3] exact learning model where some of the attributes in the examples are unspecified. While we gave some initial results for this model, many interesting questions have been raised.

An interesting direction is to see if the class of DNF formulas is efficiently learnable in the UAV model using UAV-MQ and UAV-EQ oracles as well as an EV oracle. An easier question in this direction is to see if the class of DNF formulas with constant monotone dimension is learnable in the UAV model. While we can learn general decision trees in the UAV model (see Corollary 4), an EV oracle is needed since the hypothesis class is not a decision tree. Can general decision trees be properly learned in the UAV model, thus removing the need for an EV oracle?

Another direction would be to look at a variation of the UAV model in which there is a cost for obtaining the value of an initially unspecified attribute. There has been some research in this direction (such as the work of Greiner, Grove and Roth [25]). By allowing the cost to possibly be infinite this could model the situation in which some unspecified attributes could be obtained (at some given cost), yet the values of some unspecified attributes are just not available.

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References


