Negotiation as a Resource Allocation Process

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In the first part of this work we present a brief survey of the literature on resource allocation processes in which communication among agents plays a relevant role. Then, taking into account the results of AI research, we present a model of negotiation with a strong emphasis on the underlying linguistic structure. We use the properties of that structure to derive the conditions that message exchanges must obey in order to provide Pareto-optimal allocations.

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NEGOTIATION AS A RESOURCE ALLOCATION PROCESS*

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Abstract

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KEYWORDS: negotiation, allocations, Pareto optimality, games, communication, lattices

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1 Introduction

Economics has been defined as the study of the allocation of scarce resources to competing ends [Intrigilator 71]. Although this gives a narrow account of the discipline it is far from being an inaccurate definition. There can be several interpretations of this statement, based on different notions of scarcity and ends.

Mainstream Economic Theory interprets that the ends respond to the goals of individual rational agents, seeking in their interactions to maximize their preferences, under the restriction that the endowments they trade are finite. It is assumed that agents have all the information they need to perform this task. The main theoretical problem here is to determine the properties of the collective outcome in such a scenario.

Economics is a discipline with a strong normative tone, so it seems natural ask for properties of optimality of the outcome. Pareto provided at the beginning of this century a definition of optimality that is standard in Economic Theory: an outcome is optimal if there is no other outcome that makes every agent better off. Of course this notion of optimality has raised disputes among economists of different ideological standpoints, so it is recommended to use the term efficiency to describe it (despite this observation, we will use in this work the usual expression Pareto-optimality).

Additional structure is required in order to determine if an abstract economy achieves efficiency. This additional structure concerns the linguistic properties of the interactions among economic agents.

For the basic model of economic exchange two different kinds of interactions can be defined. The first approach, a very successful one, was developed by Walras. The idea is that economic agents do not interact unless a price is defined for each commodity. Moreover, given the set of prices (actually equilibrium prices) agents trade in an anonymous way as if they were able to deposit their endowments in a common heap and to take cut of the heap the quantities and kinds of commodities that have exactly the same value as their endowments. It can be shown that the final distribution of commodities is, in this setting, optimal.

An opposite line of thought was initiated by Edgeworth, in which interactions among agents are allowed, without requiring necessarily external variables like the prices. In this case, there can be a great variety of outcomes, all of them consistent with the total quantity of initial endowments.
The possible outcomes are determined by the interactions among agents in a process that can be described as multilateral bargaining: the agents form coalitions in order to block allocations that are not the best possible for their members. Given the rationality of the agents it can be ensured that the outcome is optimal in every possible case.

The indeterminacy of the Edgeworthian approach was considered a disadvantage in comparison with the Walrasian one. So it is no wonder that the latter constitutes the backbone of contemporary economic theory. Moreover, the development of Game Theory confirmed the existing intuitions about the clarity of the Walrasian model. Arrow and Debreu proved that a general equilibrium outcome is a *Nash equilibrium* in the representation of exchange economies as non-cooperative games.

The Edgeworthian approach was seen as an instance of the general game-theoretic problem of determining the outcome of a bargaining process. Two main approaches were applied to the analysis of bargaining, one of them a purely axiomatic characterization of the outcome, studied by Nash in the '50s and the other in which bargaining is seen as a sequential process, analyzed by Rubinstein in the '80s.

The importance of the Edgeworthian approach relies on its relations with other theoretical models in which conflict and cooperation are deeply intertwined. To give a single example, it has been noted that economic development seems to arise with institutional arrangements regulating property rights among individuals in a society [North 86]. The birth of these institutions is due to processes of communication among agents trading intangible commodities like political support or legal entitlements.

This generalization falls short of a more general view of *negotiation processes* as resource allocators. The reason for that is that solutions to bargaining problems do not illuminate the *linguistic* and *computational* features of these processes. Only behavioral economics has in a certain sense highlighted these aspects [Simon 82].

On the other hand, Computer Science and particularly Artificial Intelligence as a quite autonomous discipline that studies representations of *intentional agents* shows the way to determine interactions among them, taking into account their intrinsic linguistic nature and the computational limitations inherent to these processes. Several *multiagent systems* have been developed, with special emphasis on the linguistic features of the communications among agents [Bond-Gasser 88]. Very general properties have been shown
as relevant for these systems shedding light on the computational aspects of negotiation.

The goal of this work is twofold: in the first place it is to present a survey of the relevant ideas in this area and second, to propose a model of resource allocation incorporating the insights provided by Artificial Intelligence. In this model, the agents engaged in a negotiation go through rounds of message exchange. Given any such exchange, the agents revise their beliefs about the other agents and about their own possibilities, deciding what message they should emit in order to obtain the best outcome given their beliefs in that stage of the negotiation.

The analysis of the proposed model is based on the properties of fixed-points in a non-canonical ordering of the power set of possible messages. It will be shown that certain monotonic properties of policies of negotiation (called protocols) ensure the existence of Pareto-optimal outcomes of the negotiation. Besides, the dependency of the outcomes on the beliefs of agents is shown as being relevant to determining desirable protocols.

In section 2 we will discuss the Walrasian and Edgeworthian approaches to exchange economics in the light of Game Theory. In section 3 we will review the problems of bargaining and of implementation of social choice functions. In section 4 a characterization of economic systems as message passing systems will be given, and certain results of the theory of multi-agent system will be applied to suggest what conditions should be added in the definition of economic systems in order to represent adequately a negotiation context. In section 5, a formal model of negotiation is presented and its properties discussed, particularly those that allow the existence of Pareto-optimal allocations. Finally in section 6 the conclusions and future lines of work are outlined.
2 Resource allocation and Game Theory

The standard economic model represents interactions among economic agents exclusively through the price system: every agent makes decisions based only on a globally available variable. Direct interactions among agents are not considered. Following Hildenbrandt and Kirman ([Hildenbrandt- Kirman 88]) we can call this model Walrasian and confront it with the Edgeworthian model, in which interactions among agents are given by trades. This latter model provides a basis for the notion of the core of an economy.

In Economics the formal setting for discussing interaction among agents is Game Theory, Any economic model in which interactions among agents determine the outcome seems to be able to be embedded in a game-theoretic setting. So, for example, both models, the Walrasian and the Edgeworthian, can be presented game-theoretically. The first one as a non-cooperative game (because no communication exists among agents) and the second as a cooperative game because it can be seen in coalitional terms. To be more precise lets introduce the main elements that will be used in both approaches [Blad-Keiding 90]:

Definition 1 An economic agent i can be represented by a tuple $(X_i, u_i, w_i)$ where $X_i \subseteq \mathbb{R}^l$ is a subset of the Euclidean l-dimensional space of commodities such that:

- $X_i \neq \emptyset$
- $X_i$ is bounded below
- $X_i$ is closed
- if $x_1 \in X_i$ and $\exists x_2, x_1 \leq x_2$ then $x_2 \in X_i$
- $X_i$ is convex
- $w_i \in X_i$ is finite
- $u_i : X_i \rightarrow \mathbb{R}$ is a quasi-concave, continuous utility function
- $w_i \in X_i$ is i's initial endowment.

We assume also that if $x < y$ then $u_i(x) < u_i(y)$ (monotonicity).
The condition of quasi-concavity of utility functions is very strong and is not derivable from considerations on the rationality of the agents. It means that for any $\alpha$, the set $\{x \in X_i : u_i(x) \geq \alpha\}$ is convex. The framework in which the agents interact is the following:

**Definition 2** A pure exchange economy is a tuple  
$$\varepsilon = ((X_i, u_i, w_i)_{i=1}^n, w)$$
where $n$ is the number of consumers in the economy and $w = \sum_{i=1}^n w_i$.

The notions of allocation and Pareto-optimal allocations are the following:

**Definition 3**
- An allocation is a vector $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for $i = 1, \ldots, n$; it is feasible if $\sum_{i=1}^n x_i \leq w$.
- An allocation $(x_1, \ldots, x_n)$ is Pareto-optimal if it is feasible and there is no other allocation $(x'_1, \ldots, x'_n)$ such that for all $i$, $u_i(x'_i) \geq u_i(x_i)$ and for at least one $i^*$, $u_{i^*}(x'_{i^*}) > u_{i^*}(x_{i^*})$.

### 2.1 The Walrasian Approach

In the Walrasian setting, there is an additional set of variables, the prices, $p \in \mathbb{R}_+^l = \{r : r \in \mathbb{R}^l, r \geq 0\}$, such that the problem for each agent becomes:

**Definition 4** Determine $x^0_i \in X_i$ such that $x^0_i$ maximizes locally $u_i$ on the budget set $\{x \in X_i : p \cdot x \leq p \cdot w_i\}$ where $p$ is the vector of given prices.

And for the whole economy:

**Definition 5** A market equilibrium is a vector $(x^0_1, \ldots, x^0_n, p)$ such that $(x^0_1, \ldots, x^0_n)$ is feasible and $\forall i$, $x^0_i$ maximizes $u_i$ on the set $\{x_i \in X_i : p \cdot x_i \leq p \cdot w_i\}$.

We can see the problem of the existence of a market equilibrium in game-theoretical terms:

**Definition 6** An exchange economy $\varepsilon$ can be seen as a game $\varepsilon = ((S_i)_{i=1}^{n+1}, g)$, where $S_i = \{x \in X_i : p \cdot x \leq p \cdot w_i\}$ for $i = 1 \ldots n$;
\[ S_{n+1} = \{ p \in R_+^n : \sum_{j=1}^n p_j = 1 \} \text{ and } g : \prod_{i=1}^n S_i \rightarrow R^n, \text{ such that for each } (x_1, \ldots, x_n) \in \prod_{i=1}^n S_i, \]
\[ g(x_1, \ldots, x_n, p) = (u_1(x_1), \ldots, u_n(x_n), p \cdot (\sum_{i=1}^n w_i - \sum_{i=1}^n x_i)) \]

The \((n+1)\)-player is a representation of the "Walras auctioneer", a fictitious player whose role in the game is choosing prices, reconfiguring the set of strategies of the agents. His set of strategies is confined to normalized prices due to the fact that given \( p \in S_{n+1} \) and \( \lambda \in R_+ \), for each \( i \), \( \{ x \in X_i : p \cdot x \leq p \cdot w_i \} = \{ x \in X_i : \lambda p \cdot x \leq \lambda p \cdot w_i \} \). The payoff of the fictitious player is the value of the excess supply of the economy, that is the difference between the global endowments and the sum of the choices of the agents (the global demand).

As we assume that there is no communication nor commitment among players, \( \varepsilon \) is a non-cooperative game, and the classical solution for this kind of games is:

**Definition 7** A Nash equilibrium for a game \( G = ((S_i)_{i=1}^{n+1}, g) \) is \((s_1, \ldots, s_{n+1}) \in \prod_{i=1}^{n+1} S_i, \text{ such that for every } i, \text{ and for any other } s_i' \in S_i, \]
\[ g_i(s_1, \ldots, s_i', \ldots, s_{n+1}) \leq g_i(s_1, \ldots, s_i, \ldots, s_{n+1}) \]

That is, for each \( i, s_i \) is the best answer to \((s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n+1})\).

The sufficient conditions for the existence of a Nash equilibrium in a game [Blad-Keiding 90]:

**Theorem 1** A game \( G = ((S_i)_{i=1}^{n+1}, g) \) has a Nash equilibrium if the following conditions are valid:

- for each \( i, S_i \) is a non-empty, compact and convex set
- for each \( i, g_i \) is a continuous and quasi-concave function

**Proof 1** Lets consider the global best response correspondence,
\[ \Pi : \prod_{i=1}^{n+1} S_i \rightarrow \prod_{i=1}^{n+1} S_i \]
such that for each $i$, $\Pi_i(s_1, \ldots, s_{n+1}) = s^*_i \in S_i$ such that for any other $s'_i \in S_i$, $g_i(s_1, \ldots, s'_i, \ldots, s_{n+1}) \leq g_i(s_1, \ldots, s^*_i, \ldots, s_{n+1})$. Given the conditions of the theorem it is immediate that $\Pi_{i=1}^{n+1} S_i$ is non-empty, compact and convex.

By the continuity of the $g_i$ it can be shown that if $\{s^i\}_{i=1}^\infty$ is a sequence with limit $\hat{s}$ and $\{\pi^i : \pi^i \in \Pi(s^i)\}_{i=1}^\infty$ a sequence with limit $\hat{\pi}$, then $\hat{\pi} \in \Pi(\hat{s})$. This condition and the boundeness of the images under $\Pi$ (obvious since $\text{Im}(\Pi)$ is a compact set in an Euclidean space) imply that $\Pi$ is upper hemicontinuous (a correspondence $f : X \to Y$ is upper hemicontinuous if its graph $\{(x, y) : y \in f(x)\}$ is closed).

Finally, since for every $s \in \Pi_{i=1}^{n+1} S_i$, $\Pi(s)$ maximizes $(g_1(s), \ldots, g_n(s))$ and the $g_i$ are quasi-concave, $\Pi(s)$ is convex.

Therefore, $\Pi$ is a upper hemicontinuous and convex correspondence on a non-empty, convex and compact set and its image coincides with its domain. By Kakutani's fixed point theorem (for a proof see [Balasko 88]), there must exist an $(s^*_1, \ldots, s^*_{n+1})$ such that $(s^*_1, \ldots, s^*_{n+1}) \in \Pi(s^*_1, \ldots, s^*_{n+1})$. It follows trivially that this vector is a Nash equilibrium.

It is easy to see that

**Lemma 1** The game $\varepsilon$ has a Nash equilibrium

**Proof 1** Each $S_i$ is a closed and bounded convex set in the Euclidean space $R^l$, therefore it is compact and convex. For $i = 1 \ldots n$, $g_i \equiv u_i$, which is a continuous and quasi-concave function. $g_{n+1}$ is, by its definition, a linear functional, therefore continuous and trivially quasi-concave. By the Theorem 1, $\varepsilon$ has a Nash equilibrium

The resulting equilibrium can be written in more convenient terms as $(x^*_1, \ldots, x^*_n, p^*)$ and it has by definition all the properties of a market equilibrium:

**Lemma 2** The Nash equilibrium $(x^*_1, \ldots, x^*_n, p^*)$ is a market equilibrium

**Proof 2** It is clear that for each $i$, $x^*_i$ maximizes $u_i$ on the set \{\(x_i \in X_i : p^* \cdot x_i \leq p^* \cdot w_i\)\}: suppose that for $i_0$,

$$\exists x_{i_0}^* \in \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot w_i\}$$
such that \( u_{i_0}(x^*_{i_0}) > u_{i_0}(x^*_{i_0}) \). Then

\[
(u_1(x^*_{1}), \ldots, u_{i_0}(x^*_{i_0}), \ldots, u_n(x^*_{n}), p^* \cdot (w - \sum_{i=1}^{n} x^*_i)) > \\
(u_1(x^*_{1}), \ldots, u_{i_0}(x^*_{i_0}), \ldots, u_n(x^*_{n}), p^* \cdot (w - \sum_{i=1}^{n} x^*_i))
\]

contradicting that \((x^*_1, \ldots, x^*_i, \ldots, x^*_n, p^*)\) is a Nash equilibrium. Absurd.

Now, let's see that \( p^* \cdot (x^*_1, \ldots, x^*_n) \leq p^* \cdot w \) (a property that resumes the global aspects of an equilibrium). Suppose that \( p^* \cdot (x^*_1, \ldots, x^*_n) > p^* \cdot w \), then, for each \( j = 1 \ldots l \) such that \( x_j > w_j \), be \( p_j' > p_j^* \) and for each \( j \) such that \( x_j \leq w_j \) be \( p_j \leq p_j^* \) (obviously in such a way that \( \sum_{j=1}^{l} p_j' = 1 \)). Therefore \( p' \cdot (w - \sum_{i=1}^{n} x^*_i) > p^* \cdot (w - \sum_{i=1}^{n} x^*_i) \), and so \( (u_1(x^*_{1}), \ldots, u_n(x^*_{n}), p' \cdot (w - \sum_{i=1}^{n} x^*_i)) > (u_1(x^*_{1}), \ldots, u_n(x^*_{n}), p^* \cdot (w - \sum_{i=1}^{n} x^*_i)) \) contradicting that \((x^*_1, \ldots, x^*_i, \ldots, x^*_n, p^*)\) is a Nash equilibrium. Absurd.

Therefore \((x^*_1, \ldots, x^*_i, \ldots, x^*_n, p^*)\) is a market equilibrium.

To complete the proof it has to be shown that \( p^* \in \text{Int}(S_{n+1}) \), as to avoid the possibility of the existence of a commodity \( j \) such that \( p_j^* = 0 \). The proof of this point is on the basis of the monotonicity of the utility function and does not add to the main argumentation of this work (for the details see [Debreu 59]). More relevant in this sense is to show that, given a market equilibrium, it supports a Pareto-optimal allocation:

**Proposition 1** If \((x^*_1, \ldots, x^*_n, p^*)\) is a market equilibrium, the associate allocation \((x^*_1, \ldots, x^*_n)\) is Pareto-optimal

**Proof 1** Suppose that \((x^*_1, \ldots, x^*_n)\) is not Pareto-optimal. Then, there will exist a feasible allocation \((x'_1, \ldots, x'_n)\) such that for all \( i \), \( u_i(x'_i) \geq u_i(x^*_i) \) and for at least one \( i_0 \), \( u_{i_0}(x'_i) > u_{i_0}(x^*_{i_0}) \). Then, as \( x^*_{i_0} \) maximizes \( U_{i_0} \) on \( \{ x_{i_0} \in X_{i_0} : p^* \cdot x_{i_0} \leq p^* \cdot w_{i_0} \} \), \( p^* \cdot x_{i_0} > p^* \cdot w_{i_0} \). Therefore, \( \sum_{i=1}^{n} p^* \cdot x_i > \sum_{i=1}^{n} p \cdot w_i \), but as \((x'_1, \ldots, x'_n)\) is feasible, \( \sum_{i=1}^{n} x'_i \leq \sum_{i=1}^{n} w_i \) and so \( \sum_{i=1}^{n} p^* \cdot x_i \leq \sum_{i=1}^{n} p^* \cdot w_i \) (recall that for all \( j \), \( p_j^* \neq 0 \)). Absurd. Then \((x^*_1, \ldots, x^*_n)\) is a Pareto-optimal allocation.
2.2 The Edgeworthian Approach

In this case we consider that the agents can engage in any possible agreement if it is advantageous for them. A subset $C \subseteq \{i\}_{i=1}^n$, $C \neq \emptyset$ is called a coalition. The advantageous agreements can be determined using the following characterization [Shubik 84):

**Definition 8** A coalition $C$ blocks a feasible allocation $(x_1, \ldots, x_n)$ if for every $i \in C$ there exists a $x'_i \in X_i$ such that:

- $u_i(x'_i) \geq u_i(x_i)$ and for at least one $i_0 \in C$, $u_{i_0}(x'_{i_0}) > u_{i_0}(x_{i_0})$

- $\sum_{i \in C} x'_i = \sum_{i \in C} w_i$

This means that given a feasible allocation, it is blocked by a coalition if its members can be better off redistributing their own endowments among themselves. This defines a particular set of feasible allocations, the core:

**Definition 9** A feasible allocation $(x_1, \ldots, x_n)$ is in the core $\mathcal{CO}$ if there is no coalition $C$ that can block $(x_1, \ldots, x_n)$.

The core is therefore the set of allocations such that no coalition can be formed such that their members are better off redistributing their endowments. In other words, is the set of allocations for which a collective agreement can be reached. For every coalition $C$ we can define the set of blocked allocations:

**Definition 10** Given a coalition $C$, the set of blocked allocations by $C$ is:

$\hat{C} = \{(x_1, \ldots, x_n) : \exists (x^*_1, \ldots, x^*_n); \forall i \in C, u_i(x^*_i) \geq u_i(x_i), \text{and}, \sum_{i \in C} x^*_i = \sum_{i \in C} w_i\}$

The set of blocked allocations for $\{i\}_{i=1}^n$ may be called $\hat{I}$. A definition that will be useful is the following:

**Definition 11** A family of coalitions $\{C_1, \ldots, C_n\}$ is said balanced, if there exist non-negative numbers, for each $C_j$ $c_j \in [0,1]$ such that, given $i$, $\sum_{(j: i \in C_j)} c_j = 1$

An important characterization is the following:

**Definition 12** $\{(i)^n_{i=1}, X^n, C\}$ is said a cooperative game, where:
\textbullet \ \{i\}_{i=1}^{n} \text{ is the set of players}

\textbullet \ X^{n1} \text{ is the set of feasible outcomes of the game}

\textbullet \ C : 2^{\{i\}_{i=1}^{n}} \rightarrow X^{n1} \text{ is a correspondence that for each coalition } C \text{ assigns its set of blocked allocations } \hat{C}

\{(i\}_{i=1}^{n}, X^{n1}, C) \text{ is a balanced game if for every balanced family of coalitions } \{C_{1} \ldots C_{q}\}, \cap_{j=1}^{q} \hat{C}_{j} \subseteq \hat{I}

We can show that:

\textbf{Proposition 2} \textbf{ The exchange economy } \varepsilon \textbf{ constitutes a balanced game} \\

\textbf{Proof 2} \textbf{ It is clear that } \varepsilon \textbf{ is a cooperative game as defined above and the set of feasible outcomes is of course}

\[
\{(x_{1}, \ldots, x_{n}) : \forall i, x_{i} \in R^{l}, \text{and}, \sum_{i=1}^{n} x_{i} \leq \sum_{i=1}^{n} w_{i}\}
\]

Every subset of \(2^{\{i\}_{i=1}^{n}} \) is a coalition and in particular \(\{C_{1} \ldots C_{q}\} \) such that \(\cup_{j=1}^{q} C_{j} = \{i\}_{i=1}^{n} \), and for every \(j, |C_{j}| = m, m \leq n \), is balanced since every \(i \) appears in \(n-1 \) coalitions, and \(c_{j} = 1/(n-1) \). From the several balanced families, let consider any one of them \(\{C_{1} \ldots C_{q}\} \) and \((x_{1}, \ldots, x_{n}) \in \cap_{j=1}^{q} \hat{C}_{j}\). Since \((x_{1}, \ldots, x_{n}) \in \hat{C}_{j} \) there is a \((x_{1}^{j}, \ldots, x_{n}^{j})\), such that for every \(i \in C_{j}, u_{i}(x_{i}^{j}) \geq u_{i}(x_{i}) \), and, \(\sum_{i \in C_{j}} x_{i}^{j} = \sum_{i \in C_{j}} w_{i} \). We now define a new allocation \((x_{1}^{*}, \ldots, x_{n}^{*}) = \sum_{i \in C_{j}} c_{j}(x_{1}^{j}, \ldots, x_{n}^{j})\) By the quasi-concavity of the \(u_{i}\)'s we have that \(u_{i}(x_{i}^{*}) \geq u_{i}(x_{i}) \). On the other side:

\[
\sum_{i=1}^{n}(x_{1}^{*}, \ldots, x_{n}^{*}) = \sum_{i=1}^{n} \sum_{C_{j} \in C_{j}} c_{j}(x_{1}^{*}, \ldots, x_{n}^{*}) = \sum_{j=1}^{q} c_{j}(\sum_{i \in C_{j}} x_{i}^{j}, \ldots, x_{n}^{j}) = \sum_{j=1}^{q} c_{j}(\sum_{i \in C_{j}} w_{i}) = \sum_{i=1}^{n} w_{i}
\]

This result shows that \((x_{1}, \ldots, x_{n}) \in \hat{I}\). Therefore \(\varepsilon \) is a balanced game.

An important result is the following theorem of Scarf:

\textbf{Theorem 2} \textbf{ The core of a balanced game } \((\{i\}_{i=1}^{n}, X^{n1}, C) \) \textbf{ is not empty}
Proof 2 See [Moulin 88].

It is immediate the following:

Lemma 3 The exchange economy ε has a non-empty core

Proof 3 Immediate consequence of Proposition 2 and Theorem 2

CO has a close relation with the set of Pareto-optimal allocations:

Proposition 3 If \((x_1, \ldots, x_n) \in CO\) then \((x_1, \ldots, x_n)\) is a Pareto-optimal allocation

Proof 3 Suppose \((x_1, \ldots, x_n)\) is not Pareto-optimal, then there exists another feasible allocation \((x'_1, \ldots, x'_n)\) such that \(u_i(x'_i) \geq u_i(x_i)\) for all \(i\) and for at least a \(i_0\), \(u_{i_0}(x'_{i_0}) > u_{i_0}(x_{i_0})\). We can therefore consider an alternative allocation obtained as follows: let \(r_i = (x'_{i_0} - x_{i_0})/2(n - 1)\) for \(i \neq i_0\) and \(r_{i_0} = (x'_{i_0} - x_{i_0})/2\). The new allocation is \((x''_1, \ldots, x''_{i_0}, \ldots, x''_n) = (x'_1 + r_1, \ldots, x_{i_0} + r_{i_0}, \ldots, x'_n + r_n)\). Evidently \(\sum_{i=1}^n x''_i = \sum_{i=1}^n x'_i\), and therefore \((x''_1, \ldots, x''_{i_0}, \ldots, x''_n)\) is feasible and now for every \(i\) \(u_i(x''_i) > u_i(x_i)\) (since \(x''_i > x_i\) and the \(u_i\)'s are continuous and monotone). That is, \(\{i\}_{i=1}^n\) blocks \((x_1, \ldots, x_n)\) which contradicts that \((x_1, \ldots, x_n) \in CO\). Absurd. Therefore \((x_1, \ldots, x_n)\) is Pareto-optimal.

So, in this approach, a global outcome obtains that is again Pareto-optimal, but this time considering the possible cooperative interactions among agents.
3 Bargaining and Implementation

The Edgeworthian approach above was analyzed in cooperative game-theoretic terms. The resulting resource allocation can be seen as the result of a consensus among several groups of agents. One of the main open problems in Game Theory is to provide a foundation for the basic entities of cooperative game theory, the coalitions, based in individual behavior. That is, to provide a rationale for coalitional behavior, based only in the individual characteristics of the players. An instance of this problem is to give a solution to the following one, that represents the basic problem of allocation of resources by consensus (the edgeworthian exchange economics is a particular case):

Definition 13 A Bargaining Problem arises when a set of agents $I = \{i\}_{i=1}^{n}$ must decide on the allocation of resources, $(x_1, \ldots, x_n) \in \bar{X} \subset \prod_{i=1}^{n} X_i$, given a status quo allocation $(w_1, \ldots, w_n)$ ($\bar{X} = \{(x_1, \ldots, x_n) : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i\}$). A bargaining solution is a rule that assigns a solution vector $(x_1^{*}, \ldots, x_n^{*})$ to a bargaining problem.

Two main solutions have been given. The oldest one is in axiomatic terms and the newer one can be described in terms of a refinement of Nash equilibrium. We will analyze both to show the ideas applied in the solution of the bargaining problem.

3.1 The axiomatic approach

The classical form to solve a bargaining problem is terms of an axiomatic characterization of the potential solution. J.Nash proposed a few axioms [Nash 50]:

- the bargaining solution is invariant to independent changes of units: if $(x_1^{*}, \ldots, x_n^{*})$ is a solution under the set of utility functions $\{u_i\}_{i=1}^{n}$, it is also a solution under $\{\beta_i u_i\}_{i=1}^{n}$, where for each $i$, $\beta_i \in R^+$

- the bargaining solution satisfies the Pareto property: there is no $(x_1, \ldots, x_n)$ such that for every $i$, $u_i(x_1^{*}, \ldots, x_n^{*}) \geq u_i(x_1, \ldots, x_n)$, with strict inequality for at least an $i_0$
• the bargaining solution satisfies the property of symmetry: if \( \prod_{i=1}^{n} \hat{X}_i \)
 is a symmetric set (that is, it remains unaltered by rotations around \((w_1, \ldots, w_n)\))
 then \( u_i(x_i^*) - u_i(w_i) = u_j(x_j^*) - u_j(w_j) \), for every \( i, j \in I \)

• the bargaining solution satisfies the property of individual rationality: for every \( i \),
   \( u_i(x_i^*) \geq u_i(w_i) \)

• the bargaining solution satisfies the property of independence of irrelevant alternatives: given
   the set of alternatives \( \prod_{i=1}^{n} \hat{X}_i \) if there is a \( \prod_{i=1}^{n} X'_i \subseteq \prod_{i=1}^{n} \hat{X}_i \), such that the solution of the bargaining
   problem on \( \prod_{i=1}^{n} \hat{X}_i \), \( (x_1^*, \ldots, x_n^*) \in \prod_{i=1}^{n} X'_i \), then \( (x_1^*, \ldots, x_n^*) = (x_1^*, \ldots, x_n^*) \),
   where \( (x_1^*, \ldots, x_n^*) \) is the solution to the bargaining problem on \( \prod_{i=1}^{n} X'_i \)

Implicit in this solution is a geometric approach to the bargaining problem, in such a way that
the set of possible outcomes is a Pareto frontier of a set that contains properly the status quo point
\((w_1, \ldots, w_n)\). It preserves its shape with changes in units, and lies “above” the status quo. Moreover,
if the solution point to the problem is the only point in common with a set that represents another bargaining problem, that point is a solution point.

Nash proved the following theorem:

**Theorem 3** Given a bargaining problem on \( \hat{X} = \prod_{i=1}^{n} \hat{X}_i \), and the axioms
above, a solution obtains maximizing the function

\[ (u_1(x_1) - u_1(w_1)) \times \ldots \times (u_n(x_n) - u_n(w_n)) \]

on \( \hat{X} \). Moreover, this solution is the only one that verifies the axioms.

**Proof 3** It is clear that \( \prod_{i=1}^{n} \hat{X}_i \) is a compact set (it is closed and bounded in an Euclidean space). The function

\[ (u_1(x_1) - u_1(w_1)) \times \ldots \times (u_n(x_n) - u_n(w_n)) \]

is continuous and quasi-concave (by the properties of the individual utility functions). By Weierstrass theorem, there exists a solution lying in the frontier of \( \hat{X} \). The Pareto and symmetry properties obtain immediately. The independence of irrelevant alternatives is a consequence of the strict concavity of \( (u_1(x_1) - u_1(w_1)) \times \ldots \times (u_n(x_n) - u_n(w_n)) \). Finally, the invariance to changes in units is a consequence of the form of the function to be maximized: every \( x_i^* \) must be a maximum of \( u_n(x_n) - u_n(w_n) \) on the compact set \( \hat{X} \), so it must be also a maximum for \( \beta_i(u_n(x_n) - u_n(w_n)) \).
3.2 A non-cooperative solution

Rubinstein solved the following version of the bargaining problem [Rubinstein 82]:

Definition 14 (Alternate Offers) \( I = \{a, b\}, w = w_a + w_b \in R \) is a quantity to be split (the sum of the initial endowments). Time is considered discrete and the rules are:

- \( t = 1 \): Player \( a \) offers \( x_i^1 \in [0, w] \). Player \( b \) can do the following: accept and the game finishes with outcome \( (w - x_i^1, x_i^1) \) or rejects the offer and the game continues.

- \( t = 2 \): Player \( b \) offers \( x_i^2 \in [0, w] \). Player \( a \) can do the following: accept and the game finishes with outcome \( (x_i^2, w - x_i^2) \) or rejects the offer and the game continues.

- ... 

- \( t = 2m + 1 \): Player \( a \) offers \( x_i^t \in [0, w] \). Player \( b \) can do the following: accept and the game finishes with outcome \( (w - x_i^t, x_i^t) \) or rejects the offer and the game continues.

- \( t = 2m \): Player \( b \) offers \( x_i^{t+1} \in [0, w] \). Player \( a \) can do the following: accept and the game finishes with outcome \( (x_i^{t+1}, w - x_i^{t+1}) \) or rejects the offer and the game continues.

- ... 

An additional condition is that if the agreement is reached in time \( t \) each one receives a share such that its utility is \( \delta^{t-1}u_i(x_i^t) \), where \( \delta \in (0, 1) \) is a discount factor and \( u_i \) is a linear utility function.

In order to present Rubinstein’s result we have to give several definitions [Mas-Colell et alt. 95] (this fundamental text also served as a source for unified notation in this work):

Definition 15 \( \Gamma = \langle T, A, I, pr, \alpha, I, inf, prob, u \rangle \) is said a game in extensive form, where the components are:

- A finite set of players \( I \) (including a fictitious player 0, Nature)
• A tree $T = < N, r >$, where $N$ is the set of nodes and $r$ is the root node

• A set of actions $A$

• A function $pr : N \rightarrow \{N \cup \emptyset\}$, assigning to each node its immediate predecessor. The successor correspondence is $suc \equiv pr^{-1}$.

• A function $\alpha : N - \{r\} \rightarrow A$ that to each non-root node assigns an action that lead to it from its immediate predecessor. If $n_1, n_2 \in suc(n_0), n_1 \neq n_2$, then $\alpha(n_1) \neq \alpha(n_2)$. The choices at $n_0$ are $c(n_0) = \{a \in A : a = \alpha(n'), n' \in suc(n_0)\}$

• A class of information sets $I$, such that there is a function, $inf : N \rightarrow I$, that assigns to a node $n_0$ an information set $inf(n_0)$. The information sets determine a partition of $N$. The nodes in an information set have the same choices: if $inf(n_1) = inf(n_2)$ then $c(n_1) = c(n_2)$

• A function $P : I \rightarrow I$, assigning an information set only to a single player. So the collection of information sets of player $i$ is $I_i$

• A function $prob : I_0 \times A \rightarrow [0,1]$ assigning probabilities to actions in information sets where nature moves (for each one of these information sets the sum of the probabilities should be 1)

• A payoff function $(u_1, \ldots, u_n)$ assigning an utility to each leaf node.

This definition tries to encompass the fact that the players take turns to make their moves and their choices depend on how well do they know the sequence of previous moves. Their imperfect information, obvious in games with chance or with simultaneous or private moves, is represented by means the information sets (the nodes in an information set $I_i$ are not distinguishable for $i$). The strategies in the game can be defined as follows:

**Definition 16** Given the information sets of a player $i$, $I_i$, a strategy for player $i$ is a function $s : I_i \rightarrow A$, such that $s(n_0) \in c(n_0)$

The payoff can be extended to have as a domain the strategies sets. So, for a vector of strategies $(s_1, \ldots, s_n)$, $u_i(s_1, \ldots, s_n) = u_i(n)$, where $n$ is a leaf node such that it is the outcome of the play in which each $i$ uses her strategy $s_i$
This means that a strategy assigns to every information set an element in the set of choices available at that information set. Any vector of strategies determines a path from the root to a leaf in the tree. Precisely, the tree structure of the game, suggests the definition of subgame:

Definition 17 A subgame $\Gamma_S$ of a game $\Gamma$ is a subset of $\Gamma$ with all the characteristics of an extensive form game and with the following additional properties:

- Its root is an information set with a single node, and it includes only all the the nodes that are immediate successors and the later successors of this node.

- If there is a node $n$ in the subgame, then every $n' \in \text{inf}(n)$ is such that $n' \in \Gamma_S$

A refinement of Nash equilibrium that does not allow the existence of equilibria with non-credible strategies (ones that are irrational to play) is the following:

Definition 18 A vector of strategies $(s_1, \ldots, s_n)$ in an extensive form game $\Gamma$ is a Nash equilibrium perfect in subgames if in every subgame $\Gamma'$, for all $i$, $u_i(s^*_i, \ldots, s^*_i, s_n) \geq u_i(s^*_i, \ldots, s_i, \ldots, s_n)$, for any other $s_i$

That is, for a game $\Gamma$ a vector of strategies is a Nash equilibrium perfect in subgames if it determines a path, that corresponds to an equilibrium in every subgame.

The solution given by Rubinstein to the alternate offers version of the bargaining problem was the following:

Proposition 4 The Alternate Offers game has a unique Nash equilibrium perfect in subgames: $x_a = w/(1 + \delta)$; $x_b = (\delta w)/(1 + \delta)$, attained immediately in period 1.

Proof 4 The symmetry of the problem is such that the subgame initiated in period $t + 1$ is exactly the same (with the roles of the players reversed) as the subgame initiated in $t$. So, if for player $a$, $x_a$ is the largest share that a can get in any Nash equilibrium perfect in subgames, it follows that $x_b = \delta x_a.$
where \( \hat{x}_b \) is the highest share that \( b \) can obtain in any subgame in which she can make the first offer (in particular in the subgame beginning in period 2 after rejecting a's offer). In the same sense, in the subgame that begins in period 1, a cannot get a lesser share that \( x^*_a = w - \delta \hat{x}_a \) (if a could get less than that, he could offer \( b \) in period 1 a share greater that \( \delta \hat{x}_a \), absurd because that is the highest share that \( b \) can obtain). On the other hand, by the assumption of rationality of the players, the best that player a can get in period 1 is \( w - \delta x^*_a \) (because if she makes an offer in which she gets more, \( b \) can reject it and obtain at least \( \delta x^*_a \) in her offer in period 2). So, given the definitions, this implies that \( \hat{x}_a = x^*_a \). So, \( \hat{x}_a = w - \delta \hat{x}_a \). This means that \( \hat{x}_a = w/(1 + \delta) \), being the share that a gets in a Nash equilibrium perfect in subgames, offering \( (\delta w)/(1 + \delta) \) to \( b \) in period 1. The uniqueness follows from the fact that it is the only result that \( a \) (and \( b \)) can obtain.

This solution is the same as in the axiomatic case, when \( w_a = w_b = 0 \) (that is, the status quo is no player having initial endowments), and \( \delta = 1 \) (the utility of the outcome is the same in all periods). With these features, the problem becomes almost trivial: the players should split a quantity they “found” and they can negotiate indefinitely until reaching an agreement. And the agreement is of course to split the quantity in equal parts.

### 3.3 Implementation of Social Choice Functions

Both approaches to the bargaining problem (non-cooperative or axiomatic) are concerned with finding ways to determine the outcome to a process, without discussing the associated communication process among agents. A problem related with bargaining, and in which solutions depend on certain features of that process, is that of a social outcome based on the characteristics of the agents [Sen 86]:

**Definition 19** Given a set \( I = \{i\}_{i=1}^n \) of agents, such that each agent \( i \) has preferences determined by his type \( \theta_i \in \Theta_i \), and a set of possible alternatives \( X \), the Social Choice Function is a function \( f : \Theta_1 \times \ldots \times \Theta_n \rightarrow X \) such that for every profile of types it assigns a collective outcome in \( X \).

The type of an agent can be conceived as a signal that determines the form of his utility function (in all this subsection we write \( u_i(x) \) for the utility of
outcome $x$ for agent $i$ as a shorthand for the actual utility function $u_i(x, \theta_i)$, that has as argument also the type of $i$). Moreover, this signal is private, in the sense that only the agent can observe it. The function $f$ must be implemented by means a mechanism, that is [Myerson 94]:

**Definition 20** A mechanism is a game form (of incomplete information, because of the existence of private information) $\Gamma = (S_1 \ldots S_n, p(\cdot))$ where:

- the $S_i$ are the strategy sets, that is, sets of possible choices for each agent. A strategy $s_i$ is a function $s_i : \Theta_i \to S_i$.
- $p$ is the outcome function, $p : S_1 \times \ldots \times S_n \to X$.

$\Gamma$ implements the Social Choice Function $f$ if there is an equilibrium $(s_1^* \ldots s_n^*)$ such that $p(s_1^*(\theta_1) \ldots s_n^*(\theta_n)) = f(\theta_1 \ldots \theta_n)$ for all $(\theta_1 \ldots \theta_n) \in \Theta_1 \times \ldots \times \Theta_n$.

An admissible interpretation of the elements of the $S_i$s is as “declarations”, where each $s \in S_i$ is a sentence. The simultaneous declarations of all the players would determine the social outcome. A social choice function becomes implemented if there is a “declarations policy” for each agent, such that all of them collectively determine the outcome of the social choice for every possible distribution of types among agents.

The simplest case of mechanism is of course the following:

**Definition 21** A direct revelation mechanism is a mechanism in which $S_i = \Theta_i$ and $p(\cdot) \equiv f(\cdot)$.

In this case the only declarations the agents make is their types. In this way the private information becomes public and the outcome is determined directly by means the social choice function. A problem is here to determine if agents are willing to declare their real types. More precisely, it must be determined if the implementation responds to the following definition:

**Definition 22** A social choice function is said strategy-proof or incentive compatible if, for all $i$, $s_i^*(\theta_i) = \theta_i$ and

$$p_i(\theta_1, \ldots, \theta_i, \ldots, \theta_n) \geq p_i(\theta_1, \ldots, \theta_i', \ldots, \theta_n)$$

for all $\theta_i' \in \Theta_i$ (given that i's type is $\theta_i$)
This means that in a direct revelation mechanism with an incentive compatibility, the social outcome will be supported by a Nash equilibrium. It must be determined if such a mechanism exists. Let us give certain previous definitions before presenting the main result:

**Definition 23** A social choice function \( f \) is incentive compatible in dominant strategies if

- \( s_i^*(\theta_i) = \theta_i \), for all \( \theta_i \in \Theta_i \)
- if \( i \)'s type is \( \theta_i \), then for any other \( \theta_i' \in \Theta_i \),
  \[
  u_i(f(\theta_1, \ldots, \theta_i, \ldots, \theta_n)) \geq u_i(f(\theta_1, \ldots, \theta_i', \ldots, \theta_n))
  \]

This means that for \( i \), the best outcome obtains when he declares his real type.

**Proposition 5** \( f \) is incentive compatible in dominant strategies if and only if for all \( i \), given \( (\theta_i, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \), and for any \( \theta_i, \theta_i' \in \Theta_i \), if \( i \) is \( \theta_i \)

\[
  u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i', \theta_{i+1}, \ldots, \theta_n)) < u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n))
  \]

and if \( i \) is \( \theta_i' \)

\[
  u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i', \theta_{i+1}, \ldots, \theta_n)) \leq u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i', \theta_{i+1}, \ldots, \theta_n))
  \]

**Proof 5** Immediate from the definition of incentive compatible in dominant strategies

**Definition 24** A social choice function \( f \) is dictatorial if there is a \( i_0 \in I \) such that for all \( (\theta_1, \ldots, \theta_n) \in \prod_{i=1}^n \Theta_i \),

\[
f(\theta_1, \ldots, \theta_n) \in \{ x \in X : u_{i_0}(x) \geq u_{i_0}(x'), \forall x' \in X \}\]

This means that a dictatorial social choice function choses always an alternative that is one of the most preferred by the agent \( i_0 \)

**Definition 25** A social choice function \( f \) is monotonic if for any \( (\theta_1, \ldots, \theta_n), (\theta_1', \ldots, \theta_n') \), and for all \( i \),

\[
  \{ x' : u_i(f(\theta_1, \ldots, \theta_n)) \geq u_i(x') | \theta_i \} \subseteq \{ x' : u_i(f(\theta_1, \ldots, \theta_n)) \geq u_i(x') | \theta_i' \}
  \]

then

\[
f(\theta_1, \ldots, \theta_n) = f(\theta_1', \ldots, \theta_n')
\]
This means that if all the agents change their types and no alternative becomes most preferred than the former social outcome, that outcome must continue to be the social outcome (now for the new assignation of types).

Now we can state the main result for direct revelation mechanisms [Dasgupta et alt.79]:

**Theorem 4 (Gibbard-Satterthwaite)** If \( X \) is finite, with \( \text{Card}(X) \geq 3 \) and for every \( i \) and every \( x, x' \in X, x \neq x' \), \( u_i(x) \neq u_i(x') \), then the epicyclic social choice function \( f \) is incentive compatible in dominant strategies if and only if it is dictatorial.

**Proof** 4 \( \Rightarrow \) The argument here is to reduce the conditions of this theorem to the conditions of Arrow’s Impossibility Theorem. The version to be used here is the following: Given a function \( F: \prod_{i=1}^{n} U_i \rightarrow X \), where the \( U_i \)s are sets of utility functions for each \( i \), if \( F \) verifies:

- \( F \) is monotonic, that is, given \((u_1, \ldots, u_n)\) and \((u'_1, \ldots, u'_n)\), such that \( F(u_1, \ldots, u_n) = x \) and there is no \( x' \in X \), such that for every \( i \) \( u_i(x') \leq u_i(x) \) and \( u'_i(x') \geq u'_i(x) \) then \( F(u'_1, \ldots, u'_n) = x \)

- \( F \) is weakly paretoian, that is, if, given \( x, x' \in X \), \( u_i(x) \leq u_i(x') \) for all \( i \) then \( F(u_1, \ldots, u_n) \neq x \)

- \( \text{Card}(X) \geq 3 \)

then, \( F \) is dictatorial, that is, there exists \( i_0 \) such that for every \((u_1, \ldots, u_n)\), \( F(u_1, \ldots, u_n) \in \{ x : u_{i_0}(x) \geq u_i(x') \}, \) for every \( x' \in X \) Now the main proof can be developed in stages as follows:

- \( f \) is monotonic

  To see this, consider \((\theta_1, \ldots, \theta_n), (\theta'_1, \ldots, \theta'_n)\), and for all \( i \),

  \[
  \{x' : u_i(f(\theta_1, \ldots, \theta_n)) \geq u_i(x') \} \subseteq \{x' : u_i(f(\theta'_1, \ldots, \theta'_n)) \geq u_i(x') \} \]

  , we want to prove that

  \[
  f(\theta_1, \ldots, \theta_n) = f(\theta'_1, \ldots, \theta'_n) \]

  . We know by a previous proposition that for any \( i \),

  \[
  u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta'_i, \theta_{i+1}, \ldots, \theta_n)) \leq u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n))
  \]

  (if \( i \) is \( \theta_i \)). By transitivity it follows that

  \[
  u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta'_i, \theta_{i+1}, \ldots, \theta_n)) \leq u_i(f(\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n))
  \]
(if i is \(\theta_i\)). But as no two alternatives can give the same utility being different, \(f(\theta_1, \ldots, \theta_{i-1}, \theta'_i, \theta_{i+1}, \ldots, \theta_n) = f(\theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n)\).
Iterating the argument for each i, we see that f is monotonic

• if f is epijective and monotonic, the corresponding F is weakly paretri

Suppose that given \(\theta = (\theta_1, \ldots, \theta_n)\), exists \(x \in X\), such that for each i,
\(u_i(f(\theta_1, \ldots, \theta_n)) \leq u_i(x)\). Then there exist a pair (by epijective
of f) \(\theta', \theta'' \in \prod_{i=1}^n \Theta_i\) such that \(f(\theta') = x\), and \(u_i(f(\theta)) \leq u_i(f(\theta''))\) (for
of type \(\theta''\)) but, by monotonicity it means that \(f(\theta'') = f(\theta') = f(\theta)\,
absurd.

• f, being monotonic and weakly paretri is dictatoral
by Arrow's Theorem

\(\Leftarrow\) Trivial: if f is dictatoral, for every non-dictatorial agent i it has no point
to hide her true type, because it would not change the outcome. Therefore f
is incentive compatible.

It can be argued that this negative result affects only mechanisms incentive compatible in dominant strategies. But the following result show that it affects every possible implementation in dominant strategies:

**Theorem 5 (Revelation Principle)** If there is a mechanism \(\Gamma = (S_1 \ldots S_n, p(\cdot))\) implementing a social choice function f in dominant strategies, then f is incentive compatible in dominant strategies

**Proof 5** If \(\Gamma = (S_1 \ldots S_n, p(\cdot))\) implements f in dominant strategies, then,
there exists \((s_i^*(\cdot), \ldots, s_n^*(\cdot))\) such that \(p(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = f(\theta_1, \ldots, \theta_n)\),
for all \((\theta_1, \ldots, \theta_n)\); and for all i and all \(\theta_i \in \Theta_i:\n\begin{align*}
u_i(p(s_1(\theta_1), \ldots, s_i(\theta_i), \ldots, s_n(\theta_n))) &\geq u_i(p(s_1(\theta_1), \ldots, s_i(\theta_i), \ldots, s_n(\theta_n)))
\end{align*}\)
for any \(s_i \in S_i\). In particular:
\(\begin{align*}
u_i(p(s_1^*(\theta_1), \ldots, s_i^*(\theta_i), \ldots, s_n^*(\theta_n))) &\geq u_i(p(s_1^*(\theta_1), \ldots, s_i^*(\theta_i), \ldots, s_n^*(\theta_n)))
\end{align*}\)

Therefore \(u_i(f(\theta_1, \ldots, \theta_i, \ldots, \theta_n)) \geq u_i(f(\theta_1, \ldots, \theta'_i, \ldots, \theta_n))\). On the other side, it is clear that for all i, \(s_i^*(\theta_i) = \theta_i\). Otherwise it could be defined a
\((\theta_1, \ldots, s^*_i(\theta_i), \ldots, \theta_n)\) such that

\[ u_i(f(\theta_1, \ldots, \theta_i, \ldots, \theta_n)) \leq u_i(p(\theta_1, \ldots, s^*_i(\theta_i), \ldots, \theta_n)) \]

while \(p(\theta_1, \ldots, s^*_i(\theta_i), \ldots, \theta_n) = f(\theta_1, \ldots, \theta_i, \ldots, \theta_n)\) (by hypothesis). Absurd.

Before finishing this line of argumentation lets say that there are other forms of implementation that do not use the dominant strategies approach (we will discuss some features of them in the next section) [Holmström et al. 83]. In either case our point is that message passing is an essential feature of economic interactions, despite the limitations imposed by particular "architectures of interaction".
4 Economic Systems: the importance of communications

Economic interactions among agents trying to allocate resources have communicational features that go beyond those that are implicit in the bargaining problem or the design of mechanisms. In this section we will examine first the notion of Economic System, that emphasises on the communication among agents. And then, we will provide a characterization of communication systems borrowed from Computer Science.

4.1 Economic Systems

We will begin giving the definition of economic system due to Hurwicz, Mount and Reiter [Mount-Reiter 74]:

**Definition 26** An Economic System is a communication process in which agents exchange formal messages in iterative fashion, followed by a decision process and finally a translation of decisions into real actions. Formally, given \( \Theta_1 \times \ldots \times \Theta_n \) the space of possible types of the agents, \( X \) the set of possible outcomes, then, the corresponding social choice function \( f \) can be seen, using a space of messages \( MSG = \prod_{i=1}^{n} MSG_i \), as the composition of a correspondence \( \lambda : \Theta_1 \times \ldots \times \Theta_n \to MSG \) and a function \( F : MSG \to X \), where \( f = F \circ \lambda \) if \( F \) is constant on \( \lambda(\theta) \) for all \( \theta \in \Theta_1 \times \ldots \times \Theta_n \)

The emphasis lies here more in the communication process, that constitutes a step previous to taking the actual actions. The process can be characterized as an iterative exchange of messages until a stationary message is reached. The joint action is taken only on the basis of the stationary message:

**Definition 27** The communication process in an economic system is a function

\[ \Lambda : \mathbb{Z}^+ \to MSG \]

such that \( \Lambda(t) \in \lambda(\theta_1, \ldots, \theta_n) \), for \( t \in \mathbb{Z}^+ \). A message \( a \in MSG \) is stationary if exists \( t \in \mathbb{Z}^+ \) such that for any \( t' \geq t \), \( \Lambda(t') = a \)
It can be shown that this approach seems useful for comparing different economic systems, in terms of the efficiency of the communication process.

We have to take into account the possibility of the messages being mere “babblings” without commitment and without credibility. The literature of Game Theory takes this possibility in consideration calling it *cheap talk*. But its analysis shows as a consequence that the set of possible equilibria must grow. Before discussing this we need to define a new type of equilibrium:

**Definition 28** A probability distribution $\mu$ on $\prod_{i=1}^{n} S_i$ is a correlated equilibrium in a game with a mediator that recommends strategies to the players, if

$$\sum_{\prod_{i=1}^{n} S_i} \mu(s_1(\cdot), \ldots, s_i(\cdot), \ldots, s_n(\cdot)) u_i(s_1(\cdot), \ldots, s_i(\cdot), \ldots, s_n(\cdot)) \geq \sum_{\prod_{i=1}^{n} S_i} \mu(s_1(\cdot), \ldots, s_i(\cdot), \ldots, s_n(\cdot)) u_i(s_1(\cdot), \ldots, s'_i(\cdot), \ldots, s_n(\cdot))$$

which means that in a correlated equilibrium is better for every $i$ to follow the advice of the mediator.

Now, we can present a particular version of another revelation principle [Myerson 91]:

**Proposition 6** Given a mechanism $\Gamma$ that implements the social choice function $f$ of an economic system $<\prod_{i=1}^{n} \Theta_i, X, MSG, f, \mathcal{F}>$ any equilibrium in the communication part of the system must be equivalent to a correlated equilibrium

**Proof 6** Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a probability distribution on $\prod_{i=1}^{n} MSG_i$ such that it is an equilibrium in randomized strategies for $<\prod_{i=1}^{n} MSG_i, X, \mathcal{F}, \{u_i\}_{i=1}^{n}>$ That is: for each $i$, $\sigma_i : MSG_i \to [0,1]$ is a function such that $\sum_{m \in MSG_i} \sigma_i(m) = 1$, and for any other $\sigma'_i$

$$\sum_{(m_1, \ldots, m_n) \in MSG} \sigma_1(m_1) \ldots \sigma_i(m_i) \ldots \sigma_n(m_n) u_i(\mathcal{F}(m_1, \ldots, m_n)) \geq \sum_{(m_1, \ldots, m_n) \in MSG} \sigma'_1(m_1) \ldots \sigma'_i(m_i) \ldots \sigma_n(m_n) u_i(\mathcal{F}(m_1, \ldots, m_n))$$

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Now considering the conditional probability \( p((m_1, \ldots, m_n)|(\theta_1, \ldots, \theta_n)) \), we can define the following distribution:

\[
\mu(m_1, \ldots, m_n) = \prod_{i=1}^{n} \sigma_i(m_i)p((m_1, \ldots, m_n)|(\theta_1, \ldots, \theta_n))
\]

Obviously, \( \mu \) is a correlated equilibrium. Suppose not, then, a player \( i \) could gain changing the message she emits from \( m_i \) to \( m'_i \). But doing so \( \sigma_i \) can be replaced by other distribution \( \sigma'_i \), such that \( \sigma_i(m_i) \geq \sigma_i(m'_i) \) and \( \sigma'_i(m'_i) \geq \sigma_i(m_i) \). Therefore \( (\sigma_1, \ldots, \sigma_i, \ldots, \sigma_n) \) is not an equilibrium. Absurd.

That is, the communication part of an economic system can be simulated by a game in which an mediator gives advice to the agents (this results from considering that the mediator knows the conditional probability of messages given the types of the agents.

As we said before, the message passing processes in an economic system do not reduce the set of equilibria. Now we can see why: they add a number of equilibria (babbling equilibria) consisting in messages that do not convey a commitment and are ignored by the other players. The existence of such equilibria is a matter of completeness: there are more correlated equilibria than Nash equilibria in a game. A proposal to reduce the set of correlated equilibria is based on the following definition due to Myerson [Myerson 89]:

**Definition 29** Negotiation is a communication processess in which individuals try to determine or influence the effective equilibrium that they will play thereafter in some game.

Again here we have a previous exchange of messages and then the decision is taken. Interesting is the classification of statements that can be emitted in such exchange:

- **allegations** describing some private informations
- **promises** describing how the parties plan to choose their future actions
- **requests** describing actions that the emisor wants the other parties to use

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Myerson analysis presents a game of incomplete information in order to find as correlated equilibria only the results of negotiations using credible statements. This means that all players will believe the allegations of the others and will fulfill their promises and all requests will be consistent with these facts. With this requirement, babbling equilibria become eliminated. The remaining equilibria are called the outcomes of coherent processes.

This model of negotiation is more concerned with the characteristics of the message passing process. We will not go deeper in its formalism but instead we will introduce an approach taken from other disciplines (mainly Computer Science) for the analysis of message passing systems.

4.2 Communication Among Agents

Fagin, Halpern, Moses and Vardi provide the following definition of multi-agent systems [Fagin et al. 95]:

Definition 30 A multi-agent system $S$ consists of

- a set of agents $I = \{i\}_{i=1}^n$

- for each agent $i$, $S_i$, the set of its local states (all the information to which the agent has access). Each subset of $S_i$ is called an event.

- $G = S_1 \times \ldots \times S_n$, the set of global states.

- (assuming discrete time) $R = \{r : r : Z^+ \rightarrow G\}$, where each $r$ is a function called a run of the system. Its domain is the set of the non-negative integers (representing discrete points of time).

In particular, a message passing system is a instance of a multi-agent system:

Definition 31 A message passing system is such that:

- for each $i$, $\Sigma_i$ is the set of initial states and $INT_i$ is the set of internal acts that $i$ can perform

- for each $i$, $MSG_i$ is the set of possible messages that $i$ can emit and $MSG_{-i}$ the set of messages that $i$ can receive from the other agents
• given a run \( r \) and a time point \( t \), \( r_i(t) \) is a history over \( <\Sigma_i, INT_i, MSG> \) (informally: a history includes the initial states, the reception and emission of messages by \( i \) until \( t \) and all the internal actions performed during that time)

• for every event receive(\( \mu, j, i \)) \( \in r_i(t) \) ("\( i \) received the message \( \mu \) from \( j \)"), there exists a corresponding event send(\( \mu, i, j \)) ("\( j \) sent the message \( \mu \) to \( i \)"

• \( r_i(0) \) is a sequence of length one (the initial state), and \( r_i(t+1) \) is either identical to \( r_i(t) \) or the result of appending a set of events to \( r_i(t) \)

A message passing system can be seen as a multi-agent system, considering that the internal states for each agent \( i \) in time \( t \) are his possible histories until that moment. The runs are defined by the protocols used by the agents:

**Definition 32** A protocol \( P_i \) for agent \( i \) is a description of the actions agent \( i \) may take in function of her history: \( P_i : S_i \rightarrow MSG_i \), where \( S_i \) is the set of possible histories and \( MSG_i \) is the set of messages that \( i \) can send (including the "empty" message)

A joint protocol is \( P = (P_1, \ldots, P_n) \) where each \( P_i \) is the protocol of agent \( i \)

This means that the protocol is a rule that prescribes what message must be sent by every agent according to the past exchange of messages. To relate this with the definition of an Economic System, we must remember that resource allocations are decided in function of a stationary message. A sufficient condition of stationarity in the process is given by the Agreement Theorem of Aumann. Previous definitions must be given before stating this result:

**Definition 33** A decision function for \( i \) in a message passing system is \( D_i : 2^{R \times T} \rightarrow MSG_i \) where \( (r, t) \equiv r(t) \in R \times T \)

\( D_i \) is union consistent if \( A_1 \ldots A_m \in 2^{R \times T} \) are such that \( A_1 \cap \ldots \cap A_m = \emptyset \) and for every \( j \), \( D_i(A_j) = m \) then \( D_i(\cup_j A_j) = m \)

That is, the decision function may prescribe the emission of the same message for different runs in different time points.

**Definition 34** A protocol \( P_i \) is compatible with a decision function \( D_i \) if for every history \( h \), \( P_i(h) = D_i(\{(r, t) : h = r_i(t)\}) \)
This means that the protocol prescribes the emission of the same message for the same histories, being independent of the particular run or point of time in which they appear.

**Definition 35** A context is a $\Gamma = (G_0, \tau, \Psi)$, where $G_0$ is the set of initial global states, $\tau$ is a family of functions $\{\tau_m\}_{m \in MSG}$ such that each $\tau_m : G \rightarrow G$, and $\Psi$ is a set of admissible runs. A context is recording if $\tau$ is such that given a global state $s$ and a joint message $m$, $s \subseteq \tau_m(s)$.

Informally: a context is given by the initial states of all the parties, the rules that determine the transition between global states and the feasible runs. In a recording context the state of the system in each point of time in a run is its global history for that run until the moment. A notion related to this one is that of knowledge of the agents. It is assumed that knowledge is about *states of the world*: complete descriptions of the environment in which agents interact. A state of the world $\omega$ should not be necessarily identical to a global state $s = (s_1, \ldots, s_n)$, given that by definition each $s_i$ is a set of local information for $i$, and in a state of the world there can be facts that no agent knows. In any case, we suppose that given the set of possible states of the world $\Omega$, each agent has a knowledge operator [Geanakoplos 94]:

**Definition 36** Given $i \in I$, the knowledge operator of $i$, is $K_i : 2^\Omega \rightarrow 2^\Omega$, such that given $\mathcal{E} \subseteq \Omega$, $K_i(\mathcal{E}) \subseteq \Omega$, where $K_i(\mathcal{E})$ is the set of states of the world that $i$ cannot distinguish from the states of the world in $\mathcal{E}$.

The knowledge of the agents can interact, to obtain a property called *common knowledge*:

**Definition 37** A set of states of the world $\mathcal{E} \subseteq \Omega$ is common knowledge at $\omega$, among the agents in $I$, if, for any finite sequence of agents, $(i_1, \ldots, i_m)$, $i_j \in I, j = 1, \ldots, m$,

$$\omega \in K_{i_1}(K_{i_2} \cdots (K_{i_m}(\mathcal{E}) \cdots))$$

Informally: a set of states of the world $\mathcal{E}$ is common knowledge among a set of agents if the real state of the world is in $\mathcal{E}$, and everybody knows that everybody knows that...everybody knows $\mathcal{E}$. Important aspects of an economic environment can be common knowledge, for example the structure of the environment, the rationality of the agents, etc.
Now, we are in condition to state the following theorem (applied to message passing systems):

**Theorem 6** *(Aumann)* Given a pair \((P, \Gamma)\), where \(P\) is a joint protocol implementing an union-consistent decision function and \(\Gamma\) a recording context, then, for all individual messages \(a_i \in MSG = \cap_i MSG_i\), if it is common knowledge that each agent \(i\) will perform \(a_i\) then \(a_1 = \ldots = a_n\).

**Proof 6** see [Aumann 76]

This result states that, in the conditions of the theorem, the agents can not agree to disagree. Now, we can derive the following proposition, that applies the last result to Economic Systems:

**Proposition 7** If an Economic System is such that:

1. every agent uses in his decision making a protocol that for the same history prescribes the emission of the same message in every possible situation

2. there is no noise or loss in the communication process

3. the set of common messages is not empty

4. the system can attain a situation in which there is common knowledge of the messages to be emitted, the characteristics and the rationality of every agent. Moreover, the emitted messages are in the shared message space

then a stationary message must be reached

**Proof 7** Condition 1 is equivalent to the existence of a joint protocol implementing an union-consistent decision function; condition 2 is equivalent to the fact that the message passing system is recording. So the conditions of Aumann’s theorem are fulfilled and therefore there must be a \(t\) such that a common message \(\bar{a} = (a, \ldots, a) \in \prod_{i=1}^n MSG_i\) will be emitted. Suppose that \(\bar{a}\) is not stationary, then exists \(t' > t\), such that the message \(a' \neq \bar{a}\) will be emitted, in which, for at least one \(i\), \(a'_i \neq a_i\). But the emission by \(i\) of \(a\), as well \(\theta_i\) and her rationality were common knowledge so for \(t' \geq t\), no new state of the world appears. Therefore \(i\) will emit again \(a\). Contradiction.
Now it can be asked if this result can be seen as characterizing the outcome of any negotiation on resource allocations. The answer in short is no, because it assumes complete information and common knowledge. If all these conditions were met, there would be no need of negotiation.
5 A Model of Negotiation Process

The final result in last section is useful as a basis for an informal characterization of a negotiation process:

A resource allocation negotiation is an iterative process of exchange of messages in such a way that agents change their characteristics and beliefs, according to the messages they receive. The final outcome is a resource allocation that corresponds to a stationary message.

The iterative exchange of messages has, therefore, consequences also on the characteristics of the agents. In other words, the agents are “modified” during the negotiation. The aspects that make this approach different are concerned with information and knowledge. This is consistent with the point of view of the adherents to the doctrine of procedural rationality [March 86]. This point of view emphasizes on the properties of economic processes instead of on the economic outcomes. Rationality is seen here as using the best available policy of decision making in a process: in every possible situation the a rational agent should use all the information available to him. Changes in the preferential orderings are allowed if they obey to a increment of knowledge about the environment in which the agent has to operate. Moreover, new alternatives can appear as a result of the acquisition of knowledge. Artificial Intelligence, for example, is also concerned with analogous problems (see [Dung 95], [Doyle 90], [Kraus-Wilkenfeld-Zlotkin 95]).

To be more precise, we will give here a characterization of a rational agent in a resource allocation process according to the informal discussion above:

Definition 38 A rational agent $i \in I = \{1 \ldots n\}$ in a negotiation context can be represented by a tuple $\langle X_i, w_i, C^t_i, \preceq^t_i, B^t_i, L_i, D_i, MSG_i \rangle$, where:

- $X_i \subseteq R^l$ is a bounded and convex set of potential options for $i$
- $w_i \in X_i$ represents the endowments of $i$
- $C^t_i \subseteq X_i$ is the compact set of actual options for $i$ in period $t$.
- $\preceq^t_i$ is a transitive and complete ordering of preferences on $C^t_i$. The set of maximal elements under this ordering is $c^t_i$.
• $B_i^t$ is the belief set of $i$ in $t$:

$$B_i^t = \{ (\theta_i^t, \ldots, \theta_n^t) : \theta_j^t = \psi_i^t(X_j^t), \psi_i^t(\preceq_j^t), \psi_i^t(\omega_i), \psi_i^t(B_j^{t-1}) > \}$$

where, loosely speaking, $\psi_i^t(y)$ means, for $y \in Y$ a conjecture that $i$ makes in $t$ about the value of $y$, $\psi_i^t(y) \in Y$. The conjecture is that $y = \psi_i^t(y)$ (a more precise characterization will be given below).

• $L_i$ is the linguistic function such that for each $t$ $L_i(c_i^t, B_i^t) = l_i^t \in MSG_i$, where $MSG_i$ is the language of $i$. $MSG_i$ is a set of expressions about the negotiation context, which do not need to have a truth value (for example: an order like “give me your endowments” is not true nor false). For every $i$, $\bot \in MSG_i$, where $\bot$ is the empty message.

• $D_i$ is an operator such that $D_i((l_1^t, \ldots, l_n^t), B_i^t) = C_i^{t+1}, \preceq_i^{t+1}$

Associated to this definition is the following one:

**Definition 39** A negotiation context is $\epsilon = \langle I, X, w, MSG >$, where $X = \prod_i X_i$, $w = \sum_{i=1}^n w_i$ and $MSG = \prod_i MSG_i$. A negotiation is a sequence $r = \langle c^0, B^0, l^0 >, \langle c^1, B^1, l^1 >, \cdots >$, where $c^t = (c_1^t, \ldots, c_n^t)$, $B^t = (B_1^t, \ldots, B_n^t)$ and $l^t = (L_1(c_1^t, B_1^t), \ldots, L_1(c_1^t, B_1^t))$.

This means that a rational agent in a negotiation, proceeds in an iterative fashion, observing the result of the previous step in the negotiation, reconfiguring his point of view and making decisions. Important is to note that there are certain facts that are implicit. For example, nothing indicates that $w_i \in C_i^0$. A possible interpretation of this fact is that $i$ may not see all the possibilities of transaction beforehand.

Another important implicit consequence of the definition above is that in the belief set of agent $i$ in $t$ are included conjectures about the beliefs of the other agents in $t - 1$, which in turn must include conjectures about all the agents (including $i$) in $t - 2$, and so on. This implies the following:

**Lemma 4** A belief set for $i$ in $t$ can be represented by a $(n-1)$-tree of depth $t + 1$.
Proof 4 Trivial: let $B_i^t$ be the root, the $n-1$ elements \((j \neq i)\), \(<\psi^i(X_j^i),\psi^i(\leq_j^i),\psi^i(w_i),\psi^i(B_j^{t-1})>\) are the nodes in level 1. Recursively we obtain a tree of depth $t+1$ in which the leaves have the form
\[
<\psi^i(\psi^{i-1}_j(\psi^{i-2}_k(\cdots (X^0_s)\cdots))),\psi^i(\psi^{i-1}_j(\psi^{i-2}_k(\cdots (\leq^0_s)\cdots))),
\psi^i(\psi^{i-1}_j(\psi^{i-2}_k(\cdots (w^0_s)\cdots)))>
\]
where $j, k, s \in I$. Of course, nodes in this tree may be empty sets, depending on the characterization of $\psi_i$.

It is clear that the only determinant of the actual run of the negotiation is the belief formation process used by the agents. This process obeys to a conjecture making policy (see [Loui 92] for an analysis of policies in several aspects of reasoning):

Definition 40 A conjecture making policy for an agent $i$ is a $\psi_i$ such that $\psi_i = (\psi^i_0, \ldots, \psi^i_t, \ldots)$. The set of conjecture making policies for $i$ is $\Psi_i = \{\psi_i\}$

We do not restrict here the set $\Psi_i$, which means that any possible conjecture making policy is an element of $\Psi_i$. The set of all possible belief sets for an agent $i$ is defined in terms of $\Psi_i$ as follows

Definition 41 The set of beliefs for $i$ is:
\[
B_i = \{B_i^t : \exists \psi_i \in \Psi_i, \exists t,
\]
\[
B_i^t = \{(\theta^i_1, \ldots, \theta^i_t) : \theta^i_j = <\psi^i(X^i_j), \psi^i(\leq^i_j), \psi^i(w_i), \psi^i(B_j^{t-1})>\}
\]
In this stage we can define with precision what a conjecture making operator is:

Definition 42 The conjecture making policy $\psi_i$ is such that
\[
\psi_i : Z^+ \times MSG \times B_i \rightarrow \prod_{j \neq i} 2^{\mathbb{R}^\infty} \times \mathcal{O} \times \mathbb{R}^\infty \times B_j
\]
where $\mathbb{R}^\infty$ is the Euclidean space of arbitrary dimensions and $\mathcal{O}$ is the space of orderings on $\mathbb{R}^\infty$.
In words: the conjectural operator in a time point and according the beliefs of the agent at that moment, "transforms" a received message in a conjecture on the possible set of options, the preferences, the endowments and the beliefs of the other agents. Abusing of language we use $\psi$ to represent the rank of $\psi_i$. Moreover, given $y \in Y$, where (using a BNF notation) $Y = \langle 2^{\mathbb{R}_{\infty}} | R^{\infty} | O | B_j \rangle$, the meaning of $\psi_y$ is $\text{Im}(\psi_i)_Y$ (the projection of the image of $\psi_i$ on $Y$).

The conjecture making policy may be arbitrary. Of course a particular case is given by the bayesian updating of beliefs. There exist, notwithstanding, sharp differences between this formulation and a probabilistic one. Here the beliefs are not identified with a probability distribution but with one of the several possibilities that may arise. It can be said that it implies a commitment to a state of the world, with or without further grounding.

With this point in clear we can consider the set of all possible negotiations:

**Definition 43** The space of negotiations for the context $\epsilon$ is

$$\mathcal{R} = \{r | r : Z^+ \rightarrow \prod_{i=1}^{n} X_i \times \prod_{i=1}^{n} B_i \times MSG\}$$

It follows that:

**Lemma 5** $\bigcup_{r \in \mathcal{R}} \text{Im}(r)|_X = X$, where $\text{Im}(r)|_X$ is the projection on $X$ of the image of a negotiation $r$

**Proof 5** Suppose that there is $\tilde{x} \in X$, such that $\tilde{x} \notin \bigcup_{r \in \mathcal{R}} \text{Im}(r)|_X$. We can define a negotiation $\bar{r} = (\langle \tilde{x}, B, \bot \rangle, \ldots, \langle \tilde{x}, B, \bot \rangle, \ldots)$, such that for every $i$, for any other $j$, $\psi_i(X_j) = X_j$ if $\psi_i(\bot_j) = \bot_j$ and $\psi_i(B_j) = B_j$ and $\tilde{x}_i$ is the single maximal element in $X_i$ under the ordering $\preceq_i$. Then, $\tilde{x} \notin \bigcup_{r \in \mathcal{R}} \text{Im}(r)|_X$, contradiction.

The interaction among agents is linguistic, but it has a close paralell in the resource allocations. Let's consider a previous definition to motivate this result:

**Proposition 8** For every $l \in MSG$ corresponds a $\tilde{x} \subseteq \prod_{i=1}^{n} X_i$
\textbf{Proof 8} Given \( l \in MSG \) let's consider a negotiation \( r \in R \) such that for a \( t, l = l' \). Now let's consider the vector correspondence

\[
L^{-1}(l)|_X = (L_1^{-1}(l_1)|x_1, \ldots, L_n^{-1}(l_n)|x_n) =
\]

\[
\{c' \subseteq \prod_{i=1}^n X_i : \forall i, \exists \tilde{B}_i \in B_i, L_i(c'_i, \tilde{B}_i) = l_i\}
\]

then \( L^{-1}(l)|_X = \tilde{\varepsilon} \subseteq X \)

We can define then the following notion:

\textbf{Definition 44} The excess demand for a \( l \in MSG \), is the correspondence

\[
z(l) = \{z = \sum_{i=1}^n x_i - \sum_{i=1}^n w_i : x = (x_1, \ldots, x_n) \in L^{-1}(l)|_X\}
\]

The excess demand for a message \( l \) is, therefore, the set of excess demands that are derived from the resource allocations corresponding to \( l \).

Certain algebraic properties will be useful in the following:

\textbf{Definition 45} Given \( l, l' \in MSG \),

- the join of \( l, l' \) is

\[
l \vee l' = \{l'' : (\tilde{\min}(z_1, z'_1), \ldots, \tilde{\min}(z_l, z'_l)) \in z(l'') ; z \in z(l), z' \in z(l')\}
\]

where for any pair of real numbers \( x, y \),

\[
\tilde{\min}(x, y) = \tilde{\min}(y, x)
\]

and

\[
\tilde{\min}(x, y) = \begin{cases} y & \text{if } \|x\| \geq \|y\| \\ x & \text{if } \|x\| < \|y\| \end{cases}
\]

- the meet of \( l, l' \) is

\[
l \wedge l' = \{l'' : (\tilde{\max}(z_1, z'_1), \ldots, \tilde{\max}(z_l, z'_l)) \in z(l'') ; z \in z(l), z' \in z(l')\}
\]

where for any pair of real numbers \( x, y \),

\[
\tilde{\max}(x, y) = \tilde{\max}(y, x)
\]

and

\[
\tilde{\max}(x, y) = \begin{cases} x & \text{if } \|x\| \geq \|y\| \\ y & \text{if } \|x\| < \|y\| \end{cases}
\]
The following property is straightforward:

**Proposition 9** $l \wedge l' \neq \emptyset$, for any $l, l' \in MSG$,

**Proof 9** Suppose that exists $l, l' \in MSG$, such that $l \wedge l' = \emptyset$. Let take $z \in z(l)$ and $z' \in z(l')$, $z = (z_1, \ldots, z_l); z' = (z'_1, \ldots, z'_l)$. Obviously, for $j = 1, \ldots, l$, exists $\tilde{\max}(z_j, z'_j) \in R^l$, so, there exists a vector $z'' = (\tilde{\max}(z_1, z'_1), \ldots, \tilde{\max}(z_l, z'_l)) \in R^l$, then $x'' = z'' + w$ is a vector in $X$ (by convexity of $X$), then, by lemma 5 there exists a run $r \in R$, and $t \in Z^+$ such that $r(t)|_X = x''$. Then, taking $l'' = r(t)|_{MSG}$, we see that $z'' \in z(l'')$, therefore $l'' \in l, l'$. Contradiction.

A similar argument can be applied to show that $l \vee l'$ is non-empty.

These operations can be extended to subsets of $MSG$:

**Definition 46** Given $\bar{l} \subseteq MSG$, $\bar{l} = \{l^k\}_{k \in K}$, where $K$ is an index set

- **the join of $\bar{l}$** is

$$\bigvee_{k \in K} l^k = \{l'': (\tilde{\inf}(\{x^k_1\}_{k \in K}), \ldots, \tilde{\inf}(\{x^k_l\}_{k \in K})) \in z(l''); \forall k \in K, x^k \in z(l^k)\}$$

where for any $\{x^k\}_{k \in K} \subseteq R$, $\{x^k\}_{k \in K} = \{x^k_+\} \cup \{x^k_-\}$

$$\{x^k_+\} = \{x^k : k \in K; x^k \geq 0\}; \{x^k_-\} = \{x^k : k \in K; x^k < 0\}$$

$$\tilde{\inf}(\{x^k\}_{k \in K}) = \begin{cases} \inf(\{x^k_+\}) & \text{if } \inf(\{x^k_+\}) \leq \inf(\{\|x^k_-\|\}) \\ \sup(\{x^k_+\}) & \text{if } \inf(\{x^k_+\}) > \inf(\{\|x^k_-\|\}) \end{cases}$$

- **the meet of $\bar{l}$** is

$$\bigwedge_{k \in K} l^k = \{l'': (\tilde{\sup}(\{x^k_1\}_{k \in K}), \ldots, \tilde{\sup}(\{x^k_l\}_{k \in K})) \in z(l''); \forall k \in K, x^k \in z(l^k)\}$$

where for any $\{x^k\}_{k \in K} \subseteq R$, $\{x^k\}_{k \in K} = \{x^k_+\} \cup \{x^k_-\}$

$$\{x^k_+\} = \{x^k : k \in K; x^k \geq 0\}; \{x^k_-\} = \{x^k : k \in K; x^k < 0\}$$

$$\tilde{\sup}(\{x^k\}_{k \in K}) = \begin{cases} \sup(\{x^k_+\}) & \text{if } \sup(\{x^k_+\}) \geq \sup(\{\|x^k_-\|\}) \\ \inf(\{x^k_-\}) & \text{if } \sup(\{x^k_+\}) < \sup(\{\|x^k_-\|\}) \end{cases}$$

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Proposition 10 \( \forall_{k \in K} l^k \neq \emptyset \), for every \( \{l^k\}_{k \in K} \subseteq MSG \)

**Proof 10** Suppose that for a \( \{l^k\}_{k \in K} \subseteq MSG \), \( \forall_{k \in K} l^k = \emptyset \). Let's consider the set \( \{z^k : z^k \in z(l^k); k \in K\} \), and define \( z = (\sup\{z^k\}, \ldots, \sup\{z^k\}) \in R^l \) and \( x = z + w \). \( x \) is a vector in \( X \) (because \( X \) is convex and closed), then by lemma 5 there exists an \( r \in R \) and \( t \in Z^+ \) such that \( r(t)|_X = x \). Taking \( l = r(t)|MSG \) we can see that \( z \in z(l) \) and therefore \( l \in \forall_{k \in K} l^k \). Contradiction.

Again, a similar argument can be applied to show that \( \forall_{k \in K} l^k \) is not empty, for every \( \{l^k\}_{k \in K} \subseteq MSG \)

The operations \( \lor \) and \( \land \) (and their extensions, \( \lor \) and \( \land \)) are required to apply to our problem the following mathematical structure (see for example [Davey-Priestley 91]):

**Definition 47** A lattice is an algebraic structure \( < L, \lor, \land, > \), such that \( L \) is the carrier set, and for any \( l, l' \in L \), \( l \lor l' \in L \), \( l \land l' \in L \).

A lattice is said complete if for any \( \bar{l} = \{l^k\}_{k \in K} \subseteq L \) (\( K \) a set if indexes), \( \land_{k \in K} l^k \in L \), \( \lor_{k \in K} l^k \in L \)

The following result follows immediately:

**Lemma 6** The set, \( 2^{MSG} - \emptyset \) (where \( 2^{MSG} \) is the power set of \( MSG \)), with the operations given in definition 46 constitutes a complete lattice.

**Proof 6** By definition, for any \( \bar{l} = \{l^k\}_{k \in K} \subseteq MSG \), \( \land_{k \in K} l^k \subseteq MSG \) and \( \lor_{k \in K} l^k \subseteq MSG \). Moreover, by proposition 10, \( \land_{k \in K} l^k \neq \emptyset \), and \( \lor_{k \in K} l^k \neq \emptyset \). Now consider \( \bar{M} = \{M^j\}_{j \in J} \subseteq 2^{MSG} - \emptyset \), where for each index \( j \in J \), \( M^j \subseteq MSG \), \( M^j \neq \emptyset \). Then consider the set \( M = \cup_{j \in J} M^j \). By the set-theoretical Well Ordering Axiom Every set can be well ordered (i.e. it can be given a total order such that every subset has a minimal element) [Halmos 60]. If a set is well ordered it exists an order preserving bijection between the set and an ordinal \( K \). Therefore, there exists a set of indexes \( K \), such that \( M = \{l^k\}_{k \in K} \) and

\[
\land_{j \in J} M^j = \land_{k \in K} l^k; \lor_{j \in J} M^j = \lor_{k \in K} l^k
\]

The operations \( \lor, \land \) determine an order on \( 2^{MSG} - \emptyset \):
Definition 48  For \( M, M' \in 2^{MSG} - \emptyset \), \( M \trianglelefteq M' \) iff \( M \cap M' = M \) and \( M \cup M' = M' \)

Proposition 11  The set \( 2^{MSG} - \emptyset \) with the relation \( \trianglelefteq \) is partially ordered set

Proof 11  Trivial. We have to show that \( \trianglelefteq \) is a reflexive, antisymmetric and transitive relation for any \( M, M', M'' \in 2^{MSG} - \emptyset \):

- reflexive: \( M \trianglelefteq M \) because \( M \cap M = M \cup M = M \)

- antisymmetric: \( M \trianglelefteq M' \) then \( M \cup M' = M' \) and \( M \cap M' = M \). Then, \( M \cup M' \neq M \) and \( M \cup M' \neq M' \) (unless \( M = M' \)). Therefore \( M' \trianglelefteq M \), except if \( M = M' \)

- transitive: if \( M \trianglelefteq M' \) and \( M' \trianglelefteq M'' \) then (the associativity of \( \vee \) and \( \wedge \) is immediate from the proof of lemma 6):

\[
M = M \cap M' = M \cap (M' \cap M'') = (M \cap M') \cap M'' = M \cap M''
\]

and

\[
M'' = M' \cup M'' = (M' \cup M') \cup M'' = M \cup (M' \cup M'') = M''
\]

therefore \( M \trianglelefteq M'' \)

The complete lattice \( < 2^{MSG} - \emptyset, \cup, \cap > \) can help to find a structure on the space of negotiations \( \mathcal{R} \):

Theorem 7  There exists a partition of \( \mathcal{R} \) based in elements from \( < 2^{MSG} - \emptyset, \cup, \cap > \)

Proof 7  Lets consider the set

\[
M_z = \{ M^k : k \in K ; z(l) = z(l'), \forall l, l' \in M^k \} \subseteq 2^{MSG} - \emptyset
\]

then we can consider the set of all the sequences of elements in \( M_z \):

\[
\bar{M}_z = \{ (M^0,M^1,\cdots) : M^t \in M_z, t \in Z^+ \}
\]
Given \( r, r' \in \mathcal{R} \),
\[
    r = ( < c^0, B^0, t^0 >, < c^1, B^1, t^1 >, \cdots )
\]
and
\[
    r' = ( < c'^0, B'^0, t'^0 >, < c'^1, B'^1, t'^1 >, \cdots )
\]
we will say that \( r \simeq r' \) if \( r|_{MSG} = (l^0, l^1, \cdots) \) and \( r'|_{MSG} = (l'^0, l'^1, \cdots) \) are such that \( r|_{MSG}, r'|_{MSG} \in (M^0, M^1, \cdots) \in \tilde{M}_z \), where for every \( t \in \mathbb{Z}^+ \), \( l^t, l'^t \in M^t \).

The relation \( \simeq \) is by definition, for any \( r, r', r'' \in \mathcal{R} \):

- reflexive: \( r \simeq r \)
- symmetric: if \( r \simeq r' \) then \( r' \simeq r \)
- transitive: if \( r \simeq r' \) and \( r' \simeq r'' \) then \( r \simeq r'' \)

Therefore \( \simeq \) is an equivalence relation and determines a partition on \( \mathcal{R} \).

Naturally a negotiation may constitute a singleton in the quotient space of negotiations:

**Definition 49** The canonical space of negotiations is the set
\[
\tilde{\mathcal{R}} = \mathcal{R}/\simeq = \{ \tilde{r} : \tilde{r} = \{ r' \in \mathcal{R} ; r' \simeq r \} ; r \in \mathcal{R} \}
\]
Any \( \tilde{r} \in \tilde{\mathcal{R}} \) determines a "path" in the partially ordered set \( 2^{MSG} - \emptyset \), given by the sequence of sets \( (M^0, M^1, \cdots) \). This path determines a protocol:

**Definition 50** A protocol of negotiation is, given \( \tilde{r} \in \tilde{\mathcal{R}} \),
\[
    \Phi_{\tilde{r}} : \mathbb{Z}^+ \to 2^{MSG} - \emptyset
\]
such that
\[
    \Phi_{\tilde{r}}(t) = M^t
\]
where \( \tilde{r}|_{MSG-\emptyset} = (M^0, M^1, \cdots, M^t, \cdots) \)

We can find an alternative characterization of a protocol in terms of a recursive function \( f^n = f \circ f \circ f \cdots f \), which obtains applying a function \( f \) \( n \) times:
Proposition 12
Given a protocol \( \Phi \), there exists a function \( \phi : 2^\text{MSG} \rightarrow 2^\text{MSG} \) such that for any \( t \in \mathbb{Z}^+ \)
\[
\phi_t(M^0) = M^t = \Phi(t)
\]

Proof 12 Trivial. Given \( \bar{r}_{\text{MSG-0}} = (M^0, M^1, \cdots) \), let's define
\[
\phi = \{(M^t, M^{t+1}) : t \in \mathbb{Z}^+, M^t, M^{t+1} \in \bar{r}_{\text{MSG-0}}\}
\]
Therefore, given \( M^t, \phi(M^t) = M^{t+1} \), so inductively:
\[
\phi_0^0(M^0) = M^0 = \Phi(0)
\]
and if
\[
\phi_t^0(M^0) = M^t = \Phi(t)
\]
then
\[
\phi_{t+1}(M^0) = \phi_t(\phi_{t+1}(M^0)) = \phi(M^t) = M^{t+1} = \Phi(t+1)
\]

As for Economic Systems, in which the communicational process ends when a stationary message \( l \in \text{MSG} \) obtains, here we will try to find the possible stationary messages. In this case we generalize for a \( M \in 2^\text{MSG} \), but by definition, for any message in \( M \) the excess demand is the same. So we will assume that a negotiation comes to an end (successfully or not) if attains a set of messages that determines a constant allocation of resources:

Definition 51 Given a \( \bar{r} \in \bar{R} \), a stationary message set, \( M \) is such that exists \( t \in \mathbb{Z}^+ \) and for any \( t' \geq t \), \( \phi_{t'}^0(M^0) = M \), where \( \bar{r}_{\text{MSG-0}} = (M^0, M^1, \cdots, M, M', \cdots) \)

Therefore:

Proposition 13 Given \( \bar{r} \in \bar{R} \), a message set \( M \) is stationary iff it is a fixed-point of \( \phi \)

Proof 13 Trivial. \( M \) is a stationary message set iff for \( t' \geq t \),
\[
\phi_{t'}^0(M^0) = \phi_{t'+1}^0(M^0) = M
\]
iff
\[
\phi_0(M) = \phi_t(\phi_{t'}^0(M^0)) = \phi_{t'+1}(M^0) = M
\]
We want to find a characterization of stationary messages in terms of $\phi_F$. Unfortunately we can only give sufficient conditions, according to the following property (see [Tarski 55]):

**Lemma 7** Given a complete lattice $< L, \vee, \wedge >$ and a function $f : L \rightarrow L$, such that for all $l \in L$, $l \wedge f(l) = l$ and $l \vee f(l) = f(l)$, there exists a $l^* \in L$, such that $f(l^*) = l^*$

**Proof 7** Consider the set

$$\bar{L} = \{ l \in L : l \wedge f(l) = l ; l \vee f(l) = f(l) \}$$

by definition $\bar{L} \equiv L$. $L$, being a complete lattice has an element $l^* = \vee L \in L$ and $l^* \wedge f(l^*) = l^*$, $l^* \vee f(l^*) = f(l^*)$, but $l^* = \vee L$ and $f(l^*) \in L$, so $l^* \vee f(l^*) = l^*$. Therefore $f(l^*) = l^*$

We can apply this result to show that

**Theorem 8** Given a protocol $\phi_F$, such that for each $M \in \text{Dom}(\phi_F)$

$$M \preceq \phi_F(M)$$

there exists a stationary message $M^*$

**Proof 8** We have only to show that $\text{Dom}(\phi_F)$ is a complete lattice and then by lemma 7 and proposition 13 it follows that exists a stationery message set $M^*$:

given $M^0 \in \tau_{\phi \text{msa } \phi}$, for every $t \in Z^+$, $M^{t+1} = \phi_F(M^t)$ and by hypothesis

$$M^t \preceq \phi_F(M^t) = M^{t+1}$$

So $\text{Dom}(\phi_F) = \{M^0, M^1, \cdots \}$ constitutes a totally ordered set:

- $< \text{Dom}(\phi_F), \preceq >$ is a partially ordered set by the properties of $\preceq$
- for every $M, M' \in \text{Dom}(\phi_F)$, exist $t_0, t_m \in Z^+$, say $t_0 \leq t_1 \leq \cdots \leq t_m$, such that

$$M = M^{t_0} \preceq M^{t_1} \preceq \cdots \preceq M^{t_m} = M'$$

Then, by transitivity of $\preceq$, $M \preceq M'$

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For any \( \bar{M} \subseteq \text{Dom}(\phi_r) \) we can define two operations

\[
\bigwedge_{\phi_r} \bar{M} = \{ M \in \bar{M} : \forall M' \in \bar{M}, M \preceq M' \}
\]

and

\[
\bigvee_{\phi_r} \bar{M} = \{ M \in \bar{M} : \forall M' \in \bar{M}, M' \preceq M \}
\]

We can show that

\[
\bigwedge_{\phi_r} \equiv \bigwedge_{\text{Dom}(\phi_r)}
\]

that is, that the operation \( \bigwedge_{\phi_r} \) coincides with the operation \( \bigwedge \) (defined for \( 2^{M_{SG}} - \emptyset \)) restricted to \( \text{Dom}(\phi_r) \):

- \( \subseteq \)

\[
\bigwedge_{\phi_r} \bar{M} \subseteq \bigwedge_{\text{Dom}(\phi_r)} \bar{M}
\]

Trivial, because by definition \( M^* \in \bigwedge_{\phi_r} \bar{M} \) is such that for every \( M \in \bar{M}, M^* \preceq M \) that is \( M^* \bigwedge M = M^* \). Therefore \( M^* \in \bigwedge \bar{M} \) on \( \text{Dom}(\phi_r) \).

- \( \supseteq \)

\[
\bigwedge_{\phi_r} \bar{M} \supseteq \bigwedge_{\text{Dom}(\phi_r)} \bar{M}
\]

If \( M^* = \bigwedge_{\text{Dom}(\phi_r)} \bar{M}, M^* \bigwedge M = M^*, M^* \bigvee M = M \) for any \( M \in \bar{M}, \) so \( M^* \preceq \bar{M} \) and therefore \( M^* \in \bigwedge_{\phi_r} \bar{M} \)

\( \bigwedge_{\phi_r} \bar{M} \) is a singleton because \( \bigwedge_{\text{Dom}(\phi_r)} \bar{M} \) is a singleton in \( 2^{M_{SG}} - \emptyset \). \( \bigwedge_{\phi_r} \bar{M} \in \text{Dom}(\phi_r) \) because by definition \( \bigwedge_{\phi_r} \bar{M} \in \bar{M} \subseteq \text{Dom}(\phi_r) \). As the same argument applies to \( \bigvee_{\phi_r} \), we have shown that \( \langle \text{Dom}(\phi_r), \bigwedge_{\phi_r}, \bigvee_{\phi_r} \rangle \) is a complete lattice. As for any \( M \in \text{Dom}(\phi_r), M \preceq \phi_r(M), \) by lemma 7 there exists a fixed-point \( M^* \) for \( \phi_r \) and therefore, by proposition 13, \( M^* \) is a stationary message for the negotiations in \( \bar{\tau} \).

The outcomes of the processes with the property that ensures the existence of a fixed point can be characterized by the following correspondence:
Definition 52  An integrative protocol is a partial correspondence

\[ \phi : 2^{\mathcal{M}G} - \emptyset \rightarrow 2^{\mathcal{M}G} - \emptyset \]

such that for any \( \phi_{\mathcal{R}} \) that verifies that, for any \( M \in \text{Dom}(\phi_{\mathcal{R}}) \), \( M \trianglelefteq \phi_{\mathcal{R}}(M) \), then

\[ \phi_{\mid \text{Dom}(\phi_{\mathcal{R}})} = \phi_{\mathcal{R}} \]

It is a correspondence because given \( \phi_{\mathcal{R}} \) and \( \phi_{\mathcal{R}'} \), their domains (two totally ordered sets) may intersect only in an \( M \), and so

\[ \phi(M) = \phi_{\mathcal{R}}(M) \neq \phi_{\mathcal{R}'}(M) = \phi(M) \]

A property of \( \phi \) is the following:

**Lemma 8** Given \( M \in 2^{\mathcal{M}G} - \emptyset \), there exists a \( M' \) such that \( M \trianglelefteq M' \in \phi(M) \) iff given \( z \in z(l), l \in M, \) exists \( z' \in z(l'), l' \in \phi(M), \) such that

\[ z'_j \leq z_j \text{ if } z_j \geq 0 \]

and

\[ z'_j > z_j \text{ if } z_j < 0 \]

**Proof 8** \( \bullet \Rightarrow \) if \( M \trianglelefteq M' \in \phi(M) \) then there exists a \( \bar{r} \in \mathcal{R}, \) such that \( M \trianglelefteq M' = \phi_{\mathcal{R}}(M) \). This means that (as \( M \wedge M' = M \))

\[ z = \bar{\sup}(z, z') \text{ for } z \in z(l), z' \in z(l'), l \in M, l' \in M' \]

and that (as \( M \vee M' = M' \))

\[ z' = \bar{\inf}(z, z') \text{ for } z \in z(l), z' \in z(l'), l \in M, l' \in M' \]

Then, by definition of \( \bar{\sup} \) and \( \bar{\inf} \), follows that

\[ z'_j \leq z_j \text{ if } z_j \geq 0 \]

and

\[ z'_j > z_j \text{ if } z_j < 0 \]
\( \Rightarrow \) Given \( z \in z(l), l \in M, z' \in z(l'), l' \in \phi(M) \), such that
\[
    z'_j \leq z_j \text{ if } z_j \geq 0
\]
and
\[
    z'_j > z_j \text{ if } z_j < 0
\]
we can define \( M' = \{l'\} \in \phi(M) \). It is immediate that \( M \land M' = M \) and \( M \lor M' = M' \), therefore \( M \leq M' \).

Finally we can see that the fixed-points in the integrative protocol support “equilibrium” allocations:

**Theorem 9** The fixed-points of \( \phi \), that is, the \( M^* \in 2^{MSG} - \emptyset \) such that \( M^* \in \phi(M^*) \), verify that exist \( l \in M^* \) such that \( z \equiv 0 \in z(l) \).

**Proof 9** Given \( M^* \), a fixed-point of \( \phi \), there exists \( \bar{r} \in \bar{R} \), such that \( \phi_{\bar{r}}(M^*) = M^* \), that means that for \( l, l' \in M^* \), \( z \in z(l), z' \in z(l') \) are, by lemma 8, such that \( z'_j \leq z_j \) if \( z'_j \geq 0 \) and \( z'_j > z_j \) if \( z'_j > 0 \). But by Theorem 7 we know that for \( l, l' \in M^* \), \( z(l) = z(l') \), so, taking \( z = z' \) we see that the only possibility is that \( z \equiv 0 \).

Now we see that, if the protocol is integrative the stationary messages support a feasible allocation of resources. Any negotiation that has such an outcome has the following features:

**Theorem 10** If \( r \in R \) is a negotiation such that
\[
r_{MSG} = (l^0, l^1, \ldots, l^*, l^*, l', \ldots) \in (M^0, M^1, \ldots, M^*, M^*, M^*, \ldots)
\]

(\( l^* = l^t \text{ for } t \geq t^* \)) then there exists a conjecture making policy \( \psi_i \) for each \( i \), such that the final outcome \( (x_1, \ldots, x_n) \) is a a Pareto-optimal allocation for \( (\geq_1^{**}, \ldots, \geq_n^{**}) \).

**Proof 10** It is obvious that if a stationary message \( l^* \) is reached, the belief sets must also be stationary \( B^*_i = B^*_i \). Suppose that they are not stationary, then there exists \( t > t^* \), such that \( B^*_i \neq B^*_i \) for at least one \( i \), then \( D_i(l^*, B^*_i) \neq D_i(l^*, B^*_i) \), therefore \( c^*_i \neq c^*_i \) and
\[
    l^*_i = L_i(c^*_i, B^*_i) \neq L_i(c^*_i, B^*_i) = l^*_i
\]
absurd. Therefore (given the characterization of $\Psi_i$) there exist a $\psi_i$ such that $\psi_i(t, t^*, B_i^{t^*}) = B_i^{t^*}$, for $t \geq t^*$. As $c_i^t = c_i^{t^*}$ and $x_i \in c_i^{t^*}$ is a maximal element for $X_i^{t^*} = X_i^{t^*}$ under the preferential ordering $\preceq_i^{t^*} = \preceq_i^{t^*}$ and $\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} w_i = 0$ (by Theorem 9), then (by Definition 3) $(x_1, \ldots, x_n)$ is Pareto-optimal.

The stationarity of the belief sets is only a necessary condition for Pareto-optimality, because of the following result.

**Proposition 14** Given $\overline{r} \in \overline{R}$ such that for every $i$ the sequence of belief sets $(B_i^0, B_i^1, \ldots)$ converges to a set $B_i^*$, the sequence $(M^0, lM^1, \ldots) \in \overline{r}_{\phi_{\text{MSG}}} - \emptyset$ converges to a stationary message set only if there exists a protocol $\phi_{\overline{r}}$, such that for each $M \in \text{Dom}(\phi_{\overline{r}})$

$$M \preceq \phi_{\overline{r}}(M)$$

**Proof 14** If for every $i$ the belief sets converge to $B_i^*$, there exists a $t^* \in \mathbb{Z}^+$ such that for $t \geq t^*$, $B_i^t = B_i^*$ and we can redefine the basic operators in the negotiation:

$$D_i^*(\cdot) = D_i(\cdot, B_i^*) : \text{MSG} \rightarrow \text{MSG}$$

and

$$L_i^*(\cdot) = L_i(\cdot, B_i^*) : \text{MSG} \rightarrow \text{MSG}$$

Therefore, the sequence of messages (for $t \geq t^*$) can be defined by:

$$\prod_{i=1}^{n} l_i^{t+1} = \prod_{i=1}^{n} L_i^* \circ D_i^*(l^i) \in M^{t+1}$$

that is, we can define

$$\phi_{\overline{r} \in \overline{R}} = \prod_{i=1}^{n} L_i^* \circ D_i^* : \text{MSG} \rightarrow \text{MSG}$$

$\phi_{\overline{r} \in \overline{R}}$ can be extended to $2^{\text{MSG}} - \emptyset$ as follows:

$$\phi_{\overline{r}}(M) = \{ \phi_{\overline{r} \in \overline{R}}(l) : \forall l \in M; \forall r \in \overline{r} \}$$

By definition, $\phi_{\overline{r}}$ is a protocol of negotiation. We know that stationary messages obtains for fixed points of $\phi_{\overline{r}}$ i.e. for any $M^* \in 2^{\text{MSG}} - \emptyset$ such that:

$$\phi_{\overline{r}}(M^*) = M^*$$
From Theorem 8 we know that a fixed point exists if $\phi_r$ is such that

$$M \preceq \phi_r(M)$$

for every $M \in \text{Dom}(\phi_r)$.

So, to summarize, for ensuring a Pareto-optimal outcome in a negotiation $r$ we need two conditions:

- the sequence of belief sets $(B^0_i, B^1_i, \ldots)$ must be convergent for every $i$
- given for every $i$ a stationary belief set $B^*_i$, the composition $L^*_i \cdot D^*_i \equiv L_i(D_i(. , B^*_i), B^*_i)$, must generate, for $\tilde{r}$ (the partition of $\mathcal{R}$, to which $r$ corresponds) a protocol

$$\phi_r = \prod_{i=1}^{n} L^*_i \circ D^*_i : 2^{MSG} \rightarrow 2^{MSG}$$

such that for any $M \in \text{Dom}(\phi_r)$, $M \preceq \phi_r(M)$

The last condition makes reference to constitutive properties of a rational agent. So, if we add this property to the older ones, we see that what we obtain is a stronger notion than rationality. It can be interpreted as indicating that agents are capable of, once attaining a stable set of beliefs, to act according to this beliefs, emitting a message that implies a redistribution of the endowments, giving to everyone the best possible outcome according to a final stable preferential ordering. It is implicit that this final outcome can be to give to each agent its own endowment (meaning that no agreement has been reached) or an outcome in which only a subset of $I$ agrees in an allocation of resources, because the only condition that has to be met is that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} w_i$ (This condition is also valid when one or more agents do not know their whole endowments.).
6 Conclusion

We have analyzed briefly the main economic models of resource allocation. We found that most of them do not make explicit reference to communication among agents. So, beginning with certain insights provided by Artificial Intelligence, we developed a model of resource allocation by means of negotiations.

The properties of an "optimal" negotiation were reviewed, showing that they leave lot of room for a variety of outcomes. On the other side, most properties derived are properties that are not able to be known at the agent level. So, for example, the ordering in the power set of messages can be only known by an almost omniscient agent that knows the endowments of all the agents in I. To put an example: in the negotiation between a potential buyer and a seller, one of them has to know the whole inventory of the seller (including the qualities of the products) and also all the money that the buyer can spend. Moreover, to ensure a Pareto-optimal outcome, both of them should have that information.

We have not yet studied protocols that may ensure a reasonable outcome to agents which are not fully informed about the initial characteristics of the whole group. But it seems to us that the complete framework for that analysis has been given in the last section.

So, the lines of future attack on the problem of representing negotiations is based on the relation between properties of protocols and corresponding outcomes. In any case, the possible results of a negotiation span a big set, and to detect policies for reducing that set is a main goal of this line of research.
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