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Sub 2 nm Particle Characterization in Systems with Aerosol Formation and Growth

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ABSTRACT OF THE DISSERTATION

Essays on Monetary Economics

by

Chien-Chiang Wang

Doctor of Philosophy in Economics
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Professor Stephen D. Williamson, Chair

In the first chapter, I propose a liquidity theory of yield curves to analyze the impact of quantitative easing, especially its influence on the yield curve and the inflation rate at the zero lower bound. In the model, a term premium originates from the endogenous difference in liquidity between securities of varying maturities, and the difference is generated by financial market frictions. Financial market frictions cause liquidation risk and reinvestment risk for holding assets, and households with different characteristics make different assessments of the two risks. Accordingly, different households require different term premia and endogenously participate in markets for different maturities. When the short-term interest rate reaches the zero lower bound, long-term interest rates may not reach the reservation interest rates for long-term bond buyers. Thus, central banks’ purchases of long-maturity securities can effectively decrease long-term interest rates and the term premium. Moreover, central banks’ long-term security purchases decrease inflation at the zero lower bound. These two effects together result in a distinct policy implication: quantitative easing shifts down the real yield curve at the long-maturity end but shifts it up at the short-maturity end if the households are sufficiently diverse in the term premia they require.

In the second chapter, I develop a dynamic general equilibrium model to investigate the interaction between asset market liquidity and repo haircuts. In the economy, investors finance their asset purchases through secured borrowing, and the asset is pledged
as collateral. Investors’ debt roll over before their assets mature. The maturity of assets is random, and default occurs when the borrowing limit is reached. The search and matching friction in the financial market results in delays in collateral liquidation, and therefore causes a gap between the asset price and the borrowing capacity, which is the haircut. The model reveals an endogenous feedback loop between asset market liquidity and repo haircuts. On the one hand, asset market liquidity determines the easiness of asset liquidation, which in turn determines the haircuts. On the other hand, haircuts influence entrepreneurs’ borrowing limits and leverage, which affect the probability of default and therefore influence the asset market liquidity. When an unanticipated shock on market liquidity occurs, the increase in haircuts decreases households’ borrowing limit and triggers simultaneous defaults. The liquidation of asset further decreases the liquidity of the asset market, and the impact is exacerbated by the endogenous feedback loop.

The third and final chapter studies the macroeconomic consequences of central banks’ risky asset purchases. By purchasing risky assets, central banks remove them from the financial market and inject money, which is a less risky and more liquid asset. Whereas, the removed risky assets stay in central banks’ balance sheets and increase the instability of their budgets, and thus, create inflation risk. The key friction in the model is the market segmentation between the money transaction sector and financial transaction sector. The households in money transaction sector can only use cash as a medium of exchange, but households in the financial transaction sector can use all forms of assets and asset backed securities to facilitate transaction. The central banks’ purchases of risky assets overcome the market segmentation and can improve social welfare through risk sharing between financial sector transactions and money transactions. However, because the risk in money transactions cannot be efficiently allocated between risk-averse and risk-neutral traders by financial intermediaries, central banks should make the holding of cash less risky, and it is not optimal for central banks to purchase all risky assets and completely insure the risk in the financial transactions with money transactions.
Chapter 1


1.1 Introduction

Prior to the 2008 financial crisis, the U.S. Federal Reserve primarily influenced the economy through conventional monetary policy: it adjusted the target short-term nominal interest rate and achieved this target rate through open market operations on overnight repos. Since 2008, owing to a series of interest rate cuts implemented by the Fed, the short-term nominal interest rate has been close to its effective zero lower bound, implying that there is little scope left for conventional monetary policy. However, the nominal interest rates of long-term Treasury securities remained higher than zero, and thus, central bankers turned their attention to long-term yields and conducted a series of quantitative easing operations. Under quantitative easing, central banks purchase long-term Treasury securities and other long-maturity assets such as mortgage-backed securities. The goal of the policy is to "ease" by placing downward pressure on long-term interest rates and
shifting down the yield curve. In contrast to conventional monetary policy which has been extensively studied, quantitative easing initiated a contentious debate over whether it would work, and if it does, which theory and mechanism it should work through.

In this paper, I propose a liquidity theory of yield curves to improve our understanding of the effects of quantitative easing, especially its impacts on the yield curve and the inflation rate. In my model, a term premium originates from the endogenous difference in liquidity between securities of varying maturities. The difference in liquidity is generated by financial market frictions, which limit arbitrage and raise the concern of a mismatch between the arrival of households’ liquidity demand and the maturity of securities. Households in the economy occasionally have the desire to consume goods, while money is the only asset which can be accepted as a medium of exchange to purchase goods. Other securities in the economy cannot be used in the same manner, but they pay money when they mature. Accordingly, two corresponding risks are generated: liquidation risk occurs when a household needs money, but the security it holds has not matured; reinvestment risk occurs when a security matures, but the holder of the security does not need money. Households may eliminate these two risks by trading in the financial market, but such activities are obstructed by financial market frictions. In principle, short-term securities bear greater reinvestment risk and long-term securities bear greater liquidation risk. As a result, the term premium can be positive or negative depending on the relative strength of these two risks.

Furthermore, households with different characteristics, such as the time discount rate and the financial market friction they encounter, value liquidation and reinvestment risk differently. Thus, different households require different term premia and endogenously trade securities at different maturities. A zero short-term interest rate means that the short-term yield reaches the security traders’ reservation short-term yield, and thus, purchases of short-term securities by central banks cannot further decrease the interest rate. However, the market term premium only reflects the term premium for the marginal
trader. When the short-term interest rate reaches the zero lower bound, there may exist some traders who demand smaller term premium and are willing to pay higher prices for the long-term securities. Thus, central banks’ purchases of long-maturity securities can decrease long-term interest rates and the term premium, and the effectiveness of the policy is determined by the degree of diversity in households’ term premia demand.¹

Financial market frictions have been broadly studied in the previous literature. For example, Alvarez et al. (2002) analyze the effects of money injections when the goods market and asset market are segmented: an agent who wants to transfer assets must pay a Baumol-Tobin-style fixed cost (Baumol, 1952; Tobin, 1956). In Aiyagari and Williamson (2000), individuals are, at random, subject to limited participation in the financial sector, and money is held to insure against these random circumstances.² In this paper, the financial market friction takes the form of randomly limited participation. This frictional financial market structure is closely related to that of Duffie et al. (2005), in which a model with an over-the-counter market featuring search and matching frictions is constructed, and traders can exchange assets only if they meet the counterparties or brokers.³

This paper is related to the literature on quantitative easing. Eggertsson and Woodford (2003) analyze quantitative easing in the New Keynesian framework, in which the financial market is arbitrage-free, and long-term interest rates only reflect households’ expectations concerning future short-term interest rates. As the short-term interest rate is controlled by central banks, quantitative monetary policy has no effect in their model. The arbitrage-free condition is relaxed in segmented market theory, which has been favored by policy makers as a transmission mechanism of quantitative easing.⁴ The theory posits

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¹ By employing various methods and data, most empirical studies agree that quantitative easing has helped to reduce long-term interest rates. For example, see D’Amico and King (2011), Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), Li and Wei (2013) and Wu (2014). In contrast, Bauer and Rudebusch (2011) and Christensen and Rudebusch (2012) argue that quantitative easing works mainly through the signaling channel.

² Among others, limited participation in the financial market is applied in Alvarez et al. (2009), Duffie and Sun (1990), Gabaix and Laibson (2001), and Alvarez et al. (2004).

³ The friction in the asset market is also closely related to the literature on the partial liquidity of assets. See, for example, Li et al. (2012) and Rocheteau et al. (2014).

⁴ Segmented market theory is closely related to preferred habitat theory (Modigliani and Sutch, 1966).
that markets for assets of different maturities are segmented, or financial investors have preferences for assets of particular maturities. Under these presumptions, arbitrage is limited, and the interest rate for a given maturity is determined in part by the demand and supply specific to that maturity. However, the theory lacks an explanation for why the markets for long- and short-maturity securities are segmented or why an investor prefers assets of particular maturities. Moreover, the theory severs the connection between long-term yields and short-term yields, and thus, yield curves can take any shape. However, one can observe a rigid pattern of yield curves and a comovement of long- and short-term yields in the real world, which demonstrates a strong connection between long- and short-term securities.

This paper provides an alternative to segmented market theory. In my model, the friction exists in the financial market as a whole, and a household is allowed to trade securities of all maturities if it successfully enters the financial market. The diversity in households’ portfolios does not come from market segmentation but is an endogenous outcome, and households change the maturity they trade depending on market interest rates. Without exogenous market segmentation, the connection between long- and short-term securities is preserved, and the shapes of the equilibrium market yield curves are upward and downward sloping. Accordingly, although purchases of long-term securities by the central bank can decrease long-maturity yields, such purchases are less effective at doing so when the short-term interest rate reaches the zero lower bound or is controlled by central banks.

My paper is closely related to the literature on the term premium from the perspective of asset liquidity. In the New Monetarist framework, Williamson (2016) and Gromichalos et al. (2013) construct a positive term premium by arguing that short-term bonds are more liquid than long-term bonds. The difference between long- and short-

---

Recent developments in macroeconomics include Andres et al. (2004) and Chen et al. (2012), among others.

5 An alternative, reduced-form approach is Vayanos and Vila (2009). They combine preferred habitat theory and no-arbitrage affine models and strengthen the connection between long- and short-maturity yields by introducing risk-averse arbitrageurs that can trade in all maturities.
term yields comes in part from the liquidity premium, and when the short-term yield reaches the zero lower bound, long-term yields also reach their effective lower bound. Thus, liquidity premium theory predicts that quantitative easing has limited effects on reducing long-term yields when short-term yields reach zero. In Williamson (2016), quantitative easing swaps illiquid long-term securities for money. The swap increases the quantity of money and overall liquidity and, thus, decreases the inflation rate. As a result, at the zero lower bound, quantitative easing shifts up the real yield curve because of the deflation effect.\footnote{Similar results are obtained in Braun and Oda (2010) and Herrenbrueck (2014).} In my model, a reduction in the supply of long-term bonds also decreases inflation when the government is subject to a constraint on tax levies or seigniorage revenue.

In contrast to prior approaches, the heterogeneity of households generates greater flexibility between long- and short-term interest rates and is crucial to the effectiveness of quantitative easing. As a result, at the zero short-term interest rate, if households are sufficiently diverse in the term premia they demand, the decrease in long-term nominal interest rates may dominate the decrease in inflation, and thus, long-term asset purchases shift down the real yield curve at the long-maturity end but shift up the real yield curve at the short-maturity end.

The remainder of the paper is organized as follows. In Section 2, I describe a baseline representative household model with financial frictions. In Section 3, I analyze the equilibrium and incorporate monetary policies into the model. In Section 4, I introduce the complete model with heterogeneous households. Section 5 concludes.
1.2 The Representative Household Model

1.2.1 Environment

Time $t = 1, 2, \ldots$ is discrete and infinite. There is a continuum of infinitely lived households indexed by $\omega \in [0, 1]$. There are three types of goods in the economy. The first is a consumption good, which generates utility for households and is perfectly divisible and perishable. The other two types of goods are nonperishable assets: cash, $M$, and nominal bonds with various maturities, $B = (B_1, B_2, \ldots, B_k)$. Both cash and nominal bonds are issued by the government, and the quantities of asset are defined as of the end of each period. For tractability, I assume that the maturity of bonds is random as in Leland (1994). Let $\lambda_i > 0$ denote the maturity rate of bond $B_i$. Then, at the beginning of each period, with probability $\lambda_i$, one unit of nominal bonds matures, and the household receives one unit of cash from the government; with probability $1 - \lambda_i$, one unit of nominal bonds does not mature and remains in the bondholder’s account. I denote by $\theta_i = \frac{1}{\lambda_i}$ the average maturity of bonds, and hence, longer-term bonds have larger $\theta_i$ (smaller $\lambda_i$), and shorter-term bonds have smaller $\theta_i$ (larger $\lambda_i$).

Each period is divided into two subperiods, and the subperiods are identified by their markets. In the first subperiod, the asset market opens after bonds mature. Households exchange cash and bonds in the asset market, and the government also sells new issued bonds in the asset market. In the second subperiod, the goods market opens, and the trade of goods takes place. Figure 1.1 depicts the timing of the markets, and further details are described below.

A household can be in one of two states $J = \{c, n\}$: a consumer ($j = c$) or a non-consumer ($j = n$). A consumer can produce and consume in the goods market; a non-consumer can only produce but cannot consume. A household does not consume its own output, meaning that it must exchange goods in the goods market. At the beginning of each period, before the asset market opens, the states of households are realized.
A household is in the consumer state with probability $\alpha$ and in the non-consumer state with probability $1 - \alpha$, and the state is iid. Let $C_t \subset [0,1]$ denote the set of consumers and $N_t \subset [0,1]$ denote the set of non-consumers at time $t$. Household $\omega$ is a consumer at time $t$ if $\omega \in C_t$ and is a non-consumer if $\omega \in N_t$. I assume that utility is linear in consumption: consuming $c$ units of goods generates $c$ units of utility. Producing $h$ units of consumption goods generates convex disutility $l(h)$ with $l(0) = 0$, $l'(0) = 0$. Households discount future at the rate $\frac{1}{1+\rho}$, $\rho > 0$. A household’s lifetime utility can be expressed as follows:

$$E_0 \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} [x_t c_t - l(h_t)].$$

Where $x_t$ denotes the realization of a preference shock. When $x_t = 1$, consumption generates utility, which represents the consumer state; when $x_t = 0$, consumption does not generate utility, which represents the non-consumer state.

I assume that households are anonymous, meaning that credit arrangements are not feasible in the goods market or the asset market. Added to the assumption of perishable consumption goods, households that wish to buy consumption goods in the goods mar-
ket must pay with media of exchange. I further assume that cash is the only asset that can be recognized in the goods market, and thus, cash is the only asset that can be used as a medium of exchange, while bonds cannot be so used.

The asset market is subject to random limited participation. With probability $\mu$, a household is able to participate in the asset market; with probability $1 - \mu$, the household fails to participate in the asset market but goes straight to the goods market. The asset market is competitive for the participants.\(^7\) A household realizes whether it can trade in the asset market at the beginning of each period. In the electronic world, bonds are automatically redeemed at maturity: when one unit of bonds matures, one unit of cash is directly transferred to the bondholder’s account, and the unit of bonds simultaneously disappears from its account. Note that a household does not have to participate the financial market to find the government or to contact the dealer to redeem its bonds, meaning that payment for a bond at maturity is not subject to the financial friction.

I denote the price of cash in terms of goods in the goods market by $\phi_t$. The inflation rate is

$$1 + \pi_t = \frac{\phi_{t-1}}{\phi_t}.\]

I also denote cash in real terms by $m_t = \phi_t M_t$, and nominal bonds in real terms by $b_{i,t} = \phi_t B_{i,t}$. For the sake of conciseness, $m_t$ and $b_t = (b_{1,t}, b_{2,t}, \ldots, b_{k,t})$ will be called cash and bonds hereafter.

It is worthwhile to compare this model with models that embed the over-the-counter market with search and matching frictions in the monetary-search framework (Lagos and Wright, 2005). For example, in Lagos and Zhang (2013), Geromichalos and Herrenbrueck (2014), and Geromichalos et al. (2013), households face idiosyncratic demand shocks, and

\(^7\)This asset market structure is equivalent to a simplified version of the over-the-counter market structure of Duffie et al. (2005), where a household can only meet an asset dealer in the decentralized market with probability, $\mu$, but bilateral meetings between households are excluded. Moreover, a household makes a take-it-or-leave-it offer to the dealer in the meeting, and only dealers can access a Walrasian inter-dealer market to exchange assets and cash. Consequently, the prices of assets in the bilateral meeting are equal to the prices in the inter-dealer market.
the shocks are the key factors that generate demand and supply in the over-the-counter market. However, the idiosyncratic shocks also generate heterogeneity in households’ asset holdings. In their models, there is a periodic Walrasian market in which households can exchange all goods and assets; the Walrasian market facilitates portfolio rebalancing and helps degenerate the distribution of individual’s asset holdings. However, the matching frictions in the over-the-counter market also become less sustainable because households can always engage in arbitrage trades in the Walrasian market.

In this paper, I do not assume a periodic Walrasian market for the exchange of all goods and assets, and hence, the only market in which households can exchange cash and bonds is the frictional asset market. As a result, arbitrage trading is obstructed, and the friction in the asset market can be sustained into the future. The distribution of asset holdings is not degenerate in this model; thus, households may hold different portfolios depending on the history of their states. However, the linearity of consumption utility results in constant marginal values of assets, meaning that asset holdings have no effect on households’ decisions. This property helps analyze households’ behavior under a non-degenerate asset distribution.

1.2.2 Households’ Problems

Households’ problems will be discussed in reverse order of time. I first analyze the goods market, and then consider the asset market.

The Goods Market

A household’s state variables are the amount of cash and bonds the household brings into each market, \((m_t, b_t)\). I assume that \(\frac{1}{(1+\rho)(1+\pi_t)} < 1\), or a household will defer its consumption. The government imposes a lump-sum tax, \(\tau_t\), in the form of cash in the goods market, and the tax is independent of the household’s state and asset holdings. Let \(\Phi^j_t(m_t, b_t)\) and \(U^j_t(m_t, b_t)\) denote the value of a \(j\)-state household at the beginning
of a period and the beginning of the goods market, respectively. Since a household is a consumer with probability $\alpha$ and a non-consumer with probability $(1 - \alpha)$, I denote $\Phi_{t+1}^c (m_{t+1}, b_{t+1}) \equiv \alpha \Phi_{t+1}^e (m_{t+1}, b_{t+1}) + (1 - \alpha) \Phi_{t+1}^n (m_{t+1}, b_{t+1})$ to be a household’s expected value at the beginning of the next period. $U^j_t (m_t, b_t)$ satisfies the following Bellman equation:

$$U^j_t (m_t, b_t) = \max_{c_t, h_t, m_{t+1}, \tilde{m}_t} x_t^j c_t - l(h_t) + \frac{1}{1 + \rho} \Phi_{t+1}^e (m_{t+1}, b_{t+1})$$

subject to

$$\begin{align*}
\tilde{m}_t &= h_t + m_t - \tau_t - c_t \\
c_t &\leq m_t \\
\tilde{b}_{i,t} &= b_{i,t} \\
m_{t+1} &= \frac{1}{1 + \pi_t} \left( \tilde{m}_t + \sum_i \lambda_i \tilde{b}_{i,t} \right) \\
b_{i,t+1} &= \frac{1}{1 + \pi_t} (1 - \lambda_i) \tilde{b}_{i,t} \\
c_t &\geq 0
\end{align*}$$

The first constraint is the budget constraint: consumers’ end-of-period cash holdings, $\tilde{m}_t$, must be equal to the initial cash holdings, $m_t$, plus labor income, $h_t$, minus the tax charged by the government, $\tau_t$, and minus consumption, $c_t$. The second constraint is the cash-in-advance constraint. As there is no credit in the goods market, consumption is subject to the cash holdings at the beginning of goods market trading. The third constraint states that consumers simply retain bonds until the goods market closes, because bonds cannot be employed as a medium of exchange in the goods market. Finally, a $\lambda_i$ proportion of bond $i$ matures at the beginning of the next period, and thus, after being discounted by the inflation rate, the cash holding at the beginning of the subsequent asset market phase is $\frac{1}{1 + \pi_t} \left( \tilde{m}_t + \sum_i \lambda_i \tilde{b}_{i,t} \right)$, and the holding of bond $i$ is $\frac{1}{1 + \pi_t} (1 - \lambda_i) \tilde{b}_{i,t}$.

I denote by $v^j_t$ and $w^j_{i,t}$ the marginal value of cash and bonds when the goods market phase begins. That is,
\[ v^j_t = \frac{\partial U^j_t}{\partial m_t}(m_t, b_t), \]
\[ w^j_{i,t} = \frac{\partial U^i_t}{\partial b_i}(m_t, b_t). \]

Let \( \frac{1}{1 + d_t} = \frac{1}{(1 + \rho)(1 + \pi_t)} < 1 \) denote the discount factor. The necessary conditions for consumers’ optimal decision are as follows:\(^8\)

\[ \frac{dl(h)}{dh} = \frac{1}{1 + d_t} \frac{\partial \Phi_{t+1}}{\partial m_t}(m_{t+1}, b_{t+1}), \quad (1.1) \]
\[ v^j_t = \max \left\{ x^j_t, \frac{1}{1 + d_t} \frac{\partial \Phi_{t+1}}{\partial m_t}(m_{t+1}, b_{t+1}) \right\}, \quad (1.2) \]
\[ w^j_{i,t} = \frac{1}{1 + d_t} \left[ \lambda_i \frac{\partial \Phi_{t+1}}{\partial m_i}(m_{t+1}, b_{t+1}) + (1 - \lambda_i) \frac{\partial \Phi_{t+1}}{\partial b_i}(m_{t+1}, b_{t+1}) \right]. \quad (1.3) \]

These conditions have straightforward interpretations. Equation (1.1) states that the marginal disutility of working must be equal to its marginal benefit, which is the discounted expected marginal value of cash in the next period. Equation (1.2) states that households make decisions concerning spending cash by comparing the marginal utility of consumption, \( x^j_t \), with the future value of cash, \( \frac{1}{1 + d_t} \frac{\partial \Phi_t}{\partial m_t} \). Recall that a consumer gains one unit of utility by consuming one unit of goods (\( x^c_t = 1 \)). Thus, if \( \frac{1}{1 + d_t} \frac{\partial \Phi_t}{\partial m_t} < 1 \), a consumer spends all its cash on purchasing goods (\( c_t = m_t \)); if \( \frac{1}{1 + d_t} \frac{\partial \Phi_t}{\partial m_t} > 1 \), a consumer does not purchase goods but keeps its cash for the next period (\( c_t = 0 \)). A non-consumer, however, does not gain utility from consumption (\( x^n_t = 0 \)); thus, it retains all of its cash into the next period. Because households cannot use bonds as a medium of exchange in the goods market, the marginal value of bonds in the goods market, \( w^c_{i,t} \), is equal to their discounted expected future value (equation (1.3)).

Because the utility function is linear in consumption, households may play Ponzi games when the return on saving is too high: they may not spend cash on goods and

\(^8\)See the Appendix for the proof of the necessary conditions.
Figure 1.2: Market structure and households’ value functions

Instead hold all of their cash into the next period. These Ponzi game paths are excluded by the transversality conditions:

\[
\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t v_i^j m_{t+1} = 0,
\]
\[
\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t w_{i,j} b_{i,t+1} = 0.
\] (1.4)

The Asset Market

In the first subperiod, the asset market opens. With probability \( \mu \), a household can exchange different bonds for cash in different submarkets, but there is no market for households to directly exchange different maturity bonds with one another. Thus, there exist \( k \) submarkets for bonds \( i = 1, 2, \cdots, k \), and I denote the price of bond \( i \) in terms of cash as \( q_i \).

Consumers and non-consumers face the same problem in the asset market. For a household that cannot participate in the asset market, its value in the first subperiod is equal to its value in the goods market, \( U_i^j(m_t, b_t) \). I denote \( \Omega_i^j(m_t, b_t) \) to be the value of a household that can participate in the asset market. Figure 1.2 provides an illustration of the market structure. A household’s value at the beginning of each period can be expressed as

\[
\Phi_i^j(m_t, b_t) = \mu \Omega_i^j(m_t, b_t) + (1 - \mu) U_i^j(m_t, b_t).
\] (1.5)
\( \Omega^j_t(m_t, b_t) \) can be written as the following optimization problem:

\[
\Omega^j_t(m_t, b_t) = \max_{m^+_{i,t}, b^+_{i,t}, \tilde{m}_{i,t}, \tilde{b}_{i,t}} U^j_t(\tilde{m}_t, \tilde{b}_t)
\]

Subject to

\[
\begin{align*}
\tilde{m}_t &= m_t - \sum_i m^+_{i,t} + \sum_i q_{i,t} b^+_{i,t} \\
\tilde{b}_{i,t} &= b_{i,t} + \frac{m^+_{i,t}}{q_{i,t}} - b^+_{i,t} \\
\sum_i m^+_{i,t} &\leq m_t \\
b^+_{i,t} &\leq b_{i,t} \\
m^+_{i,t} &\geq 0 \\
b^+_{i,t} &\geq 0
\end{align*}
\]

There are two types of transactions that a household can do in the asset market. First, it can sell cash for bond \( i \) in submarket \( i \), denoted by \( m^+_{i,t} \); this activity decreases its cash holdings by \( m^+_{i,t} \) and increases its bond \( i \) holdings by \( \frac{m^+_{i,t}}{q_{i,t}} \). A household can also sell bond \( i \) in submarket \( i \) for cash, denoted by \( b^+_{i,t} \); this activity decreases its bond \( i \) holdings by \( b^+_{i,t} \) and increases its cash holdings by \( q_{i,t} b^+_{i,t} \). Households are anonymous, meaning that they are not allowed to issue their own liabilities. Thus, the amount of assets for sale is subject to the assets-in-advance constraint, \( \sum_i m^+_{i,t} \leq m_t \) and \( b^+_{i,t} \leq b_{i,t} \).

Similar to the households’ problem in the goods market, we have the following equations for the marginal value of cash and bonds in the asset market.\(^9\)

\[
\frac{\partial \Omega^j_t}{\partial m_{t}}(m_t, b_t) = \max \left\{ v^j_{i,t}, \frac{w^j_{1,t}}{q_1}, \ldots, \frac{w^j_{k,t}}{q_k} \right\} \tag{1.6}
\]

\[
\frac{\partial \Omega^j_t}{\partial b_{i,t}}(m_t, b_t) = \max \left\{ q_{i,t} v^j_{i,t}, w^j_{i,t} \right\} \tag{1.7}
\]

These conditions state that a household sells its assets in the submarkets that provide the best terms of trade, or the household simply brings the assets into the goods market without selling them in the asset market. I first consider the households’ decisions

\(^9\)See appendix for the proofs of the necessary conditions
regarding selling cash. For example, suppose there is a bond $i$, such that $\frac{w_i^t}{q_i^t} > v_i^t$ and $\frac{w_i^t}{q_i^t} > \frac{w_k^t}{q_k^t}$, for all $k \neq i$, then the bond $i$ submarket provides the best terms of trade for the households. In this case, the household must sell all its cash for bond $i$ ($m_{i,t}^t = m_t$). Similarly, if $v_i^t > \frac{w_i^t}{q_i^t}$ for all $i$, the prices of bonds in all submarkets are all too high; thus the household simply retains all its cash and brings the cash into the goods market ($m_{i,t}^t = 0$ for all $i$). Households’ decisions regarding selling bonds have similar interpretations.

Let $\hat{v}_i^t$ and $\hat{w}_i^t$ denote the marginal value of assets at the beginning of each period, before households realize whether they can trade in the asset market. We have the following equations for $\hat{v}_i^t$ and $\hat{w}_i^t$, by Combining (1.5), (1.6), and (1.7):

$$\hat{v}_i^t = \partial \Phi_i^t (m_i^t, b_i^t) = \mu \max \left\{ \frac{w_i^t}{q_i^t} - v_i^t, \ldots, \frac{w_k^t}{q_k^t} - v_i^t, 0 \right\} + v_i^t,$$  \hspace{1cm} (1.8)

$$\hat{w}_i^t = \partial \Phi_i^t (m_i^t, b_i^t) = \mu \max \left\{ q_i^t v_i^t - w_i^t, 0 \right\} + w_i^t.$$  \hspace{1cm} (1.9)

Equations (1.8) and (1.9) state that the value of assets in the asset market is equal to their value in the goods market plus the trading surplus in the asset market, and the trading surplus is subject to asset market frictions, $\mu$.

Finally, the transversality conditions in the asset market also need to be satisfied:

$$\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t \hat{v}_i^t m_{i,t+1} = 0,$$

$$\lim_{t \to \infty} \left( \frac{1}{1+\rho} \right)^t \hat{w}_i^t b_{i,t+1} = 0.$$  \hspace{1cm} (1.10)

1.2.3 The Government

The government is the only issuer of cash and bonds. Let $\bar{M}_i^t$ and $\bar{B}_i^t$ denote the aggregate bond supply issued by the government at the end of each period. The supply of cash and
bonds has the same growth rate $\gamma_t$, and thus,

$$\frac{\bar{M}^g_t}{\bar{M}^g_{t-1}} = \frac{\bar{B}^g_{i,t}}{\bar{B}^g_{i,t-1}} = 1 + \gamma_t.$$ 

To achieve the target quantity of bonds, $\bar{B}^g_{i,t}$, the government sells new bonds in the asset market. Because $\lambda_i$ proportion of old bonds $i$ mature at the beginning of each period, the government issues $\bar{B}^g_{i,t} - (1 - \lambda_i)\bar{B}^g_{i,t-1}$ units of new bond $i$. I restrict my attention to the case in which the government is a bond seller in the asset market; that is,

$$\bar{B}^g_{i,t} - (1 - \lambda_i)\bar{B}^g_{i,t-1} > 0,$$

Because the government is the issuer of cash and bonds and faces no asset-in-advance constraint, I assume that the government has no limited participation problem and can always sell the new bonds in the asset market. The government must satisfy the following budget constraint:

$$\phi_t T_t + \left\{ \phi_t \bar{M}^g_t - \phi_t \bar{M}^g_{t-1} \right\} + \left\{ \sum_i q_{i,t} \phi_t \left[ \bar{B}^g_{i,t} - (1 - \lambda_i)\bar{B}^g_{i,t-1} \right] - \sum_i \lambda_i \phi_t \bar{B}^g_{i,t-1} \right\} = 0.$$

There are three components in the government’s budget constraint. The first is lump-sum taxes, $\phi_t T_t$. The second is the seigniorage revenue, $\phi_t \bar{M}^g_t - \phi_t \bar{M}^g_{t-1}$. The third is the surplus from the government’s debt management, $\sum_i q_{i,t} \phi_t \left[ \bar{B}^g_{i,t} - (1 - \lambda_i)\bar{B}^g_{i,t-1} \right] - \sum_i \lambda_i \phi_t \bar{B}^g_{i,t-1}$, which equals the revenue from bond issuances minus the payments at bond maturity. The government’s balanced budget constraint can be rewritten in real terms as follows:

$$\tau_t + \left\{ \bar{m}^g_t - \frac{1}{1 + \pi_t} \bar{m}^g_{t-1} \right\} + \left\{ \sum_i q_{i,t} \left[ \bar{b}^g_{i,t} \frac{(1 - \lambda_i)}{1 + \pi_t} \bar{b}^g_{i,t-1} \right] - \sum_i \frac{1}{1 + \pi_t} \lambda_i \bar{b}^g_{i,t-1} \right\} = 0.$$

(1.11)

A quantitative monetary policy instrument is an adjustment of the composition of
the central bank’s balance sheet and outstanding government liabilities. In the baseline model, I assume that the government controls the growth rate of the asset supply, $\gamma_t$, and chooses the outstanding bond supply $(\bar{b}_{1,t}^g, \bar{b}_{2,t}^g, \ldots, \bar{b}_{k,t}^g)$. The amount of cash, $\bar{m}_i^g$, and the tax, $\tau_t$, are endogenously determined.\(^{10}\)

### 1.2.4 Market Clearing Conditions

As noted in Section 2.1, the linearity of consumption utility results in no wealth effect, and hence, it is sufficient to analyze the aggregate asset holdings without considering the asset distributions among households.

Let $\bar{m}_i^j$ and $\bar{b}_i^j$ denote the total amount of cash and bonds held by state-$j$ households at the beginning of the asset market phase and $\bar{m}_i^{j,l}$ and $\bar{b}_i^{j,l}$ denote the total amount of asset holdings at the beginning of the goods market phase. I also denote by $\bar{m}_i^{j,l}$ and $\bar{b}_i^{j,l}$ the total amount of cash and bonds sold by state-$j$ households in the market for bond $i$.

The following asset market clearing condition is thus obtained for the bond $i$ submarket:

$$\frac{1}{q_{i,t}} \left[ \bar{m}_{i,t}^{tc} + \bar{m}_{i,t}^{tn} \right] = \bar{b}_{i,t}^{tc} + \bar{b}_{i,t}^{tn} + \left[ \bar{b}_{i,t}^g - \left( \frac{1 - \lambda_i}{1 + \pi} \right) \bar{b}_{i,t-1}^g \right].$$

(1.12)

The left-hand side is the demand for bond $i$, which consists of the cash sold by consumers and non-consumers in the bond $i$ submarket divided by the price of bond $i$. The right-hand side is the supply of bond $i$, which consists of the quantity of bond $i$ sold by consumers and non-consumers plus the new quantity of bond $i$ issued by the government, $\bar{b}_{i,t} - \left( \frac{1 - \lambda_i}{1 + \pi} \right) \bar{b}_{i,t-1}$. Notice that only proportion $\mu$ of households can trade in the asset market, and thus, it must be the case that $\bar{m}_i^{j,l} \leq \mu \bar{m}_i^j$ and $\bar{b}_i^{j,l} \leq \mu \bar{b}_i^j$.

In the goods market, the resource constraint must hold: the goods produced by the households must be equal to the amount of goods consumed by consumers. Thus,

\(^{10}\) Alternative policy regimes would be to assume that the government is committed to a certain flow of taxes and transfers, or also including debt management. Such policy regimes will be discussed in Section 1.3.6
\[ h_t = \int c_t(\omega) 1_{C_t}(\omega) d\omega, \quad (1.13) \]

where \( 1_A(\omega) \) is the indicator function

\[
1_A(\omega) := \begin{cases} 
1 & \text{if } \omega \in A \\
0 & \text{if } \omega \notin A 
\end{cases}
\]

### 1.2.5 Asset Distributions

At the beginning of each period, before the asset market opens, the existing bonds mature, and households draw their new states. Let \( \tilde{m}_i^\prime \) and \( \tilde{b}_i^\prime \) denote the total amount of cash and bonds held by the households at the end of a period. Then, the aggregate cash holdings at the beginning of period \( t \), after bonds mature, are \( \frac{1}{1 + \pi_t}(\tilde{m}_i^\prime_{t-1} + \sum_i \lambda_i \tilde{b}_i^\prime_{t-1}) \), and the aggregate real bond \( i \) holdings equal \( \frac{1}{1 + \pi_t}(1 - \lambda_i)\tilde{b}_i^\prime_{t-1} \). A proportion \( \alpha \) of assets are held by consumers, and a proportion \( (1 - \alpha) \) of assets are held by non-consumers. Let \( \tilde{m}_i^\prime \) and \( \tilde{b}_i^\prime \) denote the cash and bond holdings at the beginning of a period, before the asset market open; then

\[
\tilde{m}_i^\prime = \frac{\alpha}{1 + \pi_t}(\tilde{m}_{i-1}^\prime + \sum_i \lambda_i \tilde{b}_{i-1}^\prime); \quad \tilde{b}_i^\prime = \frac{\alpha}{1 + \pi_t}(1 - \lambda_i)\tilde{b}_{i-1}^\prime; \\
\tilde{m}_i^n = \frac{1 - \alpha}{1 + \pi_t}(\tilde{m}_{i-1}^\prime + \sum_i \lambda_i \tilde{b}_{i-1}^\prime); \quad \tilde{b}_i^n = \frac{1 - \alpha}{1 + \pi_t}(1 - \lambda_i)\tilde{b}_{i-1}^\prime.
\]

In the asset market, households exchange cash and bonds. Cash holdings at the close of the asset market should be equal to cash holdings at the opening of the asset market, \( \tilde{m}_i^\prime \), minus the aggregate amount of cash sold in all bond submarkets, \( \sum_i \tilde{m}_{i,t}^\prime \), plus the amount of cash gained from selling bonds, \( \sum_i q_{i,t} \tilde{b}_{i,t}^\prime \). The bond \( i \) holdings at the opening of the goods market have similar interpretations. Let \( \tilde{m}_i^\prime \) and \( \tilde{b}_i^\prime \) denote the cash and bond holdings at the end of the asset market phase; then,

\[
\tilde{m}_i^\prime = \tilde{m}_i^c - \sum_i \tilde{m}_{i,t}^c + \sum_i q_{i,t} \tilde{b}_{i,t}^c; \quad \tilde{b}_i^c = \tilde{b}_i^c + \frac{1}{q_{i,t}} \tilde{m}_{i,t}^c - \tilde{b}_{i,t}^c; \\
\tilde{m}_i^n = \tilde{m}_i^n - \sum_i \tilde{m}_{i,t}^n + \sum_i q_{i,t} \tilde{b}_{i,t}^n; \quad \tilde{b}_i^n = \tilde{b}_i^n + \frac{1}{q_{i,t}} \tilde{m}_{i,t}^n - \tilde{b}_{i,t}^n.
\]
At the beginning of the goods market phase, households’ asset holdings are equal to the amount held at the close of the asset market. In the goods market, households’ cash holdings increase because of the sale of consumption goods, $h_t$. Moreover, consumers’ cash holdings decrease because of the purchases of consumption goods, $-\int c_t(\omega)1_{c_t}(\omega)d\omega$, and tax, $\tau$. Therefore, the total amount of households’ cash holdings at the end of a period, $\bar{m}_t'''$, satisfies

$$\bar{m}_t''' = \bar{m}_t'' + \bar{m}_t^c + h_t - \tau_t - \int c_t(\omega)1_{c_t}(\omega)d\omega.$$  

Because households cannot exchange bonds in the goods market, the end-of-period bond holdings are equal to the bond holdings at the opening of the goods market. Thus,

$$\bar{b}_{i,t}''' = \bar{b}_{i,t}'' + \bar{b}_{i,t}^c.$$  

Finally, the total amount of cash and bonds held by the households in the end of each period must be consistent with the total amount of outstanding cash and bonds issued by the government. Thus

$$\bar{m}_t''' = \bar{m}_t^g,$$

$$\bar{b}_{i,t}''' = \bar{b}_{i,t}^g.$$

### 1.3 Equilibrium

I focus on the stationary equilibrium in which asset prices, asset values, and the inflation rate are constant over time. Recall that the distribution of asset holdings is not degenerate in this model. Moreover, bonds with different maturities are perfect substitutes for households, and thus, the equilibrium distribution of households’ asset allocations may not be unique. The stationary monetary equilibrium is defined as follows.
1.3.1 Definition of Equilibrium

Definition 1. A stationary monetary equilibrium is a vector \( x = (q, c, h, m^g, b^g, \pi, \tau, v^j, w^j, \hat{v}^j, \hat{w}^j) \) with corresponding asset allocations \( (\bar{m}^j, \bar{m}^j, \bar{m}^j, \bar{m}^j, \bar{b}^j, \bar{b}^j, \bar{b}^j, \bar{b}^j) \) such that:

1. Households optimize in the goods market and asset market.
2. The government budget constraint holds.
3. The goods market and asset market clear.

In the definition above, government debt, \( \bar{b}^g \), and the rate of money growth, \( \gamma \), are chosen by the policy maker; government transfers, \( \tau \), and the supply of cash, \( \bar{m}^g \), are endogenously determined. In a stationary equilibrium, \( \bar{m}^g_t = \bar{m}^g_{t-1} \), and thus, \( \phi_t M^S_t = \phi_{t+1} M^S_{t+1} \), which implies that the inflation rate must be equal to the growth rate of the asset supply, \( \pi = \gamma \).

1.3.2 Households’ Trading Strategy

In a stationary equilibrium, all of the time subscripts are dropped. To exclude the Ponzi game paths, the continuation value of cash in the goods market must be smaller than one. That is,

\[
\frac{1}{1 + d} [\alpha \delta^c + (1 - \alpha) \delta^n] < 1, \tag{\!*}
\]

which guarantees that consumers spend all of their cash on consumption in the goods market but do not accumulate cash for the future; therefore,

\[
\int c(\omega)1_C(\omega)d\omega, = \bar{m}^c, \tag{1.14}
\]

and the goods market resource constraint can be rewritten as

\[
\bar{m}^c = h. \tag{1.15}
\]
Moreover, combining (1.2) and (*), we have

\[ v^c = 1. \]  

(1.16) states that the value of cash for a consumer comes from the purchases of consumption goods, and because one unit of consumption goods generates one unit of utility, the value of cash for a consumer is one. First, because bonds can not be used as media of exchange in the goods market, consumers and non-consumers value bonds equally in the goods market, \( w^c_i = w^n_i \equiv w_i \) (see equation(1.3)). Second, the benefit of selling cash in different bond markets for the representative household must be the same. That is,

\[ \frac{w_i}{q_i} = \frac{w_{i'}}{q_{i'}} \text{ for all } i, i' \in 1, \ldots, k. \]

The reason is simple. Suppose that there exist bond \( i \) and bond \( i' \) such that \( \frac{w_i}{q_i} > \frac{w_{i'}}{q_{i'}} \); then, because bond \( i \) market provides better terms of trade for the representative household than the bond \( i' \) market does, there will be no demand for bond \( i' \). However, there is positive supply in the bond \( i' \) market because of the new bond \( i' \) issued by the government. Thus, the market clearing condition for the bond \( i' \) market does not hold. Therefore, I assume that \( \frac{w_i}{q_i} = \frac{w_{i'}}{q_{i'}} = V \) for all \( i, i' \in 1, \ldots, k \). Thus, consumers’ and non-consumers’ valuations of assets can be simplified as the following equations:

1. Asset values in the asset market (1.8), (1.9) can be rewritten as follows:

\[
\begin{align*}
\hat{v}^c &= \mu \max \{v^c, V\} + (1 - \mu)v^c \quad \text{(V1)} \\
\hat{v}^n &= \mu \max \{v^n, V\} + (1 - \mu)v^n \quad \text{(V2)} \\
\hat{w}^c_i &= \mu \max \{v^c, V\} + (1 - \mu)V \quad \text{(V3)} \\
\hat{w}^n_i &= \mu \max \{v^n, V\} + (1 - \mu)V \quad \text{(V4)}
\end{align*}
\]
2. Asset values in the goods market (1.2), (1.3) can be rewritten as follows:

\[ v^c = 1 \] \hspace{1cm} (V5)

\[ v^n = \frac{1}{1+d} [\alpha \hat{v}^c + (1-\alpha) \hat{v}^n] \] \hspace{1cm} (V6)

\[ V = \frac{1}{1+d} \left\{ \frac{\lambda_i}{q_i} [\alpha \hat{v}^c + (1-\alpha) \hat{v}^n] + (1-\lambda_i) \left[ \alpha \frac{\hat{w}^c_i}{q_i} + (1-\alpha) \frac{\hat{w}^n_i}{q_i} \right] \right\} \] \hspace{1cm} (V7)

and

\[ \frac{w_i}{q_i} = V \] \hspace{1cm} (V8)

Note that \( V \) represents the exchange value of cash for a household in the asset market: on the one hand, a household can sell one unit of cash for \( \frac{1}{q_i} \) units of bond \( i \) in the asset market and obtain \( \frac{w_i}{q_i} = V \) units of utility; on the other hand, a household can also obtain one unit of cash by selling \( \frac{1}{q_i} \) units of bond \( i \) in the asset market, which costs the household \( \frac{w_i}{q_i} = V \) units of utility. As a result, households' decisions on asset exchanges are determined by the relative value between the exchange value of cash \( V \) and the use value of cash in the goods market, \( v^n \) and \( v^c \).

Given the exchange value of cash, \( V \), the following asset values and bond prices \( v^n \), \( v^c \), \( \hat{v}^n \), \( \hat{v}^c \), \( w^n_i \), \( w^c_i \), \( \hat{w}^n_i \), \( \hat{w}^c_i \), \( q_i \) can be solved from (V1) to (V8). Note that (\( \ast \)) implies \( v^c > v^n \), and thus, we can divide (V1) to (V8) into three cases:

1. \( V < v^n < v^c \): In this case, the exchange value of cash in the asset market, \( V \), is lower than both \( v^c \) and \( v^n \), meaning that a household does not purchase bonds in either the consumer state or the non-consumer state. Therefore, the asset market provides no value added for cash, and the value of cash in the asset market is equal to its value in the goods market; that is,

\[ \hat{v}^c = v^c, \] \hspace{1cm} (1.17)

\[ \hat{v}^n = v^n. \]
Combining (V6) and (1.17), one finds that $v^n = \frac{\alpha}{\alpha + d}$ and $v^c = 1$. Because of the requirement that $V < v^n$, this case holds if and only if $V < \frac{\alpha}{\alpha + d}$. Note that in this case, there is no demand for any bonds in the asset market. However, there is always a positive supply of bonds issued by the government, and thus, the market clearing conditions cannot be satisfied, and this case cannot occur in a stationary equilibrium.

2. $v^n < V < v^c$: In this case, the exchange value of cash $V$ is higher than non-consumers’ reservation value of cash but lower than consumers’ reservation value of cash. As a result, when a household is in the non-consumer state, it sells all of its cash for bonds if it can participate in the asset market. When the household is in the consumer state, it sells all of its bonds for cash if it is able to trade in the asset market. Note that $v^n < V < v^c$ implies that $\frac{w^n_i}{v^n} > q_i > \frac{w_i}{v^c}$, where $\frac{w^n_i}{v^n}$ and $\frac{w_i}{v^c}$ are the reservation bond prices for non-consumers and consumers. As $\frac{w^n_i}{v^n} > \frac{w_i}{v^c}$, non-consumers value bonds relatively higher than consumers do, and thus, the asset market facilitates asset reallocations between the high-liquidity demanders (the consumers) and the low-liquidity demanders (the non-consumers). (V1) and (V2) can be rewritten as

$$\hat{v}^c = v^c$$
$$\hat{v}^n = \mu V + (1 - \mu) v^n$$

3. $V > v^c > v^n$: In this case, the exchange value of cash $V$ is higher than the value of cash in the goods market for both the consumers and non-consumers. Therefore, a household sells all of its cash for bonds in both consumer and non-consumer state, and it never sells bonds but holds them to maturity. (V1) and (V2) can be rewritten as

$$\hat{v}^c = \mu V + (1 - \mu) v^c$$
$$\hat{v}^n = \mu V + (1 - \mu) v^n$$ (1.18)
Because $\nu^c = 1$, this case holds only if $V > 1$. Note that it is also necessary to verify that $\star$ holds, or the transversality condition will be violated. Combining (1.18), (V6), and $\star$, the transversality condition holds if only if $V < \frac{\mu + d}{\mu}$. To summarize, for $V > 1$, the return in the asset market is sufficiently high that even a consumer would desire to sell cash for bonds in the asset market. However, there is a probability $\mu$ that a consumer fails to participate in the asset market and holds a positive amount of cash when entering the goods market. If $V < \frac{\mu + d}{\mu}$, the continuation value of cash in the goods market is not excessively high ($\star$ holds), and hence, consumers in the goods market will spend all their cash on consumption in the goods market. If $V > \frac{\mu + d}{\mu}$, the continuation value of cash is greater than one, and thus, consumers defer their consumption in the goods market, and the transversality condition is violated.

The following lemma summarizes the discussion above.

**Lemma 1.** There exists $\bar{V} = \frac{\alpha}{\kappa + \delta}$ and $\bar{V} = \frac{\mu + d}{\mu}$ such that

1. For $V \in (0, \bar{V}) : V < \nu^u(V) < \nu^c(V)$. (Region 0)
2. For $V \in [\bar{V}, 1) : \nu^u(V) \leq V < \nu^c(V)$. (Region 1)
3. For $V \in [1, \bar{V}) : \nu^u(V) < \nu^c(V) \leq V$. (Region 2)
4. For $V \in (\bar{V}, \infty) : \text{the transversality conditions are violated}$. (Region 3)

**Proof.** See the Appendix.

Figure 1.3 illustrates the relationship between the exchange value of cash $V$ and households’ decisions in the asset market. Recall that Region 0 and Region 3 are excluded from the stationary equilibrium because of violations of market clearing conditions and the transversality conditions, and thus, I focus on the case in which $V \in \left[ \frac{\alpha}{\kappa + \delta}, \frac{\mu + d}{\mu} \right]$. For $V \in \left[ \frac{\alpha}{\kappa + \delta}, \frac{\mu + d}{\mu} \right]$, I can derive bond prices $q_i = q(\lambda_i; V, \pi)$ from (V1) to (V8). The following
<table>
<thead>
<tr>
<th>Region 0</th>
<th>Region 1</th>
<th>Region 2</th>
<th>Region 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Households buy bonds in the non-consumer state and sell bonds in the consumer state</td>
<td>Households buy bonds in both states and never sell</td>
<td>TVCs are violated</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$V &lt; v^n &lt; v^c$</th>
<th>$v^n &lt; V &lt; v^c$</th>
<th>$v^n &lt; v^c &lt; V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^n = V &lt; v^c$</td>
<td>$v^n &lt; V = v^c$</td>
<td>$v^n &lt; V &lt; v^c$</td>
</tr>
</tbody>
</table>

Figure 1.3: Exchange value of cash and households’ trading decisions

Lemma 2. For $V \in \left[ \frac{a}{a + d}, \frac{\mu + d}{\mu} \right]$ and $\lambda_i \in (0, 1)$, there is a unique $(v^n, v^c, \hat{v}^n, \hat{v}^c, w^n_i, w^c_i, \hat{w}^n_i, \hat{w}^c_i, q_i)$ that solves (V1) to (V8).

1. $\frac{\partial q_i}{\partial V} < 0$

2. $\frac{\partial (V - v^n)}{\partial V} > 0$, $\frac{\partial (V - v^c)}{\partial V} > 0$.

Proof. See the Appendix.

First, Lemma 2 shows that $q_i$ and $V$ are negatively related. When the asset prices are lower, households can purchase more bonds in the asset market by selling cash; thus, a higher exchange value of cash, $V$, implies lower bond prices. Second, the net surplus from selling cash in the asset market, $V - v^l$ (a negative value represents the surplus from purchasing cash), increases as $V$ increases. Combining Lemma 2.1 and Lemma 2.2, we obtain that an increase in bond prices, $q_i$, increases the surplus from cash selling in the asset market, $V - v^l$, and decreases the surplus from cash purchases in the asset market, $v^l - V$.

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1.3.3 Asset Market Clearing

Given the properties of households’ behavior discussed above, we can simplify the asset market clearing conditions (1.12). Recall that households must gain the same surplus from selling cash in different bond markets, and thus, households allocate their cash to different bond markets until the equilibrium bond prices make them indifferent. Therefore, the bond prices are determined by the overall quantity of cash and bonds. I thus summarize the market clearing conditions for all bond markets:

$$\sum_i \left( \bar{m}_i^c + \bar{m}_i^n \right) = \sum_i q(\lambda_i; V, \pi) \left[ \bar{b}_i^c + \bar{b}_i^n + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}_i^g \right].$$ (1.19)

I first consider the case in which the equilibrium bond prices are located between the reservation prices of consumers and non-consumers: $q_i \in \left( \frac{w_i^c}{\alpha}, \frac{w_i^n}{v} \right)$. In this case, consumers sell all of their bonds and do not sell cash in the asset market, and thus, $\bar{b}_i^c = \mu \bar{b}_i^c$ and $\bar{m}_i^c = 0$; the non-consumers sell all of their cash and do not sell bonds in the asset market, so $\sum_i \bar{m}_i^n = \mu \bar{m}^n$ and $\bar{b}_i^n = 0$. Therefore, the aggregate market clearing condition (1.19) becomes

$$\mu \bar{m}^n = \sum_i q(\lambda_i; V, \pi) \left[ \bar{b}_i^c + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}_i^g \right],$$ (1.20)

The equilibrium $V \in (\frac{\alpha}{\alpha + d}, 1)$, and the equilibrium bond prices $q_i \in \left( \frac{w_i^c}{\alpha}, \frac{w_i^n}{v} \right)$ solve (1.20).

Recall that the bond prices cannot be higher than the reservation prices for non-consumers, and thus, when the amount of cash is sufficiently large relative to the amount of bonds such that

$$\mu \bar{m}^n > \sum_i \frac{w_i^n}{\sigma^n} \left[ \bar{b}_i^c + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \bar{b}_i^g \right],$$

the equilibrium bond prices are equal to their upper bound, the reservation prices for the non-consumers ($V = v^n = \frac{\alpha}{\alpha + d}, q_i = \frac{w_i^n}{v}$). In this case, the non-consumers are indifferent between buying bonds and not, and they only sell part of their cash to the bond markets and are rationed ($\bar{m}_i^n < \mu \bar{m}^n$).
Similarly, when the amount of cash is sufficiently small relative to bonds such that

$$\mu \tilde{m}^n < \sum_i \frac{w_i^c}{v_i^c} \left[ \tilde{b}_i^c + \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \tilde{b}_i^g \right],$$

the equilibrium bond prices are equal to or smaller than the reservation prices for the consumers ($V \geq 1, q_i \leq \frac{w_i^c}{v_i^c}$), and it is the case of region 2. Specifically, if $q_i = \frac{w_i^c}{v_i^c}$, the bond prices are equal to the reservation prices of consumers, and the consumers are indifferent between buying bonds and selling bonds and are rationed. This case holds require that $\mu \tilde{m}^n + \mu \tilde{m}^c \geq \sum_i \frac{w_i^c}{v_i^c} \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \tilde{b}_i^g$. If $\mu \tilde{m}^n + \mu \tilde{m}^c < \sum_i \frac{w_i^c}{v_i^c} \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \tilde{b}_i^g$, the equilibrium bond prices will be lower than the reservation prices of consumers ($q_i < \frac{w_i^c}{v_i^c} < \frac{w_i^c}{v_i^c}$). Thus, households are bond buyers in both the consumer and the non-consumer state, and the bond prices solve $\mu \tilde{m}^n + \mu \tilde{m}^c = \sum_i q_i \left( \frac{\lambda_i + \pi}{1 + \pi} \right) \tilde{b}_i^g$.

### 1.3.4 Equilibrium Yield Curves

In this section, I characterize the equilibrium yield curves under the fixed asset growth rate policy regime. Specifically, the government chooses the growth rate of cash and bond issuance, $\gamma$, and the bond supply, $\tilde{b}_i$. The exchange value of cash, $V^*$, and quantity of cash, $\tilde{m}^*$, can be solved endogenously from the asset market clearing condition and goods market clearing condition. Once $V^*$ is pinned down, the equilibrium bond prices in all maturities, $q_i^* = q(\lambda_i; V^*, \pi)$, are also pinned down. I denote the implied nominal interest rate, $R_i$, as the solution of the price function $q_i = \frac{1}{1 + R_i} [\lambda_i + (1 - \lambda_i)q_i]$. Therefore, $R$ can also be expressed as a function of $V^*$, $\pi$ and $\lambda$:

$$R(\lambda; V^*, \pi) = \frac{\lambda}{q(\lambda; V^*, \pi)} - \lambda.$$
Figure 1.4: Yield curves: $\lambda_s = 1, \lambda_l = 0.025, \mu = 0.8, \rho = 0.05, \pi = 0.05$. $R$: equilibrium yield curves. $R^c \equiv \frac{\lambda}{\rho} - \lambda$: reservation yields for consumers. $R^n \equiv \frac{\lambda}{\theta \gamma} - \lambda$: the reservation yields for non-consumers.

Because yield curves are arranged as functions of the expected maturities of bonds, which are inverse functions of maturity rates, I define

$$\tilde{R}(\theta; V^*, \pi) \equiv R\left(\frac{1}{\theta^*}; V^*, \pi\right).$$

Then $\tilde{R}(\theta; V^*, \pi)$ is the equilibrium yield curve, which collects the yields for different maturity bonds generating the same equilibrium exchange value of cash, $V^*$.

I then characterize the relationship between the bond supply and equilibrium yield curves. For simplicity, in Figure 1.4, I demonstrate the case in which the government issues bonds with only two different maturities: a short-term bond with maturity rate $\lambda_s = 1$, and a long-term bond with maturity rate $\lambda_l < 1$. As in standard models, we can still price assets that are in zero supply. First, as described in Lemma 2.1, the bond yields shift up and down together. Second, the yield curves are upward sloping when the yields are low; the slope of the yield curve decreases as the yield curve shifts up and become negative when the yields are high enough. The following lemma helps analyze the mechanism driving these results.

**Lemma 3.** For $V \in \left[\frac{\alpha}{\alpha + d}, \frac{\mu + d}{\mu}\right]$
Lemma 3 states that the slopes of yield curves are determined by the asset market friction, \( \mu \), and asset market trading surplus, \( V - v^n \) and \( V - v^c \). When the asset market is frictionless (\( \mu = 1 \)), the interest rates are irrelevant to the maturity of bonds (\( \frac{\partial R}{\partial \lambda} = 0 \)), meaning that the yield curves are flat lines. This result is straightforward because a household can always engage in arbitrage trading if it faces no friction, and therefore, its required yields for different maturities to be the same.

When households face financial frictions (\( \mu < 1 \)), yield curves can be upward sloping, downward sloping, or flat. From the discussion at the end of Section 2.7, when the bond yields are close to their lower bound (\( V \rightarrow v^n \)), \( V - v^n \) is close to zero, and \( V - v^c \) is negative. In this case, according to Lemma 3.1, \( \frac{\partial R}{\partial \lambda} < 0 \), and the yield curve is upward sloping. When bond yields are high (\( V \) high), \( V - v^c \) and \( V - v^c \) become positive; thus, according to Lemma 3, \( \frac{\partial R}{\partial \lambda} > 0 \), and the slope of the yield curve becomes downward sloping. This feature is attributed to two risks caused by the mismatch between the arrival of liquidity demand and bond maturity. The first is liquidation risk, which occurs when the bondholder needs cash but the bond does not mature; the bondholder may sell bonds for cash in the financial market to eliminate such risks, but the financial market frictions limit the bondholders’ ability to do so. The second is reinvestment risk, which occurs when a bond matures, but the bondholder does not need cash; the bondholder may reinvest by
selling cash for bonds in the financial market, but such activities are also obstructed by the financial market frictions.

In principle, short-term bonds bear greater reinvestment risk, and long-term bonds bear greater liquidation risk. For example, in this paper, one unit of short-term bonds with maturity rate \( \lambda = 1 \) becomes one unit of cash in the next period with certainty; thus, the short-term bond bears no liquidation risk but bears the greatest reinvestment risk. However, one unit of long-term bonds (small \( \lambda \)) is highly unlikely to be converted to cash in a given period. As a result, the holders of the long-term bonds are not concerned with reinvestment risk but face greater liquidation risk.

I first discuss the case of region 1 in Lemma 1 \((v^c \geq V \geq v^n)\), in which households purchase bonds in the non-consumer state and sell bonds in the consumer state. When bond supply is low, bond yields are low \((V \rightarrow v^n\) Figure 1.4 Case 1). On the one hand, the bond yield is close to the reservation yield for bond buyers. The surplus from bond purchasing \((v^c - V)\) is small, meaning that a household in the non-consumer state (a bond buyer) is indifferent between purchasing bonds and holding cash. Thus, the financial friction affecting the buy side of the market, which causes reinvestment risk, does not matter. On the other hand, the bond prices are much higher than the reservation prices of bond sellers, and the surplus from bond selling \((v^c - V)\) is large. Thus, the shape of the yield curve is influenced by the friction on the sell side of the financial market, which causes liquidation risk. As a household faces greater liquidation risk when it holds long-term bonds, it requires higher long-term yields than short-term yields, and hence, the yield curve is upward sloping. As discussed in the last subsection, when the bond supply is sufficiently small, the interest rates of bonds are equal to non-consumers’ reservation interest rates. Note that non-consumers’ reservation interest rate for a short-term bond with maturity rate \( \lambda = 1 \) is always zero, because non-consumers are indifferent between holding cash and holding the short-term bond which matures next period. However, when the short-term interest rate is zero, bonds with maturity rate \( \lambda < 1 \) all have positive
interest rates because of the liquidation risk generated by the financial market friction.

As bond supply increases, bond prices decrease. The decrease in bond prices leads to an increase in the surplus from cash selling and a decrease in the surplus from bond selling. Consequently, the importance of liquidation risk decreases, and the importance of reinvestment risk increases, and thus, the slope of the yield curve decreases. When the bond yields are close to the reservation prices of consumers \((V \rightarrow \nu^c, \text{Figure 1.4 Case 3})\), a consumer is indifferent with respect to selling bonds or not, and thus, liquidation risk does not matter. However, a non-consumer gains high surplus from selling cash \((V - \nu^n)\), meaning that the financial friction affecting the buy side of the market, which causes the reinvestment risk, determines the shape of the yield curve. Therefore, the yield curve is downward sloping.

When the bond supply is sufficiently high, bond yields are higher than the reservation yields of consumers \((V \geq \nu^c > \nu^n)\), and region 2 is reached (Figure 1.4 Case 4). In this region, a household purchases bonds in both the consumer state and the non-consumer state and never sells bonds. As a result, households are influenced by reinvestment risk in both states, but liquidation risk is irrelevant. Therefore, the yield curve is downward sloping. These properties of yield curves are summarized in Proposition 1.

**Proposition 1.** For \(V \in \left[\frac{\alpha}{\alpha + d}, \frac{\mu + d}{\mu}\right]\)

1. If \(\mu = 1\) : \(\frac{\partial R(\theta; V, \pi)}{\partial \theta} = 0\)

2. If \(\mu < 1\) : There exists \(\tilde{V} \in \left(\frac{\alpha}{\alpha + d}, 1\right)\) such that

   \(a\) \(\frac{\partial R(\theta; V, \pi)}{\partial \lambda} > 0\) for \(V < \tilde{V}\)

   \(b\) \(\frac{\partial R(\theta; V, \pi)}{\partial \lambda} < 0\) for \(V > \tilde{V}\)

   \(c\) \(\frac{\partial^2 R(\theta; V, \pi)}{\partial \lambda \partial V} < 0\)

**Proof.** See the Appendix.  

\(\square\)

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Recall that the shape and position of the yield curve is determined by the aggregation of bond supply at all maturities. A different composition of bond supply may generate the same yield curve, and a change in the supply of a security with a particular maturity not only changes its own interest rate, but also shifts up the whole yield curve. Moreover, an increase in bond supply, either long- or short-term bonds, decreases the term premium.

It is worthwhile to analyze how the inflation rate changes the shapes of yield curves. Figure 1.5 demonstrates the possible outcomes of equilibrium yield curves under different inflation rates. The inflation rate represents the cost of delaying consumption, and thus, a decrease in the inflation rate decreases the importance of liquidation risk but increases the importance of reinvestment risk. Consequently, the term premium decreases.

### 1.3.5 Comparative Statics at The Zero Lower Bound

Because of the asset market friction, the effects of changes in long- and short-term bond supply are disparate, especially when the short-term interest rate reaches the zero lower bound. In this section, I discuss the comparative statics regarding changes in long- or short-term bond supply. As in Section 1.3.4, I assume that the government issues bonds with two different maturities: a short-term bonds with maturity rate $\lambda_s = 1$, and a long-term bond with maturity rate $\lambda_l < 1$. The government determines the growth rate of the cash and bond issuance, $\gamma$, and the quantity of long- and short-term bonds, $(\bar{b}_s^g, \bar{b}_l^g)$;
but the amount of outstanding cash, $\bar{m}^g$, and the bond prices, $(q_s, q_l)$, are endogenously determined by the goods market clearing and asset market clearing conditions. Moreover, the lump-sum tax, $\tau$, adjusts to satisfy the government budget constraint. Because the research interest in this paper is the policy effects around the zero lower bound, I focus on the cases of region 1 in Lemma 1. Recall that in region 1, consumers sell all their bonds for cash whenever they are able to participate in the asset market, and thus, we can rewrite the goods market resource constraint (1.13) as

$$\frac{\alpha}{1+\pi} \left[ \bar{m}^g + \bar{b}_s^g + \lambda_l \bar{b}_l^g \right] + \frac{\alpha \mu}{1+\pi} q_l (1 - \lambda_l) \bar{b}_l^g = h. \tag{1.21}$$

The left-hand side represents consumers’ cash holdings in the goods market. The first part is the cash holding of consumers at the beginning of the asset market, and the second part represents consumers’ cash gain from selling long-term bonds in the asset market. When the government issues only long- and short-term bonds, the government budget constraint can be expressed as follows:

$$\tau + \left[ 1 - \frac{1}{1+\pi} \right] \bar{m}^g + \left[ q_s - \frac{1}{1+\pi} \right] \bar{b}_s^g + \left[ q_l \left( 1 - \frac{1 - \lambda_l}{1+\pi} \right) - \frac{\lambda_l}{1+\pi} \right] \bar{b}_l^g = 0 \tag{1.22}$$

In the first scenario, I analyze the effect of a change in short-term bond supply when the long-term bond supply is fixed; second, I analyze the effect of a change in long-term bond supply when the short-term bond supply is fixed (Figure A.1 and A.2 in the Appendix provide illustrations). The policy effects of decreases in short- and long-term bonds at the zero lower bound are summarized in Proposition 2.

**Proposition 2.** In the scenario of a fixed asset growth rate, if the financial market is frictional, the following properties hold when the short-term interest rate is zero

1. A decrease in long- and short-term bond supply increases the quantity of cash: $\frac{\partial \bar{m}^g}{\partial \bar{b}_s^g} = -1$,
\[ \frac{\partial m^c}{\partial b} \in (-1,0) \]

2. The government budget satisfies

\[ \tau + \frac{\pi}{\tilde{\alpha}} h - \Theta_l \lambda_l \tilde{b}_l = 0 \]

where

\[ \Theta_l = \left[ 1 - \frac{\lambda_l (1 + d) + (1 - \lambda_l) (1 - \mu) (d - \rho)}{\lambda_l (1 + d) + (1 - \lambda_l) (1 - \mu) d} \right] \]

3. A decrease in short-term bond supply has no impact on the tax: \[ \frac{\partial \tau}{\partial b_s} = 0 \]

4. A decrease in long-term bond supply decreases lump-sum the tax: \[ \frac{\partial \tau}{\partial b_l} > 0 \]

Proof. See Appendix.

At the zero lower bound, cash is plentiful and bond buyers are rationed. A change in bond supply on the margin does not change bond prices or the continuation value of cash in the goods market, and therefore, production, \( h \), is not affected. Thus, according to the goods market resource constraint (1.21), a reduction in bond supply increases outstanding cash, and this is referred to as the "substitution effect". Specifically, at the zero lower bound, a decrease in short-term bond supply only decreases non-consumers’ short-term bond purchases and increases the cash they retain and bring to the goods market on a one-to-one basis, but consumers’ cash holdings in the goods market are not affected. Because the price of a short-term bond is one at the zero lower bound, from the government budget constraint (1.22), a change in short-term bond supply has no impact on the lump-sum tax.

However, a change in the long-term bond supply changes the lump-sum tax at the zero lower bound. I analyze this result by decomposing the government budget constraint into three terms as in Proposition 2.2. The first term is the lump-sum tax, \( \tau \); the second term is the revenue due to inflation, or inflation tax, \( \frac{\pi}{\tilde{\alpha}} h \); the third term, \(-\Theta_l \lambda_l \tilde{b}_l\), is the cost of
issuing illiquid long-term bonds, where $\Theta_l$ represents the degree of inefficiency. We can observe from Proposition 2.2 that the inefficiency of long-term bond issuance is caused by the financial market friction. When the financial market is frictional ($\mu < 1$), $\Theta_l$ is greater than zero, and a reduction in long-term bond supply can improve the surplus from the government’s cash and debt management and, therefore decreases tax. Moreover, because of the financial market friction, one-unit decrease in long-term bond supply will increase cash by less than one unit (see equation (1.21)). If the asset market is frictionless ($\mu = 1$), a long-term bond is as liquid as a short-term bond, and the long-term interest rate is the same as the short-term interest rate. In this case, a decrease in long-term bond supply has no impact on tax at the zero lower bound, and it also increases outstanding cash on a one-to-one basis.

When the short-term interest rate is above zero, a change in the bond supply has different effects. I first discuss its impacts on the quantity of cash. A positive short-term interest rate means that cash is relatively scarce, and in this case, a reduction in bond supply can increase bond prices through asset market clearing (1.20); this is referred to as the "scarcity effect". First, because of the inverse relationship between bond prices and bond supply, the substitution effect between cash and bonds is smaller than when the short-term interest rate is zero (see equation (1.21)). Second, an increase in bond prices decreases the continuation value of cash, $v^n$, which decreases output, $h$, and therefore, the demand for cash decreases. These two effects jointly makes the quantity of cash less responsive to a reduction in bond supply. Moreover, if the elasticity of the disutility function for working is sufficiently large, the increase in output dominates the substitution effect, meaning that the quantity of cash may increase following an increase in bond supply.

Similarly, the supply of bonds has different impacts on the lump-sum tax when the short-term interest rate is above the zero lower bound. As the bond supply decreases, bond prices increase through the scarcity channel, meaning that the cost of bond issuance decreases, which decreases the lump-sum tax; however, the decrease in output decreases
the base for the inflation tax, which increases the lump-sum tax. As a result, the influence of bond supply on the lump-sum tax also depends on the elasticity of working. If working is completely inelastic to the return of working, a decrease in bond supply more significantly decreases the lump-sum tax because of the scarcity effect; otherwise, a decrease in bond supply may increase the lump-sum tax because of the decrease in output.

1.3.6 Non-Ricardian and Ricardian Policy Regimes

All of the previous analysis is based on the assumption that the government is committed to a certain rate of growth in asset issuance, and it adjusts the tax to satisfy its budget constraint. While in the real world, the government can be constrained to a certain level of taxes and transfers, or also include the deficit of debt management. A long line of research emphasizes that how the government meets its budget constraint, or the interaction between fiscal and monetary policy, is important for determining policy effects.\textsuperscript{11}

To evaluate the policy effects from a more comprehensive perspective, I follow Aiyagari and Gertler (1985) and distinguish between the polar non-Ricardian and Ricardian policy regimes.

In the polar non-Ricardian regime, the government is committed by its tax levies and therefore takes the lump-sum tax $\tau$ as given. The central bank fully accommodates a fiscal deficit by financing the government debt with money creation, and thus, a change in the bond supply implies a change in seigniorage revenue.

In the polar Ricardian regime, the fiscal authority fully accommodates the deficit/surplus of debt management through tax levies. Therefore, the seigniorage revenue, $\frac{\pi}{1+\pi} m^S$, is not affected by debt issuance but is fixed at an exogenously determined value. Note that, in contrast to the fixed asset growth rate scenario, in these two regimes, the government can no longer control the growth rate of asset issuance, and the inflation rate, $\pi$, is endogen-

ously determined to meet the policy constraint.

The Non-Ricardian Regime (Fixed Tax)

Figures A.3 and A.4 in the Appendix illustrate how changes in long- and short-term bond supply affect nominal interest rates, real interest rates, and inflation in the non-Ricardian regime. When the bond supply is small, the short-term yield reaches zero \( q_s = 1 \), and the scarcity channel, through the market clearing condition, is not effective. According to (1.21) and (1.22), the short-term bond, \( \bar{b}_g \), and cash, \( \bar{m}_g \), are equivalent. Similar to the fixed asset growth rate scenario, a decrease in the short-term bond supply only decreases non-consumers’ short-term bond purchases and increases the cash they retain and bring to the goods market, but consumers’ cash holdings in the goods market is not affected. Since only consumers are engaged in goods purchases, a change in the short-term bond supply has no effect on the price level or inflation rate and is therefore neutral at the zero lower bound.

However, a decrease in the long-term bond supply is not neutral at the zero lower bound. Recall that in the fixed inflation rate case (Section 1.3.5), a decrease in long-term bond supply improves government debt management and decreases the lump-sum tax (Proposition 2.4); thus, in the fixed tax regime, a decrease in the long-term bond supply requires an adjustment of the inflation rate to increase the tax and again meet the target \( \Lambda_\tau \). According to Proposition 2.2, a decrease in the inflation rate influences the seigniorage through two channels. First, a decrease in \( \pi \) increases the cost of long-term bond issuance, \( \Theta \), which increases \( \tau \). Moreover, a decrease in \( \pi \) influences the inflation tax \( \pi h \). If the elasticity of output with respect to \( \pi \) is smaller than one, a decrease in \( \pi \) will decrease \( \pi h \) and increase \( \tau \). As the elasticity of output with respect to the inflation rate should be smaller than one in normal cases, a reduction in long-term bond supply results in a decrease the inflation rate in the non-Ricardian regime.\(^\text{12}\) This deflation effect de-

\(^\text{12}\)In the case of isoelastic working disutility \( l(h) = \frac{h^{\frac{1}{2}}}{1+\tau} \), the sufficient and necessary condition for \( \frac{d\pi h}{d\pi} > 0 \)
creases households’ cost of holding bonds and therefore decreases the long-term nominal interest rate; however, this effect is small at the zero lower bound, and the real long- and short-term interest rates increase because of the direct effect of deflation.

Figure A.5 depicts how implied market yield curves change with respect to a reduction in the long-term bond supply when the short-term interest rate is zero. The supply of long-term bonds is cut into a third comparing to the original quantity, but the yield curve only shifts down slightly, and the real yield curve shifts up because of the decrease in the inflation rate.

The Ricardian Regime (Fixed Seigniorage Revenue)

In the Ricardian regime, the seigniorage revenue is not affected by the issuance of government debt. I focus on the case in which the fiscal authority runs a net deficit; thus, the seigniorage revenue has to be positive to balance the government’s budget \( \frac{\pi}{1+\bar{\pi}} \tilde{m} > 0 \), and a positive inflation rate is required.

The policy effect in the Ricardian regime is similar to that in the non-Ricardian regime (see Figure A.6 and A.7 in the Appendix); however, a reduction in short-term bond supply is not neutral but deflationary at the zero lower bound. Note that in the fixed inflation rate scenario, a reduction in both long- and short-term bond supply increases the quantity of cash at the zero lower bound (Proposition 2.1), and therefore, increases the seigniorage revenue. Similar to the mechanism in the non-Ricardian regime, the seigniorage revenue target will be re-achieved by decreasing the inflation rate if the output is not too responsive to inflation. Because of this deflation effect, a reduction in long- or short-term bond supply decreases the nominal long-term interest rate and increases the real long- and short-term interest rates at the zero lower bound.\(^{13}\)

\[ \alpha + d - v\pi(1 + \rho) > 0. \] This condition holds in the proper parameter space. For example, for \( \alpha = 0.8, \pi = 0.05, \rho = 0.05 \), we have \( \frac{d\pi}{d\pi} > 0 \) if and only if \( v < 17.2 \). While in the literature, \( v \) is estimated between zero and one. (see, for example, Christiano et al. (2010))

\(^{13}\)When the government is running a net surplus, the seigniorage revenue is negative, and the conclusion is reversed.
In summary, under both policy regimes, a reduction in the long-term bond supply shifts down the yield curve at the zero lower bound, but the level is rather small; as a result, the real yield curve shifts up because of the deflation effect. The key reason for this result is the substitutability between long- and short-term bonds: when the short-term yield reaches the zero lower bound, the long-term yield also reaches its effective lower bound. In Section 1.4, I introduce heterogeneous types of households into the baseline model, and the rigid relationship between long- and short-term yields is relaxed.

1.3.7 Monetary Policy and Welfare

In this section, I analyze how monetary policy influences social welfare. In this model, the first-best allocation is achieved when \( \frac{dl(h)}{dh} = 1 \). By (1.1) and (1.2), the marginal disutility of labor is equal to the expected discounted value of cash, \( v^n \): 

\[
\frac{dl(h)}{dh} = v^n = \frac{1}{1 + d} [\alpha \hat{v}^c + (1 - \alpha) \hat{v}^d]
\]

Therefore, an increase in \( v^n \) implies an increase in social welfare as long as the No-Ponzi game condition is not violated \( (v^n \leq 1) \), and the first-best is achieved when \( v^n = 1 \). In this model, the first-best can be achieved by the Friedman rule \( 1 + d = 0 \). Under the Friedman rule, it is costless to hold cash and delay consumption, and thus the quantity of bonds has no effect on social welfare. However, the Friedman rule may not be achievable. Depending on the policy regime, the bond supply has different impacts on welfare.

Figure 1.6 illustrates the relationship between the long-term bond supply and social welfare in the non-Ricardian and Ricardian regime. In the non-Ricardian regime, taxation is predetermined. Because bonds cannot be used as a medium of exchange in this model, and when the nominal interest rate of a bond is above zero, the bond becomes an inferior asset to cash. As the issuance of cash and bonds is entirely funded by the government’s tax/transfer, zero nominal interest rates are optimal, and a reduction in the issuance of
bonds with positive interest rates improves welfare. Because of the asset market friction, a zero long-term nominal interest rate is not achievable, and thus, given a tax target, the optimal policy is to issue no long-term bonds and only issue zero or a small amount of short-term bond such that the short-term nominal interest rate reaches the zero lower bound.

In the Ricardian fiscal regime, bond issuance is fully supported by the government’s tax levies. Unlike the non-Ricardian regime, the issuance of government securities is not restricted by a fixed tax, but an increase of bond supply implies an increase in taxes, which enlarge the size of the funding pool for the issuance of cash and bonds. When the short-term interest rate is above the zero lower bound, an increase in bond supply increases the interest rates and hence increases the surplus from selling cash in the asset market. As a result, a larger bond supply generates higher cash value, $v^n$, and thus, improves social welfare. However, at the zero lower bound, the trading surplus from selling cash is zero, and an increase in bond supply does not increase the exchange value of cash on the margin. In this case, cash in the goods market is valued at its fundamental value, $v^n = \frac{\alpha}{\alpha + d}$. As discussed in Section 1.3.6, an increase in bond supply increases inflation at the zero lower bound and, therefore, decreases the fundamental value of cash and decreases social welfare.¹⁴

1.4 The Heterogeneous Households Model

In this section, I introduce households with different characteristics. A household’s characteristics include the asset market friction it faces, $\mu$, the time discount rate, $\rho$, and the arrival rate of liquidity demand, $\alpha$, and I denote the vector of household characteristics by $A = (\mu, \rho, \alpha)$. Note that different households value assets differently, which depends the

¹⁴Note that, in this model, bonds are not essential to achieve the social optimum. On the necessity of illiquid bonds for improving social welfare, readers are referred to Kocherlakota (2003), Shi (2005), and Paola and Camera (2006).
household’s individual characteristics. Given a combination of equilibrium bond yields, a household would purchase the bonds that offer it the highest value, and thus, different households may trade bonds with different maturities. Consequently, even if the short-term bond yield reaches the reservation short-term yield of short-term bond buyers, the long-term yield may not reach the reservation long-term yield of long-term bond buyers, meaning that a reduction in the long-term bond supply can still effectively decrease the nominal long-term interest rate even if the short-term interest rate hits the zero lower bound.

Suppose that there is a household $i$ with characteristic set $A_i = (\mu_i, \rho_i, \alpha_i)$; I denote $R(\lambda; V, \pi, A_i)$ to be the interest rates that generate the same exchange value of cash, $V$, for the household. Different households own different such functions, and in a heterogeneous household model, the equilibrium market yields do not necessarily make a household indifferent between purchasing different bonds. Thus, I call $R(\lambda; V, \pi, A_i)$ the subjective yield curve for household $i$, in contrast to the actual equilibrium market yield curve.$^{15}$

I first analyze how a household’s characteristics affect its subjective yield curves. Let $\tilde{R}_i(R_s; A)$ denote the long-term yield that makes the household indifferent between purchasing long-term bonds and short-term bonds when the short-term yield is $R_s$. Sup-
pose that there are two types of households with characteristics $A_1, A_2$ and $\tilde{R}_l(R^*_s; A_1) < \tilde{R}_l(R^*_s; A_2)$. Then, when the short-term interest rate is $R^*_s$, a type $A_2$ household demands a higher term premium than a type $A_1$ household does.

First, I compare the subjective yield curves for households with different time discount rates, $\rho$. In Figure A.14, the left and the middle panels illustrate the subjective yield curves for impatient households ($\rho_1 = 0.075$) and patient households ($\rho_2 = 0.025$), respectively. The right panel demonstrates the function $\tilde{R}_l(R^*_s; \rho_i)$ for these two households. Note that $\tilde{R}_l(R^*_s; \rho_1) > \tilde{R}_l(R^*_s; \rho_2)$ for all $R^*_s$, meaning that given a short-term interest rate, $R^*_s$, impatient households require a higher term premium than patient households. As a result, impatient households prefer short-term bonds, and patient households prefer long-term bonds, relatively. Moreover, $R(\lambda; \bar{V}, \pi, \rho_1) < R(\lambda; \bar{V}, \pi, \rho_2)$ for all $\lambda$ and $\pi$, meaning that a patient household requests lower interest rates to enter Region 2, and thus, patient households are more likely to hold bonds to maturity. The patient traders behave analogously to insurance companies and pension funds, which purchase long-term securities and hold them to maturity; impatient traders are participants in the shorter-term securities market, and they frequently purchase and sell assets.

Asset market frictions influence households’ subjective yield curves in a different way. Figure A.15 illustrates the subjective yield curves for frictional households ($\mu_1 = 0.75$) and frictionless households ($\mu_2 = 1$). As Proposition 1 states, the yield curves are upward sloping when yields are low and downward sloping when yields are high, but the yield curves are flatter for households that face smaller asset market frictions. Therefore, $\tilde{R}_l(R^*_s; \mu_1) > \tilde{R}_l(R^*_s; \mu_2)$ when $R^*_s$ is small and $\tilde{R}_l(R^*_s; \mu_1) < \tilde{R}_l(R^*_s; \mu_2)$ when $R^*_s$ is large. Thus, when yields are low, frictional households prefer short-term bonds more than frictionless households, meaning that frictional households tend to be short-term bond buyers; when yields are high, frictional households prefer long-term bonds more than frictional households, and the frictional households tend to be long-term bond buyers.

16 The notation is slightly abused. I denote $\tilde{R}_l(R^*_s; \rho_i)$ by $\tilde{R}_l(R^*_s; (\mu, \rho, \alpha))$, where $\mu$ and $\alpha$ are the same across households.
Finally, Figure A.16 illustrates the yield curves for households facing different arrival rates of liquidity shocks, $\alpha$. Observe from the right panel that an increase in $\alpha$ increases the slope of $\tilde{R}_l(R_s;\alpha)$ but has no effect on the intercept. Therefore, $\tilde{R}_l(R_s;\alpha_1) > \tilde{R}_l(R_s;\alpha_2)$ for all $R_s > 0$. That is, $\alpha$ has no influence on the yield curve at the zero lower bound, but when the short-term yield is above the zero lower bound, households that have higher $\alpha$ demand a higher term premium.

Similar to segmented market theory, different households participate in markets for securities at different maturities. However, in contrast to segmented market theory, in this model, the maturity a household would like to purchase are endogenously determined. Moreover, a household’s preferred assets and its portfolio may change depending on the market interest rates.

Lemma 4. Given $\pi$, let $R_s$ and $R_l$ be the equilibrium yields of short-term and long-term bonds; then, $R_l \in [\check{R}_l(R_s), \bar{R}_l(R_s)]$, where $\check{R}_l(R_s) = \min_k \{ \tilde{R}_l(R_s; A_k) \}$ and $\bar{R}_l(R_s) = \max_k \{ \tilde{R}_l(R_s; A_k) \}$.

The proof is straightforward. If $R_l > \check{R}_l(R_s)$, there will be no demand for short-term bonds. If $R_l < \check{R}_l(R_s)$, there will be no demand for long-term bonds. Note that in a representative household model, $\check{R}_l(R_s) = \bar{R}_l(R_s)$, and thus, the relationship between short-term and long-term bonds is one-to-one. However, in the model with multiple types of households, given a short-term interest rate, any interest rate located between $\check{R}_l(R_s)$ and $\bar{R}_l(R_s)$ can be a possible outcome of the equilibrium long-term interest rate. Consequently, the rigid relationship between long-term and short-term interest rate is relaxed in a heterogeneous households model.

1.4.1 The Model with Two Types of Households

In this section, I assume that there are two types of households in the economy, and their characteristics are $A_1$ and $A_2$. Let $R^*_s$ and $R^*_l$ be the equilibrium short-term and long-term yields, and without loss of generality, I assume that $\tilde{R}_l(R^*_s; A_1) > \tilde{R}_l(R^*_s; A_2)$. According to
Lemma 4, the equilibrium long-term yield, $R_l^*$, must satisfy $\hat{R}_l(R_s^*; A_1) \geq R_l^* \geq \hat{R}_l(R_s^*; A_2)$. Therefore, given an equilibrium short-term yield, the equilibrium long-term yields can be classified into three cases.

In the first case, $\hat{R}_l(R_s^*; A_1) = R_l^* > \hat{R}_l(R_s^*; A_2)$ (Figure 1.7 left), the long-term yield is equal to its highest possible value, and this is referred to as the "High $R_l$" case. In this case, type 1 households are indifferent between purchasing short- and long-term bonds; type 2 households only purchase long-term bonds because purchasing long-term bonds provides them with higher marginal utility than purchasing short-term bonds. In the second case, $\hat{R}_l(R_s^*; A_1) > R_l^* > \hat{R}_l(R_s^*; A_2)$ (Figure 1.7 middle). Type 1 households only purchase short-term bonds, and type 2 households only purchase long-term bonds. This case is referred to as the transition regime. In the last case, $\hat{R}_l(R_s^*; A_1) > R_l^* = \hat{R}_l(R_s^*; A_2)$ (Figure 1.7 right). The long-term yield is equal to its lowest possible value, and this is referred to as the "Low $R_l$" case. In this case, type 1 households only purchase short-term bonds, but type 2 households are indifferent between purchasing short- and long-term bonds.

In the rest of this section, I analyze the policy effects when the short-term interest rate reaches the zero lower bond. Because a change in the short-term bond supply has similar effects as in the representative household model, I focus on the policy effects of a change in the long-term bond supply. In the following policy experiment, I assume that half of the households face low financial frictions ($\mu = 0.95$), and half of the households face higher financial frictions ($\mu = 0.75$).

Figures A.8 and A.9 in the Appendix illustrate the effects of a change in long-term bond supply in the non-Ricardian and Ricardian regimes. When the short-term interest rate reaches zero, if the long-term bond supply is relatively high, the long-term yield will be equal to the reservation long-term yields of frictional households. This is the case of the high $R_l$ region. In this case, the frictionless households gain positive surplus from purchasing long-term bonds, and thus, they sell all of their cash in the asset market for
long-term bonds only. However, the frictional households are indifferent among purchasing long-term bonds, purchasing short-term bonds and retaining their cash, and they are buyers of both long- and short-term bonds. In this region, the reduction in the long-term bond supply only reduces the amount of cash frictional households sell to the long-term bond market, and the scarcity channel is ineffective at decreasing long-term interest rate. The long-term interest rate only slightly decreases through deflation channel.

When the frictional households’ long-term bond purchases have been reduced to zero, the transition region is reached. In the transition region, frictionless households are the only long-term bond buyers, and frictional households are the only short-term bond buyers. The reduction in the long-term bond supply significantly decreases the long-term yield through the scarcity channel, and the trading surplus for frictionless households is also diminished.

Finally, when the long-term yield reaches frictionless households’ reservation yield for long-term bonds, the low $R_l$ region is reached. In the low $R_l$ region, frictionless households are indifferent among purchasing long-term bonds, purchasing short-term bonds and retaining their cash holdings. In this region, the scarcity channel is, again, not effective. If the supply of long-term bonds decreases further, frictionless households will reduce the amount of cash they sell in the long-term bond market, and the long-term interest rate only decreases slightly through the deflation channel.

Figure 1.7: Subjective yield curve and market interest rates
To summarize, at the zero lower bound, a reduction in the long-term bond supply has a limited effect on nominal interest rates in the high $R_l$ and the low $R_l$ regions. This policy effect is similar to a reduction in the long-term bond supply at the zero lower bound in the representative household model, and these two regions are also referred to as the "ineffective" regions. However, there also exists a transition region in which a reduction in the long-term bond supply effectively decreases the long-term interest rate even if the short-term interest rate is zero. This transition region results from the heterogeneity of households and is referred to as the "effective" region.

### 1.4.2 The Model with Multiple Types of Households

In this section, I introduce households that face various financial frictions and time discount rates. To capture the observation that the majority of the households face small asset market frictions, I assume that the financial friction that households face follows a left-skewed beta distribution. The distribution of time discount rates is assumed to be uniform. The joint distribution of households’ characteristics is depicted in Figure A.13. The mechanism here is similar to that discussed in Section 1.4.1. The only difference is that there are multiple types of households, meaning that monetary policy encounters a mixture of multiple effective and ineffective regions.

Figures A.10 and A.11 in the Appendix illustrate the policy effect of a change in the amount of outstanding long-term bonds. Unlike the representative household model, a reduction in the long-term bond supply significantly decreases the nominal long-term interest rate when the short-term interest rate reaches zero. This is because the heterogeneity of households loosens the one-to-one relationship between the long-term yield and the short-term yield. Although the short-term yield is at the zero lower bound, the long-term interest rate may not reach its effective lower bound. There exist long-term bond buyers who gain positive trading surplus from purchasing long-term bonds, and thus, a reduction in the long-term bond supply can effectively diminish the trading surplus of
those households and decrease the nominal long-term interest rate.

However, the substitution effect between long- and short-term bonds for individual households makes the reduction in the supply of long-term bonds less effective at the zero lower bound. When the short-term rate reaches zero, there exist marginal traders whose reservation long-term interest rate is equal to the equilibrium long-term rate. Those marginal traders are indifferent among trading long- and short-term bonds and retaining their cash. When the long-term bond supply decreases, the marginal traders are unwilling to pay higher prices for the long-term bonds, but they decrease the amount of cash they sell in the long-term bond market. The existence of these marginal traders places pressure on the rise in long-term bond prices. As a result, the impact of a reduction in the long-term bond supply on the long-term interest rate is less effective when the short-term rate reaches zero.

I also consider whether a reduction in the long-term bond supply can decrease real interest rates given a zero short-term rate. In this policy experiment, the households are highly diversified in both market frictions and time discount rates, and thus, the effective region is large. As a result, a reduction in the supply of long-term bonds can more effectively decrease the nominal long-term interest rate. The decrease in the nominal long-term interest rate dominates the decrease in the inflation rate, and thus, the long-term real interest rate decreases. Another polar example is the representative household model of Section 2, where households are identical, and thus, there is no such effective region at the zero lower bound; hence, the long-term real interest rate increases. In short, quantitative easing decreases or increases long-term real interest rates depending on the diversity of households. Note that since the short-term interest rate has already reached the zero lower bound, the reduction in the long-term bond supply always increases real short-term interest rate because of deflation.

Note that in a heterogeneous household model, the shape and position of the yield curve is determined by both the overall supply of the bond as well as the relative sup-
ply of long- and short-term bonds. For example, when the short-term interest rate is above zero, a decrease in the long-term bond supply may increase the slope of the yield curve because of a decrease in overall bond supply. However, a decrease in the long-term bond supply also decreases the relative supply of long- to short-term bonds, and thus, the policy places more downward pressure on the long-end of the yield curve and may decrease its slope.

Figure A.12 demonstrates the impact of a reduction in long-term bond supply on the implied market yield curve at the zero lower bound.\(^{17}\) When households are sufficiently diverse, a reduction in the long-term bond supply at the zero lower bound results in a significant downward shift of the nominal yield curve. Moreover, because of the decrease in the inflation rate, the real yield curve shifts down at the long-maturity end but shifts up at the short-maturity end. Consequently, the reduction in the long-term bond supply results in a clockwise rotation of the real yield curve.

1.5 Conclusion

In this paper, I develop a dynamic general equilibrium model that has the potential to enhance our understanding of the liquidity channel of quantitative easing. The model features financial market frictions and heterogeneous households and provides a theoretical basis for the term structure of interest rates. Notably, I do not impose maturity-specific frictions in the model. Instead, a general friction in the financial market allows me to analyze the importance of the fundamental difference between long- and short-maturity bonds, which is their maturity. The mismatch between households’ liquidity demand and bonds’ maturity, along with the financial market friction, is sufficient to generate upward- and downward-sloping yield curves. Note that, in reality, maturity-specific frictions may still exist, and incorporating these frictions can help to calibrate the model to better fit the

\(^{17}\)Note that in the heterogeneous household model, the implied market yield curve equals the subjective yield curve for the marginal traders.
An extension could be to consider the role of financial intermediaries and other technologies in the financial market. For example, a household may expend search effort or fixed costs to increase the probability of a successful trade in the financial market. Moreover, a frictionless investor could provide middleman services, which help frictional investors trade in the asset market, in exchange for a fee. Households can also form a coalition to share their liquidity demand risk and risks in the financial market. These technologies and intermediaries may alleviate the financial friction and even alter the way financial frictions influence the financial market, and therefore, the relationship between long- and short-term interest rates could be changed.

In this paper, I focus on the Treasury bond market, but an important issue could be the transmission mechanism from the term structure of Treasury bond yields to the private asset and capital markets. As discussed in Section 4.3, quantitative easing decreases long-term real interest rates but increases the short-term real interest rate in the Treasury bond market. If assets are better substitutes for assets with similar maturities, the maturity structure of liabilities and investment should also be influenced. Thus, quantitative easing could increase long-term investment but reduce short-term investment. A model incorporating long- and short-term private investment and liabilities could help clarify the mechanism.

Finally, this paper focuses on the impact of bond supply on households, which represent the demand side of the Treasury bond market. I do not take a stand on the supply side of the market in considering why the government issues bonds with different maturities. For example, short- and long-term government bonds may play different roles in the government’s debt management, and thus, the term structure of Treasury bonds may influence the risk of sovereign default. I leave these issues for future research.
Chapter 2

Haircuts and Asset Market Liquidity

2.1 Introduction

When a trader purchases a security, the trader may finance the purchases by using the security as collateral to borrow. However, the trader may not borrow the entire price, and the difference between the security’s price and the borrowing capacity is the haircut. Our model shows that the haircut affects – and is affected by – asset market liquidity in a profound way. Asset market liquidity captures the ease with which financial assets can be traded, which limits the amount of funding that can be channeled to traders with financing constraints. As the borrowing limit affects the default rate and influences the quantity of assets liquidated, the asset market liquidity is also influenced by traders’ borrowing limits.

In this paper, we develop a dynamic general equilibrium model featuring endogenous asset market liquidity and endogenous haircuts. In our model, investors require assets to produce dividends. Investors finance their asset purchases through secured loans, and the assets are pledged as collateral. The maturity of projects is random, and investors’ debts roll over until their assets mature. Assets are traded on a search market where only a fraction of quoted orders is successfully matched each period. This fraction is
endogenously determined by the participation intensity on either side of the market. For instance, the more asset buyers there are relative to asset sellers, the sooner the buyer will match with a seller. Default occurs endogenously if the borrowing limit is reached before the project matures, and the collateral is liquidated in the frictional asset market to repay the debt. The delay in asset resale drives a wedge between the transaction asset prices and borrowing limit, which is the haircut. On the one hand, the asset market liquidity determines the time delay of asset liquidation, and thus, determines the level of haircuts. On the other hand, the haircut influences borrowers’ borrowing limits and leverage, which determines the default rate of investors and the number of assets on sale in the market. As a result, the asset market tightness and the salability of assets are inversely affected.

In the literature, there are studies of the relationship between borrowers’ default and the amplification mechanism in the repo market. Dang et al. (2011) shed light on the channel whereby larger haircuts reduce the amount of lending to the borrower, which reduces the borrower’s liquidity and increases the default probability, thereby increasing haircuts and, in turn, further reducing the amount of lending. Zhang (2013) emphasizes the channel whereby a borrower’s default makes the lender’s portfolio less liquid and makes the lender tend to default. This contagion effect amplifies systematic shocks and causes overshooting. While the literature has focused on the relationship between a borrower’s default related to a counterparty’s funding risk, we discuss the dynamic relationship between a borrower’s default and the salability of the collateral, which is determined by the liquidity of the asset market.

In this paper, we demonstrate the existence of a fire sale generated through sudden illiquidity in the asset market. When the probability of a successful matching in the asset market decreases, the salability of the asset decreases and the haircut increases. The increase in the haircut decreases investors’ borrowing limit and forces borrowers who have borrowed over the new limit to default. The collateralized asset is forced to be li-
liquidated in the asset market, thereby increasing the number of assets on sale. This fire sale decreases the salability of the asset and further increases haircuts and forces more borrowers to default. This amplification mechanism leads to overshooting in asset prices and haircuts and causes repo runs.

This paper also analyzes the role of assets as a liquidity provision. In the model, we incorporate liquidity considerations by adding a decentralized service market similar to the one in Lagos and Wright (2005) in which traders exchange services financed with collateralized loans. This decentralized market aims to create the liquidity demand in the wholesale financial markets. It can also be interpreted as a market where households finance idiosyncratic consumption opportunities or firms finance investment opportunities with collateralized loans or means of payment. Assets create private money in various forms. First, assets as collateral are essential to create loans. Second, idle assets that are not used to produce can be valued for the future revenue coming from selling them in the asset market. The secured loans and idle assets are private money that can serve as a medium of exchange and store of value. Moreover, the government issues fiat money or Treasury bonds as public liquidity. This public liquidity competes against private money in the liquidity market and allows the government to influence the asset prices and financial market through monetary policy.

2.1.1 Related Literature

The literature has discussed haircuts caused by different factors. For example, a haircut can be caused by the price volatility of the collateral between regular margining dates, as it allows lenders to hedge against price risk (see Gottardi et al., 2015). Another factor can be information asymmetry when the lender does not know the true quality of the collateral. For example, Li et al. (2012) show that fraud activity generates endogenous resalability constraints and rationalizes retention mechanisms in markets for asset-backed securities or haircuts in repo markets. The third factor can be the cost of liquidating col-
lateral following a default event, which may come from transaction costs or time delays (see Lee, 2013).

The literature highlighted the relationship between repo contracts and the fragility of borrowers’ funding liquidity. He and Xiong (2012) and Acharya et al. (2011) emphasize short maturity as a source of fragility. Martin et al. (2014) show that short-term collateralized borrowing may be highly unstable because of market-wide changes in expectations. Infante (2013) argues that repo contracts’ exemption from automatic stay allows lenders to easy access to the collateral from defaulted loans, which enhances market liquidity but creates the potential for fire sales. In Brunnermeier and Pedersen (2008), when funding liquidity is tight, traders become reluctant to take on positions and therefore lower market liquidity. Further, low future market liquidity increases the risk of financing a trade, thus increasing margins. This paper contributes to the literature by considering the fragility caused by a decrease in asset market liquidity and simultaneous default of investors.

The search literature provides a theory of endogenous asset liquidity following the pioneering work of Duffie et al. (2005), who apply search-theory to model trading frictions on OTC markets. Search frictions have been used to study the features of markets for a wide range of financial assets, such as asset-backed securities, corporate bonds, private equity and housing, amongst others. Alternative approaches include the competitive search literature (Moen (1997)), and proceeding studies also discuss information frictions, such as adverse selection models in Guerrieri et al. (2010) and Guerrieri and Shimer (2012).

This paper is also related to the monetary-search literature such as Lagos and Wright (2005), in which money serves as a medium of exchange in bilateral transactions. Privately liquidity created by claims on capital and firms’ profit, and deposits are discussed in Lagos and Rocheteau (2008), Rocheteau and Rodriguez-Lopez (2014) and Williamson (2012).

We proceed by describing the model in the next section and solving the stationary

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1The reader is referred to Rocheteau and Weill (2011) for a survey.
equilibrium in Section 3. In section 4, we characterize the transition dynamics of a sudden market illiquidity shock and interest rate shock. Section 5 concludes.

2.2 The Model

Time is continuous with an infinite horizon. The economy is populated by two groups of agents: households and investors. Each of the groups has mass of unit measure, and agents in both groups discount time utility at rate \( \rho > 0 \). There are two forms of divisible and perishable goods. The first is the numeraire, which can be produced and consumed by households and investors. The second is service, which is produced and consumed by households only. There are two forms of assets. The first are durable and indivisible trees with fixed mass \( A \), and trees are of different types, \( i \sim U[0,1] \). The second form of asset is government debt with supply \( B \), and the interest payments on government debt are financed through lump-sum taxation imposed on the households.

An investor is born with a non-transferable, indivisible project but with no tree. A project matures with the Poisson arrival rate \( \delta > 0 \). Similar to trees, each project also has a type \( \tau \sim U[0,1] \), and the type of the project is known by its owner. An investor can implement his or her project on a tree to produce: when the investor’s project matures, if the investor has implemented his or her project on a tree which has same type as the project, the project produces \( y \) units of numeraire as dividend. I assume that an investor can only implement the project once. If the project is implemented on a tree with a wrong type, the project can not be implemented again and will never generate the dividend. The types of trees are unknown in advance by investors, but investors can meet trees bilaterally in the decentralized market and observe their types. When an investor meets a tree, he or she can immediately contact the tree owner. The investor and the tree owner bargain to determine the terms of trade. The number of successful matchings, defined as that an investor matches with a tree which has the same type as his or her project, is
characterized by a Cobb-Douglas matching function $M(N_b, N_s) = \Lambda N_b^{\gamma} N_s^{1-\gamma}$, where $N_b$ is the number of investors who are searching, $N_s$ is the number of trees for sale, and $\Lambda$ is the parameter which captures the intensity of the matching process, or the liquidity of the tree market. Let $\sigma \equiv \frac{N_b}{N_s}$ denote the market tightness; then, the arrival rate of a buyer meeting a tree, $\mu$, and the arrival rate for a tree to meet a buyer, $\eta$, satisfy

$$
\mu(\sigma) = M(1, \sigma^{-1}) = \Lambda \sigma^{\gamma-1}
$$

$$
\eta(\sigma) = M(\sigma, 1) = \Lambda \sigma^{\gamma}
$$

An investor’s life time utility can be expressed as

$$
E \left\{ \int_0^{\min(\tau_a, \tau)} -e^{-\rho t} dw_t + e^{-\rho \tau} x(\tau) \right\}
$$

(2.1)

Where $\tau$ is the time the investor’s project matures, and $\tau_a$ is the time the household receives a tree. An investor can work to produce the numeraire good before it receives a tree, and producing $w$ units of numeraire generates $w$ units of disutility. When an investor receives a tree, he or she can no longer work, and the only thing the investor can do is wait for the project to mature. An investor can only consume the numeraire good at the point in time when its project matures, and consumption generates linear utility $x(\tau) > 0$. After an investor’s project matures, the investor leaves the economy forever. To make the total number of investors constant, we assume that new investors enter the economy at rate $\delta$ in every period $t$.

A household occasionally has an opportunity to exchange services in the service market, which is decentralized with bilateral matching and bargaining. The lifetime expected utility of a household is

$$
E \left\{ \int_0^\infty e^{-\rho t} dC_t + \sum_{n=1}^{\infty} e^{-\rho T_n} \{ u[c(T_n)] - y(T_n) \} \right\}
$$

(2.2)
The first term accounts for the utility from net consumption of the numeraire good, while the second term accounts for the utility from service market trades. The process $\{T_n\}$ is Poisson with arrival rate $\alpha > 0$, and indicates the times at which the trader is matched with another trader. Upon a bilateral match being formed, a trader is chosen at random to be either a supplier of services or a user of services. The utility from consuming $c$ units of services is $u(c)$, where $u$ is strictly concave, $u(0) = 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. The disutility from producing $y$ units of services is $y$. At all $t \neq \{T_n\}_{n=1}^{\infty}$, households can consume and produce the numeraire good, $dC(t) \in \mathbb{R}$. The technology to consume/produce the numeraire good is not available at times $T_n$ when traders are matched. We assume that there is no commitment or record keeping technology, and added to the assumption and that the numeraire good is perishable implies that the buyer in the service market cannot finance $c$ with the production of the numeraire good, thereby creating a need for liquid assets to serve as means of payment from service market participants. The liquid asset in this model includes government debt, trees, and loans secured by trees.

There is a **centralized liquidity market** for the exchange of all forms of liquid asset, and all investors and households can participate at any point in time. In the liquidity market, investors and households can only exchange the *ownership* of liquid assets, but they cannot identify the type of the trees. Consequently, if an investor wishes to utilize a tree to produce dividends, the investor must participate in the decentralized tree market to search for a tree which matches the type of its project. Let $r$ be the rate of return in the centralized liquidity market. The price of trees, $f$, in the liquidity market is equal to the discounted expected return of holding a tree and selling it in the decentralized tree market. As in Williamson (2012) and Rocheteau and Rodriguez-Lopez (2014), in equilibrium, households are on the demand side of the liquidity market: they purchase liquid assets in order to facilitate the bilateral service market transactions. Investors are on the supply side of the liquidity market: they create private liquidity by issuing asset-based loans to finance their tree purchases in the tree market.
We focus on loan contracts in the following form. One unit of loan contract promises to repay 1 unit of the numeraire good at time \( t + \triangle t \). The price of the contract in terms of the numeraire is \( \frac{1}{(1 + rt \triangle t)} \), and \( l \) proportion of the tree are pledged as collateral. A borrower can issued \( b \) units of secured loan when he or she pledged one unit of trees as collateral. If the borrower fails to repay, the tree must be liquidated in the centralized liquidity market; \( bl \) proportion of the liquidation return is transferred to the lender, and the remaining \((1 - bl)\) proportion goes to the borrower. As in Kiyotaki and Moore (1997), the borrower can issue at most \( f \) units of tree secured loan when he or she pledges one unit of trees as collateral, otherwise the borrower will not repay the debt and instead chooses to default.

### 2.3 Stationary Equilibrium

In this section, we solve for the stationary equilibrium of the dynamic competitive economy. We will analyze in turn the supply of private liquidity arising from investors, the demand of liquidity by households, and the determination of the real interest rate to clear the market for liquid assets.

#### 2.3.1 Investors and the Tree Market

Investors stand on the supply side of the liquidity market. They borrow to finance their purchases of trees in the tree market and issue loans secured by trees. Investors take the interest rate \( r \) and borrowing limit \( f \) as given. We first assume that \( r < \rho \), and will show that this condition holds in equilibrium. We discuss the investor’s problem in backward order. First, we analyze the problem when an investor already purchased a tree from the tree market and, for notational purposes, we name this kind of investors as entrepreneurs. Investors who have not yet received a tree are named workers. Second, we discuss the bargaining problem in the bilateral tree market meeting.
Entrepreneurs

Let \( z \) denote the initial saving of an entrepreneur. Then the dynamics for savings (\( z \geq 0 \)) or debt (\( z < 0 \)) is

\[
z_t = z e^{r t}
\]

If an entrepreneur has net savings, \( z \geq 0 \), it does not worry about the borrowing limit. The entrepreneur waits for the project to mature and consumes all its wealth, which includes the dividend \( y \) of running the project, plus the return of tree liquidation \( f \), plus the interest payment for initial saving, \( z e^{r t} \). Because the process of a project’ maturity is Poisson, the entrepreneur’s value function is

\[
\tilde{W}(z) = \int_0^\infty e^{-(\rho+\delta)t} \delta \left[ y + f + z e^{r t} \right] dt
\]

If an entrepreneur has net borrowings, \( z < 0 \), the debt rolls over, and the entrepreneur defaults when the borrowing limit is reached. Because entrepreneurs cannot work, there is no way for the entrepreneur to repay the debt before the project matures. Let \( \bar{\tau}(z) \) denote the maximum time that an entrepreneur can roll over the debt when he or she has real balance \( z < 0 \). Then \( \bar{\tau}(z) \) solves

\[
z e^{r \bar{\tau}} = -f
\]

Entrepreneurs make decisions about defaulting or rolling over the debt. First, we assume that an entrepreneur does not default but keeps rolling over his or her debt until the project matures or the borrowing limit is reached. The value function of an entrepreneur with net borrowing (\( z < 0 \)) is

\[
\tilde{W}(z) = \int_0^{\bar{\tau}(z)} e^{-(\rho+\delta)t} \delta \left[ y + f + z e^{r t} \right] dt
\]

(2.3)
Figure 2.1: Debt Rollover

Equation (2.3) states that if the project matures before the borrowing limit is reached, the entrepreneur receives $y + f + ze^{rt}$, which consists of the dividend from running the project, plus the return of tree liquidation $f$, minus the debt repayment. If the borrowing limit is reached before the project matures, the entrepreneur defaults and is forced to liquidate the tree to repay his or her debt. In this case, there is no dividend payment, and $ze^{rt} = -f$, meaning that the revenue of liquidating trees are fully repayed to the lender, but the entrepreneur receives no return (see Figure 2.1 for an illustration). $\tilde{W}(z)$ can be solved as follows (see Appendix):

$$\tilde{W}(z) = \begin{cases} 
\frac{\delta}{\rho + \delta} (y + f) + \frac{\delta}{\rho + \delta - r} z & \text{if } z < 0 \\
\frac{\delta}{\rho + \delta} (y + f) + \frac{\delta}{\rho + \delta - r} z & \text{if } z \geq 0
\end{cases}$$

(2.4)

Second, we discuss the case in which an entrepreneur defaults before the borrowing limit is reached. In this case, the tree will be liquidated on the centralized liquidity market, and the entrepreneur repays his or her debt $b = -z$ to the lender and receives the rest, $f - b$. Recall that the entrepreneur still has to wait for the project to mature in order to
consume. Let $W^f(z)$ denote the value function of a defaulting entrepreneur; then,

$$W^f(z) = \int_0^\infty e^{-\rho t} \left[(f + z)e^{rt}\right] \delta e^{-\delta t} \, dt.$$ 

Solve for the equation, we have

$$W^f(z) = \frac{\delta}{\rho + \delta - r}(f + z).$$

An entrepreneur can decide to default or not at any point in time, and thus, the value of an entrepreneur is $W(z) = \max\{\bar{W}(z), W^f(z)\}$. The following lemma shows that if the liquidation price is not too high, an entrepreneur will not default before the borrowing limit is reached.

**Lemma 5.** If $f \leq \frac{\rho + \delta - r}{r} y$, the following properties hold:

1. $\bar{W}(z) > W^f(z)$. An entrepreneur does not default before the borrowing limit is reached, and thus, $W(z) = \bar{W}(z)$.

2. $W(z)$ is strictly concave for $z < 0$.

3. $W(-f) = 0$.

4. $\frac{d}{dz} W(-f) = \frac{\delta y}{r} f$.

5. $\frac{d}{dz} W(z) = \frac{\delta}{\rho + \delta - r} < 1$ for $z \geq 0$.

Lemma 1.1 is a direct result of the property that $W^f(z) < \bar{W}(z)$ for all $z > -f$ when $f \leq \frac{\rho + \delta - r}{r} y$. This property is intuitive because an entrepreneur only operates the project if its return is high enough. Lemmas 1.2 to 1.5 characterize an entrepreneur’s value function. When the debt is close to the borrowing limit ($z \to -f$), an entrepreneur defaults very soon, and thus, its wealth is close to zero. In this case, a decrease in borrowing not only decreases the interest payment but also increases the length of time to operate the
project. When $z > 0$, the investor is a saver, and an increase in wealth only increases the interest payment but does not improve the surplus from operating the project; thus, $\frac{d}{dz} W(z)$ decreases as $z$ increases, so $W(z)$ is concave in $z$. (see Figure 2.2) These properties given by Lemma 1 will help us solve the bargaining problem for meetings in a decentralized tree market.

**Tree Market Bargaining**

Before an investor receives a tree, it participates the tree market for a bilateral meeting. When an investor meet a tree in a bilateral tree market meeting, the investor and the tree owner bargain to determine the terms of trade. In a bilateral meeting, an investor has two ways to finance the tree purchases: it can work, or it can use the tree as collateral to borrow. An investor works less if it borrows more, but a higher level of initial borrowing means higher interest payments and greater likelihood of reaching the borrowing limit before the project matures. The investor and the tree owner bargain over the terms of trade $(p, x, a)$, where $a$ captures whether there is a trade or not. If $a = 1$, the trade happens, and the tree is transferred from the tree owner to the investor; if $a = 0$, there is no trade,
and the tree owner retains the tree. If there is a trade, \( p \) denotes the price of the tree, and \( x \) denotes the investor’s level of work. We assume that the terms of trade are determined by proportional bargaining, in which the buyer obtains a constant fraction of the surplus from the exchange in the bilateral meeting. Let \( \theta \in [0,1] \) denote the fraction of surplus received by the buyer, and let \( \vartheta \) represent the bargaining power of the consumer. The proportional bargaining problem is expressed as follows:

\[
\max_{x, p \in \mathbb{R}, a \in \{0,1\}} \ [W(x - p) - x - V]a + [p - f]a \\
\text{s.t.} \quad [W(x - p) - x - V]a = \frac{\theta}{1 - \theta} [p - f]a
\]

(2.5)

\begin{align*}
W(x - p) - x - V & \geq -f \\
\end{align*}

(2.6)

Where \( [W(x - p) - x - V] \) is the surplus for an investor in the decentralized meeting, and \( W(x - p) - x \) is the investor’s return in the meeting, which is equal to its expected return from operating the project with initial real balance \( x - p \), minus the disutility of working \( x \). \( V \) is the continuation value of the investor, and the investor’s continuation value is equal to the investor’s expected return from a future decentralized meeting. Therefore, \( V^* \) solves

\[
(\rho + \delta)V^* = \mu [W(x^* - p^*) - x^* - V^*].
\]

(2.7)

Therefore,

\[
V^* = \frac{\mu}{\rho + \mu + \delta} [W(x^* - p^*) - x^*].
\]

(2.8)

We now turn to the selling side of the meeting, where \([p - f]\) represents the trading surplus of a tree seller. Recall that \( f \) is the liquidation price of the tree, and it is also the reservation value of the tree for the tree seller. We first take \( f \) as given, and \( f \) will be determined in equilibrium. The bargaining problem is to maximize total surplus in the exchange subject to the proportional bargaining constraint (2.5) and the investor’s borrowing constraint (2.6).
Notice that because one unit of labor generates one unit of disutility, and the surplus of the tree seller also increases one-for-one in \( p \), two participants in the bargaining game can linearly transfer their surplus through working and paying the tree price. Therefore, we can divide the bargaining problem into two stages. In the first stage, the buyer and the seller maximize the total surplus by choosing the net savings of the buyer, \( z^* \), which solves \( \max_z [W(z) - z] \) subject to \( z \geq -f \). In the second stage, they determine the allocation of the surplus by choosing \( x \) and \( p \) based on the bargaining power, \( \theta \), and matching rate, \( \mu \).

The first-order condition with respect to \( z \) is

\[
\begin{align*}
\frac{d}{dz} W(z^*) &\leq 1, \\
\frac{d}{dz} W(z^*) &= 1 \quad \text{if } z^* > -f
\end{align*}
\]

The first order condition states that if there is a trade, the marginal benefit from working should be equal to the marginal cost. First, from Lemma 1.5, \( \frac{d}{dz} W(z) = \frac{\delta}{\rho + \delta - r} < 1 \), meaning that it is not optimal for the tree buyer to work too much and hold a positive real balance, and this is because the interest rate is lower than the time discount rate. Second, from Lemma 1.4, \( \frac{d}{dz} W(-f) = \frac{\delta y}{r f} \), and therefore, if \( y > \frac{r}{\delta} f \), the return from operating the project is large enough, and a positive surplus can be created in the decentralized meeting by transferring trees from the seller to the buyer, which implies that trees will be traded \((a = 1)\). Combining Lemmas 1.4 and 1.5, if \( y > \frac{r}{\delta} f \), the investor should start its project, and it borrows to finance the tree purchase, \( x^* - p^* \in (-f, 0) \). If \( y < \frac{r}{\delta} f \), there is no surplus in the decentralized meeting, and thus it is irrelevant whether the tree is traded.

Henceforth, we assume that \( f < \frac{\delta}{\tau} y \) and will show that the condition holds in equilibrium. Notice that \( r < \rho \) and \( f \leq \frac{\delta}{\tau} y \) imply \( f \leq \frac{\rho + \delta - r}{\tau} y \), and thus, according to Lemma 1.1, an entrepreneur does not default until the borrowing limit is reached. The properties of the bargaining solution are summarized in Lemma 6.

**Lemma 6.** If \( f < \frac{\delta}{\tau} y \), the following properties hold.

1. An tree is traded in a decentralized meeting \((a = 1)\).
2. An tree buyer always borrows to buy the tree. \((x^* - p^* < 0)\)

3. Whenever a buyer obtains an tree from the decentralized meeting, it will not default before the borrowing limit is reached.

Finally, given \(r, \mu\) and \(f\), we can obtain the analytical solution of \(p^*, x^*\) and \(V^*\) from equations (2.5), (2.8) and (2.9).

**Liquidation Price and Haircuts**

We define \(N_0, N_1, N_d\) as the number of workers, the number of entrepreneurs and the number of defaulting entrepreneurs. The number of buyers in the asset market equals the number of workers, \(N_0\). The number of sellers in the decentralized asset market should be equal to the number of trees that are not used to operate projects, \(A - N_1\). Then, asset market tightness satisfies

\[
\sigma = \frac{N_0}{A - N_1} \quad (2.10)
\]

We now turn to the liquidation price of trees, \(f\). A tree that is not used to operate a project is waiting to be sold in the decentralized asset market. These trees are equivalent to securities that give the tree holders \(p\) units of the numeraire good with a Poisson arrival rate \(\eta(\sigma)\). Therefore, \(f\) solves

\[
rf = \eta(\sigma) [p(f, \sigma, r) - f]
\]

That is, the liquidation price of the tree is the expected discounted value of its cash flow:

\[
f = \frac{\eta(\sigma)}{r + \eta(\sigma)} p(f, \sigma, r) \quad (2.11)
\]

We denote \(h(\sigma) = \left(1 - \frac{\eta(\sigma)}{r + \eta(\sigma)}\right)\); then, \(h(\sigma)\) is the proportional difference between the price of the tree and the borrowing capacity when the tree is used as collateral, which is the haircut of the secured loan. The haircut is caused by tree market search and matching.
friction, and as the salability of assets in tree market, $\eta(\sigma)$, decreases, the haircut increases.

We can rewrite (2.11) as

$$ f = (1 - h(\sigma)) p(f, \sigma, r) $$

(HE)

**Wealth Distribution**

Let the distribution of entrepreneurs’ borrowing at time $t$ be $G_t(b)$. The dynamics of $G_t(b)$ must satisfy the following equation

$$ G_{t+\Delta t}((1 + r\Delta t)b) = G_t(b)(1 - \delta \Delta t) + \mu(\sigma) \Delta t N_0 $$

(2.12)

The left-hand side is the number of entrepreneurs at time $t + \Delta t$ who borrow less than $(1 + r\Delta t)b$. This should be equal to the number of entrepreneurs who borrow less than $b$ at time $t$, $G_t(b)$, minus the entrepreneurs whose projects mature between time $t$ and $t + \Delta t$, $G_t(b)\delta \Delta t$, plus the number of new entrepreneurs who just matched in the decentralized market, $\mu(\sigma) \Delta t N_0$.

Let $\Delta t \to 0$, equation (2.12) becomes

$$ rbG'(b) = \dot{G}(b) - \delta G(b) + \mu(\sigma) N_0 $$

(2.13)

In a stationary equilibrium $\dot{G}(b) = 0$. Thus, we rewrite (2.13) as

$$ rbG'(b) = -\delta G(b) + \mu(\sigma) N_0 $$

(2.14)

Because an entrepreneur’s initial borrowing is $p^* - x^*$, we also have an initial condition for the ordinary differential equation

$$ G(p^* - x^*) = 0 $$

(2.15)

Solving for the ordinary differential equation (2.14) with the initial condition (2.15), we
have

\[ G(b) = \frac{\mu(\sigma)}{\delta}N_0 \left[ 1 - \left( \frac{p^* - x^*}{b} \right)^{\frac{1}{\gamma}} \right], \quad b \geq p^* - x^* \]  

(2.16)

Since entrepreneurs can borrow up to the borrowing limit, \( f \), the total number of entrepreneurs satisfies

\[ N_1 = G(f) = \frac{\mu(\sigma)}{\delta}N_0\Phi(f, r) \]  

(2.17)

Where

\[ \Phi(f, r) = \left[ 1 - \left( \frac{p^* - x^*}{f} \right)^{\frac{1}{\gamma}} \right] \]  

(2.18)

Moreover, in a stationary equilibrium, the number of workers must be constant, so

\[ \delta N_0 + \mu(\sigma)N_0 = \delta \]  

(2.19)

The left-hand side of the equality is the outflow of workers, which is equal to the number of workers whose project mature and leave the economy plus the number of workers who obtain a decentralized meeting and become entrepreneurs. The right-hand side is the inflow of workers, who are the new entrants in the economy.

Finally, the sum of workers, entrepreneurs, and defaulting entrepreneurs are equal to 1.

\[ N_0 + N_1 + N_d = 1 \]  

(2.20)

By equations (2.17), (2.19), (2.20), we can solve \( N_0, N_1, N_d \) as functions of \( r, \sigma \) and \( f \). Substitute \( N_0 \) and \( N_1 \) into (2.10), the condition for a stationary equilibrium is obtained

\[ \Phi(f, r) = \frac{\mu(\sigma) + \delta}{\mu(\sigma)}A - \frac{\delta}{\sigma \mu(\sigma)} \]  

(ST)
Figure 2.3:

2.3.2 Tree Market Partial Equilibrium

**Definition 2.** Given an interest rate $r$, a stationary asset market partial equilibrium is a set $(f, p, \sigma, a)$ and corresponding $N_0, N_1, N_d$ such that

1. Entrepreneurs’ debt rollover is rational

2. Given $f$ and $\sigma$, the bargaining problem in the decentralized meeting is solved by $a$ and $p$

3. The expected return from holding trees is equal to the liquidation price $f$

4. The distribution of investors is endogenously determined by laws of motion.

5. Asset market tightness $\sigma$ is consistent with the number of households.

The stationary equilibrium can be solved by equations (HC) and (ST), which are depicted in Figure 4. The monotonicity of (HC) and (ST) provides the following lemma.

**Lemma 7.** Given the interest rate $r < \rho$, there exists a unique pair $(f^*, \sigma^*)$ which solves equation (HC) and (ST). Moreover, $f^* < \frac{\hat{\delta}}{r} y$.

**Proof.** See the Appendix.
Combining Lemma 3 and Lemma 4, we can characterize the following characteristics in equilibrium. Lemma 7 demonstrates the uniqueness of the stationary equilibrium. Moreover, because $f^* < \frac{\delta}{\tau} y$, the properties in Lemma 6 must hold in a stationary equilibrium. This allows me to characterize the behavior of investors in the asset market when $r$ is given.

**Comparative Statics**

In this subsection, we obtain comparative statics for the stationary equilibrium to examine the effects of changes in the parameters on the endogenous variables. We first analyze the effect of changes in the dividend payment $y$. Figure B.1 in the Appendix depicts the effect of changes in $y$ on the variables listed above. Note that the dividend payment is the source of the asset value of the surplus of asset market trades, and the household gains linear utility/disutility from the consumption of the dividend and working. Therefore, a decrease in the dividend payment, $y$, only proportionally decreases the asset price, $p$, and the liquidation price, $f$, but the haircut, the asset market tightness, and households’ leverage are not affected.

However, a change in asset market liquidity $\Lambda$ affects the haircut, market tightness, and households’ leverage (see Figures B.2 and B.3 in the Appendix). First, a decrease in asset market liquidity directly decreases the probability of selling a tree in the decentralized market after a borrower defaults and, therefore, increases the haircut and decreases the borrowing limit. This effect also increases investors’ default rate and causes deleveraging. Moreover, a decrease in asset market liquidity decreases the a investor’s probability of obtaining a decentralized asset market meeting. Therefore, the number of entrepreneurs, $N_e$, decreases, and the number of idle trees, $A - N_e$, increases. However, the decrease in asset market liquidity also increases the number of workers, $N_0$. Note that according to (2.10), when trees are scarce relative to the number of investors, the increase in idle trees dominates the increase in workers, and thus, market tightness decreases, and
the probability of selling a tree decreases further (Figure B.2). When trees are plentiful relative to the number of investors, the increase in workers dominates the increase in idle trees, and thus, market tightness increases (Figure B.3). To summarize, a change in asset market liquidity results in an increase in the haircut and the number of idle trees, and it also causes deleveraging. Moreover, this effect is especially amplified through a decrease in market tightness when the number of trees is low.

2.3.3 Households' Problem

We now turn to the demand for liquidity by households. Let $H(a_0)$ denote the lifetime expected discounted utility of a household holding $a_0$ units of liquid assets (asset based loan, idle trees, and government bonds). The household solves

$$H(a_0) = \max_{a(t), c(t), \Delta T_n} \int_0^\infty e^{-(\rho + \alpha)t} \{c(t) + \alpha \Omega [a(t)]\} dt - \sum_{n=1}^\infty e^{-(\rho + \alpha)T_n} \Delta_n$$

s.t. $\dot{a} = ra - c - \tau$

$$\Delta T_n = a(T_n^+) - a(T_n^-)$$

$$a(0) = a_0$$

where $T_n$ is the random time at which the trader is matched with another trader. According to (2.21) the trader chooses his asset holdings, $a(t)$, and consumption path, $c(t)$, so as to maximize the discounted sum of his consumption $c$ plus the present continuation value of a trading opportunity in the service market, $\Omega [a(t)]$. From (2.23), the last term on the right side of (2.21) represents discrete jumps in asset holdings, $\Delta_n$, financed by lumpy production of the numeraire good. The trader chooses both the sizes of these discrete adjustments, $\Delta_n$, and their timing, $T_n$. Equation (2.22) is a budget identity according to which the trader produces the numeraire good to finance the change in asset holdings ($\dot{a}$) and taxes ($\tau$) net of the return on those assets ($ra$).
The current-value Hamiltonian is \( \Pi = c + a\Omega(a) + \lambda [ra - c - \tau] \). Taking the derivatives with respect to \( c \) and \( a \) yields

\[
\begin{align*}
\lambda &= 1 \\
(\alpha + \rho)\lambda &= \alpha\Omega'(a) + \lambda r + \dot{\lambda}
\end{align*}
\]

Combining the above equations, we have

\[
\Omega'(a) = 1 + \frac{\rho - r}{\alpha} \geq 0
\]

(2.25)

The left side of (2.25) is the benefit to a trader from holding an additional unit of asset. The right side of (2.25) is the cost of purchasing assets worth one unit of numeraire good augmented by the expected holding cost of the asset until the next trading opportunity in the service market. This holding cost is equal to the difference between the rate of time preference and the real interest rate, \( \rho_h - r \), multiplied by the average time until the next trading opportunity in the bilateral service meeting, \( \frac{1}{\alpha} \).

In the bilateral service meeting, households can produce or consume service. The household is a consumer with probability 1/2 and a producer with probability 1/2. The bargaining is proportional, and the buyer has bargaining power \( \epsilon \in [0,1] \). Let the consumer hold \( a_c \) units of the liquid asset and the producer hold \( a_p \) units of liquid asset. The bargaining problem in a bilateral meeting is

\[
\max_{x,m} \left[ u(x) - m + H(a_c) \right] + \left[ -x + m + H(a_p) \right]
\]

Subject to \( u(x) - m = \epsilon [u(x) - x] \)

\[
m \leq a_c
\]
Where \( u(x) - m + H(a_b) \) represents the trading surplus of the consumer, and \(-x + m + H(a_p)\) represents the trading surplus of the producer. In a bilateral meeting, the buyer and seller choose the consumption of services, \(c\), and a transfer of liquid assets to the seller, \(m\). The first constraint is the proportional bargaining constraint, and the second constraint states that the transfer in the bilateral meeting is limited by the real balance of the consumer. We first assume that the real balance constraint \((m \leq a_c)\) does not bind. Let \(\tilde{x}\) solve \(u'(\tilde{x}) = 1\); then, the bargaining solution is \(x = \tilde{x}\), and the transfer quantity must be \(m = \epsilon \tilde{x} + (1 - \epsilon)u(\tilde{x})\). We need to check whether the real balance constraint binds, which requires \(a_c \geq \epsilon \tilde{x} + (1 - \epsilon)u(\tilde{x})\). If \(a_c < \epsilon \tilde{x} + (1 - \epsilon)u(\tilde{x})\), then let \(x(a_c)\) solve. Then

\[
\begin{cases}
  x^* = \tilde{x}, m^* = \epsilon \tilde{x} + (1 - \epsilon)u(\tilde{x}) & \text{for } a_b \geq \tilde{x} \\
  x^* = x(a_c), m^* = a_c & \text{for } a_b < \tilde{x}
\end{cases}
\]  

(2.26)

Let \(\Omega_b(a)\), \(\Omega_s(a)\) be the value of buyers and sellers in a DM meeting. \(\Omega(a) = \frac{1}{2}\Omega_b(a) + \frac{1}{2}\Omega_s(a)\) is the household’s value when it obtains a decentralized meeting, before her state as a seller or a buyer is realized. Then,

\[
\frac{\partial \Omega_c}{\partial a}(a) = \begin{cases} 
  \frac{u'(x(a))}{\epsilon + (1 - \epsilon)u'(x(a))} & \text{if } a \geq \tilde{x} \\
  1 & \text{if } a < \tilde{x}
\end{cases}
\]

\[
\frac{\partial \Omega_s}{\partial a}(a) = 1
\]

\[
\frac{\partial \Omega}{\partial a}(a) = \frac{1}{2} \frac{\partial \Omega_b}{\partial a}(a) + \frac{1}{2} \frac{\partial \Omega_s}{\partial a}(a)
\]

Let \(a(r)\) solve (2.25). Then, liquidity demand as a function of \(r\) can be expressed as follows:

\[
L^d(r) = \begin{cases}
  a(r) & \text{for } r < \rho_h \\
  \tilde{x}, \infty & \text{for } r = \rho_h
\end{cases}
\]  

(2.27)
To summarize, a household consumes the interest payment at each point in time and holds the value of its real balance constant as $a(r)$. It waits for an opportunity to participate in a decentralized meeting. Once it engages on the buy side of the DM meeting, it spends all of its assets on DM goods. Immediately after the DM meeting, the household works and restores $a(r)$ units of its real balance from the liquidity market to prepare for the next DM meeting. If the household participates on the sell side of DM meeting, it produces DM goods and earns $a(r)$ units of real balance. Immediately after the DM meeting, it consumes the extra $a(r)$ units of real balance and restores its original real balance, which is also $a(r)$.

### 2.3.4 The Liquidity Market

In this subsection, we solve the liquidity market equilibrium and the equilibrium interest rate $r$. Trees provide liquidity in various ways. First, when entrepreneurs use trees as collateral to borrow, the secured loans become a risk-free security. Second, when a tree is not used to operate a project, it remains in the tree market and is waiting to be sold, which also a liquid asset. Moreover, we introduce the government into the liquidity market. The government can influence the interest rate through open market operations and, therefore, influences the tree market and liquidity market.

**Private Liquidity**

First of all, when an tree is used to run the project, it is pledged as collateral to secure the loan. In this model, the aggregate liquidity supply created by secured lending, $L^e(r)$, is equal to the total amount of entrepreneurs’s borrowing.

\[
L^e(r) = \int_{p^*-x^*}^{f^*} b \, dG(b) \\
= (p^* - x^*) \left[ \frac{\mu(\sigma)}{r - \delta} \right] \left[ \frac{\delta}{\mu(\sigma) + \delta} \right] \left[ \left( \frac{p^* - z^*}{f^*} \right)^{\frac{\sigma}{r - \delta}} - 1 \right]
\]
Where $p^* - x^*$ is entrepreneurs’ initial borrowing, $f^*$ is the borrowing limit, and $G(b)$ is the distribution of entrepreneurs’ borrowing. Second, trees which are not matched with entrepreneurs are waiting to be sold in the asset market. The liquidity provided by one unit of the tree is equal to its liquidation price, $f$. Therefore, the aggregate liquidity provided by trees remaining in the market is

$$L^f(r) = f^*(A - N_1)$$

Public Liquidity

In addition to private liquidity, the government provides public liquidity by issuing bonds. We assume that there is a supply, $B$, of government debt, and the interest payment on bonds is financed through lump-sum taxation imposed on the households. Therefore, the government’s budget constraint is $\tau = rB - \dot{B}$. Combining private liquidity and public liquidity, the aggregate liquidity supply in the credit market is

$$L^s(r) = L^e(r) + L^f(r) + B$$

Liquidity Market Clearing

The clearing condition for the liquidity market

$$L^d(r) = L^s(r)$$

Since liquidity demand increases monotonically with the interest rate, and liquidity supply decreases monotonically with the interest rate, there exists a unique $r$ that clears the liquidity market. The uniqueness of the market clearing interest rate $r$ also guarantees the uniqueness of the stationary equilibrium.

Figure 2.4 depicts how changes in parameters shift liquidity demand and supply. First, an increase in the dividend $y$ and maturity rate $\delta$ increases the value of trees, and thus,
the asset price \( p \) and the liquidation price \( f \) will increase. Therefore, the aggregate value of asset-backed loans and unused trees in the asset market increases, which increases the supply of liquidity. Second, an increase in the matching rate \( \Lambda \) decreases the haircut and thus increases the liquidation price, meaning that \( L_f(r) \) increases. Moreover, this increase also speeds up tree circulation and increases the number of entrepreneurs, and therefore, \( L_f(r) \) increases. Combining these two effects, the aggregate liquidity supply increases. Finally, an increase in tree buyers’ bargaining power \( \theta \) decreases the tree price \( p \), and thus liquidity supply decreases. The influence of parameters on liquidity demand is intuitive: the demand for media of exchange increases when the arrival rate of the liquidity shock, \( \alpha \), increases and decreases when the consumers’ bargaining power, \( \epsilon \), increases.

### 2.4 Transitional dynamics

We assume that the government establishes an interest rate target, and it achieves the target by changing the bond issuance and tax levies. The transition dynamics are simulated as follows. Before \( t = 0 \), the economy is in a steady-state equilibrium. At time \( t = 0 \),
an unexpected, permanent shock arrives. Once the shock arrives, all households and investors in the economy instantly have full information about the state. There will be no remaining uncertainty, making the transition path deterministic.

In the first exercise, we simulate an unexpected decrease in asset market liquidity Λ. In the second exercise, we simulate monetary policy shock: a change in the interest rate target, r.

### 2.4.1 The Dynamic Equilibrium

The dynamic equilibrium is defined as follows.

**Definition 3.** Given r, an equilibrium along a deterministic transition path is a tuple \( \{ p_t, z_t, f_t, \sigma_t \} \), borrowing distribution of entrepreneurs \( G_t(b) \), and population \( N_{0t}, N_{1t} \) such that

1. \( \{ p_t, z_t \} \) solves the bargaining problem.
2. The liquidation price \( f_t \) satisfies the law of motion \( r f_t = \eta(\sigma_t)p_t + \dot{f}_t - \eta(\sigma_t)f_t \)
3. The borrowing distribution \( G_t(b) \) is endogenously determined by the laws of motion.

\[
\left\{ \begin{array}{l}
rbG_t'(b) = -\dot{C}_t(b) - \delta G_t(b) + \mu(\sigma_t)N_{0t}^0I_{\{b \geq p_t - z_t\}} & \text{for } b < f_t \\
G_t'(b) = 0 & \text{for } b \geq f_t
\end{array} \right.
\]

4. The number of entrepreneurs satisfies \( G_t(f_t) = N_{1t} \)
5. Asset market tightness satisfies \( \sigma_t = \frac{N_{0t}^0}{\Lambda - N_{1t}} \)

The algorithm to compute the transition dynamics is presented in Appendix B.7.

### 2.4.2 Asset market liquidity shock

In the first exercise, we simulate the case of a decrease in asset market liquidity \( \Lambda \) (Figure B.4). At time 0, asset market liquidity \( \Lambda \) decreases unexpectedly. This effect directly reduces the probability of being able to liquidate the tree in the asset market and, therefore, increases the haircut. The increase in the haircut decreases the borrowing limit of a
collateralized contract, and entrepreneurs whose borrowing exceeds the new borrowing limit are forced to default. A defaulter’s tree must be liquidated in the asset market. This massive default dramatically decreases asset market tightness, θ, and further decreases the salability of the tree, increases the haircut and results in more default. This feedback loop generates a fire sale in tree and causes an overshooting in the tree prices, market liquidity and haircut.

Although the tree price and liquidation price rebound after the shock, the number of entrepreneurs recovers slowly and remains in a low state for a longer time. The decline in the number of entrepreneurs indicates a sustainable decrease in collateralized borrowing and output. However, the low market tightness also makes tree buyers more likely to buy an tree. The new entrepreneurs pay a lower price and start their projects with lower initial levels of borrowing. The number of entrepreneurs and asset market tightness recover gradually as the trees are sold in the DM meetings. The tree price and liquidation price also increase as the asset market becomes more liquid and ultimately converge to the new equilibrium. The temporary illiquidity in the asset market is the key friction that causes the fire sale and overshooting of the tree price and borrowing limit.

### 2.4.3 Interest Rate Shock

In this subsection, the model is simulated as an increase in the target interest rate r. (Figure B.5) The direct effect decreases the tree price. This is because the increase in r increases entrepreneurs’ borrowing cost and makes them more likely to default. Moreover, the increasing of interest rate increases the cost of delayed payment and, therefore, increases the haircut. These two effects jointly decrease the liquidation price, f. Similar to the effect of a shock to tree market liquidity, the sudden decrease in the borrowing limit causes a massive default, and the mutually reinforcing effects lead to overshoots in the tree price, market tightness, and the haircut.
2.5 Conclusion

In this paper, we constructed a dynamic model to study the feedback loop between haircuts and asset market liquidity. In the model, the borrowing limit is equal to the liquidation asset price, and a haircut results from the search friction in the asset market. When an unanticipated shock arrives, the simultaneous defaults make the asset market illiquid. The mutually reinforcing effects between the borrowing limit and search friction result in a fire sale and repo run. Moreover, the asset price and borrowing limit overshoot because of the temporary illiquidity in the asset market.

There are several possible avenues for extensions of the present work in future research. First, instead of unanticipated shocks, one could consider anticipated shocks. For example, asset market liquidity decreases with some probability, and the probability is common knowledge. Second, one could also assume that there is a probability that investors’ projects fail, and therefore, the investor must default. In those cases, a repo contract is no longer risk-free to lenders, and lenders must request a positive repo spread as a risk premium.
Chapter 3

The Liquidity Theory of Risky Asset Purchases

3.1 Introduction

Since the 2008 financial crisis, governments have engaged in large-scale, unconventional purchases of government debt and private assets. For the Fed, such purchases, often referred to as the large-scale asset purchases program (LSAP) or quantitative easing (QE), have included purchases of over $1 trillion in long-maturity Treasury securities, mortgage-backed securities and agency securities. The US Treasury Department also implemented the Troubled Asset Relief Program (TARP), which involves purchases of risky assets and equity from financial institutions of over $400 billion. Those policies involve swaps of "risky" private assets for "safe" public assets, and the goal is to lower risk premia and improve liquidity.

By purchasing private assets, central banks remove them from the financial market and inject government liability. The removed private assets stay on central banks’ balance sheets, and, especially, when the private assets are considered risky, the purchased risky private assets can potentially influence the stability of the government budget and
influence the pay off of government securities.

In this paper, we construct a model of government swaps of public liquidity for risky assets. The basic structure follows Lagos and Wright (2005), and some details of the model are closely related to models constructed in Williamson (2012) and Williamson (2016), particularly in terms of the structure of financial intermediation and the relationship between fiscal and monetary policy. In the model, the basic assets are currency and trees. Trees are owned by households and pay random dividends. Currency can only be issued by the government. Consumers use financial intermediary liabilities and currency in decentralized exchanges. The financial intermediation arrangement is a type of insurance arrangement in the spirit of Diamond and Dybvig (1983), and banks act to efficiently allocate liquid assets in exchange. Limited commitment in the model requires that private debt be secured, either by currency or trees.

There are two forms of transaction in the decentralized meeting, financial transactions and currency transactions. In the financial transactions, sellers are well informed and able to recognize all forms of assets and liabilities issued by the financial intermediaries, and therefore, they are willing to accept all forms of assets and secured liabilities as media of exchange. However, in the currency transactions, sellers are less informed, and they can only recognize and accept currency as a medium of exchange. Because of the market segmentation between currency and financial transactions, the risks in the value of trees only influence the financial transactions but not the currency transactions. This paper considers government purchases of the risky asset as aggregate liquidity and risk management. When the government purchases of risky assets swap the risky assets for currency, the quantity of risky assets outstanding decreases but the risk in currency increases. The policy overcomes the market segmentation and reallocates the risks from financial transactions into currency transactions, and social welfare is improved because of the risk sharing between those two transactions.

The goal of this paper is to discuss the optimal level of risky asset purchases. If house-
holds in the economy are identical, the optimal policy is to purchase all risky assets and completely share the risk between financial transactions and currency transactions. However, when households are heterogeneous in their risk aversion, purchasing all risky assets may not be optimal. This is because, unlike the risks in financial transactions, which can be well allocated by banks across households with different levels of risk aversion, in currency transactions, banks’ liabilities are not accepted as a medium of exchange, and thus, banks cannot efficiently reallocate the risks in currency transactions.

The government can potentially issue more than one type of currency, each of which has a different risk structure, to facilitate risk allocation in currency transactions. However, the issuance of currencies, the assets that are recognizable by every seller, can be costly. When the issuance cost is high, it is not worthwhile for the government to issue too many currencies. As a result, the risks in currency transactions cannot be well allocated, and the government purchasing all risky assets and completely sharing the risk between financial transaction and currency transactions may not be socially optimal. The optimal policy requires the government to purchase only a part of the risky assets and ensure that the currency remains less risky.

3.2 Related Literature

This paper is closely related to the literature on unconventional monetary policy and private asset purchases. In Williamson (2012), private asset purchases by the central bank are either irrelevant or reallocate credit and redistribute income between firms and households. Williamson (2014) consider asymmetric information in private assets, and the central bank’s asset purchasing program works by increasing the value of the stock of collateralizable wealth and relaxing banks’ collateral constraints. Other related work on central bank intervention and collateral includes Kiyotaki and Moore (2012), Gertler and Kiyotaki (2011), and Gertler and Karadi (2011).
This paper contributes to this strand of literature by considering an environment with heterogenous risk aversion and risk sharing among households with different levels of risk aversion. In Caballero and Farhi (2016), swaps of private risky assets for safe public debt can stimulate aggregate demand and output. The government’s capacity to increase the total supply of safe assets by issuing safe public debt depends on its fiscal capacity, that is, its ability to raise additional taxes after the bad Poisson event. He et al. (2015) emphasize that the public supply of safe assets is determined not only by social capacity but also by self-fulfilling expectations supported by strategic complementarities among investors arising in the presence of default decisions.

This paper is closely related to the literature that discusses the risk and liquidity of assets in monetary search models. In Lagos (2010), risky assets are valued by the liquidity premium and the risk premium. In the asymmetric information literature, Li et al. (2012) capture the liquidity of assets by considering a counterfeiting problem. In Lester et al. (2012), households are required to invest in asset recognition. This paper considers an environment in which the government pays a fixed cost to issue assets that are difficult to counterfeit and can be recognized by every household.

The remainder of the paper is organized as follows. In Section 3, I describe the baseline model. The equilibrium is characterized and analyzed in Section 4. In Section 5, the policy effects of the government’s risky asset purchases are analyzed. Section 6 concludes.

3.3 The Model

Time is indexed by $t = 0, 1, 2, \ldots$, and in each period there are two sub-periods — the decentralized market (DM) and the subsequent centralized market (CM). There is a continuum of buyers and a continuum of sellers, each with unit mass. Buyers consume in the DM and consume and produce in the CM. Sellers produce in the DM and consume and produce in the CM. One unit of labor input produces one unit of the perishable con-
sumption good, in either the CM or the DM. An individual buyer has preferences

\[ E_0 \sum_{t=0}^{\infty} \beta^t [u_t(x_t) + X_t] \]

where \( X_t \in (-\infty, \infty) \) is net consumption in the CM, and \( x_t > 0 \) is consumption in the DM. \( u_t(x_t) \) is assumed to be CRRA, and \( 0 < \beta < 1 \). Each seller has preferences

\[ E_0 \sum_{t=0}^{\infty} \beta^t [-h_t + X_t] \]

where \( X_t \in (-\infty, \infty) \) is net consumption in the CM, and \( h_t > 0 \) is labor supply in the DM.

There are two assets in this economy: trees and currency. In each CM, sellers produce \( k \) units of divisible trees at zero cost. A tree yields fruit in the next CM, but this payment is subject to an aggregate shock. With probability \( (1 - \epsilon) \), all trees in the economy deliver one unit of fruit, \( d_t = 1 \), and this state is referred to as the high state; with probability \( \epsilon \), all trees in the economy yield no fruit, \( d_t = 0 \), which is referred to as the low state. Trees are fully depreciate after the fruit and payments are delivered in the CM, and they sell for \( q^k_t \) units of consumption goods in the CM. Currency is issued by the government. The government has \( m_t \) units of currency outstanding (in real term) in period \( t \), and the inflation rate is \( \pi^h_t \) and \( \pi^l_t \) in the high and low states, respectively.

In the DM, there are random matches between buyers and sellers. Each buyer is matched with a seller, and the buyer makes a take-it-or-leave-it offer to the seller. All DM matches have the property that there is no memory or record-keeping, and thus, the lack of memory implies that there is no unsecured credit. If a seller were to extend an unsecured loan to a buyer, the buyer would default.

I assume that there is geographic segregation between markets. There are three islands in the economy, \( \{C, D^1, D^2\} \). The CM opens on island \( C \), and the DM opens on the other two islands, \( D^1 \) and \( D^2 \). In the DM, a fraction \( \gamma \) of buyers and sellers goes to island \( D^1 \), and sellers on island \( D^1 \) cannot verify that the buyer possesses any asset other than
currency, and thus, currency is the only means of payment in exchanges. The transactions on island $D^1$ are denoted currency transactions. A fraction $1 - \gamma$ of buyers goes to island $D^2$. All forms of assets and credit held by the buyer can be verified by the seller on island $D^2$ and, therefore, can be accepted as means of payment by the sellers. Recall that credit arrangements must be secured by some sort of collateral, and I assume that the buyer cannot abscond with any proportion of the asset that is pledged as collateral. The transactions on island $D^2$ are denoted financial transactions.

I first assume that the government can issue at most one form of currency (i.e., all units of currency have the same inflation in the high and low states), and it costs $e^\delta$ units of CM goods for the government to issue the currency. The cost can come from preventing counterfeiting or decreasing the cost of screening to make households easily recognize the currency. In most economies in the real world, there is only one currency, and in our paper, this one-currency policy generates particularly interesting results regarding the optimal risky asset purchases by the government. I will relax this one-currency assumption in Section 3.5.3 and, by comparing the costs or indirect social costs of issuing multiple currencies and the social benefit thereof, I will discuss under what circumstances the one-currency environment emerges and how it influences the optimal policy.

In the DM, a buyer encounters idiosyncratic preference shocks that affect his or her degree of relative risk aversion, $\theta$. That is,

$$u_t(x) = \begin{cases} \frac{x^{1-\theta_a}}{1-\theta_a} & \text{with probability } \alpha \\ \frac{x^{1-\theta_u}}{1-\theta_u} & \text{with probability } 1 - \alpha \end{cases}, \text{ and } \theta_a \geq \theta_u.$$

At the beginning of the CM, a buyer does not know whether he or she will go to island $D^1$ or $D^2$ in the DM or his or her level of risk aversion in the subsequent DM. Buyers learn this at the end of the CM, after consumption and production have taken place in the CM. Once a buyer learns his or her type at the end of the CM, he or she can meet with at most one other agent (of his or her choice) before the end of the CM.
3.3.1 Banks

As in Williamson (2012), there is a banking arrangement that arises endogenously to efficiently allocate assets to the appropriate transactions. A bank is able to insure buyers against the need for different types of assets in different types of exchange and different levels of risk aversion. Without banks, individual buyers would acquire a portfolio of currency and trees before knowing whether they will be in a currency transaction or financial transaction in the subsequent sub-period and before they know their level of risk aversion. A banking arrangement essentially permits assets to be efficiently allocated among different buyers.

In the model, any agent – a buyer or a seller in the CM – can operate a bank. The timing of events is crucial in this model. I assume that a bank can only issues deposits in the CM before buyers learn their type of transaction and level of risk aversion in the subsequent DM. Banks can impose some sort of withdraw technology in the DM, such as an ATM, for buyers to withdraw assets. Specifically, Banks are able to place different ATMs on different DM islands, and this technology allows banks to differentiate between buyers in currency transactions and financial transactions. After the types are revealed, deposit-holders go to the islands and redeem their deposits at the ATM for a quantity of currency or tradable claims that will be redeemed by the bank in the CM in the next period for a specified quantity of consumption goods. A bank can issue different claims to target households with different levels of risk aversion. However, because buyers’ types are private information, it must be incentive compatible for the household to choose that claim. A bank has the same limited commitment problem that any individual agent has, and thus, the bank’s claims must be backed by trees as collateral. There exist asymmetric information problems in the sense that a buyer’s level of risk aversion is private information and cannot be observed by banks.

The bank contract is chosen in equilibrium to maximize the expected utility of the
buyer in the CM:

\[
\max -D_t + \beta(1 - \epsilon) \begin{cases} 
\gamma \left[ aV^u(c_{t+1}^{a,h}) + (1 - a)V^n(c_{t+1}^{n,h}) \right] \\
+ (1 - \gamma) \left[ aV^u(x_{t+1}^{a,h}) + (1 - a)V^n(x_{t+1}^{n,h}) \right]
\end{cases}
\]

\[
+ \beta \epsilon \begin{cases} 
\gamma \left[ aV^u(c_{t+1}^{a,l}) + (1 - a)V^n(c_{t+1}^{n,l}) \right] \\
+ (1 - \gamma) \left[ aV^u(x_{t+1}^{a,l}) + (1 - a)V^n(x_{t+1}^{n,l}) \right]
\end{cases}
\]

(3.1)

All quantities are expressed in units of the CM consumption goods. \(D_t\) denotes the quantity of goods deposited by the representative buyer. Note that because assets other than currency cannot be used as a medium of exchange in the currency transaction, it is weakly dominant for banks to allocate no trees or claims to buyers in the currency transaction. Therefore, I assume that banks only transfer currency to buyers in currency transactions. Let \(c_{t+1}^{ij}\) denote the value of currency that can be withdrawn by a depositor at the end of the CM if he or she encounters a currency transaction and is of type \(i\) in state \(j\). \(x_{t+1}^{ij}\) is the value of claims to consumption goods in the next CM that the buyer can trade in the DM. The objective function follows from the assumption of take-it-or-leave-it offers by the buyer in the DM. \(V^i(w)\) is a type \(-i\) buyer’s value in a DM meeting, which solves

\[
V^i(w) = \max u^i(y) + a
\]

subject to \(y + a = w\)

Where \(y\) is the liquidity used to facilitate DM trading, and \(a\) is the amount of liquidity retained by the CM. Let \(y^*\) solve \(u^i(y^*) = 1\); then,

\[
\begin{cases} 
\quad y = w \text{ if } w < y^* \\
\quad y = y^* \text{ if } w \geq y^*
\end{cases}
\]

If the medium of exchange is scarce (i.e., \(w < y^*\)) such that buyers’ marginal utility from
consuming DM goods is greater than one, a buyer gives all of his or her medium of ex-
change to the seller for DM goods; if the medium of exchange is plentiful, (i.e., \( w \geq y^* \)) the
buyer purchases up to \( y^* \) and retains the rest of his or her medium of exchange, \( w - y^* \),
into the next CM.

Banks are subject to the following individual rationality constraint:

\[
\{ D_t - m_t - q^t k_t \} + \beta (1 - \epsilon) \left\{ \begin{array}{c}
- \gamma \left[ ac_{t+1}^a + (1 - \alpha) c_{t+1}^{a,h} \right] \\
- (1 - \gamma) \left[ as_{t+1}^{a,h} + (1 - \alpha) s_{t+1}^{n,h} \right] + \left[ \frac{1}{1+\pi_{t+1}} m_t + k_t \right] \\
+ \beta \epsilon \left\{ \begin{array}{c}
- \gamma \left[ ac_{t+1}^d + (1 - \alpha) c_{t+1}^{n,d} \right] \\
- (1 - \gamma) \left[ as_{t+1}^{n,d} + (1 - \alpha) s_{t+1}^{d} \right] + \left[ \frac{1}{1+\pi_{t+1}} m_t \right] \\
\end{array} \right\} \geq 0
\]

(3.2)

The left-hand side of inequality (3.2) is the expected net payoff from banking activity.
Equation (3.2) states that bank must receive a nonnegative net return over the current
and the next CM, which is also referred to the individual rationality constraint. In the CM
of period \( t \), the bank receives \( D_t \) in deposits and acquires a portfolio of currency and trees
at market prices — the \( m_t \) and \( q^t k_t \), respectively. The bank pays currency to a fraction \( \gamma \)
of depositors who learn that they will need currency and each withdraw currency with
value \( c_{t+1}^{i,j} \). The remaining fraction \( 1 - \gamma \) of depositors trades its deposit claims in the DM,
and the holders of the deposit claims are paid \( x_{t+1}^{i,j} \) in the CM of period \( t + 1 \). Moreover,
the bank receives the payoffs from the remainder of its asset portfolio in the CM of period
\( t + 1 \). The total payoffs on currency and trees are \( \frac{1}{1+\pi_{t+1}} m_t + k_t \) in high state and \( \frac{1}{1+\pi_{t+1}} m_t \)
in low state.

Currency has two usages: it can be used as a medium of exchange in the currency
transactions, or it can be stored by banks and converted into reserves to back the issuance
of bank deposits and facilitate trading in financial transactions. Banks allocate a part of
the currency, \( m_t^c \), to buyers in currency transactions; the remainder of the currency, \( m_t^x \), is
converted into reserves, which can be used as collateral for the bank claims. Thus,

\[ m_i^c + m_i^x = m_t \left( \phi_{t+1} \right), \]  

where the variables in the parentheses are denoted by the associated Lagrangian multipliers hereafter. The currency allocated to currency transactions is further divided into \( m_i^a \) and \( m_i^n \), which represent currency transferred to type-\( a \) and type-\( n \) buyers at the end of the CM (equation (3.4)).

\[ m_i^a + m_i^n = m_i^c \left( \mu_{t+1} \right) \]  

The aggregate currency value for a type-\( i \) buyer in currency transactions in high and low states are \( \frac{1}{1+\pi_i^h} m_i^a \) and \( \frac{1}{1+\pi_i^l} m_i^a \). Such liquidity is equally distributed to buyers for their consumption in the DM. That is,

\[
\begin{align*}
\gamma a c_{t+1}^{a,h} &= \frac{1}{1+\pi_i^h} m_i^a \left( \eta_{t+1}^{a,h} \right) \\
\gamma (1-a) c_{t+1}^{n,h} &= \frac{1}{1+\pi_i^h} m_i^n \left( \eta_{t+1}^{n,h} \right) \\
\gamma a c_{t+1}^{a,l} &= \frac{1}{1+\pi_i^l} m_i^a \left( \eta_{t+1}^{a,l} \right) \\
\gamma (1-a) c_{t+1}^{n,l} &= \frac{1}{1+\pi_i^l} m_i^n \left( \eta_{t+1}^{n,l} \right)
\end{align*}
\]  

Because buyers’ types are private information, we require the following incentive compatibility constraint for currency transactions:

\[
\begin{align*}
\left( 1-\epsilon \right) V^n(c_{t+1}^{n,h}) + \epsilon V^n(c_{t+1}^{n,l}) & \geq \left( 1-\epsilon \right) V^n(c_{t+1}^{a,h}) + \epsilon V^n(c_{t+1}^{a,l}) \\
\left( 1-\epsilon \right) V^a(c_{t+1}^{a,h}) + \epsilon V^a(c_{t+1}^{a,l}) & \geq \left( 1-\epsilon \right) V^a(c_{t+1}^{n,h}) + \epsilon V^a(c_{t+1}^{n,l})
\end{align*}
\]  

Inequality (3.6) is the incentive compatibility constraint for type-\( n \) buyers, which states that the expected utility of type-\( n \) buyers from choosing the currency allocated to type \( n \) must be greater than or equal to their expected utility from choosing the currency allocated to type \( a \); otherwise, they will pretend to be type-\( a \) buyers. Inequality (3.6) is the
incentive compatibility constraint for type-\(a\) buyers and has a symmetric explanation.

Because all types of assets and secured liabilities are accepted as means of payment in the financial transactions, banks can issue various secured claims that are contingent on the aggregate shocks to better allocate the risks. Banks can issue different contracts targeting buyers of different types in their relative risk aversion \(\theta\). \(b_t^{a_j}\) and \(b_t^{n_j}\) represent the overall bank liability allocated to type-\(a\) and type-\(n\) buyers in state \(j\); therefore,

\[
\begin{align*}
(1 - \gamma)ax_{t+1}^{a,h} &= b_t^{a,h} \\
(1 - \gamma)(1 - \alpha)x_{t+1}^{n,h} &= b_t^{n,h} \\
(1 - \gamma)ax_{t+1}^{a,l} &= b_t^{a,l} \\
(1 - \gamma)(1 - \alpha)x_{t+1}^{n,l} &= b_t^{n,l}
\end{align*}
\]

Moreover, the following collateral constraints on banks deposit liabilities must be satisfied:

\[
\begin{align*}
\frac{1}{1 + \pi_t} m_t^x &\geq \frac{1}{1 + \pi_t} m_t^x + k_t \left(\delta_t^h\right) \\
\frac{1}{1 + \pi_t} m_t^x &\geq \frac{1}{1 + \pi_t} m_t^x \left(\delta_t^l\right)
\end{align*}
\]

The equations state that the value of collateral cannot be lower than the value of deposit liability, which is equal to \(\frac{1}{1 + \pi_t} m_t^x + k_t\) in the high state and \(\frac{1}{1 + \pi_t} m_t^x\) in the low state; otherwise, the bank will default. Similar to the currency allocations, bank claims are also subject to the incentive compatibility constraints for households with different levels of risk aversion:

\[
\begin{align*}
(1 - \epsilon)V^n(x_{t+1}^{n,h}) + \epsilon V^n(x_{t+1}^{a,l}) &\geq (1 - \epsilon)V^n(x_{t+1}^{a,h}) + \epsilon V^n(x_{t+1}^{n,l}) \\
(1 - \epsilon)V^a(x_{t+1}^{a,h}) + \epsilon V^a(x_{t+1}^{a,l}) &\geq (1 - \epsilon)V^a(x_{t+1}^{n,h}) + \epsilon V^a(x_{t+1}^{n,l})
\end{align*}
\]

In summary, banks maximize (3.1) subject to (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8),
Banks play two roles in the economy. First, banks allocate liquidity among buyers with different levels of risk aversion. Second, banks allocate the liquidity provided by currency between currency transactions and financial transactions. However, the limitation of banks is that they cannot allocate trees to currency transactions as a medium of exchange. The government’s privilege is to circumvent this limitation and facilitate liquidity and risk transfer from financial transactions to currency transactions by purchasing risky trees.

### 3.3.2 The Government

I first write the government’s budget constraint when it does not purchase any trees:

$$\tau_t = m_t - \frac{1}{1 + \pi_t} m_t - e^s.$$  

The budget constraint states that the revenue from currency issuance, $m_t - \frac{1}{1 + \pi_t} m_t$, minus the issuance cost, $e^s$, must be equal to the transfer to buyers, $\tau_t$. I assume that the government sets the currency value in the DM, which is also referred to as public liquidity, to be equal to $V$. That is,

$$\frac{1}{1 + \pi_t} m_t = V.$$  

We will confine our attention to stationary equilibria, and thus, all time subscripts are dropped hereafter.\(^1\)

The aggregate liquidity in state $j$ consists of the public liquidity provided by currency, $V$, plus the private liquidity provided by the trees, $k$ in high state and 0 in low state. In this paper, I consider the purchases of risky trees by the government as swaps between public liquidity and private liquidity. The policy reallocates the risks and pay off of trees to the currency but holds the aggregate liquidity unchanged.\(^2\) That is, the government

---

\(^1\)Currency can also be represented by fiat money; then, the quantity of cash is $m = q_z$, and $\frac{q_z}{T} \equiv 1 + \pi$ is the inflation rate.

\(^2\)Other policy rules include fixing the value of the transfer in period 0, as in Williamson 2016, and an
purchases \( k^g \) units of trees, where \( k^g \leq k \), and the trees circulated in the economy become \( k - k^g \), and the amount of private liquidity provided by trees becomes

\[
p = \begin{cases} 
  k - k^g & \text{in high state} \\
  0 & \text{in low state}
\end{cases}
\]

Then, the public liquidity provided by currency is equal to

\[
l^g = \begin{cases} 
  \frac{1}{1+\pi} m = V + k^g & \text{in high state} \\
  \frac{1}{1+\pi} m = V & \text{in low state}
\end{cases}
\]

The government’s budget constraint after purchasing \( k^g \) units of trees becomes

\[
\tau_j = m - \frac{1}{1 + \pi_j} m - q_j k^g + d_j k^g - e^g \quad \text{in state } j
\]

Note that because \(- \frac{1}{1+\pi} m + d_h k^g = - \frac{1}{1+\pi} m + d_l k^g = -V\) is a constant, the transfer is not affected by the aggregate state, and thus, \( \tau_h = \tau_l \).

We can also consider other policy rules. For example, the government may be allowed to change tax levies to reduce the risk of the currency return: it can increase the tax in the low state and decrease the tax in the high state to balance the changes in the dividends of risky assets. The question is how the tax capacity is determined. Moreover, if the government can levy higher taxes to issue more liquidity, why does it not do so in the high state? I discuss an environment in which tax capacity is endogenously determined by the household incentive compatibility constraint in the appendix.

endogenous tax capacity, as in Williamson and Sanches. Those policy rules show similar results and are discussed in the appendix.
3.4 Equilibrium

In this section, we solve the bank’s problem in Section 3.1 and analyze how banks allocate assets and bank liabilities among households of different types and in different transactions.

3.4.1 Individual Rationality and Collateral Constraints

First, free entry guarantees that the individual rationality constraint (3.2) binds, as the objective function is strictly increasing in both $c_{t+1}^{ij}$ and $x_{t+1}^{ij}$, and the left-hand side of (3.2) is strictly decreasing in $c_{t+1}^{ij}$ and $x_{t+1}^{ij}$. Second, (3.9) and (3.10) must bind when currency and assets are scarce such that $V_i(x_{t+1}^{ij}, h_{t+1}) > 1$ and $V_i(x_{t+1}^{ij}, l_{t+1}) > 1$, respectively. If (3.9) or (3.10) does not bind, then $D_t - q_t^x z_t - q_t^k k_t < 0$. $V_i(x_{t+1}^{ij}) > 1$ implies that an increase in $x_{t+1}^{ij}$ for $\Delta x$ units and an increase in $D_t$ of $\beta \Delta x$ units can improve a buyer’s utility while continuing to ensure that the individual rationality constraint holds. When the incentive compatibility constraint does not bind, this means that banks purchase and hold assets in excess of the collateral required. This occurs only if assets are abundant and banks are indifferent between issuing extra deposits and not doing so. Without loss of generality, I assume that banks always issue the largest amount of deposits, meaning that (3.9) and (3.10) bind.

3.4.2 Incentive Compatibility Constraints

I first discuss the incentive compatibility constraints for currency transactions. Equation (3.6) implies that $\frac{m_t^{i+1}}{(1-\alpha)} \geq \frac{m_t^{i+1}}{\alpha}$, and equation (3.6) implies that $\frac{m_t^n}{\alpha} \leq \frac{m_t^n}{(1-\alpha)}$. Therefore, we can rewrite the incentive compatibility constraint in currency transactions as

$$\frac{m_t^{i+1}}{\alpha} = \frac{m_t^n}{(1-\alpha)}$$
where $\frac{m^a_{t+1}}{\alpha}$ is the amount of currency available for a type-$a$ buyer, and $\frac{m^n_{t+1}}{(1-\alpha)}$ is the currency available for a type-$n$ buyer. The equation states that banks must allocate the same amount of currency to different types of buyers; otherwise, the types offered a smaller amount currency will pretend to be of another type to obtain a larger amount of currency. Second, I discuss the incentive compatibility problem for the financial transactions. Because $\theta_a > \theta_n$, suppose that the bank allocates a given amount of liquidity to all buyers; the marginal utility of a type-$a$ buyer is then higher than that of a type-$n$ buyer when liquidity is not scarce (i.e., the liquidity allocated to a buyer is smaller than one, and the marginal utility is higher than one). Suppose that there were no incentive compatibility constraint; the bank would have allocated more liquidity to type-$a$ buyers as an efficient allocation. However, in this case, type-$n$ buyers would pretend to be type-$a$ buyers to receive higher utility. As a result, the incentive compatibility constraint for type-$n$ buyers (3.11) binds, but the incentive compatibility constraint for type-$a$ buyers (3.12) is slack.

We can simplify (3.11) and (3.12) as

$$\left[ (1 - \epsilon) V^n(x^n_{t+1}) + \epsilon V^n(x^n_{t+1}) \right] = \left[ (1 - \epsilon) V^n(x^n_{t+1}) + \epsilon V^n(x^n_{t+1}) \right]$$

The necessary conditions for an optimal, incentive-compatible financial transaction allocation are as follows:

$$\frac{\partial}{\partial x} V^a(x^a_{t+1}) - \frac{\rho^n}{(1-\gamma)\alpha} \frac{\partial}{\partial x} V^n(x^a_{t+1}) = \frac{\partial}{\partial x} V^n(x^a_{t+1}) + \frac{\rho^n}{(1-\gamma)(1-\alpha)} \frac{\partial}{\partial x} V^n(x^a_{t+1})$$

The explanation for the equations is as follows. Without considering the incentive compatibility constraint, that is, if $\rho^n = 0$, the equations merely state that an optimal allocation requires the marginal utility for type-$a$ and type-$n$ buyers to be the same in both the high and low states. However, when we consider the incentive compatibility constraint, an increase in liquidity for type-$a$ buyers tightens the constraint while an increase in liquidity
for type-\(n\) buyers relaxes it. As a result, the banks have to reduce the amount of liquidity allocated to type-\(a\) buyers and increase the amount of liquidity allocated to type-\(n\) buyers to prevent buyers from disguising their types. Rearranging (3.15), we obtain

\[
\frac{\alpha}{\alpha} \frac{\partial}{\partial x} V^a(x_{t+1}^{a,h}) - \frac{\partial}{\partial x} V^a(x_{t+1}^{a,l}) = \frac{\partial}{\partial x} V^a(x_{t+1}^{n,l}) - \frac{\partial}{\partial x} V^a(x_{t+1}^{n,l})
\]

Given \((m_{t+1}^x, k_{t+1})\), we can solve for the liquidity allocation in financial transactions \((x_{t+1}^{a,h}, x_{t+1}^{a,l}, x_{t+1}^{n,h}, x_{t+1}^{n,l})\) from (3.8), (3.9), (3.10), (3.14) and (3.16). Figure 3.1 depicts the risk allocation in financial transactions and currency transactions given a pair of liquidity values in the high and low states. In financial transactions, banks allocate more risk (higher variance) in liquidity to type-\(n\) buyers and allocate less risk (lower variance) to type-\(a\) buyers. However, bank deposits cannot be accepted as media of exchange in currency transactions, and thus, banks cannot reallocate the risks in currency transactions. Compared to the constrained efficient transactions in the financial transactions, in the currency transactions, type-\(a\) buyers consume too much in the high state and consume too little in the low state. Which means that type-\(a\) buyers bear too much risk while type-\(n\) buyers bear too little risk in the currency transactions.
Finally, the first-order conditions with respect to currency allocation, $s^c$ and $s^x$, require that the marginal benefit of currency be the same in both types of transactions. That is,

$$
\frac{1}{1 + \pi^h_{t+1}} [a \eta^{a,h}_{t+1} + (1 - \alpha) \eta^{n,h}_{t+1}] + \frac{1}{1 + \pi^l_{t+1}} [a \eta^{a,l}_{t+1} + (1 - \alpha) \eta^{n,l}_{t+1}] = \frac{1}{1 + \pi^h_{t+1}} \delta^h_{t+1} + \frac{1}{1 + \pi^l_{t+1}} \delta^l_{t+1}
$$

### 3.5 Policy Experiment

In this section, I explore the policy effects of risky asset purchases by the government on social welfare. Suppose that we place equal weight on all agents in the economy. This implies that all activity in the centralized market cancels out in our welfare measure, as for each unit of goods consumed by sellers, there is one unit of labor supply by buyers. Thus, our welfare measure is

$$
E_0 \sum_{t=0}^{\infty} \beta^t [u_i(x_t) - x_t],
$$

which is total surplus in a representative meeting in the DM. Given the funding pool of public liquidity issuance, $V$, the government’s goal is to determine the amount of risky asset purchases, $k^g$, to maximize (3.17). In the following subsection, I first discuss an economy with homogeneous households and then an economy with heterogeneous households.

### 3.5.1 Homogeneous Households

I first discuss the optimal policy when buyers have identical risk aversion, $\theta_a = \theta_n$. Figure 3.2 demonstrates the optimal asset purchases by the government, where $\omega$ represents the ratio of asset purchases, $\omega = \frac{k^g}{k}$. First, if public liquidity is plentiful such that the currency itself is sufficient to provide the liquidity needed in the DM ($V \geq y^*$), there will not be a risk sharing problem, and it becomes irrelevent whether the government purchases risky assets.

In the following discussion, I focus on the case of scarce public liquidity, which means
that $V < y^*$, such that the currency itself is not sufficient to provide the liquidity needed in all DM transactions. First, when the private assets are also scarce ($k < y^* - V$, Figure 3.2 left), purchasing all of risky assets generates the highest social welfare. This is because the purchases of private assets can facilitate risk and liquidity reallocation between currency and financial transactions. When purchasing all of the risky assets, the currency becomes the only asset available in the DM, and its return includes the public liquidity, $V$, plus the return on trees, $d/k$. As banks allocate the liquidity provided by currency between currency transactions and financial transactions, the risks in currency and financial transactions become the same, and risks in private assets are perfectly shared between these two transactions.

When trees are plentiful ($k > y^* - V$, Figure 3.2 right), the liquidity in the high state exceeds the maximum amount of the medium of exchange demanded for DM trades. In this case, the maximum social welfare is achieved as long as the government purchases sufficient risky assets and injects sufficient liquidity to facilitate currency transactions in the high state. The excessive liquidity, in either currency or financial transactions, will not be used as a medium of exchange in the DM transactions but retained by the buyers for use in the subsequent CM, which generates unit utility. As a result, it does not matter whether the government places the excessive liquidity in the financial transactions or currency transactions.\footnote{We need to verify that the welfare in the currency economy is higher than in autarky, in which the buyers can only use trees to trade but do not need to pay for the cost of currency issuance, $e^\delta$. If $e^\delta$ is too high, issuing no currency can be the optimal policy.}

### 3.5.2 Heterogeneous Risk Aversion

In this section, we discuss the case in which buyers are heterogeneous in their risk aversion, $\theta_a > \theta_n$. Figure 3.3 depicts the optimal asset purchases by the government in the heterogeneous buyers case. First, the optimal ratio of risky asset purchases is smaller than one when the risky assets are scarce. This is because sellers in the currency transac-
tion do not accept a bank’s claims as a medium of exchange, and therefore, banks cannot efficiently allocate the risks in currency transactions between type-\(a\) and type-\(n\) buyers, as discussed in Section 3.4.2. As a result, purchasing all of the risky assets and completely sharing risk between financial transactions and currency transactions is not optimal, and the government should place less risk in the currency transactions and leave greater risk in the financial sector.

When trees are plentiful, the liquidity in the high state exceeds the maximum amount of the medium of exchange demanded by both type-\(a\) and type-\(n\) buyers in the DM transactions. The excessive liquidity generates unit utility for both type-\(a\) and type-\(n\) buyers, and the difference in the curvature of their utility functions vanishes in the high state. Similar to the case of homogeneous buyers in the previous section, it does not matter whether the excessive liquidity is placed in the currency transactions or the financial transactions, and the social welfare is maximized as long as the government has purchased a sufficient amount of private assets (Figure 3.3 right).
3.5.3 Discussion

In the above discussion, I assume that the government is restricted to issue only one form of currency. However, we can generalize the model to allow the government to issue multiple currencies with different payment structures. In Kareken and Wallace (1981), the rate of return on the two currencies must be identical for both of them to circulate. However, in their model, households are identical, and there is no aggregate uncertainty. When aggregate shocks are introduced, households not only concern the expected return, but also the risk of the payment. Consequently, households with different risk aversions would prefer to hold different currencies with different risk payment structure (i.e., different currencies have different payoffs in the high and low states). In this section, I assume that the government can issue a second form of currency with an additional fixed cost. The government’s budget constraint becomes

\[
\tau_j = q^z,1 + q^z,2 - r^1_j - r^2_j - q_kk^g + d_jk^g - e^g,1 - e^g,2
\]  

(3.18)
where \( q_z^i \) and \( r_j^i \) are the prices of currencies; \( e_{\delta,1} \) and \( e_{\delta,2} \) are the corresponding costs of issuing the first and second currencies.

If the government issues a second currency with a different payment structure, the government can more efficiently allocate the risks in currency transactions, as the banks do in the financial transactions, and thus, welfare could be higher. Figure 3.4 shows the difference in optimal social welfare between issuing one currency and the two-currency policy under different quantities of risky assets, before considering the issuance costs. The figure shows that the improvement in social welfare generated by issuing the second currency increases as the quantity of risky assets increases until liquidity is plentiful in the high state. If the difference in welfare is smaller than the cost of issuing the second currency, \( e_{\delta,2} \), the government should issue only one currency and purchase only a part of the risky assets when overall liquidity is limited, as we discussed in the previous section.

If the difference is greater than the cost of issuing the second currency, \( e_{\delta,2} \), the government should issue the second currency to facilitate risk sharing. Because there are only two types of buyers and two states of aggregate shocks in the economy, the government can perfectly share the aggregate shocks between these buyers by issuing two currencies. In this case, the government should purchase all risky assets and completely share the risks between financial transactions and currency transactions. If buyers have more than two types of risk aversion and the aggregate shock has more shock states, an efficient allocation requires the government to issue more different currencies to allocate the risks among different buyers. In equilibrium, the government may not issue too many currencies because of the fixed cost, and thus, the risk cannot be perfectly shared.

There are two important frictions in the economy. The first is market segmentation: a fraction of sellers are well informed, and they can recognize all types of securities; another fraction of sellers has restricted information and can only recognize currency that is issued by the government. The second friction is the cost of issuing currency. The assumption here is as follows: it is costless to issue liabilities that can only be recognized by the
informed households (people in the financial sector), and banks can easily issue such liabilities to facilitate transactions for this group of people. Consequently, risks in financial transactions can be better allocated. However, it is costly to issue liabilities that can be recognized and accepted by every household. The risks in this form of liability cannot take advantage of the banking sector, and the government has to pay a higher cost to issue more currency and share the risks (multi-metal system). In the model, the government can improve social welfare by overcoming market segmentation and allocate the risk in the financial assets into currencies. However, its ability to do so is restricted by the cost of issuing an additional currency. In most economies, there is only one type of currency that can be verified and accepted by everyone. A possible reason for this that there is a high cost of operating the currency system.

In summary, risky asset purchases represent government risk management of aggregate liquidity. Purchases of risky assets are swaps between liquidity that is cheap but restricted with well-diversified risk and expensive liquidity that cannot be well-diversified but is broadly accepted. The government either pays a higher fixed cost to issue two currencies and purchases all of the risky assets to completely share the risk, or it issues one currency and has only limited ability to diversify the risk and purchases only some of the risky asset and incompletely shares the risk.

3.6 Conclusion

In this paper, I construct a model of risky asset purchases by the government. There are two key frictions in the economy. The first is the market segmentation between the financial transaction sector and the currency transaction sector. The second friction is the cost of issuing a well-recognized currency. Banks in the economy issue costless but less-recognizable liquidity, and the government issues costly but highly recognizable liquidity. By purchasing risky trees, the government is able to improve welfare by facil-
Figure 3.4: Improvement in social welfare by issuing a second currency

Emitting liquidity transfer and risk sharing between financial transactions and currency transactions. However, because issuing the highly recognizable currency is costly, the government cannot issue too many currencies and properly allocate the risks in currency transactions. As a result, when aggregate liquidity is limited, the optimal policy requires the government to purchase only a part of the risky assets and ensure that the currency remains less risky.

There are several important issues that are not discussed in this paper. The first concerns how the government is authorized to have a monopoly in issuing currency, the asset that is wildly accepted as a medium of exchange. The second concerns how the liquidity pool is determined and whether the transfer be changed according to the aggregate states. In this paper, we assume that the government’s budget is stable, namely, that the funding available for public liquidity is constant over time. However, the available funding can be affected by the aggregate state of the economy, and the government’s tax capacity may change depending on policy. The comovement between public and private liquidity can influence policy decisions.
Bibliography


Chen, H., Curdia, V., and Ferrero, A. (2012). The macroeconomic effects of large-scale asset purchase programs. FEDERAL RESERVE BANK OF SAN FRANCISCO WORKING PAPER SERIES.


Appendix A

Appendix for Chapter 1

A.1 Households’ Problem in The Goods Market

The households’ problem in the goods market can be rewritten as follows,

\[ U_t(m_t, b_t) = \max_{c_t, h_t, m_t, m_{t+1}} x_t c_t - l(h) + \frac{1}{1+\rho} \left\{ \alpha \Phi^e_{t+1}(m_{t+1}, b_{t+1}) + (1 - \alpha) \Phi^p_{t+1}(m_{t+1}, b_{t+1}) \right\} \]

subject to

\[
\begin{align*}
\hat{m}_t &= m_t - c_t \quad & (\gamma_{m_t}) \\
\hat{b}_{i,t} &= b_{i,t} \quad & (\gamma_{b_{i,t}}) \\
m_{t+1} &= \frac{1}{1+\rho_t} (\tau + h_t + \hat{m}_t + \sum_i \lambda_i \hat{b}_{i,t}) \quad & (\psi_{m_t}) \\
b_{i,t+1} &= \frac{1}{1+\rho_t} (1 - \lambda_i) \hat{b}_{i,t} \quad & (\psi_{b_{i,t}}) \\
\hat{m}_t &\geq 0 \quad & (\delta_t) \\
c_t &\geq 0 \quad & (\delta_t)
\end{align*}
\]

Where \(\gamma_{m_t}, \gamma_{b_{i,t}}, \psi_{m_t}, \psi_{b_{i,t}}, \delta_t, \delta_t\) are the corresponding Lagrangian multipliers. The first-
order conditions are as follows:

\[ c_t : \gamma_{mt} = x^j + \delta_t \]

\[ h_t : l'(h) = \frac{1}{1 + \pi_t} \psi_{mt} \]

\[ \hat{m}_t : \gamma_{mt} = \frac{1}{(1 + \pi_t)} \psi_{mt} + \delta_t \]

\[ b_{i,t} : \gamma_{b_{i,t}} = \frac{\lambda_i}{(1 + \pi_t)} \psi_{mt} + \frac{(1 - \lambda_i)}{(1 + \pi_t)} \psi_{b_{i,t}} \]

\[ m_{t+1} : \psi_{mt} = \frac{1}{(1 + \rho)} \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] \]

\[ b_{i,t+1} : \psi_{b_{i}} = \frac{1}{(1 + \rho)} \left[ \alpha \frac{\partial \Phi^c}{\partial b_{i}}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial b_{i}}(m_{t+1}, b_{t+1}) \right] \]

The first order conditions imply

\[ \gamma_{mt} = x^j + \delta_t \]

\[ l'(h) = \frac{1}{(1 + \rho)(1 + \pi_t)} \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] \]

\[ \gamma_{mt} = \frac{1}{(1 + \rho)(1 + \pi_t)} \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] + \delta_t \]

\[ \gamma_{b_{i,t}} = \left\{ \begin{array}{l} \lambda_i \left[ \alpha \frac{\partial \Phi^c}{\partial m}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial m}(m_{t+1}, b_{t+1}) \right] \\ + (1 - \lambda_i) \left[ \alpha \frac{\partial \Phi^c}{\partial b_{i}}(m_{t+1}, b_{t+1}) + (1 - \alpha) \frac{\partial \Phi^p}{\partial b_{i}}(m_{t+1}, b_{t+1}) \right] \end{array} \right\} \]

The Kuhn-Tucker conditions are

\[ \delta_t c_t = 0, \delta_t > 0 \]

\[ \bar{\delta}_t m'_t = 0, \bar{\delta}_t \geq 0 \]
The envelope theorem implies

\[ \frac{\partial U_i(m_t, b_t)}{\partial m} = \gamma_m, \]
\[ \frac{\partial U_i(m_t, b_t)}{\partial b_i} = \gamma_{b_i}. \]

**Lemma 8.**

1. \( \frac{\partial U_i(m_t, b_t)}{\partial m} = \max \left\{ x_t, \frac{1}{1+\eta_t} \frac{\partial \Phi}{\partial m} (m_{t+1}, b_{t+1}) \right\} \)
2. \( \frac{\partial U_i(m_t, b_t)}{\partial b_i} = \frac{1}{1+\eta_t} \left[ \lambda_i \frac{\partial \Phi_i}{\partial m} (m_{t+1}, b_{t+1}) + (1 - \lambda_i) \frac{\partial \Phi_i}{\partial b_i} (m_{t+1}, b_{t+1}) \right] \)

To prove the above Lemma, I first prove the following claims.

**Claim 1.** \( \delta_t \cdot \tilde{\delta}_t = 0 \)

**Proof.** Suppose that \( \delta_t \cdot \tilde{\delta}_t > 0 \); then, \( \delta_t > 0 \) and \( \tilde{\delta}_t > 0 \), so \( c_t = 0 \) and \( \hat{m}_t = 0 \). Then \( \hat{m}_t = m_t - c_t \) is violated. \( \square \)

\( \delta_t \cdot \tilde{\delta}_t = 0 \) and \( \delta_t \geq 0, \tilde{\delta}_t \geq 0 \) implies that \( \gamma_{m_t} = \max \left\{ x^j, \frac{1}{1+\eta_t} \psi^m \right\} \), so \( \frac{\partial U_i(m_t, b_t)}{\partial m} = \max \left\{ x_t, \frac{1}{1+\eta_t} \frac{\partial \Phi}{\partial m} (m_{t+1}, b_{t+1}) \right\}. \)

**A.2 Households’ Problem in The Asset Market**

\[ \Omega^i(m_t, b_t) = \max_{\hat{m}_t, \hat{b}_t, m_{i,t}^+, b_{i,t}^+} U^i(\hat{m}_t, \hat{b}_t) \]

Subject to

\[
\begin{align*}
\hat{m}_t &= m_t - \sum_i m_{i,t}^+ + \sum_i q_i b_{i,t}^+ \quad (\gamma_{m_t}) \\
\hat{b}_{i,t} &= b_{i,t} + \frac{m_{i,t}^+}{q_i} - b_{i,t}^+ \quad (\gamma_{b_{i,t}}) \\
\sum_i m_{i,t}^+ &\leq m_t \quad (\epsilon_{m_t}) \\
b_{i,t}^+ &\leq b_{i,t} \quad (\epsilon_{b_{i,t}}) \\
m_{i,t}^+ &\geq 0 \quad (\sigma_{m_{i,t}}) \\
b_{i,t}^+ &\geq 0 \quad (\sigma_{b_{i,t}})
\end{align*}
\]
The first order conditions are

\[
m^+_t : -\gamma_{m_t} + \frac{\gamma_{b_{i,t}}}{q_i} - \epsilon_{m,t} + \sigma_{m_{i,t}} = 0 \tag{B1}
\]
\[
b^+_t : q_i \gamma_{m_t} - \gamma_{b_{i,t}} - \epsilon_{b_{i,t}} + \sigma_{b_{i,t}} = 0 \tag{B2}
\]
\[
m_t : \frac{\partial U}{\partial m_t} = \gamma_{m_t} \tag{B3}
\]
\[
b_{i,t} : \frac{\partial U}{\partial b_{i,t}} = \gamma_{b_{i,t}} \tag{B4}
\]

Because I have denoted \(\nu^j_t = \frac{\partial U^j}{\partial m_t}\) and \(w^j_{i,t} = \frac{\partial U^j}{\partial b_{i,t}}\), I can rewrite (B3) and (B4) as

\[
\nu^j_t = \gamma_{m_t}
\]
\[
w^j_{i,t} = \gamma_{b_{i,t}}
\]

The Kuhn-Tucker conditions are

\[
\epsilon_{m_t} \left(m_t - \sum_i m^+_{i,t}\right) = 0, \epsilon_{m_t} \geq 0; \quad \sigma_{m_{i,t}} m^+_{i,t} = 0, \sigma_{m_{i,t}} \geq 0
\]
\[
\sum_i \epsilon_{b_{i,t}} \left(b_{i,t} - b^+_{i,t}\right) = 0, \epsilon_{b_{i,t}} \geq 0; \quad \sigma_{b_{i,t}} b^+_{i,t} = 0, \sigma_{b_{i,t}} \geq 0
\]

The Envelope Theorem is as follows

\[
\frac{\partial \Omega^j}{\partial m_t}(m_t, b_t) = \gamma_{m_t} + \epsilon_{m_t}
\]
\[
\frac{\partial \Omega^j}{\partial b_{i,t}}(m_t, b_t) = \gamma_{b_{i,t}} + \epsilon_{b_{i,t}}
\]

Lemma 9. 1. \(\frac{\partial \Omega^j}{\partial m_t}(m_t, b_t) = \max \left\{ \nu^j_t, \frac{w^j_{i,t}}{q_i}, \ldots, \frac{w^j_{k,t}}{q_k} \right\} \)

2. \(\frac{\partial \Omega^j}{\partial b_{i,t}}(m_t, b_t) = \max \left\{ q_i \nu^j_t, w^j_{i,t} \right\} \)

To prove Lemma 9, I first prove the following claims.

Claim 2. There exists \(x \in \{\epsilon_{m,t}, \sigma_{m_{1,t}}, \sigma_{m_{2,t}}, \ldots, \sigma_{m_{k,t}}\}\) such that \(x = 0\).
Proof. Suppose that $\epsilon_{m_t} > 0$ and $\sigma_{m_{i,t}} > 0$ for all $i$. Then $m_t - \sum_i m_{i,t}^+ = 0$ and $m_{i,t}^+ = 0$ for all $i$. Since $m_t > 0$, contradiction. \[\square\]

Substitute $v_i^j = \gamma_{m_t}$, and $w_{i,t}^j = \gamma_{b_{i,t}}$ into (B1); we have

$$\sigma_{m_{i,t}} + \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right) = \epsilon_{m_t}$$  \hspace{2cm} (B5)

Claim 3. $\epsilon_{m_t} = \max \left\{ \max_i \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right), 0 \right\}$.

Proof. 1. First, suppose that $\max_i \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right) < 0$ for all $i$, then, from (B5), $\sigma_{m_{i,t}} > \epsilon_{m_t}$ for all $i$. Since $\epsilon_{m_t} \geq 0$ and at least one of $\{\epsilon_{m_t}, \sigma_{m_{i,t}}\}$ is equal to zero, and thus, $\epsilon_{m_t} = 0$. In this case $\sigma_{m_{i,t}} > 0$ for all $i$, so $m_{i,t}^+ = 0$ for all $i$ and $m_t - \sum_i m_{i,t}^+ = m_t$.

2. Suppose that there exists $\left( \frac{w_{i,t}^j}{q_i} - v_i^j \right) \geq 0$ for some $i$. Since $\sigma_{m_{i,t}} \geq 0$, from (B5), $\epsilon_{m_t} \geq \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right)$ for all $i$. Suppose that $\epsilon_{m_t} > \max_i \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right)$, then $\epsilon_{m_t} > 0$ from the presumption. Moreover, by (B5), $\sigma_{m_{i,t}} > 0$ for all $i$. Claim 2 is violated. Thus, $\epsilon_{m_t} = \max_i \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right)$.

3. Combine 1 and 2, $\epsilon_{m_t} = \max \left\{ \max_i \left( \frac{w_{i,t}^j}{q_i} - v_i^j \right), 0 \right\} = \max \left\{ \frac{w_{i,t}^j}{q_i} - v_i^j, \frac{w_{i,t}^j}{q_i} - v_i^j, \ldots, \frac{w_{i,t}^j}{q_i} - v_i^j, 0 \right\}$.

Claim 3 implies:

$$\frac{\partial \Omega^j}{\partial m_i} (m_t, b_t) = v_i^j + \epsilon_{m_t} = \max \left\{ v_i^j, \frac{w_{i,t}^j}{q_1}, \ldots, \frac{w_{i,t}^j}{q_k} \right\}$$

The symmetric argument shows that

$$\frac{\partial \Omega^j}{\partial b_i} (m_t, b_t) = \max \left\{ q_i v_i^j, w_{i,t}^j \right\}$$
A.3 Comparative Statics with respect to Bond Supply

In the following policy experiment, I assume that $\lambda_s = 1$, $\lambda_l = 0.025$, and $\alpha = 0.8$. The disutility of working is assumed to be isoelastic: $I(h) = \frac{h^{1+\nu}}{1+\nu}, \nu = 0.5$.

A.3.1 The Representative Household Model – Fixed Inflation Rate (Section 1.3.5)

Figure A.1: Changes in short-term bond supply: $\pi = 0.05, \rho = 0.05, \mu = 0.8, b_l = 0.34$

Figure A.2: Changes in long-term bond supply:: $\pi = 0.05, \rho = 0.05, \mu = 0.8, b_s = 0.04$
A.3.2 The Representative Household Model – Non-Ricardian Regime

(Section 1.3.6)

Figure A.3: Changes in short-term bond supply: \( \rho = 0.05, \mu = 0.8, b_l = 0.34 \)

Figure A.4: Changes in long-term bond supply: \( \rho = 0.05, \mu = 0.8, b_s = 0.04 \)

Figure A.5: The impact of long-term bond reduction at the zero lower bound.
A.3.3 The Representative Household Model – Ricardian Regime (Section 1.3.6)

Figure A.6: Changes in short-term bond supply: $\rho = 0.05, \mu = 0.8, b_l = 0.34$

Figure A.7: Changes in long-term bond supply: $\rho = 0.05, \mu = 0.8, b_s = 0.04$
A.3.4  The Heterogeneous Household Model – Two Types (Section 1.4.1)

Figure A.8: Changes in long-term bond supply (non-Ricardian regime): $\mu_1 = 0.75, \mu_2 = 0.95, \rho = 0.05, b_s = 0.03$

Figure A.9: Changes in long-term bond supply (Ricardian regime): $\mu_1 = 0.75, \mu_2 = 0.95, \rho = 0.05, b_s = 0.03$
A.3.5 The Heterogeneous Household Model – Multiple Types (Section 1.4.2)

Figure A.10: Changes in long-term bond supply (Non-Ricardian Regime): $\rho \sim U(0.02,0.08), \mu \sim Beta(13.5,1.5), b_s = 0.03$

Figure A.11: Changes in long-term bond supply (Ricardian Regime): $\rho \sim U(0.02,0.08), \mu \sim Beta(13.5,1.5), b_s = 0.03$
Figure A.12: The impact of long-term bond reduction on the yield curve at the zero lower bound.

A.4 The Distribution of Households – Multiple Types

Figure A.13: Household Distribution: $\mu \sim Beta(1.5, 13.5)$, $\rho \in U(0.02, 0.08)$
A.5 Individual Characteristics and The Shape of Yield Curves

The left and middle panels show the subjective yield curves for $V = \gamma V + (1 - \gamma) \bar{V}$, where $\gamma = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

![Subjective yield curves: $\pi = 0.05, \mu = 0.75, \alpha = 0.5$](image1)

![Subjective yield curves: $\pi = 0.05, \rho = 0.05, \alpha = 0.5$](image2)
A.6 Proofs

A.6.1 Proof of Lemma 1 and Lemma 2

In the first part of the proof, I solve the relationship between the exchange value of cash, $V$, and the goods market values of cash, $v^c$ and $v^n$. $v^c$ and $v^n$ can be solved as functions of $V$ from the following equations:

\[
\hat{v}^c = \mu \max\{v^c, V\} + (1 - \mu)v^c \tag{V1}
\]
\[
\hat{v}^n = \mu \max\{v^n, V\} + (1 - \mu)v^n \tag{V2}
\]
\[
v^c = 1 \tag{V5}
\]
\[
v^n = \frac{1}{1 + d} [\alpha \hat{v}^c + (1 - \alpha)\hat{v}^n] \tag{V6}
\]

Since $v^c > v^n$, I discuss the solutions in the following three cases:

1. For $V < v^n < v^c$ (Region 0): From (V1), (V2)

\[\hat{v}^c = v^c,\]
\[\hat{v}^n = v^n,\]
Thus, we can obtain from (V6) that

$$\hat{\theta}^n = \nu^n = \frac{\alpha}{\alpha + d}.$$  

$$\nu^n, \nu^c, \hat{\nu}^n, \hat{\nu}^c$$ are independent of $$V$$ in this region, and hence

$$\frac{\partial \hat{\theta}^n}{\partial V} = \frac{\partial \hat{\theta}^c}{\partial V} = \frac{\partial \nu^n}{\partial V} = \frac{\partial \nu^c}{\partial V} = 0.$$

Thus,

$$\frac{\partial (V - \nu^n)}{\partial V} > 0, \frac{\partial (V - \nu^c)}{\partial V} > 0.$$  

Since $$\nu^n = \frac{\alpha}{\alpha + d}$$, this case holds if and only if $$V < \frac{\alpha}{\alpha + d}$$.

2. For $$\nu^n < V < \nu^c$$ (Region 1): From (V1), (V2)

$$\hat{\nu}^c = \nu^c = 1$$

$$\hat{\nu}^n = \mu V + (1 - \mu) \nu^n$$

Substitute into (V6),

$$\hat{\nu}^n = \frac{\mu (1 + d) V + (1 - \mu) \alpha}{(1 + d) - (1 - \mu)(1 - \alpha)}$$  \hspace{1cm} (B6)

$$\nu^n = \frac{(1 - \alpha) \mu V + \alpha}{(1 + d) - (1 - \mu)(1 - \alpha)}$$  \hspace{1cm} (B7)

The partial derivatives with respect to $$V$$:

$$\frac{\partial \hat{\nu}^n}{\partial V} > 0, \frac{\partial \hat{\nu}^c}{\partial V} > 0, \frac{\partial \nu^n}{\partial V} = \frac{\partial \nu^c}{\partial V} = 0$$

We require that $$V < \nu^c = 1$$ and $$\nu^n < V$$. Form (B7) $$\nu^n < V$$ if and only if $$V > \frac{\alpha}{\alpha + d}$$.  

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Therefore, $v^n < V < v^c$ if and only if $V \in \left[\frac{\alpha}{\alpha + d}, 1\right]$. Moreover,

$$\frac{\partial v^n}{\partial V} = \frac{(1 - \alpha)\mu}{\alpha + d + (1 - \alpha)\mu} < 1, \frac{\partial v^c}{\partial V} = 0$$

Thus,

$$\frac{\partial (V - v^n)}{\partial V} > 0, \frac{\partial (V - v^c)}{\partial V} > 0$$

3. For $V > v^c > v^n$ (Region 2): From (V1), (V2)

$$\hat{v}^c = \mu V + (1 - \mu)$$
$$\hat{v}^n = \mu V + (1 - \mu)v^n$$

Substitute into (V6),

$$\hat{v}^n = \frac{\mu [(1 + d) + (1 - \mu)\alpha] V + \alpha(1 - \mu)^2}{(1 + d) - (1 - \mu)(1 - \alpha)}$$
$$\hat{v}^n = \frac{\mu V + (1 - \mu)\alpha}{(1 + d) - (1 - \mu)(1 - \alpha)}$$

The partial derivatives with respect to $V$:

$$\frac{\partial \hat{v}^n}{\partial V} > 0, \frac{\partial \hat{v}^c}{\partial V} > 0, \frac{\partial \hat{v}^c}{\partial V} > 0, \frac{\partial v^c}{\partial V} = 0$$

Moreover,

$$\frac{\partial v^n}{\partial V} = \frac{\mu}{\alpha(1 - \mu) + d + \mu} < 1, \frac{\partial v^c}{\partial V} = 0$$
$$\Rightarrow \frac{\partial (V - v^n)}{\partial V} > 0, \frac{\partial (V - v^c)}{\partial V} > 0$$

This region holds if $V > 1$. Moreover, one must verify that $v^n = \frac{1}{1 + d} [\alpha \hat{v}^c + (1 - \alpha)\hat{v}^n] < 1$, or the transversality conditions will be not violated. Therefore, we also require that $V < \frac{d + \mu}{\mu}$. 121
In the second part of the proof, I solve the relationship between $V$ and bond prices $q_i$ from (V1) to (V8)

1. In region 1, we can solve $q_i$ as a function of $V$ as follows

$$q_i = \frac{\lambda_i (\alpha + (1 - \alpha) \hat{v}^n(V))}{[(1 + d) - (1 - \lambda_i)(1 - \alpha \mu)] V - (1 - \lambda_i) \alpha \mu}$$

Where

$$\hat{v}^n(V) = \frac{\mu (1 + d) V + (1 - \mu) \alpha}{[(1 + d) - (1 - \mu)(1 - \alpha)]}.$$ 

One can verify that

$$\frac{dq_i}{dV} < 0.$$ 

2. In region 2, we can solve $q_i$ as a function of $V$ as follows

$$q_i = \frac{\lambda (1 + d)}{(1 + d) - (1 - \mu)(1 - \alpha)} \left[ V (1 - \hat{v}^n) + (1 - \mu) \alpha \right]$$

and one can verify that

$$\frac{\partial q_i}{\partial V} < 0.$$ 

### A.6.2 Proof of Lemma 3

1. Region I:

$$\hat{v}^c = v^c; \quad \hat{v}^n = \mu (V - v^n) + v^n; \quad \frac{\hat{v}^c}{\hat{v}^n} = \mu (v^c - V) + V; \quad \frac{\hat{v}^n}{\hat{v}^n} = V.$$ 

$$V = \frac{1}{1 + d} \left\{ \frac{1}{\hat{v}^c} \left\{ \frac{\lambda}{\hat{v}^c} \left[ \alpha v^c + (1 - \alpha) [\mu (V - v^n) + v^n] \right] \right\} \right\}$$
\[
R = \frac{\lambda - \lambda}{q} = \frac{(1 + d)V - (1 - \lambda)\{\alpha \mu (v^c - V) + V\}}{(1 + d)V + \alpha \mu (V - v^c) - V + \lambda \alpha (1 - \mu) (V - v^c) + \lambda (1 - \alpha)(1 - \mu)(V - v^n)} - \lambda
\]
\[
= \frac{(1 + d)V + \alpha \mu (V - v^c) - V + \lambda \alpha (1 - \mu) (V - v^c) + \lambda (1 - \alpha)(1 - \mu)(V - v^n)}{V - \alpha (V - v^c) - (1 - \alpha)(1 - \mu)(V - v^n)}
\]
\[
\frac{\partial R}{\partial \lambda} = (1 - \mu) \frac{(1 - \alpha)[V - v^n] + \alpha [V - v^c]}{V - \{(1 - \mu)(1 - \alpha)[V - v^n] + \alpha [V - v^c]\}}
\]

2. Region II: \(v^c \geq V \geq v^n\)

\[
\dot{v}^c = \mu (V - v^c) + v^c; \quad \dot{v}^n = \mu (V - v^n) + v^n;
\]
\[
\frac{\dot{v}^c}{q} = V; \quad \frac{\dot{v}^n}{q} = V.
\]

\[
V = \frac{1}{1 + d} \left\{ \frac{\lambda}{q} \left[ \alpha [\mu (V - v^c) + v^c] + (1 - \alpha) [\mu (V - v^n) + v^n] \right] \right\} + (1 - \lambda) V
\]

\[
R = \frac{\lambda - \lambda}{q} = \frac{(1 + d)V - (1 - \lambda)V}{\alpha [\mu (v^c - V) + v^c] + (1 - \alpha) [\mu (V - v^n) + v^n]} - \lambda
\]
\[
= \frac{(1 + d)V - V - \lambda \alpha (1 - \mu) (V - v^c) - \lambda (1 - \alpha)(1 - \mu)(V - v^n)}{V - \alpha (1 - \mu)(V - v^c) - (1 - \alpha)(1 - \mu)(V - v^n)}
\]
\[
\frac{\partial R}{\partial \lambda} = (1 - \mu) \frac{(1 - \alpha)[V - v^n] + \alpha [V - v^c]}{V - \{(1 - \mu)(1 - \alpha)[V - v^n] + \alpha [V - v^c]\}}
\]

A.6.3 Proof of Proposition 1

In the proof of Lemma 1 and Lemma 2, I solve \(q\) as a function of \(\lambda\) and \(V\), that is, \(q = q(\lambda; V)\). In this part, I apply \(q(\lambda; V)\) and discuss the relationship between \(R\) and \(\lambda\).

1. In Region 1,

\[
q(\lambda; V) = \frac{\lambda [\alpha + (1 - \alpha)\dot{v}^n(V)]}{[(1 + d) - (1 - \lambda)(1 - \alpha \mu)]V - (1 - \lambda)\alpha \mu}
\]
Where
\[
\hat{v}_n(V) = \frac{\mu(1+d)V + (1-\mu)\alpha}{[(1+d) - (1-\mu)(1-\alpha)]}.\]

Thus,
\[
R(\lambda; V) = \frac{[(1+d) - (1-\lambda)(1-\alpha\mu)] V - (1-\lambda)\alpha\mu}{[\alpha + (1-\alpha)\hat{v}_n(V)]} - \lambda
\]

(a) For \(\mu = 1: \hat{v}_n(V) = V\)

\[
q(\lambda; V) = \frac{\lambda [\alpha + (1-\alpha)V]}{[(1+d) - (1-\lambda)(1-\alpha)]V - (1-\lambda)\alpha}
\]

\[
R(\lambda; V) = -\frac{\alpha + (\alpha + d)V}{\alpha + (1-\alpha)V}
\]

\[
\frac{\partial R}{\partial \lambda}(\lambda; V) = 0
\]

(b) For \(\mu < 1: I \text{ first show that } \frac{\partial R}{\partial \lambda} < 0 \text{ at the lower bound of region 1 } (V = V = \frac{a}{\alpha+d}), \text{ and } \frac{\partial R}{\partial \lambda} > 0 \text{ at the upper bound of region 1 } (V = 1). \text{ Second, I show that } \frac{\partial^2 R}{\partial \lambda \partial V} > 0 \text{ for } V \in [V, 1]. \text{ If } V = V = \frac{a}{\alpha+d}, \text{ we have } \hat{v}_n(V) = \frac{a}{\alpha+d}. \text{ Thus,}
\[
\begin{cases}
R(\lambda; V) = \frac{[(1+d) - (1-\lambda)(1-\alpha\mu)] \frac{a}{\alpha+d} - (1-\lambda)\alpha\mu}{[\alpha + (1-\alpha)\frac{a}{\alpha+d}]} - \lambda \\
\frac{\partial R}{\partial \lambda}(\lambda; V) = -\frac{(1-\alpha\mu) \frac{a}{\alpha+d} + \alpha\mu}{[\alpha + (1-\alpha)\frac{a}{\alpha+d}]} - 1 < 0
\end{cases}
\]

If \(V = 1\), we have \(\hat{v}_n(1) = \frac{\mu(1+d)/(1-\mu)\alpha}{[(1+d) - (1-\mu)(1-\alpha)]}. \text{ Thus,}
\[
\begin{cases}
R(\lambda; 1) = \frac{[(1+d) - (1-\lambda)(1-\alpha\mu)] - (1-\lambda)\alpha\mu}{[\alpha + (1-\alpha)\mu(1+d)/(1-\mu)(1-\alpha)]} - \lambda \\
\frac{\partial R}{\partial \lambda}(\lambda; 1) = -\frac{1}{[\alpha + (1-\alpha)\mu(1+d)/(1-\mu)(1-\alpha)]} - 1 > 0
\end{cases}
\]
For $V \in [V, 1]$

$$
R(\lambda; V) = \frac{[(1 + d) - (1 - \lambda_i)(1 - \alpha \mu)] V - (1 - \lambda_i) \alpha \mu - \lambda}{(1 - \mu)(1 - \alpha)(1 - \mu)(1 - \alpha)} \frac{(1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} V + \frac{(1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} \frac{(1 + (1 + d \mu)) (1 + (1 + (1 + d)(1 - \mu)(1 - \alpha))}{1 + (1 + (1 + (1 + d)(1 - \mu)(1 - \alpha))}

\frac{\partial R}{\partial \lambda} = \frac{(1 - \lambda \alpha \mu) V + \alpha \mu}{(1 - \mu)(1 - \alpha)(1 - \mu)(1 - \alpha)} \frac{(1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} V + \frac{(1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} \frac{(1 + (1 + d \mu)) (1 + (1 + (1 + d)(1 - \mu)(1 - \alpha))}{1 + (1 + (1 + (1 + d)(1 - \mu)(1 - \alpha))}

\frac{\partial^2 R}{\partial \lambda^2 V} > 0

Since $\frac{\partial R}{\partial \lambda}$ is a continuous and monotonic function in $V$ for $V \in \left[\frac{\alpha}{\alpha + d}, 1\right]$, there exists a unique $\tilde{V} \in \left[\frac{\alpha}{\alpha + d}, 1\right]$ such that $\frac{\partial R}{\partial \lambda}(\tilde{V}) = 0$, and $\frac{\partial R}{\partial \lambda}(V) < 0$ for $V < \tilde{V}$, and $\frac{\partial R}{\partial \lambda}(V) > 0$ for $V > \tilde{V}$.

2. In region 2, $V \in \left[1, \frac{\mu + d}{\mu}\right]$

$$
q = \frac{\lambda (1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} \frac{[\mu V + (1 - \mu) \alpha]}{(\lambda + d)V}

R = \frac{(1 + d)}{(1 + d - (1 - \mu)(1 - \alpha))} [\mu V + (1 - \mu) \alpha] - \lambda
$$

(a) For $\mu = 1$:

$$
q(\lambda; V) = \frac{\lambda}{(\lambda + d)}
R(\lambda; V) = d
$$

Thus,

$$
\frac{\partial R}{\partial \lambda}(\lambda; V) = 0
$$

(b) For $\mu < 1$:

$$
\frac{\partial R}{\partial \lambda} = \frac{V}{(1 + d - (1 - \mu)(1 - \alpha)) [\mu V + (1 - \mu) \alpha]} - 1 > 0
$$
Thus,
\[ \frac{\partial^2 R}{\partial \lambda \partial V} > 0 \]

**A.6.4 Proof of Proposition 2**

1. At the zero lower bound, the value of cash is not affected by the supply of bonds in the fixed inflation rate scenario. Therefore, a change in bond supply has no effect on \( h \). According to (1.21), a decrease in short-term bond supply increase outstanding cash one-to-one. However, because \( \mu < 1 \) and \( q_l < 1 \), one unit increase in the long-term bond supply increase the quantity of cash for less than one unit.

2. Lemma 4.2 can be directly inferred from (1.21) and (1.22) and \( q_s = 1 \).

3. Substitute (1.21) into (1.22) and let \( q_s = 1 \)

\[
\tau = -\frac{\pi}{\dot{\alpha}} h - \left[ 1 - \frac{(1 + \mu \pi)}{1 + \pi} (1 - \lambda_i) \right] q_i \bar{b}_l^s + \lambda_i \bar{b}_l^s
\]

At the zero lower bound,

\[
q_l = \frac{\lambda (1 + \pi)}{\lambda (1 + \pi) + (1 - \lambda)(1 - \mu) \left[ (1 + \pi) - \frac{1}{(1+\rho)} \right]}
\]

Thus,

\[
\tau = -\frac{\pi}{\dot{\alpha}} h + \left[ 1 - \frac{\lambda (1 + d) + (1 - \lambda) (1 - \mu) (d - \rho)}{\lambda (1 + d) + (1 - \lambda) (1 - \mu) d} \right] \lambda \bar{b}_l^s
\]

\[
\frac{\partial \tau}{\partial \bar{b}_l^s} = 1 - \frac{\lambda (1 + d) + (1 - \lambda) (1 - \mu) (d - \rho)}{\lambda (1 + d) + (1 - \lambda) (1 - \mu) d} > 0
\]
Appendix B

Appendix for Chapter 2

B.1 Proofs

B.1.1 The Value Function of Entrepreneurs

1. Suppose that $z > 0$; then,

$$W(z) = \int_{0}^{\infty} e^{-\rho \tau} [y + f + ze^{\rho \tau}] \delta e^{-\delta \tau} d\tau$$

$$= \left[ \frac{-\delta}{\rho + \delta} e^{-(\rho+\delta)\tau} (y + f) - \frac{\delta}{\rho + \delta - r} e^{-(\rho+\delta-r)\tau} z \right]_{0}^{\infty}$$

$$= \frac{\delta}{\delta + \rho} (y + f) + \frac{\delta}{\rho + \delta - r} z$$

2. Suppose that $z < 0$; there exists $\bar{\tau}(z)$ that solves

$$ze^{\rho \bar{\tau}} = -f$$

Then,
\[ W(z) = \int_{0}^{\tilde{\tau}(z)} e^{-\rho \tau} [y + f + ze^{r \tau}] \delta e^{-\delta \tau} d\tau \]

\[ = \frac{-\delta}{\rho + \delta} e^{-(\rho + \delta) \tau} (y + f) - \frac{\delta}{\rho + \delta - r} e^{-(\rho + \delta - \rho) \tau} z \] 0

\[ = \frac{-\delta}{\rho + \delta} (e^{r \tau}) \frac{-(\rho + \delta)}{r} (y + f) - \frac{\delta}{\rho + \delta - r} (e^{r \tau}) \frac{-(\rho + \delta - \rho)}{r} z \]

\[ = \frac{-\delta}{\rho + \delta} (y + f) - \frac{\delta}{\rho + \delta - r} \left[ -\delta \left( \frac{y + f}{\rho + \delta} - \frac{\delta}{\rho + \delta - r} \right) \right] \]

\[ = \left[ \frac{\delta}{\rho + \delta} (y + f) + \frac{\delta}{\rho + \delta - r} \right] - \left[ \frac{\delta}{\rho + \delta} (y + f) - \frac{\delta}{\rho + \delta - r} \left( \frac{y + f}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f \right] \left( \frac{z}{f} \right) \]

\[ = \delta + \rho \left( y + f \right) + \frac{\delta}{\rho + \delta - r} \left( \frac{y + f}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f \]

\[ = \left[ \frac{\delta}{\rho + \delta} (y + f) + \frac{\delta}{\rho + \delta - r} \right] - \left[ \frac{\delta}{\rho + \delta} (y + f) - \frac{\delta}{\rho + \delta - r} \left( \frac{y + f}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f \right] \left( \frac{z}{f} \right) \]

\[ B.1.2 \text{ Proof of Lemma 2} \]

We first prove the following claim.

**Claim 4.** Given \((f, \sigma)\) such that \(f < \delta \frac{\delta}{\tau} y\), \((x^*, p^*)\) has the following properties:

1. \(x^* - p^* = -\frac{(\rho + \delta - r) f}{\rho - r} \frac{\delta}{\rho + \delta - r} \)

2. \(W(x^* - p^*) = \frac{\delta}{\rho + \delta} (y + f) + \frac{\rho + \delta}{\rho + \delta} (x^* - p^*)\)

3. \(W(x^* - p^*) - (x^* - p^*) > f\)

4. \(p^* > f\)

5. \(x^* < f\)

6. \(\chi'(f) < 1\)

7. \(\frac{\partial \chi(f)}{\partial f} < 0\)

8. \(\frac{\partial p^*}{\partial \sigma} > 0\)
\[ \frac{\partial p^*}{\partial f} < 1 \]

**Proof.**

1. The first-order condition (2.9) and (2.4) imply

\[
\frac{\delta}{\rho + \delta - r} + \frac{\rho + \delta}{r} \left[ \frac{\delta y - \left( \frac{\delta}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f}{\rho + \delta} \right] \left( \frac{p^* - x^*}{f} \right)^{\frac{p + \delta - r}{r}} = 1 \quad (B.1)
\]

Solve for \( x^* - p^* \), and we obtain Claim 1.1.

2. Rearrange equation (B.1), and we have

\[
\left[ \frac{\delta}{\rho + \delta} y - \left( \frac{\delta}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f \right] \left( \frac{p^* - x^*}{f} \right)^{\frac{p + \delta}{r}} = \frac{r}{\rho + \delta} \frac{\rho - r}{\rho + \delta} (p^* - x^*) \quad (B.2)
\]

Substitute \( W(z) = \left[ \frac{\delta}{\rho + \delta} (y + f) + \frac{\delta}{\rho + \delta - r} z \right] - \left[ \frac{\delta}{\rho + \delta} y - \left( \frac{\delta}{\rho + \delta - r} - \frac{\delta}{\rho + \delta} \right) f \right] \left( \frac{z}{f} \right)^{\frac{p + \delta}{r}} \) into (B.2), and we obtain Claim 1.2.

3. From Lemma 1, we have that \( W(z) \) is differentiable and strictly concave for \( z < 0 \).

From \( f < \frac{\delta}{\rho + \delta} y \), we have \( W'(-f) > 1 \). Moreover, \( W(-f) = f \), \( W'(x^* - p^*) = 1 \) and \( x^* - p^* > -f \) imply

\[
W(x^* - p^*) - (x^* - p^*) > f. \quad (B.3)
\]

4. Combining (2.5) and (2.8), we have a modified proportional bargaining constraint

\[
W(x^* - p^*, f, r) - x^* = \omega(\mu)(p^* - f^*) \quad (B.4)
\]

where \( \omega(\mu) = \frac{\rho + \mu + \delta}{\rho + \delta} \frac{\theta}{1 - \theta} \)

Let

\[
\chi(f, r) \equiv W(f, r) - (x^* - p^*)(f, r), \quad (B.5)
\]
we can solve $p$ as a function of $(\mu, f)$

$$p^* = \frac{1}{1 + \omega(\mu)}(\chi(f, r) + \omega(\mu) f). \quad (B.6)$$

From (B.6) and (B.3), we observe that $\chi(f) > p^* > f$.

5. The borrowing constraint $x^* - p^* > -f$ and Claim 4 imply that $x^* < f$.

6. By (B.5), Claim 1.1 and Claim 1.2 we have

$$\chi(f) = \frac{\delta}{\rho + \delta} (y + f) + \frac{\rho - r}{\rho + \delta} \left[ \frac{\rho + \delta - r}{\rho - r} \right]^{\frac{\rho + \delta}{\rho + \delta - r}} f$$

Take derivative w.r.t. $f$

$$\chi'(f) = \frac{\delta}{\rho + \delta} + \frac{(\rho + \delta) \frac{\delta y}{\rho + \delta} - \delta}{\rho + \delta} \left[ \frac{(\rho + \delta - r) \frac{\delta y}{\rho - r} - \delta}{\rho - r} \right]^{\frac{\rho + \delta}{\rho + \delta - r} - 1} f$$

We need to show $\frac{\delta}{\rho + \delta} + \frac{(\rho + \delta) \frac{\delta y}{\rho + \delta} - \delta}{\rho + \delta} \left[ \frac{(\rho + \delta - r) \frac{\delta y}{\rho - r} - \delta}{\rho - r} \right]^{\frac{\rho + \delta}{\rho + \delta - r} - 1} < 1$. It is equivalent to show

$$\left[ \frac{(\rho + \delta - r) \frac{\delta y}{\rho - r} - \delta}{\rho - r} \right]^{\frac{\rho + \delta}{\rho + \delta - r}} > \left[ \frac{(\rho + \delta) \frac{\delta y}{\rho} - \delta}{\rho} \right]$$

Since $\frac{\rho + \delta}{\rho + \delta - r} > 1$, it is sufficient to show $\left[ \frac{(\rho + \delta - r) \frac{\delta y}{\rho - r} - \delta}{\rho - r} \right] > \left[ \frac{(\rho + \delta) \frac{\delta y}{\rho} - \delta}{\rho} \right]$, and it holds if and only if $\frac{\delta y}{\rho} > 1$.

7. Take partial derivative, $\frac{\partial}{\partial f} [\chi'(f) f - \chi(f)]$. By Claim 6, $\chi'(f) f - \chi(f) < f - \chi(f)$, and by Claim 3, $f - \chi(f) < 0$. We conclude that $\frac{\partial}{\partial f} [\chi'(f) f - \chi(f)] < 0$. 

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8. Take partial derivative w.r.t. $f$ on both sides of (B.6)

$$\frac{\partial p^*}{\partial f} = \frac{1}{(1 + \omega)} \left[ \chi'(f) + \omega \right]$$

Since $\chi'(f) < 1$, we have $\frac{\partial p^*}{\partial f} < 1$

9. Take partial derivative w.r.t. $\omega$ on both sides of (B.6):

$$\frac{\partial p^*}{\partial \omega} = \frac{1}{(1 + \omega)^2} \left[ -\chi(f) + f \right]$$

We have $\chi(f) > f$ by claim 3, and thus, $\frac{\partial p^*}{\partial \omega} < 0$. Since $\frac{\partial p}{\partial \sigma} = \frac{\partial p}{\partial \omega} \frac{\partial \omega}{\partial \sigma}$ and $\frac{\partial \omega}{\partial \mu} > 0, \frac{\partial \mu}{\partial \sigma} < 0$, claim 8 is proved.

\[\square\]

Claim 1.3 shows that $W(x^* - p^*) - (x^* - p^*) > f$, so there is always a positive surplus in the bargaining, and thus there is always a trade in the bargaining ($a = 1$). Claim 1.1 shows that $x^* - p^* < 0$, so the investor always borrow to finance the purchase of tree.

\subsection*{B.1.3 Proof of Lemma 3}

We first characterize equation (HC) by the following lemma.

\textbf{Lemma 10.} Let $f(\sigma)$ solves (HC).

1. $f(\sigma)$ is monotone increasing.

2. $f(0) = 0$.

3. There exists $\bar{f} \in (0, \frac{\delta}{r})$ such that $\lim_{\sigma \to \infty} f(\sigma) = \bar{f}$.

\textbf{Proof.} 1. Rearranging (HC), we have

$$[r + \eta(\sigma)] f - \eta(\sigma) p(f, \sigma) = 0$$

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Let

\[ H = [r + \eta(\sigma)] f - \eta(\sigma) p(f, \sigma) \]

By implicit differentiation:

\[
\frac{\partial H}{\partial f} = \eta(\sigma) \left[ 1 - \frac{\partial p}{\partial f} \right] \\
\frac{\partial H}{\partial \sigma} = \eta'(\sigma)(f - p) - \eta(\sigma) \frac{\partial p}{\partial \sigma}
\]

Since \( \eta(\sigma) > 0, \eta'(\sigma) > 0 \) and \((f - p) < 0, \frac{\partial p}{\partial \sigma} > 0, \frac{\partial p}{\partial f} < 1\), we have \( \frac{\partial H}{\partial f} > 0 \) and \( \frac{\partial H}{\partial \sigma} < 0 \), and therefore, \( \frac{\partial f}{\partial \sigma} > 0 \).

2. Since \( \sigma = 0 \) implies \( \eta(\sigma) = 0 \), substitute \( \eta(\sigma) = 0 \) into equation (HC), we have \( f = 0 \).

3. From equation (HC), we have

\[
\frac{p}{f} \to 1 \text{ as } \sigma \to \infty \tag{B.7}
\]

Since \( \lim_{\sigma \to \infty} \mu(\sigma) = 0 \), from the proportional bargaining constraint, we have

\[
\frac{p}{f} \to (1 - \theta) \frac{\chi(f)}{f} + \theta \text{ as } \sigma \to \infty \tag{B.8}
\]

Combine (B.7) and (B.8), as \( \sigma \to \infty \), we have

\[
1 = (1 - \theta) \frac{\chi(f)}{f} + \theta \tag{B.9}
\]

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Since \( \chi(0) = \frac{\delta}{\bar{p} + \sigma} y \) and \( \chi(\frac{\delta}{r} y) = \frac{\delta}{r} y \), we have

\[
\lim_{f \to 0} \frac{\chi(f)}{f} = \infty
\]
\[
\lim_{f \to \frac{\delta}{r} y} \frac{\chi(f)}{f} = 1
\]

Therefore, the range of the RHS of equation (B.9) is \((1 - \theta, \infty]\). By Claim 7, \( \frac{\chi(f)}{f} \) is monotone decreasing in \( f \), and thus, there exists a unique \( \bar{f} \in (0, \frac{\delta}{r} y) \) solves (B.9).

**Lemma 11.** Let \( \sigma(f) \) solve equation (ST).

1. \( \sigma(f) \) is monotone decreasing.

2. There exist \( \sigma > 0, \bar{\sigma} > 0 \) such that \( \sigma(0) = \bar{\sigma} \) and \( \sigma(\bar{f}) = \sigma \).

**Proof.**

1. (ST) is expressed as follows

\[
\Phi(f, r) = \frac{\mu(\sigma) + \delta}{\mu(\sigma)} A - \frac{\delta}{\sigma \mu(\sigma)} \tag{ST}
\]

Where

\[
\Phi(f, r) = \left[ 1 - \left( \frac{p^* - x^*}{f} \right)^\delta \right]
\]
\[
\Theta(\sigma) = \frac{\mu(\sigma) + \delta}{\mu(\sigma)} A - \frac{\delta}{\sigma \mu(\sigma)} \tag{B.10}
\]

Claim 4.1 implies that \( \frac{\partial \Phi}{\partial f}(f, r) < 0 \). Moreover \( \frac{\partial \Theta}{\partial \sigma}(\sigma) > 0 \). Thus, \( \frac{d\sigma}{df} < 0 \).

2. We have \( \lim_{f \to 0} \Phi(f, r) = 1 \), and \( \lim_{f \to \frac{\delta}{r} y} \Phi(f, r) = 0 \). Moreover, \( \lim_{\sigma \to 0} \Theta(\sigma) = -\infty \) and \( \lim_{\sigma \to \infty} \Theta(\sigma) = \infty \) and the continuity of \( \Theta(\sigma) \) implies that there exists \( \sigma \in (0, \infty) \) such that \( \Theta(\sigma) = x \) for all \( x \in (-\infty, \infty) \). Therefore, there exist \( \sigma > 0, \bar{\sigma} > 0 \) such that \( \sigma(0) = \bar{\sigma} \) and \( \sigma(\bar{f}) = \sigma \).
Now we can prove Lemma 3. By Lemma 10 and Lemma 11, there exists a unique intersection of equations (HC) and (ST), $(\sigma^*, f^*)$, and we have $\sigma^* \in (\varrho, \bar{\varrho})$ and $f^* \in (0, \frac{\delta}{r} y)$. □
B.2 Comparative statics: changes in dividend payment

Figure B.1: Parameters: $\delta = 0.2, \Lambda = 1, \rho = 0.08, r = 0.05$
B.3 Comparative statics: Scarce Asset

Figure B.2: Parameters: $A = 0.8, \Lambda = 1, \delta = 0.2, y = 5, \rho = 0.08, r = 0.05$
B.4 Comparative statics: Plentiful Asset

![Graphs showing asset price, liquidation price, haircut, leverage, number of idle assets, and market tightness.](image)

Figure B.3: Parameters: $A = 0.98$, $\delta = 0.2$, $y = 5$, $\rho = 0.08$, $r = 0.05$
B.5 Transition Dynamics – Market Liquidity Shock

Figure B.4: Market Liquidity Shock
B.6 Transition Dynamics – Interest Rate Shock

Figure B.5: Interest Rate Shock
B.7 Numerical Algorithm For Dynamic Transitions

Solving for a competitive equilibrium along a deterministic transition path is similar to solving a fixed-point problem. The numerical algorithm to compute the transition path is described as follows. Let \( T = \{t_0, t_1, t_2, \ldots \} \), \( \Delta t = t_i - t_{i-1} \). The value of the exogenous variable is \( k_0 \) before \( t_0 \) and is \( k_1 \) for \( t \geq t_0 \).

1. Compute the variables \( (\sigma, f, V) \) and asset distribution \( G(b) \) in the stationary equilibrium for \( k = k_0 \) and \( k = k_1 \).

2. Assume that the variables converge to the stationary equilibrium for \( k = k_1 \) when \( t > t^* \), \( t^* \) is large. Denote \( T^* = \{t_0, t_1, t_2, \ldots t^* \} \)

3. Propose a sequence of market tightness, liquidation asset prices, and the reservation value of workers \( (\sigma_t, f_t, V_t) \) for \( t \in T^* \).

4. Solve \( W_t(b) \): let \( \tau_t(b) \) be the time when the borrowing limit is reached. Then,

\[
W_t(b) = \delta \Delta t [y + f_t - b] + \frac{(1 - \delta \Delta t) \delta \Delta t}{1 + \rho \Delta t} \left[ y + f_{t+\Delta t} - (1 + r \Delta t)b \right] \\
+ \frac{(1 - \delta \Delta t)^2 \delta \Delta t}{(1 + \rho \Delta t)^2} \left[ y + f_{t+2\Delta t} - (1 + r \Delta t)^2b \right] + \ldots \\
+ \frac{(1 - \delta \Delta t)^{\tau_t(b)} \delta \Delta t}{(1 + \rho \Delta t)^{\tau_t(b)}} \left[ y + f_{t+\tau_t(b)\Delta t} - (1 + r \Delta t)^{\tau_t(b)}b \right]
\]

5. Given \( (f_t, \sigma_t, W_t, V_t) \), solve \( (p_t, z_t) \) from the bargaining problem.

6. Given \( \sigma_t \) and \( N_0^0 \), solve \( N_t^0 \) for all \( t \in T^* \).

7. Taking \( (p_t, z_t, f_t, \sigma_t, G_0(b)) \) as given, solve the distribution of entrepreneurs’ borrowing over time

\[
G_t(b) = \begin{cases} 
G_{t-\Delta t}(\frac{b}{1+r\Delta t})(1 - \delta \Delta t) + \mu_t N_t^0 \Delta t I_{\{b \geq p_t - z_t\}} & \text{for } b < \frac{f_{t+\Delta t}}{1+r\Delta t} \\
G_t(\frac{f_{t+\Delta t}}{1+r\Delta t}) & \text{for } b \geq \frac{f_{t+\Delta t}}{1+r\Delta t}
\end{cases}
\]
8. \( N_i^1 = G_i(f_i) \)

9. Update \( \sigma'_t = \frac{N_0^t}{A-N_i^t} \)

10. Update

\[
    f'_t = \eta'_t \Delta tp_t + \frac{(1 - \eta'_t \Delta t)}{(1 + r \Delta t)} \eta'_{t+\Delta t} \Delta t p_{t+\Delta t} + \frac{(1 - \eta'_t \Delta t)(1 - \eta'_{t+\Delta t} \Delta t)}{(1 + r \Delta t)^2} \eta'_{t+2\Delta t} \Delta t p_{t+2\Delta t} + \frac{(1 - \eta'_t \Delta t)(1 - \eta'_{t+\Delta t} \Delta t)(1 - \eta'_{t+2\Delta t} \Delta t)}{(1 + r \Delta t)^3} \eta'_{t+3\Delta t} \Delta t p_{t+3\Delta t} + ...
\]

11. Update

\[
    V'_t = \frac{(1 - \delta \Delta t)}{1 + \rho \Delta t} [W_{t+\Delta t}(z_{t+\Delta t} - p_{t+\Delta t}) - z_{t+\Delta t}] + \frac{(1 - \delta \Delta t)^2 (1 - \mu'_{t+\Delta t} \Delta t)}{(1 + \rho \Delta t)^2} [W_{t+2\Delta t}(z_{t+2\Delta t} - p_{t+2\Delta t}) - z_{t+2\Delta t}] + ...
\]

12. Repeat steps (4) to (9) until all variables converge.
Appendix C

Appendix for Chapter 3

C.1 Policy Regime: Endogenous Tax Capacity

In this section, I develop an endogenous tax capacity model, in which the upper bound of tax levies is determined by the buyers’ incentive compatibility constraint. The buyers’ budget constraint is

\[ a + H + \tau = D \]  
(C.1)

Where \( D \) is the deposit. If the IR and collateral constraints bind in the banks’ problem, there is no reserve and bank deposits are fully used to purchase assets, and thus,

\[ D = q^k k^d + m \]  
(C.2)

Combining (C.1) and (C.2), we obtain

\[ a + H + \tau = q^k k^d + m \]  
(C.3)

Where \( a \) is the asset retained from the last period. \( k^d \) represents the assets purchased by banks, and \( k^d = k - k^s \). In autarky, there is no government and no currency, meaning that
a household’s budget constraint is

\[ a + \tilde{H} = \tilde{q}^k k \]  

(C.4)

The incentive compatibility constraint for a household to be willing to pay tax is

\[
-H + \beta E[u(x)] + \frac{\beta}{1 - \beta} E\{-H + \beta E[u(x)]\} \\
\geq -\tilde{H} + \beta E[u(\tilde{x})] + \frac{\beta}{1 - \beta} E\{-\tilde{H} + \beta E[u(\tilde{x})]\}
\]

Where the left-hand side is its CM expected lifetime utility in an economy with government-issued currency, and the left-hand side is its CM lifetime utility in autarky. Substituting the budget constraint (C.3) and (C.4) into the above incentive compatibility constraint, we have

\[
a + \tau - m - q^k d^d - e^g + \beta E[u(x)] + \frac{\beta}{1 - \beta} E\{a + \tau - m - q^k d^d - e^g + \beta E[u(x)]\} \\
\geq a - \tilde{q}^k + \beta E[u(\tilde{x})] + \frac{\beta}{1 - \beta} E\{\tilde{a} - \tilde{q}^k + \beta E[u(\tilde{x})]\}
\]

Note that because of the quasi-linear utility, there is no wealth effect in the CM. Therefore, given the payment of assets, \((r_l, r_h, d_l, d_h)\), the prices \((q^z, q^k)\) are uncorrelated with the current state. We can rewrite the incentive compatibility constraint as

\[
\tau^* - m - q^k d^d - e^g + \beta E[u(x) + a] \geq -\tilde{q}^k + \beta E[u(\tilde{x}) + \tilde{a}] 
\]

(C.5)

**C.1.1 Optimal Policy**

In a stationary equilibrium, the government’s problem is to maximize the social welfare function subject to the incentive compatibility constraint (C.5). In a stationary equilib-
rium, the government’s problem is equivalent to

\[
\max E [u(x) - x]
\]

Subject to (C.5)

We can observe from the government’s budget constraint (3.13) that more tax levies, \( \tau_j \), implies a higher currency return, \( r_j \), which generates more liquidity and (weakly) increases social welfare. When liquidity is scarce, a binding incentive compatibility constraint (C.5) is an optimal strategy. That is, \( \tau_l = \tau_h = \tau^* \), where \( \tau^* \) solves (C.5). When liquidity is abundant, the level of liquidity does not affect social welfare on the margin, and thus, changing taxes at the margin does not affect welfare. In that case, the risk in asset return is irrelevant, and it does not improve the welfare if the government imposes different taxes in different states to decrease the risk in the currency.

Note that the tax capacity is determined by the expected improvement in future social welfare provided by the government. In the Lagos-Wright framework, the future social welfare is not affected by the current state, and therefore, the tax capacity is not related to the current state. Because the only use of taxes is to issue public liquidity, increasing tax levies weakly improves social welfare by increasing the aggregate liquidity. As a consequence, the government should impose taxes equal to its capacity to tax, and the tax levy is not affected by the current state; thus variations in dividend payments are fully covered by changes in currency returns.