Dilations, inclusions of matrix convex sets, and completely positive maps

Abstract

In this talk I will present a part of a recent joint work with Davidson, Dor-On, and Solel (a complementary talk will be given by Adam Dor-On in the Multivariable Operator Theory special session).

If \( A = (A_1, \ldots, A_d) \) is a tuple of operators on \( H \) and \( B = (B_1, \ldots, B_d) \) is a tuple of operators on \( K \), then \( B \) is said to be a \textit{dilation} of \( A \), denoted \( A \prec B \), if \( A_i = P_H B_i \big|_H \) for all \( i \). For a long time it seemed that the name of the game was: given a commuting tuple of operators \( A \), find a commuting tuple of \textit{normal} operators \( B \) such that \( A \prec B \) (usually with additional conditions on the joint spectrum \( \sigma(B) \), and requiring the dilation to hold for powers as well). Quite recently, Helton, Klep, McCullough and Schweighofer changed the rules, and started dilating tuples of \textit{noncommuting} operators to commuting tuples of normal operators. They showed that there is a universal constant \( \vartheta_n \), such that given a tuple of \( n \times n \) selfadjoint contractions \( A \), there exists a tuple of commuting selfadjoints \( B \), such that \( \sigma(B) \subseteq [-1,1]^d \) and \( \frac{1}{\vartheta_n} A \prec B \). This result had deep implications to spectrahedral inclusion problems.

The constant \( \vartheta_n \) behaves roughly like \( \sqrt{n} \), and was shown to be the best constant possible. We were led to ask whether it is possible to obtain such a dilation result with a constant that does not depend on \( n = \text{rank } A \) (necessarily fixing \( d \)). Moreover, we sought a normal dilation \( B \) with more precise control on the joint spectrum \( \sigma(B) \). As a representative of our results, I will present the following theorem, as well as some applications.

\textbf{Theorem.} Let \( K \) be a convex set in \( \mathbb{R}^d \) satisfying some reasonable conditions. Then for every \( d \)-tuple \( A \) of selfadjoint operators with a joint numerical range contained in \( K \), there is a \( d \)-tuple of commuting selfadjoint operators \( B \) with joint spectrum \( \sigma(B) \subseteq K \), such that

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\frac{1}{d} A \prec B.
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