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Pick Interpolation for free holomorphic functions

Jim Agler

John E. McCarthy
Washington University in St Louis, mccarthy@wustl.edu

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Jim Agler *
U.C. San Diego
La Jolla, CA 92093

John E. McCarthy †
Washington University
St. Louis, MO 63130

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1 Introduction

A holomorphic function in d (scalar) variables behaves locally like a polynomial. Given such a function ϕ, one can evaluate it also on d-tuples of commuting matrices whose joint spectrum lies in the domain of ϕ. A free holomorphic function is a generalization of this notion, where the matrices are no longer required to commute. We replace polynomials by free polynomials, *i.e.* polynomials in non-commuting variables, and consider functions that are locally limits of free polynomials.

To make this precise, let us first define $\mathbb{M}_n^d$ to be the set of all d-tuples of n-by-n complex matrices, and $\mathbb{M}^{[d]} = \cup_{n=1}^{\infty} \mathbb{M}_n^d$. Let $\mathbb{P}^d$ denote the algebra of free polynomials in d variables. A *graded function* defined on a subset of $\mathbb{M}^{[d]}$ is a function ϕ with the property that if $x \in \mathbb{M}_n^d$, then $\phi(x) \in \mathbb{M}_n$.

Definition 1.1. An *nc-function* is a graded function ϕ defined on a set $D \subseteq \mathbb{M}^{[d]}$ such that

i) If $x, y, x \oplus y \in D$, then $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$.

ii) If $s \in \mathbb{M}_n$ is invertible and $x, s^{-1}xs \in D \cap \mathbb{M}_n^d$ then $\phi(s^{-1}xs) = s^{-1}\phi(x)s$.

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Nc-functions have been studied for a variety of reasons: by Taylor [25], in the context of the functional calculus for non-commuting operators; Voiculescu [26, 27], in the context of free probability; Popescu [19, 20, 21, 22], in the context of extending classical function theory to d-tuples of bounded operators; Ball, Groenewald and Malakorn [7], in the context of extending realization formulas from functions of commuting operators to functions of non-commuting operators; Alpay and Kalyuzhnyi-Verbovetzkii [4] in the context of realization formulas for rational functions that are J-unitary on the boundary of the domain; Helton [10] in proving positive matrix-valued functions are sums of squares; and Helton, Klep and McCullough [11, 12] and Helton and McCullough [13] in the context of developing a descriptive theory of the domains on which LMI and semi-definite programming apply. Recently, Kaliuzhnyi-Verbovetskyi and Vinnikov have written a monograph [16] that gives a panoramic view of the developments in the field to date, and establishes their Taylor-Taylor formula for nc-functions.

There are two topologies that we wish to consider on $\mathbb{M}^{[d]}$. The first is called the *disjoint union topology*: a set U is open in the disjoint union topology if and only if $U \cap \mathbb{M}_n^d$ is open for every n. This topology is too fine for some purposes; for example, compact sets must have a bound on the size of the matrices that they contain. The other topology we wish to consider is the *free topology*, which is most conveniently defined by giving a basis. A basic free open set in $\mathbb{M}^{[d]}$ is a set of the form

$$G_\delta \;=\; \{x \in \mathbb{M}^{[d]} : \|\delta(x)\| < 1\},$$

where δ is a J-by-J matrix with entries in $\mathbb{P}^d$. We define the free topology to be the topology on $\mathbb{M}^{[d]}$ which has as a basis all the sets G_δ, as J ranges over the positive integers, and the entries of δ range over all polynomials in $\mathbb{P}^d$. (Notice that $G_{\delta_1} \cap G_{\delta_2} = G_{\delta_1 \oplus \delta_2}$, so these sets do form the basis of a topology). The free topology is a natural topology when considering semi-algebraic sets.

Definition 1.2. A free holomorphic function ϕ on a free open set U in $\mathbb{M}^{[d]}$ is a function that is locally a bounded nc-function, *i.e.* for every x in U there is a basic free open set $G_\delta \subseteq U$ that contains x and such that $\phi|_{G_\delta}$ is a bounded nc-function.

It is a principal result of [1] that free holomorphic functions are locally approximable by free polynomials (see Theorem 2.1 below).

The main result of this note is a criterion for solving a Pick interpolation problem on a basic open set, Theorem 1.3 below, and its generalization to extending bounded free holomorphic functions off free varieties, Theorem 1.5.

Let $H^\infty(U)$ denote the bounded free holomorphic functions on a free open set U with the supremum norm, and let $H_1^\infty(U)$ denote the closed unit ball of $H^\infty(U)$. For $1 \leq i \leq N$, let $\lambda_i \in G_\delta \cap \mathbb{M}_{n_i}^d$ and let $w_i \in \mathbb{M}_{n_i}$. The Pick problem is to determine whether or not there is a function in $H_1^\infty(U)$ that maps each λ_i to the corresponding w_i.

Note first that if U is closed under direct sums, then by letting $\Lambda = \oplus_{i=1}^N \lambda_i$ and $W = \oplus_{i=1}^N w_i$, the original N point problem is the same as solving the one point Pick problem of mapping Λ to W. Secondly, unlike in the scalar case, one cannot always solve the Pick problem if one drops the norm constraint. For example, no holomorphic function maps

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{to} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

To state the theorem, let us make the following definitions for Λ in $\mathbb{M}^{[d]}$. Define

$$\mathcal{I}_\Lambda \ = \ \{p \in \mathbb{P}^d \ : \ p(\Lambda) = 0\}$$

and

$$V_\Lambda \ = \ \{x \in \mathbb{M}^{[d]} \ : \ p(x) = 0 \text{ whenever } p \in \mathcal{I}_\Lambda\}.$$

Let

$$\mathbb{M}_\Lambda \ = \ \{p(\Lambda) \ : \ p \in \mathbb{P}^d\}.$$

Note that since $\mathbb{M}_\Lambda$ is a finite dimensional vector space, it is closed.

Theorem 1.3. Let $\Lambda \in G_\delta \cap \mathbb{M}_n^d$ and $W \in \mathbb{M}_n$. There exists a function ϕ in the closed unit ball of $H^\infty(G_\delta)$ such that $\phi(\Lambda) = W$ if and only if

(i) $W \in \mathbb{M}_\Lambda$, so there exists $p_0 \in \mathbb{P}^d$ such that $p_0(\Lambda) = W$.

(ii) $\sup\{\|p_0(x)\| : x \in V_\Lambda \cap G_\delta\} \ \leq \ 1$.

We prove this theorem in Section 3. Note that when $d = 1$, the question of whether p_0 can be found satisfying $p_0(\Lambda) = W$ can be resolved by looking at the Jordan canonical form of Λ. In this basis, the algebra $\mathbb{M}_\Lambda$ has a straightforward description. When $d > 1$, the determination of $\mathbb{M}_\Lambda$ is more delicate; generically[1], however, the algebra $\mathbb{M}_\Lambda$ will be all of $\mathbb{M}_n$.

[1]For example, if Λ^1 has n distinct eigenvalues and Λ^2 has no non-zero entry when the matrix is expressed in the basis given by the eigenvectors of Λ^1.

In Section 4, we give a description in Theorem 4.6 of all the solutions of a (solvable) Pick problem — this is called the Nevanlinna problem. Our approach is indebtted to the solution in the scalar case by J. Ball, T. Trent and V. Vinnikov [6].

Theorem 1.3 has a remarkable corollary. Suppose $\mathfrak{A}$ is an algebra in $\mathbb{P}^d$, and let $\mathfrak{V} = \mathrm{Var}(\mathfrak{A})$ be given by

$$\mathfrak{V} \;=\; \{x \in \mathbb{M}^d : p(x) = 0 \;\forall\; p \in \mathfrak{A}\}. \tag{1.4}$$

If Λ is in $\mathfrak{V}$, then $\mathfrak{A} \subseteq \mathcal{I}_\Lambda$, and $V_\Lambda \subseteq \mathfrak{V}$. Let U be a free open set in $\mathbb{M}^{[d]}$; we shall say that a function f defined on $\mathfrak{V} \cap U$ is free holomorphic if, for every point x in $\mathfrak{V} \cap U$ there is a basic free open set $G_\delta \subseteq U$ containing x and a free holomorphic function ψ defined on G_δ such that $\psi|_{\mathfrak{V}\cap G_\delta} = f|_{\mathfrak{V}\cap G_\delta}$.

In the scalar case, every holomorphic function defined on an analytic variety inside a domain of holomorphy extends to a holomorphic function on the whole domain, by a celebrated theorem of H. Cartan [8]. The geometric conditions that guarantee that all bounded holomorphic functions extend to be bounded on the whole domain have been investigated by Henkin and Polyakov [15] and Knese [17]; however, even when bounded extensions exist, the extension is almost never isometric [3]. But in the matrix case, any bounded free holomorphic function on $\mathfrak{V} \cap G_\delta$ does extend to a free holomorphic function on G_δ with the same norm.

Theorem 1.5. Let $\mathfrak{V}$ be as in (1.4), and let δ be a matrix of free polynomials such that $\mathfrak{V} \cap G_\delta$ is non-empty. Let f be a bounded free holomorphic function defined on $G_\delta \cap \mathfrak{V}$. Then there is a free holomorphic function ϕ on G_δ that extends f and such that

$$\|\phi\|_{H^\infty(G_\delta)} \;=\; \sup_{x \in \mathfrak{V}\cap G_\delta} \|f(x)\| \tag{1.6}$$

We prove this in Section 5. In Section 6 we give some applications.

The definition (1.4) naturally leads one to ask what the ideal of $\mathfrak{V}$, the set

$$I_{\mathfrak{V}} \;=\; \{p \in \mathbb{P}^d : p(x) = 0 \;\forall\; x \in \mathfrak{V}\},$$

is. In the complex case, the answer is simpler than in the scalar case, at least if $\mathfrak{A}$ is finitely generated. In [14], Bergman, Helton and McCullough proved that $I_{\mathfrak{V}}$ is the smallest ideal containing $\mathfrak{A}$, provided this ideal is finitely generated. The real (self-adjoint) case is more subtle — see *e.g.* [9].

2 Background material

We shall need some results from [1]. The first we have already referenced:

Theorem 2.1. Let D be a free domain and let ϕ be a graded function defined on D. Then ϕ is a free holomorphic function if and only if ϕ is locally approximable by polynomials.

The second, [1, Thm 8.1], says that a function is in $H^\infty_1(G_\delta)$ if and only if it has a free δ-realization.

Definition 2.2. Let ϕ be a graded function on G_δ, where δ is a J-by-J matrix of free polynomials. A free δ-realization of ϕ is a Hilbert space $\mathcal{L}$, an isometry $V : \mathbb{C} \oplus (\mathbb{C}^J \otimes \mathcal{L}) \to \mathbb{C} \oplus (\mathbb{C}^J \otimes \mathcal{L})$ that can be written

$$V \;=\; \begin{array}{c} \\ \mathbb{C} \\ \mathbb{C}^J \otimes \mathcal{L} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{C} & \mathbb{C}^J \otimes \mathcal{L} \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array},$$

and such that

$$\begin{aligned} \phi(x) \;=\; & \mathrm{id}_{\mathbb{C}^n} \otimes A \,+ \\ & (\mathrm{id}_{\mathbb{C}^n} \otimes B)(\delta(x) \otimes \mathrm{id}_{\mathcal{L}})[\mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathbb{C}^J \otimes \mathcal{L}} - (\mathrm{id}_{\mathbb{C}^n} \otimes D)(\delta(x) \otimes \mathrm{id}_{\mathcal{L}})]^{-1} \mathrm{id}_{\mathbb{C}^n} \otimes C \end{aligned}$$

for all $x \in G_\delta \cap \mathbb{M}^d_n$.

We call ϕ the transfer function of V (where δ is understood).

Theorem 2.3. Let ϕ be a graded function on G_δ. Then ϕ is in $H^\infty_1(G_\delta)$ if and only if ϕ has a free δ-realization.

The third is a Montel theorem.

Theorem 2.4. Let $(\phi_i)_{i=1}^\infty$ be a sequence in $H^\infty_1(U)$. Then there is a subsequence $(\phi_{i_j})_{j=1}^\infty$ and a function ϕ in $H^\infty_1(U)$ such that $(\phi_{i_j})_{j=1}^\infty$ converges to ϕ locally uniformly on U in the disjoint union topology.

3 Proof of Theorem 1.3

Let $E = V_\Lambda \cap G_\delta$, and let

$$E^{[2]} \;=\; \{(x, y) \,:\, x, y \in V_\Lambda \cap G_\delta \cap \mathbb{M}^d_m, \text{ for some } m\}.$$

Let us start with some lemmata.

Lemma 3.1. Let $\Lambda, x \in \mathbb{M}^{[d]}$. The following are equivalent:

(i) $x \in V_\Lambda$.

(ii) There is a homomorphism $\alpha : \mathbb{M}_\Lambda \to \mathbb{M}_x$ such that $\alpha(\Lambda^r) = x^r$ for $r = 1, \dots, d$.

(iii) The map $p(\Lambda) \mapsto p(x)$ is a well-defined map from $\mathbb{M}_\Lambda$ to $\mathbb{M}_x$.

(iv) The map $p(\Lambda) \mapsto p(x)$ is a completely bounded homomorphism.

PROOF: The equivalence of *(i) - (iii)* is by definition. That *(iii)* is equivalent to *(iv)* is because every bounded homomorphism defined on a finite dimensional space is automatically completely bounded [18]. □

Lemma 3.2. Let ϕ be in $H^\infty(G_\delta)$. Then there exists a polynomial $p_0 \in \mathbb{P}^d$ so that

$$\phi(x) \;=\; p_0(x) \qquad \forall x \in V_\Lambda \cap G_\delta. \tag{3.3}$$

PROOF: By Theorem 2.1, the free function ϕ can be uniformly approximated on a free neighborhood of Λ by free polynomials. In particular, since $\mathbb{M}_\Lambda$ is closed, there is a polynomial p_0 such that $\phi(\Lambda) = p_0(\Lambda)$.

Fix $x \in V_\Lambda \cap G_\delta$. By another application of the same theorem, there is a free polynomial p_1 such that $\phi(\Lambda \oplus x) = p_1(\Lambda \oplus x)$. Therefore $p_0(\Lambda) = p_1(\Lambda)$, so by the definition of V_Λ, we also have $p_0(x) = p_1(x)$. Therefore (3.3) holds, as desired. □

We let $\mathcal{V}$ denote the vector space of nc-polynomials on E, where we identify polynomials that agree on E; and we let $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ denote the vector space of $\mathcal{L}(\mathcal{H},\mathcal{M})$-valued nc-polynomials on E. As any such polynomial on E is uniquely determined by its values on Λ, the space of such functions is finite dimensional, if $\mathcal{H}$ and $\mathcal{M}$ are finite dimensional.

Consider the following vector spaces of functions on $E^{[2]}$, where all sums are over a finite set of indices:

$$\begin{aligned}
H(E) &= \{h(y,x) = \sum g_i(y)^* f_i(x) : f_i, g_i \in \mathbb{P}^d\} \\
R(E) &= \{h \in H(E) : h(x,y) = h(y,x)^*\} \\
C(E) &= \{h(y,x) = \sum u_i(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u_i(x) \; : \\
&\qquad u_i \text{ is } \mathcal{L}(\mathbb{C},\mathbb{C}^J) - \text{valued nc polynomial}\} \\
P(E) &= \{h(y,x) = \sum f_i(y)^* f_i(x) : f_i \in \mathbb{P}^d\}
\end{aligned}$$

We topologize $H(E)$ with the norm

$$\|h(y,x)\| \;=\; \|h(\Lambda,\Lambda)\|.$$

Lemma 3.4. Let $\mathcal{H}, \mathcal{M}$ be finite dimensional Hilbert spaces, and let $F(y,x)$ be an arbitrary graded $\mathcal{L}(\mathcal{M})$-valued function on $E^{[2]}$. Let $N_0 = \dim(\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})})$. Then if G can be represented in the form

$$G(y,x) = \sum_{i=1}^{m} g_i(y)^* F(y,x) g_i(x), \qquad (x,y) \in E^{[2]}$$

where $m \in \mathbb{N}$ and $g_i \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ for $i = 1, \ldots, m$, then G can be represented in the form

$$G(y,x) = \sum_{i=1}^{N_0} f_i(y)^* F(y,x) f_i(x), \qquad (x,y) \in E^{[2]} \tag{3.5}$$

where $f_i \in \mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$ for $i = 1, \ldots, N_0$.

PROOF: Let $\langle e_l(x)\rangle_{l=1}^{N_0}$ be a basis of $\mathcal{V}_{\mathcal{L}(\mathcal{H},\mathcal{M})}$. For each $i = 1, \ldots, m$, let

$$g_i(x) = \sum_{l=1}^{N_0} c_{il}\; e_l(x).$$

Form the $m \times N_0$ matrix $C = [c_{il}]$. As C^*C is an $N_0 \times N_0$ positive semidefinite matrix, there exists an $N_0 \times N_0$ matrix $A = [a_{kl}]$ such that $C^*C = A^*A$. This leads to the formula,

$$\sum_{i=1}^{m} \overline{c}_{il_1} c_{il_2} = \sum_{k=1}^{N_0} \overline{a}_{kl_1} a_{kl_2},$$

valid for all $l_1, l_2 = 1, \ldots, N_0$. If $(x,y) \in E^{[2]}$, then

$$\begin{aligned} G(y,x) &= \sum_{i=1}^{m} g_i(y)^* F(y,x) g_i(x) \\ &= \sum_{i=1}^{m} \big(\sum_{l=1}^{N_0} c_{il}\ e_l(y)\big)^* F(y,x) \big(\sum_{l=1}^{N_0} c_{il}\ e_l(x)\big) \\ &= \sum_{l_1,l_2=1}^{N_0} (\sum_{i=1}^{m} \overline{c}_{il_1} c_{il_2}) e_{l_1}(y)^* F(y,x) e_{l_2}(x) \\ &= \sum_{l_1,l_2=1}^{N_0} (\sum_{k=1}^{N_0} \overline{a}_{kl_1} a_{kl_2}) e_{l_1}(y)^* F(y,x) e_{l_2}(x) \\ &= \sum_{k=1}^{N_0} \big(\sum_{l=1}^{N_0} a_{kl}\ e_l(y)\big)^* F(y,x) \big(\sum_{l=1}^{N_0} a_{kl}\ e_l(x)\big). \end{aligned}$$

This proves that (3.5) holds with $f_i = \sum_{l=1}^{N_0} a_{il}\ e_l$. □

Lemma 3.6. $C(E)$ is closed.

PROOF: By Lemma 3.4, every element in $C(E)$ can be represented in the form

$$\sum_{i=1}^{N_0} u_i(y)^* [\mathrm{id} - \delta(y)^* \delta(x)] u_i(x), \tag{3.7}$$

where $N_0 = \dim \mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$. Suppose a sequence of elements of the form (3.7) approaches some h in $H(E)$ at the point (Λ,Λ):

$$\sum_{i=1}^{N_0} u_i^{(k)}(\Lambda)^* [\mathrm{id} - \delta(\Lambda)^* \delta(\Lambda)] u_i^{(k)}(\Lambda) \to h(\Lambda,\Lambda) \text{ as } k \to \infty.$$

Since $\Lambda \in G_\delta$, there is a constant M such that, for each i and k,

$$\|u_i^{(k)}(\Lambda)\| \le M.$$

Passing to a subsequence, one can assume that each $u_i^{(k)}(\Lambda)$ converges to some $u_i(\Lambda)$ (since $u_i^{(k)}$ is a graded $\mathcal{L}(\mathbb{C},\mathbb{C}^J)$ valued function and $J < \infty$). By Lemma 3.1, we have

$$u_i^{(k)}(x) \to u_i(x) \qquad \forall x \in E.$$

Therefore, for all $(x,y) \in E^{[2]}$, we have

$$\begin{aligned}\sum_{i=1}^{N_0} u_i^{(k)}(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u_i^{(k)}(x) &\to \sum_{i=1}^{N_0} u_i(y)^*[\mathrm{id} - \delta(y)^*\delta(x)]u_i(x) \\ &= h(y,x).\end{aligned}$$

□

Lemma 3.8. We have $P(E) \subseteq C(E)$.

PROOF: We have

$$f(y)^*f(x) - \sum_{k=0}^{m-1} f(y)^*\delta(y)^{k*}[\mathrm{id} - \delta(y)\delta(x)]\delta(x)^k f(x) \;=\; f(y)^*\delta(y)^m\delta(x)^m f(x). \tag{3.9}$$

As $m \to \infty$, the right-hand side of (3.9) goes to zero for every $(x,y) \in E^{[2]}$. Since $C(E)$ is closed by Lemma 3.6, this proves that $f(y)^*f(x) \in C(E)$, and hence so are finite sums of this form. □

Lemma 3.10. Suppose $\sup\{\|p_0(x)\| : x \in E\} \;\leq\; 1$. Then the function

$$h(y,x) \;=\; \mathrm{id} - p_0(y)^* p_0(x)$$

is in $C(E)$.

PROOF: This will follow from the Hahn-Banach theorem [23, Thm. 3.3.4] if we can can show that $L(h(y,x)) \geq 0$ whenever

$$L \in R(E)^* \qquad \text{and} \qquad L(g) \geq 0 \;\forall\, g \in C(E). \tag{3.11}$$

Assume (3.11) holds, and define $L^\sharp \in H(E)^*$ by the formula

$$L^\sharp(h(y,x)) = L\Big(\frac{h(y,x) + h(x,y)^*}{2}\Big) + iL\Big(\frac{h(y,x) - h(x,y)^*}{2i}\Big),$$

and then define sesquilinear forms on $\mathcal{V}$ and $\mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$ by the formulas

$$\begin{aligned}\langle f,g\rangle_{L_1} &= L^\sharp(g(y)^*f(x)), \qquad f,g \in \mathcal{V} \\ \langle F,G\rangle_{L_2} &= L^\sharp(G(y)^*F(x)), \qquad F,G \in \mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}.\end{aligned}$$

Observe that Lemma 3.8 implies that $f(y)^*f(x) \in C(E)$ whenever $f \in \mathcal{V}$ or $\mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$. Hence, (3.11) implies that $\langle f,f\rangle_{L_1} \;\geq\; 0$ for all $f \;\in\; \mathcal{V}$, and

$\langle F, F\rangle_{L_2} \geq 0$ for all $F \in \mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$, i.e., $\langle\cdot,\cdot\rangle_{L_1}$ and $\langle\cdot,\cdot\rangle_{L_2}$ are pre-inner products on $\mathcal{V}$ and $\mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$ respectively.

To make them into inner products, choose $\varepsilon > 0$ and define

$$\langle f, g\rangle_1 = L^\sharp(g(y)^* f(x)) + \varepsilon \operatorname{tr}(g(\Lambda)^* f(\Lambda)), \qquad f, g \in \mathcal{V} \tag{3.12}$$

$$\langle F, G\rangle_2 = L^\sharp(G(y)^* F(x)) + \varepsilon \operatorname{tr}(G(\Lambda)^* F(\Lambda)), \qquad F, G \in \mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)} \tag{3.13}$$

We let $\mathrm{H}^2_{L_1}$ and $\mathrm{H}^2_{L_2}$ denote the Hilbert spaces $\mathcal{V}$ and $\mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$ equipped with the inner products (3.12) and (3.13).

The fact that L is non-negative on $C(E)$ means that

$$\langle F, F\rangle_{L_2} \geq \langle \delta F, \delta F\rangle_{L_2} \tag{3.14}$$

for all F in $\mathcal{V}_{\mathcal{L}(\mathbb{C},\mathbb{C}^J)}$. Since $\|\delta(\Lambda)\| < 1$, we also have

$$\operatorname{tr}(F(\Lambda)^* F(\Lambda)) > \operatorname{tr}(F(\Lambda)^* \delta(\Lambda)^* \delta(\Lambda) F(\Lambda)), \tag{3.15}$$

if $F \neq 0$, and combining (3.14) and (3.15) we get that multiplication by δ is a strict contraction on $\mathrm{H}^2_{L_2}$.

Let M denote the d-tuple of multiplication by the coordinate functions x^r on $\mathrm{H}^2_{L_1}$. We have just shown that $\|\delta(M)\| < 1$, so M is in G_δ. As M is also in V_Λ, we have that M is in E. Therefore $\|p_0(M)\| \leq 1$, by hypothesis. Therefore

$$\mathrm{id} - p_0(M)^* p_0(M) \geq 0,$$

and so for all f in $\mathcal{V}$ we have

$$L^\sharp(f(y)^* f(x)) + \varepsilon \operatorname{tr}(f(\Lambda)^* f(\Lambda)) \geq \\ L^\sharp(f(y)^* p_0(y)^* p_0(x) f(x)) + \varepsilon \operatorname{tr}(f(\Lambda)^* p_0(\Lambda)^* p_0(\Lambda) f(\Lambda)).$$

Letting f be the function 1 and letting $\varepsilon \to 0$, we get

$$L(\mathrm{id} - p_0(y)^* p_0(x)) \geq 0,$$

as desired. □

We can now prove the theorem.

Proof of Theorem 1.3: (Necessity). Condition *(i)* follows from Lemma 3.2. Condition *(ii)* follows because $p_0(x) = f(x)$ for $x \in V_\Lambda \cap G_\delta$, and f is in the unit ball of $H^\infty(G_\delta)$, so $\|f(x)\| \leq 1$ for every x in G_δ.

(Sufficiency). Suppose *(i)* and *(ii)* hold. By Lemma 3.10, the function

$$h(y,x) \;=\; \mathrm{id} - p_0(y)^* p_0(x)$$

is in $C(E)$. By Lemma 3.4, there is some positive integer $N \leq \dim(\mathcal{V}_{\mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)})$ and an $\mathcal{L}(\mathbb{C},\mathbb{C}^{JN})$-valued nc polynomial u such that, for $x, y \in E \cap \mathbb{M}_n^d$,

$$\begin{aligned} h(y,x) &= \mathrm{id}_{\mathbb{C}^n} - p_0(y)^* p_0(x) \\ &= u(y)^* \left[\mathrm{id}_{\mathbb{C}^{nJN}} - (\delta(y)^* \otimes \mathrm{id}_{\mathbb{C}^N})\ (\delta(x) \otimes \mathrm{id}_{\mathbb{C}^N})\right] u(x). \end{aligned} \tag{3.16}$$

Replace x in (3.16) with sxs^{-1} where s is invertible in $\mathbb{M}_n$ and sxs^{-1} is in G_δ to get

$$\begin{aligned} & s - p_0(y)^* s p_0(x) \;= \\ & \quad u(y)^* \left[s \otimes \mathrm{id}_{\mathbb{C}^{JN}} - (\delta(y)^* \otimes \mathrm{id}_{\mathbb{C}^N})\ s \otimes \mathrm{id}_{\mathbb{C}^{JN}} (\delta(x) \otimes \mathrm{id}_{\mathbb{C}^N})\right] u(x). \end{aligned} \tag{3.17}$$

Equation (3.17) is true for all s in a neighborhood of the identity, and as linear combinations of such elements span $\mathbb{M}_n$, we get that (3.17) actually holds for all s in $\mathbb{M}_n$. For $k = 1, \ldots, n$, define $\pi_k : \mathbb{C}^n \to \mathbb{C}$ by the formula

$$\pi_k(v) = v_k, \qquad v = (v_1, \ldots, v_n) \in \mathbb{C}^n.$$

Letting $s = \pi_l^* \pi_k$ in (3.17) and applying to v and taking the inner product with w, where v and w are in $\mathbb{C}^n$, leads to

$$\begin{aligned} & \langle [\pi_l^* \pi_k - p_0(y)^* \pi_l^* \pi_k p_0(x)] v, w \rangle \;= \\ & \quad \langle [\pi_l^* \pi_k \otimes \mathrm{id} - (\delta(y)^* \otimes \mathrm{id})(\pi_l^* \pi_k \otimes \mathrm{id})(\delta(x) \otimes \mathrm{id})] u(x) v, u(y) w \rangle. \end{aligned} \tag{3.18}$$

For each $v \in \mathbb{C}^n$ define vectors p_v and q_v in $\mathbb{C}^{n(1+NJ)}$ by

$$\begin{aligned} p_v &= \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \mathrm{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} v \\ q_v &= \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix} v. \end{aligned}$$

For each $1 \leq k \leq n$, define vectors $p_{k,v}$ and $q_{k,v}$ in $\mathbb{C}^{1+NJ}$ by

$$\begin{aligned} p_{k,v} &= [\pi_k \otimes \mathrm{id}_{\mathbb{C}^{1+NJ}}] p_v \\ q_{k,v} &= [\pi_k \otimes \mathrm{id}_{\mathbb{C}^{1+NJ}}] q_v. \end{aligned}$$

Then (3.18), with Λ in place of both x and y, becomes

$$\langle p_{k,v}, p_{l,w} \rangle \ = \ \langle q_{k,v}, q_{l,w} \rangle \qquad \forall v, w \in \mathbb{C}^n, \ \forall 1 \leq k, l \leq n. \tag{3.19}$$

So by (3.19), there is an isometry V that maps each $p_{k,v}$ to $q_{k,v}$. If the span of the vectors $\{p_{k,v}\}$ is not all of $\mathbb{C}^{1+NJ}$, we can extend V to the orthocomplement so that it becomes an isometry (indeed, a unitary) from all of $\mathbb{C}^{1+NJ}$ to $\mathbb{C}^{1+NJ}$.

With respect to the decomposition $\mathbb{C} \oplus \mathbb{C}^{JN}$, write

$$V \ = \ \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} [\pi_k \otimes \mathrm{id}_{\mathbb{C}^{1+NJ}}] \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \mathrm{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} \ = \ [\pi_k \otimes \mathrm{id}_{\mathbb{C}^{1+NJ}}] \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \tag{3.20}$$

Since (3.20) holds for each k, we get that

$$\begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \otimes A & \mathrm{id}_{\mathbb{C}^n} \otimes B \\ \mathrm{id}_{\mathbb{C}^n} \otimes C & \mathrm{id}_{\mathbb{C}^n} \otimes D \end{bmatrix} \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \mathrm{id}_{\mathbb{C}^N}] u(\Lambda) \end{bmatrix} \ = \ \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \tag{3.21}$$

For x in $G_\delta \cap \mathbb{M}_n^d$, define

$$\begin{aligned} \phi(x) \ = \ & \mathrm{id}_{\mathbb{C}^n} \otimes A \ + \ [(\mathrm{id}_{\mathbb{C}^n} \otimes B)(\delta(x) \otimes \mathrm{id}_{\mathbb{C}^N})] \\ & [\mathrm{id}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathbb{C}^{JN}} - (\mathrm{id}_{\mathbb{C}^n} \otimes D)(\delta(x) \otimes \mathrm{id}_{\mathbb{C}^N})]^{-1} \, \mathrm{id}_{\mathbb{C}^n} \otimes C. \end{aligned}$$

Then ϕ is in the unit ball of $H^\infty(G_\delta)$ by Theorem 2.3. Moreover, by(3.21),

$$\phi(\Lambda) \ = \ p_0(\Lambda),$$

as desired. □

4 The Nevanlinna Problem

There are two sources of non-uniqueness in the solution of the Pick interpolation problem. The first is the choice of u in (3.16); the second is in the extension of V. This problem has been analyzed in the scalar case by J. Ball, T. Trent and V. Vinnikov [6]; their ideas extend to our situation.

Let us suppose throughout this section that

$$\Lambda \mapsto p_0(\Lambda) \tag{4.1}$$

is a solvable Pick problem, and we have found a finite-dimensional space $\mathcal{L}$, an $\mathcal{L}(\mathbb{C}, \mathbb{C}^J \otimes \mathcal{L})$-valued nc polynomial u satisfying

$$\mathrm{id}_{\mathbb{C}^n} - p_0(\Lambda)^* p_0(\Lambda) \ = \ u(\Lambda)^* \left[\mathrm{id}_{\mathbb{C}^{nJ}\otimes\mathcal{L}} - (\delta(\Lambda)^*\delta(\Lambda) \otimes \mathrm{id}_{\mathcal{L}})\right] u(\Lambda), \tag{4.2}$$

and V satisfying (3.20):

$$V[\pi_k \otimes \mathrm{id}_{\mathbb{C}\oplus\mathbb{C}^J\otimes\mathcal{L}}] \begin{bmatrix} \mathrm{id}_{\mathbb{C}^n} \\ [\delta(\Lambda) \otimes \mathrm{id}_{\mathcal{L}}]u(\Lambda) \end{bmatrix} \ = \ [\pi_k \otimes \mathrm{id}_{\mathbb{C}\oplus\mathbb{C}^J\otimes\mathcal{L}}] \begin{bmatrix} p_0(\Lambda) \\ u(\Lambda) \end{bmatrix}. \tag{4.3}$$

Let $\mathcal{L}_0 = \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, and

$$\mathcal{N}_2 \ := \ \vee_{k=1}^n \vee_{v\in\mathbb{C}^n} \begin{bmatrix} \pi_k\, v \\ [(\pi_k \otimes \mathrm{id}_{\mathbb{C}^J\otimes\mathcal{L}})(\delta(\Lambda) \otimes \mathrm{id}_{\mathbb{C}^N})]u(\Lambda)\, v \end{bmatrix} \subseteq \mathcal{L}_0.$$

Let

$$\mathcal{N}_1 \ := \ \vee_{k=1}^n \vee_{v\in\mathbb{C}^n} \begin{bmatrix} \pi_k p_0(\Lambda)\, v \\ (\pi_k \otimes \mathrm{id}_{\mathbb{C}^J\otimes\mathcal{L}})u(\Lambda)\, v \end{bmatrix} \subseteq \mathcal{L}_0,$$

and define $\mathcal{M}_2 = \mathcal{L}_0 \ominus \mathcal{N}_2$ and $\mathcal{M}_1 = \mathcal{L}_0 \ominus \mathcal{N}_1$. Then V is an isometry from $\mathcal{N}_2$ onto $\mathcal{N}_1$. Define a unitary

$$\begin{aligned} U : \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2 \quad &\to \quad \mathcal{M}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \\ \begin{bmatrix} m_1 \\ m_2 \\ n_2 \end{bmatrix} \quad &\mapsto \quad \begin{bmatrix} m_2 \\ m_1 \\ V n_2 \end{bmatrix}. \end{aligned} \tag{4.4}$$

By identifying $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2 \cong \mathcal{M}_1 \oplus \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$ and $\mathcal{M}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_1 \cong \mathcal{M}_2 \oplus \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, we can think of U as a unitary from $\mathbb{C} \oplus \mathcal{M}_1 \oplus \mathbb{C}^J \otimes \mathcal{L}$ to $\mathbb{C} \oplus \mathcal{M}_2 \oplus \mathbb{C}^J \otimes \mathcal{L}$, and it has a corresponding transfer function G that is a free $\mathcal{L}(\mathbb{C} \oplus \mathcal{M}_1, \mathbb{C} \oplus \mathcal{M}_2)$-valued rational function (since all the spaces are finite dimensional). Write this G as

$$G \ = \ \begin{array}{c} \\ \mathbb{C} \\ \mathcal{M}_2 \end{array} \begin{array}{c} \quad\mathbb{C} \qquad \mathcal{M}_1 \\ \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \end{array}. \tag{4.5}$$

Theorem 4.6. The function ϕ in $\mathrm{ball}(H^\infty(G_\delta))$ satisfies $\phi(\Lambda) = p_0(\Lambda)$ if and only if, for some u satisfying (4.2) and G the transfer function of U in (4.4), there is a function Θ in $\mathrm{ball}(H^\infty_{\mathcal{L}(\mathcal{M}_1,\mathcal{M}_2)}(G_\delta))$ such that

$$\phi \;=\; G_{11} + G_{12}\Theta(I_{\mathcal{M}_1} - G_{22}\Theta)^{-1}G_{21}. \tag{4.7}$$

PROOF: ($\Leftarrow$) This is a straightforward calculation.

($\Rightarrow$) By Theorem 2.3, ϕ has a free δ-realization, and by Lemma 3.4, we can assume that $\{u(x) : x \in V_\Lambda\}$ lie in a finite dimensional space that we can embed in $\mathbb{C}^J \otimes \mathcal{L}$. So we can assume that ϕ is the transfer function of some unitary $X : \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{K} \to \mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{K}$, and that $\mathcal{L} \subseteq \mathcal{K}$. For $x \in G_\delta \cap \mathbb{M}^d_m$ we have

$$[\mathrm{id}_{\mathbb{C}^m} \otimes X] \begin{bmatrix} \mathrm{id}_{\mathbb{C}^m} \\ (\delta(x) \otimes \mathrm{id}_{\mathcal{K}})\xi(x) \end{bmatrix} \;=\; \begin{bmatrix} \phi(x) \\ \xi(x) \end{bmatrix}. \tag{4.8}$$

Let $\mathcal{K}' = \mathcal{K} \ominus \mathcal{L}$. Then

$$X \;=\; \begin{array}{c} \\ \mathcal{N}_1 \\ \mathcal{M}_1 \oplus \mathbb{C}^J \otimes \mathcal{K}' \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{N}_2 & \mathcal{M}_2 \oplus \mathbb{C}^J \otimes \mathcal{K}' \end{array} \\ \begin{pmatrix} V & 0 \\ 0 & Y \end{pmatrix} \end{array}. \tag{4.9}$$

Let Θ be the transfer function of Y. Then we claim that (4.7) holds.

Let $x \in G_\delta \cap \mathbb{M}^d_m$ and $v \in \mathbb{C}^m$ be fixed for now. Let

$$\begin{aligned} p \;&=\; v \oplus (\delta(x) \otimes \mathrm{id}_{\mathcal{K}})\xi(x)v \;=\; n_2 \oplus m_2 \oplus h_2 \\ q \;&=\; \phi(x)v \oplus \xi(x)v \;=\; n_1 \oplus m_1 \oplus h_1 \end{aligned} \tag{4.10}$$

where $n_2 \in \mathbb{C}^m \otimes \mathcal{N}_2, m_2 \in \mathbb{C}^m \otimes \mathcal{M}_2, n_1 \in \mathbb{C}^m \otimes \mathcal{N}_1, m_1 \in \mathbb{C}^m \otimes \mathcal{M}_1$ and $h_2, h_1 \in \mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{K}'$. Note from (4.8) that

$$[\mathrm{id}_{\mathbb{C}^m} \otimes X]p \;=\; q. \tag{4.11}$$

Let P' be the projection from $\mathbb{C}^J \otimes \mathcal{K}$ to $\mathbb{C}^J \otimes \mathcal{K}'$. As $\delta(x) \otimes \mathrm{id}_{\mathcal{K}}$ commutes with $\mathrm{id}_{\mathbb{C}^m} \otimes P'$, we get from (4.10) that

$$[\delta(x) \otimes \mathrm{id}_{\mathcal{K}'}]h_1 \;=\; h_2.$$

Therefore

$$[\mathrm{id}_{\mathbb{C}^m} \otimes Y](m_2 \oplus [\delta(x) \otimes \mathrm{id}_{\mathcal{K}'}]h_1) \;=\; m_1 \oplus h_1. \tag{4.12}$$

As Θ is the transfer function of Y, (4.12) implies that

$$\Theta(x)\, m_2 \;=\; m_1. \tag{4.13}$$

Let P be the projection from $\mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{K}$ onto $\mathbb{C}^m \otimes \mathbb{C}^J \otimes \mathcal{L}$, and let $\eta = P\xi(x)v$. Then under the identifications of $\mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{N}_2 \oplus \mathcal{M}_2$ with $\mathbb{C} \oplus \mathbb{C}^J \otimes \mathcal{L}$, we get

$$\begin{aligned} n_1 \oplus m_1 &= \phi(x)v \oplus \eta \\ n_2 \oplus m_2 &= v \oplus (\delta(x) \otimes \mathrm{id}_{\mathcal{L}})\eta. \end{aligned}$$

Then from (4.4)

$$U : \ v \oplus m_1 \oplus (\delta(x) \otimes \mathrm{id}_{\mathcal{L}})\eta \ \mapsto \ \phi(x)v \oplus m_2 \oplus \eta.$$

By (4.13) this gives

$$\begin{pmatrix} G_{11}(x) & G_{12}(x) \\ G_{21}(x) & G_{22}(x) \end{pmatrix} \begin{pmatrix} v \\ \Theta(x)m_2 \end{pmatrix} = \begin{pmatrix} \phi(x)v \\ m_2 \end{pmatrix}. \tag{4.14}$$

As (4.14) holds for all choices of x and v, we get (4.7), as desired. □

5 Extending functions defined on varieties

Proof of Theorem 1.5: Without loss of generality, assume that

$$\sup_{x \in \mathfrak{V} \cap G_\delta} \|f(x)\| = 1. \tag{5.1}$$

Choose a sequence $(\lambda_j)_{j=1}^\infty$ in $G_\delta \cap \mathfrak{V}$ that is dense in the disjoint union topology, so for all $\varepsilon > 0$, for all $x \in G_\delta \cap \mathfrak{V}$, there exists some λ_j such that $\max_{1 \le r \le d} \|\lambda_j^r - x^r\| < \varepsilon$.

Let $\Lambda_n = \oplus_{j=1}^n \lambda_j$. By Theorem 2.1, f is locally approximable by polynomials, and so has the property that

$$\forall x \in \mathfrak{V} \cap G_\delta, \ f(x) \in \mathbb{M}_x.$$

Therefore there is some polynomial $p_n \in \mathbb{P}^d$ such that

$$p_n(\Lambda_n) = f(\Lambda_n). \tag{5.2}$$

Moreover, if $x \in \mathfrak{V} \cap G_\delta$, then by Theorem 2.1 again, one can approximate f at $x \oplus \Lambda_n$ by a sequence of free polynomials, and so by Lemma 3.1

$$\forall x \in \mathfrak{V} \cap G_\delta, \ f(x) = p_n(x). \tag{5.3}$$

As $V_\Lambda \subseteq \mathfrak{V}$, putting (5.2), (5.3) and (5.1) together, the hypotheses of Theorem 1.3 are satisfied, so there exists ϕ_n in $H_1^\infty(G_\delta)$ such that

$$\phi_n(\Lambda_n) \;=\; f(\Lambda_n).$$

By Theorem 2.4, some subsequence of ϕ_n converges locally uniformly (in the disjoint union topology) to a function ϕ in $H_1^\infty(G_\delta)$. Moreover, for each j, $\phi(\lambda_j) = f(\lambda_j)$, so by continuity, ϕ is an extension of f. □

6 Examples

Example 6.1 Let $q_1, \ldots, q_m$ be polynomials in d commuting variables, and let $V = \{z \in \mathbb{C}^d : q_i(z) = 0, i = 1, \ldots, m\}$. Let f be a (scalar-valued) holomorphic function defined on $V \cap \mathbb{D}^d$.

Let T be a d-tuple of commuting matrices that are strict contractions, and such that $q_i(T) = 0$ for $i = 1, \ldots, m$. If they are simultaneously diagonizable, then their joint eigenvalues lie in $V \cap \mathbb{D}^d$, and it makes sense to define $f(T)$ by applying f to the diagonal entries, in the basis of joint eigenvectors. If the matrices are not simultaneously diagonizable, then one can still define $f(T)$, either by the Taylor functional calculus [24], or, more constructively, as in [2].

Let us write $\mathcal{F}$ for the set of all $T = (T^1, \ldots, T^d)$ of commuting matrices such that $q_i(T) = 0, i = 1, \ldots, m$, and such that $\|T\| < 1$. Note that $\mathcal{F} = \mathfrak{V} \cap G_\delta$, where $\mathfrak{A}$ is the algebra generated by $q_1, \ldots, q_m$ and the polynomials $\{x^i x^j - x^j x^i : 1 \leq i < j \leq d\}$, $\mathfrak{V} = \mathrm{Var}(\mathfrak{A})$, and $\delta(x)$ is the diagonal matrix with entries $x^1, x^2, \ldots, x^d$. Define a norm on holomorphic functions on $V \cap \mathbb{D}^d$ by

$$\|f\|_{\mathfrak{V} \cap G_\delta} \;=\; \sup\{\|f(T)\| : T \in \mathcal{F}\}.$$

To apply Theorem 1.5, we need to know that f is a free holomorphic function on $\mathfrak{V} \cap G_\delta$, in other words that locally in G_δ it extends to a free holomorphic function (*i.e.* it can be applied to non-commuting matrices). This is true, and is proved in [2]. Then Theorem 1.5 asserts that there is a bounded extension ϕ of f, defined on the set $\{R \in \mathbb{M}^d : \|R\| < 1\}$, if and only if $\|f\|_{\mathfrak{V} \cap G_\delta}$ is finite. Moreover, if this quantity is finite, then ϕ can be found with exactly this norm. In particular, an extension to the non-commuting ball G_δ can always be found with the same norm as is attained by evaluating on commuting matrices in the variety.

Example 6.2 Specializing the previous example to the case $d = 2$, and using Andô's inequality [5], we conclude the following: if we wish to extend a polynomial p_0 off $V \cap \mathbb{D}^2$, where V is the joint sero set of the q_i's, then the minimum norm of the extension ϕ is the same when calculated as a scalar-valued function in $H^\infty(\mathbb{D}^2)$, as a function on pairs of commuting contractive matrices, or as a function on pairs of contractive matrices. The norm is attained, and is given by

$$\sup_{n \in \mathbb{N}} \sup\{\|p_0(T)\| \ : \ T \in \mathbb{M}_n^2, \ \|T^1\| < 1, \|T^2\| < 1, \\ T^1 T^2 = T^2 T^1, \ q_i(T) = 0 \, \forall \, 1 \leq i \leq m\}. \tag{6.3}$$

Unless $V \cap \mathbb{D}^2$ is a retract of $\mathbb{D}^2$, one can by [3] always find some p_0 so that (6.3) is strictly greater than

$$\sup\{|p_0(z)| \ : \ z \in \mathbb{D}^2 \cap V\}.$$

Example 6.4 Suppose $\delta(x)$ has first column $x^1, \ldots, x^d$ and its other entries zero, so

$$G_\delta \ = \ \{T \ : \ T^{1*}T^1 + \cdots T^{d*}T^d < 1\}.$$

(This is called the row ball). Suppose $\Lambda, H \in G_\delta \cap \mathbb{M}_n^d$ and one wishes to solve the interpolation problem

$$\begin{aligned} \phi(\Lambda) &= W \\ D\phi(\Lambda)[H] &= X, \end{aligned} \tag{6.5}$$

where $D\phi(\Lambda)[H]$, the derivative of ϕ at Λ in the direction H, is defined by

$$D\phi(\Lambda)[H] \ = \ \lim_{t \to 0} \frac{\phi(\Lambda + tH) - \phi(\Lambda)}{t}.$$

A necessary condition to find a function $\phi \in H^\infty(G_\delta)$ solving this problem is that there is some free polynomial p_0 with $p_0(\Lambda) = W$ and $Dp_0(\Lambda)[H] = X$. The minimum norm of a solution can be found from Theorem 1.5 by letting

$$\mathfrak{A} \ = \ \{p \in \mathbb{P}^d \ : \ p(\Lambda) = 0, \ Dp(\Lambda)[H] = 0\},$$

$\mathfrak{V} = \mathrm{Var}(\mathfrak{A})$, and calculating

$$\sup_{x \in \mathfrak{V} \cap G_\delta} \|p_0(x)\|.$$

The problem can also be solved using Theorem 1.3, as (6.5) is the same as solving the one point problem

$$\begin{pmatrix} \Lambda & H \\ 0 & \Lambda \end{pmatrix} \mapsto \begin{pmatrix} W & X \\ 0 & W \end{pmatrix},$$

since by [12, Prop 2.5], for any continuous nc-function f, one has

$$f \begin{pmatrix} \Lambda & H \\ 0 & \Lambda \end{pmatrix} = \begin{pmatrix} f(\Lambda) & Df(\Lambda)[H] \\ 0 & f(\Lambda) \end{pmatrix}.$$

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