7-27-2014

Thin sequences and the Gram matrix

Pamela Gorkin

John E. McCarthy
Washington University in St Louis, mccarthy@wustl.edu

Sandra Pott

Brett D. Wick

THIN SEQUENCES AND THE GRAM MATRIX

PAMELA GORKIN, JOHN E. MCCARTHY, SANDRA POTT, AND BRETT D. WICK

Abstract. We provide a new proof of Volberg's Theorem characterizing thin interpolating sequences as those for which the Gram matrix associated to the normalized reproducing kernels is a compact perturbation of the identity. In the same paper, Volberg characterized sequences for which the Gram matrix is a compact perturbation of a unitary as well as those for which the Gram matrix is a Schatten-2 class perturbation of a unitary operator. We extend this characterization from 2 to p, where $2 \leq p \leq \infty$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ the unit circle. Given $\{\alpha_j\}$, a Blaschke sequence of points in $\mathbb{D}$, we let B denote the corresponding Blaschke product and B_n denote the Blaschke product with the zero α_n removed. Further, we let $\delta_j = |B_j(\alpha_j)|$, $k_j = \frac{1}{1-\overline{\alpha_j}z}$ denote the Szegő kernel (the reproducing kernel for H^2) at α_j, $g_j = k_j/\|k_j\|$ the H^2-normalized kernel, and G the Gram matrix with entries $G_{ij} = \langle g_j, g_i\rangle$. In the second part of [10, Theorem 2], Volberg's goal was to develop a condition ensuring that $\{g_n\}$ is near an orthogonal basis; by this, one means that there exist U unitary and K compact such that

$$g_n = (U + K)e_n,$$

where $\{e_n\}$ is the standard orthogonal basis for ℓ^2. By [7, Section 3] or [4, Proposition 3.2], this is equivalent to the Gram matrix defining a bounded operator of the form $I + K$ with K compact. Following Volberg and anticipating the connection to the Schatten-p classes, we call such bases $\mathcal{U} + \mathcal{S}_\infty$ bases. Volberg showed that $\{g_n\}$ is a $\mathcal{U} + \mathcal{S}_\infty$ basis if and only if $\lim_n \delta_n = 1$; in other words, if and only if $\{\alpha_n\}$ is a thin sequence. Assuming $\{g_n\}$ is a $\mathcal{U}+\mathcal{S}_\infty$ basis, it is not difficult to show that the sequence $\{\alpha_n\}$ must be thin. But Volberg's proof of the converse is more difficult and depends on the main lemma of a paper of Axler, Chang and Sarason [2, Lemma 5], estimating the norm of a certain product of Hankel operators as well as a factorization theorem for Blaschke products. The lemma in [2] uses maximal functions and a certain distribution function inequality. A more direct proof of Volberg's result is desirable, and we provide a simpler proof of this result in Theorem 3.5 of this paper.

In a second theorem, letting $\mathcal{S}_2$ denote the class of Hilbert-Schmidt operators, Volberg showed (see [10, Theorem 3]) that $\{g_n\}$ is a $\mathcal{U}+\mathcal{S}_2$ basis if and only if $\prod_{n=1}^\infty \delta_n$ converges. We are interested in estimates for the "in-between" cases. We provide a new proof of Volberg's theorem for $p = \infty$ and prove the following theorem.

Theorem 1.1. *For $2 \leq p < \infty$, the operator $G - I \in \mathcal{S}_p$ if and only if $\sum_n (1 - \delta_n^2)^{p/2} < \infty$.*

Volberg's theorem covered the cases $p = 2$ and $p = \infty$, but our proofs differ in the following way: Instead of using the results of [2] and theorems about Hankel operators, we use the

P.G. Partially supported by Simons Foundation Grant 243653.
J.M. Partially supported by National Science Foundation Grant DMS 1300280.
B.D.W Partially supported by National Science Foundation Grant DMS 0955432.

relationship between growth estimates of functions that do interpolation on thin sequences (see [5], [6]) and the norm of the Gram matrix. This simplifies previous proofs and provides the best estimates available.

2. Preliminaries and notation

Let $\{\alpha_j\}$ be a sequence in $\mathbb{D}$ with corresponding Blaschke product B, and B_j be the Blaschke product with zeroes at every point in the sequence except α_j and $\delta_j = |B_j(\alpha_j)|$. The separation constant δ is defined to be $\delta := \inf_j \delta_j$. Carleson's interpolation theorem says that the sequence $\{\alpha_j\}$ is interpolating if and only if $\delta > 0$, [3]. The sequence $\{\alpha_j\}$ is said to be *thin* if $\lim_{j\to\infty} \delta_j = 1$. Given a thin sequence we may arrange the δ_j in increasing order and rearrange the zeros of the Blaschke product accordingly.

Recall that if T is an operator on a Hilbert space $\mathcal{H}$ and λ_n is the nth singular value of T, then given p with $1 \leq p < \infty$ the Schatten-p class, $\mathcal{S}_p$, is defined to be the space of all compact operators with corresponding singular sequence in ℓ^p, the space of p-summable sequences. Then $\mathcal{S}_p$ is a Banach space with norm

$$\|T\|_p = \left(\sum |\lambda_n|^p\right)^{1/p}.$$

For $p = \infty$, we let $\mathcal{S}_\infty$ denote the space of compact operators.

Recall that k_j denotes the Szegő kernel, $g_j = k_j/\|k_j\|$, and G the Gram matrix with entries $G_{ij} = \langle g_j, g_i\rangle$. (The Gram matrix depends of course on the sequence $\{\alpha_j\}$, but we suppress this in the notation). For $\{\alpha_j\}$ interpolating, we let D be the diagonal matrix with entries $1/B_j(\alpha_j)$. It is known (see, for example, formula (26) of [8]) that

$$G^{-1} \;=\; D^* G^t D. \tag{2.1}$$

For a given sequence $\{\alpha_j\}$, the interpolation constant is the infimum of those M such that for any sequence $\{a_j\}$ in ℓ^∞, one can find a function f in H^∞ with $f(\alpha_j) = a_j$ and $\|f\|_\infty \leq M\, \|a\|_{\ell^\infty}$. We shall let $M(\delta)$ denote the supremum of the interpolation constants over all sequences $\{\alpha_j\}$ with separation constant δ.

The following result is due essentially to A. Shields and H. Shapiro [9]. See [1, Proposition 9.5] for a proof of this version.

Proposition 2.1. *Let $\{\alpha_j\}$ be an interpolating sequence in $\mathbb{D}$.*
(i) If the interpolation constant is M, then both $\|G\|$ and $\|G^{-1}\|$ are bounded by M^2.
(ii) If $\|G\| = C_1$ and $\|G^{-1}\| = C_2$, then the interpolation constant is bounded by $\sqrt{C_1 C_2}$.

We shall use the following estimate of J.P. Earl (see [5] or [6]) to obtain our results.

Theorem 2.2 (Earl's Theorem)**.** *The interpolation constant $M(\delta)$ satisfies*

$$M(\delta) \leq \left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^2.$$

3. Schatten-p classes

In this section we provide estimates on the Schatten-p norm of $G - I$. We will need the theorem and lemma below.

Theorem 3.1. *(see e.g.* [11, Theorem 1.33]*) Let T be an operator on a separable Hilbert space, $\mathcal{H}$.*

If $0 < p \le 2$ then

$$\|T\|_{\mathcal{S}_p}^p = \inf\left\{\sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H}\right\}$$

and if $2 \le p < \infty$

$$\|T\|_{\mathcal{S}_p}^p = \sup\left\{\sum_n \|Te_n\|^p : \{e_n\} \text{ is any orthonormal basis in } \mathcal{H}\right\}.$$

We say that two sequences $\{x_n\}$ and $\{y_n\}$ of positive numbers are equivalent if there exist constants c and C, independent of n, such that $cy_n \le x_n \le Cy_n$ for all n. We write $x_n \asymp y_n$. We will also write $A \lesssim B$ to indicate that there exists a constant C such that $A \le CB$.

Lemma 3.2. *Let $\{e_n\}$ denote the standard orthonormal basis for ℓ^2 and $\{\alpha_j\}$ be an interpolating sequence in $\mathbb{D}$ with corresponding δ_j. Then*

$$\|(G-I)e_n\| \asymp \sqrt{1-\delta_n^2}.$$

Proof. We have

$$\begin{aligned}
\|(G-I)e_n\|^2 &= \langle (G^*-I)(G-I)e_n, e_n\rangle \\
&= \left\langle \begin{pmatrix} \langle g_n, g_1\rangle \\ \cdots \\ \langle g_n, g_{n-1}\rangle \\ 0 \\ \langle g_n, g_{n+1}\rangle \\ \cdots \end{pmatrix}, \begin{pmatrix} \langle g_n, g_1\rangle \\ \cdots \\ \langle g_n, g_{n-1}\rangle \\ 0 \\ \langle g_n, g_{n+1}\rangle \\ \cdots \end{pmatrix} \right\rangle \\
&= \sum_{j\neq n} \langle g_n, g_j\rangle \overline{\langle g_n, g_j\rangle} \\
&= \sum_{j\neq n} \left| \frac{\sqrt{1-|\alpha_j|^2}\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha_n}\alpha_j} \right|^2 \\
&= \sum_{j\neq n} 1 - \left| \frac{\alpha_j - \alpha_n}{1-\overline{\alpha_n}\alpha_j} \right|^2 .
\end{aligned}$$

But $-\log x \ge 1-x$ for $x > 0$ and $-\log x < c(1-x)$ for $x < 1$ bounded away from 0 and some constant c independent of x, so $-\log x \asymp 1-x$ for x bounded away from 0. Consequently,

$$\begin{aligned}
\|(G-I)e_n\|^2 &\asymp \sum_{j\neq n} -\log \left| \frac{\alpha_j - \alpha_n}{1-\overline{\alpha_n}\alpha_j} \right|^2 \\
&= -\log \prod_{j\neq n} \left| \frac{\alpha_j - \alpha_n}{1-\overline{\alpha_n}\alpha_j} \right|^2 \\
&= -\log \delta_n^2 \\
&\asymp 1-\delta_n^2.
\end{aligned}$$

Note that the constants involved do not depend on n. □

Combining the lemma with Theorem 3.1, we obtain the following theorem.

Theorem 3.3. *The following estimates hold:*

- *If* $2 \le p < \infty$ *then*
$$\sum_n (1-\delta_n)^{\frac{p}{2}} \lesssim \|G-I\|_{\mathcal{S}_p}^p\,;$$
- *If* $0 < p \le 2$ *then*
$$\|G-I\|_{\mathcal{S}_p}^p \lesssim \sum_n (1-\delta_n)^{\frac{p}{2}};$$
- *If* $p = 2$ *then*
$$\sum_n (1-\delta_n) \asymp \|G-I\|_{\mathcal{S}_2}^2\,.$$

Lemma 3.4. *Let* $\{\alpha_j\}$ *be an interpolating sequence and* G *the corresponding Gram matrix. Let* $C = \|G^{-1}\|$. *Then* $\|G-I\| \le C-1$.

Proof. By (2.1), we have $G \le CI$, and as G is a positive operator, we have $G \ge (1/C)I$. Therefore
$$\left(\frac{1}{C}-1\right) I \;\le\; G-I \;\le\; (C-1)I,$$
and as $C > 1$, we get $\|G-I\| \le C-1$. □

In what follows, for a positive integer N, we let G_N denote the lower right-hand corner of the Gram matrix obtained by deleting the first N rows and columns of G. Thus,

$$G_N = \begin{pmatrix} 1 & \langle g_{N+1}, g_N\rangle & \cdots & \langle g_{N+j}, g_N\rangle & \cdots \\ \langle g_N, g_{N+1}\rangle & 1 & \cdots & \langle g_{N+j}, g_{N+1}\rangle & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and $\lambda_n \ge 0$ denotes the n-th singular value of $G-I$, where the singular values are arranged in decreasing order.

We are now ready to provide our simpler proof of Volberg's result [10, Theorem 2, p. 215].

Theorem 3.5. *The sequence* $\{\alpha_n\}$ *is a thin sequence if and only if the Gram matrix* G *is the identity plus a compact operator.*

Proof. ($\Rightarrow$) Suppose $\{\alpha_n\}$ is thin. By discarding finitely many points in the sequence (α_n), we can assume that the sequence has a positive separation constant, and hence is interpolating.

Let G_N be the Gram matrix of $\{g_j\}$ for $j \ge N$. We shall let $\delta_{N,j}$ denote the δ_j defined for G_N (that is, corresponding to the Blaschke sequence $\{\alpha_j\}_{j\ge N}$). Note that $\delta_{N,j} \ge \delta_{N+j}$ for $j = 0, 1, 2, \ldots$, and so we have that $\delta(N) := \inf_j \delta_{N,j} \ge \inf_j \delta_{N+j} = \delta_N$. By Theorem 2.2 and Proposition 2.1,
$$\|G_N^{-1}\| \;\le\; (M(\delta(N))^2 \;\le\; \frac{\left(1+\sqrt{1-\delta(N)}\right)^4}{\delta(N)^4} \;\le\; \frac{\left(1+\sqrt{1-\delta_N^2}\right)^4}{\delta_N^4}.$$

Applying Lemma 3.4

$$\|G_N - I_N\| \leq \left(\frac{\left(1+\sqrt{1-\delta_N^2}\right)^4}{\delta_N^4} - 1 \right) \leq C\sqrt{1-\delta_N},$$

where C is a constant independent of N. Since $\sqrt{1-\delta_N} \to 0$ as $N \to \infty$, we conclude that $G - I$ is compact.

($\Leftarrow$) From (2.1), we have

$$G^{-1} - I = D^*(G^t - I)D + [D^*D - I]. \tag{3.1}$$

If $G - I$ is compact, then so are $G^t - I$ and $G^{-1} - I = G^{-1}(I - G)$. Therefore from (3.1), we have $D^*D - I$ is compact, which means $\lim_{j\to\infty} \delta_j^2 = 1$. Consequently, the sequence is thin. □

Theorem 3.6. *For $2 \leq p < \infty$, the operator $G - I \in \mathcal{S}_p$ if and only if $\sum_n (1-\delta_n^2)^{p/2} < \infty$.*

Proof. By Theorem 3.3, if $G - I \in \mathcal{S}_p$, then the sum is finite.

Now suppose the sum is finite. Using Lemma 3.4 as in Theorem 3.5, we have

$$\|G_N - I_N\| \leq C\sqrt{1-\delta_N},$$

where C is independent of N.

By [11, Theorem 1.4.11],

$$|\lambda_{N+1}| \leq \inf\{\|(G - I) - F\| : F \in \mathcal{F}_N\},$$

where $\mathcal{F}_N$ is the set of all operators of rank less than or equal to N. Therefore, taking F to be the matrix with the same first N rows and columns as $G - I$, which is of rank at most $2N$, we have

$$|\lambda_{2N+1}| \leq \|G_{N+1} - I_{N+1}\| \leq C\sqrt{1-\delta_{N+1}},$$

by our computation above. Therefore

$$|\lambda_{2N+1}|^p \leq C^p(1-\delta_{N+1})^{p/2}.$$

Since the singular values are arranged in decreasing order, $|\lambda_{2n+1}| > |\lambda_{2n}|$ for each n. Thus, if $\sum_N (1-\delta_N)^{p/2} < \infty$, then $\sum_n |\lambda_{2n}|^p \leq 2\sum_n |\lambda_{2n+1}|^p < \infty$ and we conclude that $G - I \in \mathcal{S}_p$. □

We conclude by remarking that it is possible to trace through the proofs above to determine constants c and C , which depend only on $\delta = \inf_n \delta_n$, such that for $2 \leq p \leq \infty$,

$$c\|\sqrt{1-\delta_n}\|_{\ell^p} \;\leq\; \|G - I\|_{\mathcal{S}_p} \;\leq\; C\|\sqrt{1-\delta_n}\|_{\ell^p}. \tag{3.2}$$

In particular, by choosing δ close enough to 1, one can choose c and C in (3.2) arbitrarily close to $\sqrt{2}$ and $4\sqrt{2}(2^{1/p})$, respectively.

Question 3.7. *Is Theorem 3.6 true for $p < 2$?*

Acknowledgement. We thank the referee for his or her careful reading of this manuscript as well as helpful comments.

References

[1] J. Agler and J.E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, American Mathematical Society, Providence, 2002. ↑2

[2] Sheldon Axler, Sun-Yung A. Chang, and Donald Sarason, *Products of Toeplitz operators*, Integral Equations Operator Theory **1** (1978), no. 3, 285–309. ↑1, 2

[3] L. Carleson, *An interpolation problem for bounded analytic functions*, American J. Math. **80** (1958), 921–930. ↑2

[4] I. Chalendar, E. Fricain, and D. Timotin, *Functional models and asymptotically orthonormal sequences*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 5, 1527–1549. ↑1

[5] J. P Earl, *On the interpolation of bounded sequences by bounded functions*, J. London Math. Soc. **2** (1970), 544–548. ↑2

[6] J. P. Earl, *A note on bounded interpolation in the unit disc*, J. London Math. Soc. (2) **13** (1976), no. 3, 419–423. ↑2

[7] Emmanuel Fricain, *Bases of reproducing kernels in model spaces*, J. Operator Theory **46** (2001), no. 3, suppl., 517–543. ↑1

[8] Paul Koosis, *Carleson's interpolation theorem deduced from a result of Pick*, Complex analysis, operators, and related topics, Oper. Theory Adv. Appl., 113, Birkhäuser, Basel, 2000, pp. 151–162. ↑2

[9] H.S. Shapiro and A.L. Shields, *On some interpolation problems for analytic functions*, American J. Math. **83** (1961), 513–532. ↑2

[10] A. L. Vol′berg, *Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason*, J. Operator Theory **7** (1982), no. 2, 209–218. ↑1, 4

[11] Kehe Zhu, *Operator theory in function spaces*, Second, Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007. ↑3, 5

Pamela Gorkin, Department of Mathematics, Bucknell University, Lewisburg, PA USA 17837

E-mail address: pgorkin@bucknell.edu

John E. McCarthy, Dept. of Mathematics, Washington University in St. Louis, St. Louis, MO 63130

E-mail address: mccarthy@wustl.edu

Sandra Pott, Faculty of Science, Centre for Mathematical Sciences, Lund University, 22100 Lund, Sweden

E-mail address: sandra@maths.lth.se

Brett D. Wick, School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA USA 30332-0160

E-mail address: wick@math.gatech.edu